Types in locally o-minimal structures

前園久智 (Hisatomo Maesono)
早稲田大学グローバルエデュケーションセンター
(Global Education Center, Waseda University)

概要

abstract Locally o-minimal structures are some local adaptation from o-minimal ones. These structures were treated, e.g. in [1], [2]. O-minimal structures have been studied extensively, in particular, they are characterized by means of behavior of types. We try analogous argument in locally o-minimal structures.

1. Introduction

We recall some definitions and fundamental results at first.

Definition 1 A linearly ordered structure $M = (M, <, \cdots)$ is $o-minimal$ if every definable subset of M^1 is a finite union of points and intervals.

A linearly ordered structure $M = (M, <, \cdots)$ is $weakly\ o-minimal$ if every definable subset of M^1 is a finite union of convex sets.

Definition 2 Let $M = (M, <, \cdots)$ be a densely linearly ordered structure.

M is $locally\ o-minimal$ if for any $a \in M$ and any definable set $A \subset M^1$, there is an open interval $I \ni a$ such that $I \cap A$ is a finite union of points and intervals.

M is $strongly\ locally\ o-minimal$ if for any $a \in M$, there is an open interval $I \ni a$ such that whenever A is a definable subset of M^1, then $I \cap A$ is a finite union of points and intervals.

M is $uniformly\ locally\ o-minimal$ if for any formula $\varphi(x, \bar{y})$ over $\emptyset$ and any $a \in M$, there is an open interval $I \ni a$ such that $I \cap \varphi(M, \bar{b})$ is a finite union of points and intervals for any $\bar{b} \in M^n$, where $\varphi(M, \bar{b})$ is the realization set of $\varphi(x, \bar{b})$ in M.

Example 3 The following examples are shown in [1] and [2].

$(\mathbb{R}, +, <, \mathbb{Z})$ where $\mathbb{Z}$ is the interpretation of a unary predicate, and $(\mathbb{R}, +, <, \sin)$ are (strongly) locally o-minimal structures.

Let a language $L = \{<\} \cup \{P_i : i \in \omega\}$ where P_i is a unary predicate. Let $M = (\mathbb{Q}, <^M, P_0^M, P_1^M, \dots)$ be the structure defined by $P_i^M = \{a \in M : a < 2^{-i}\sqrt{2}\}$. Then M is uniformly locally o-minimal, but it is not strongly locally o-minimal.

Theorem 4 [1] *Weakly o-minimal structures are locally o-minimal.*

Theorem 5 [1] *A structure* $\mathcal{M} = (M, <, \dots)$ *expanding a dense linear order* $(M, <)$ *without endpoints is locally o-minimal if and only if for any* $a \in M$ *and any definable set* $X \subset M$*, there are* $c, d \in M$ *such that* $c < a < d$ *and either* $X \cap (c, d)$ *or* $(c, d) \setminus X$ *is equal to one of the following :* (1) $\{a\}$, (2) $(c, a]$, (3) $[a, d)$, *or* (4) *the whole interval* (c, d).

Corollary 6 [1] *Local o-minimality is preserved under elementary equivalence. But, strongly local o-minimality is not preserved under elementary equivalence.*

2. Types in locally o-minimal structures

From the beginning, o-minimal structures are defined by the property of definable sets of $1-$variable formulas. And they are characterized by means of behavior of $1-$types. They consider two kinds of $1-$types by the way to cut linear orders of parameter sets, e.g. in [5].

Definition 7 Let M be a densely linearly ordered structure and $p(x) \in S_1^{or}(M)$, that is, $p(x)$ is complete over M with respect to the order relation.

We say that $p(x)$ is *cut (irrational) over* M if for any $a \in M$, if $a < x \in p(x)$, then there is $b \in M$ such that $a < b < x \in p(x)$, and similarly, if $x < a \in p(x)$, then there is $c \in M$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1^{or}(M)$ is *noncut (rational) over* M if $q(x)$ is not a cut type.

Here we consider nonisolated types only.

Definition 8 Let M be locally o-minimal and $p(x) \in S_1^{or}(M)$ be noncut.

There are four kinds of noncut types ;

$p(x) = \{m < x < a \,:\, m < a \in M\}$ for some fixed a,

or $\{a < x < m \,:\, a < m \in M\}$ for some fixed a.

Here we call these types *bounded* noncut types.

$p(x) = \{m < x \,:\, m \in M\}$ or $\{x < m \,:\, m \in M\}$.

We call these types *unbounded* noncut types.

3. Basic property of types in locally o-minimal structures

We can characterize locally o-minimal structures by means of types defined as above to some extent.

At first we recall some basic result from [3].

Theorem 9 [3] *Let M be a linearly ordered structure.*

Then M is o-minimal

if and only if

Any $p(x) \in S_1^{or}(M)$ is complete over M, that is, $p(x)$ is extended to the unique $1-$type over M.

We can show the next lemma.

Lemma 10 *Let M be a densely linearly ordered structure.*

Then M is locally o-minimal

if and only if

Any bounded noncut type $p(x) \in S_1^{or}(M)$ is complete over M.

Proof ;

$(\Longrightarrow)$

Let $p(x) \in S_1^{or}(M)$ be bounded noncut, that is, $p(x) = \{m < x < a : m < a \in M\}$ for some fixed $a \in M$. The other cases are proved similarly.

For any formula $\varphi(x, \overline{m})$ over M, there is an interval $I \subset M$ such that $a \in I$ and "$I \cap \varphi(M, \overline{m})$ is a union of finite points and intervals". (We call this property "I has $o-minimal$ property", "$OM-property$" for short in the following.)

Thus there is $b \in I$ such that either for any $c \in I$ with $b < c < a$, $M \models \varphi(c, \overline{m})$, or for any $c \in I$ with $b < c < a$, $M \models \neg\varphi(c, \overline{m})$. If for any $c \in I$ with $b < c < a$, $M \models \varphi(c, \overline{m})$, then $p(x) \vdash \varphi(x, \overline{m})$. Suppose that $p(x) \cup \{\neg\varphi(x, \overline{m})\}$ is consistent, then the formula "$b < x < a \wedge \neg\varphi(x, \overline{m})$" is satisfied in M. Contradiction.

$(\Longleftarrow)$

We must show that for any formula $\varphi(x, \overline{m})$ over M and any $a \in M$, there is an interval $I \subset M$ such that $a \in I$ and I has $OM-property$ with respect to $\varphi(x, \overline{m})$.

Suppose that for any $b < a$, there are convex sets $\{D_i : i < \omega\}$ in the interval $I' = \{b < x < a\}$ such that ;

for any $d_i \in D_i$ and $d_j \in D_j$, if $i < j < \omega$, then $d_i < d_j$, and

for any $d_j \in D_{2i}$ and $d_k \in D_{2i+1}$, $M \models \varphi(d_j, \overline{m}) \wedge \neg\varphi(d_k, \overline{m})$.

Thus for the noncut type $p(x)$, both $p(x) \cup \{\varphi(x, \overline{m})\}$ and $p(x) \cup \{\neg\varphi(x, \overline{m})\}$ are consistent. It contradicts to the hypothesis.

So there is a convex set C such that for any $e \in C$, either if $e < g < a$, then $M \models \varphi(g, \overline{m})$ or if $e < g < a$, then $M \models \neg\varphi(g, \overline{m})$. If C has no left boundary point in M, then we can take it *in C for the interval I. The reverse argument holds in the right side of a. Then M is locally o-minimal.* ∎

For the next argument, we recall some definition.

Definition 11 Let M be a linearly ordered structure.
M is *definably complete* if every definable unary set has both a supremum and an infimum in $M \cup \{\pm\infty\}$.

This condition is equivalent to the fact that every open definable unary set in M is a disjoint union of open intervals. We can show the next lemma.

Lemma 12 *Let M be a locally o-minimal structure and $A \subset M$ with $A \neq \emptyset$. And let M be definably complete.*
Then the isolated types of $Th(M, a)_{a \in A}$ are dense.

Next, we refer to results in [6]. We recall some definitions.

Definition 13 Let M be a structure.
A type $p(\overline{x}) \in S_n(M)$ is *definable* if for any $\varphi(\overline{x}, \overline{y})$ (over $\emptyset$), there is a formula $d\varphi(\overline{y})$ over M such that for all $\overline{a} \in M$, $\varphi(\overline{x}, \overline{a}) \in p(\overline{x})$ iff $M \models d\varphi(\overline{a})$.
Let $M \subset N$ be linearly ordered structures.
M is *Dedekind complete* in N if no cut in $S_1(M)$ is realized in N (where a cut $p(x) \in S_1(M)$ is a complete type over M which contains the cut $p(x) \restriction < \in S_1^{or}(M)$).

Theorem 14 [6] *Let M be an o-minimal structure and let $p(\overline{x}) \in S_n(M)$.*
Then $p(\overline{x})$ is definable if and only if for any realization $\overline{a}$ of $p(\overline{x})$, M is Dedekind complete in $M(\overline{a})$ where $M(\overline{a})$ is the prime model over $M \cup \{\overline{a}\}$.
In particular, let $q(x) \in S_1^{or}(M)$.
Then $q(x)$ is definable if and only if $q(x)$ is noncut.

Non-definability of cut types is easily checked in o-minimal structures. They used the cell decomposition theorem to prove the theorem above. I can not clearly show that the cell decomposition theorem holds in locally o-minimal structures on what condition. But the next fact is easily confirmed.

Fact 15 *Let M be a locally o-minimal structure and $p(x) \in S_1^{or}(M)$ be bounded and noncut.*
Then $p(x)$ is definable.

4. Some characterization of strongly locally o-minimal structures by types

In this section, the property of 1−types is used for characterizing strongly locally o-minimal structures. It is suggested by the argument in [5] and [6]. First we recall some result from [2].

Theorem 16 [2] *Let M be strongly locally o-minimal. And let D be a definable set of M and $f : D \longrightarrow M$ be a definable function.*

Then for any $a \in D$, there are open intervals $I \subset M$ containing a and $J \subset M$ containing $f(a)$ such that, by putting $f^* = f \cap (I \times J)$,

the domain of f^* can be broken up into a finite union of points and intervals, on each of which f^* is constant, strictly increasing and continuous, or strictly decreasing and continuous.

We can show the next propositions. In general, there are many examples of locally o-minimal structures which are not definably complete and have incomplete cut types.

Proposition 17 *Let M be a locally o-minimal structure and $p(x)$, $q(x) \in S_1^{or}(M)$. And let $p(x)$ be noncut and $q(x)$ be cut, and $q(x)$ be incomplete over M.*

Then there are no realizations a of $p(x)$ and b of $q(x)$ such that a and b have a common interval $I \subset N$ such that $\{a, b\} \subset I$ and for any formula $\varphi(x, \overline{n})$ over N, $\varphi(N, \overline{n}) \cap I$ is a finite union of points and intervals in any strongly locally o-minimal structure $N \succ M$.

(In the following, we say that the interval I above has "strongly locally $o-$ minimal property", "$SLOM-property$" for short.)

Sketch of proof ;

Suppose not, that is, there are a strongly locally o-minimal structure $N \succ M$ and realizations a of $p(x)$ and b of $q(x)$, and an interval $I \subset N$ such that $a, b \in I$ and I has $SLOM-property$. Thus we can consider $tp^{or}(b/acl(Ma)) \vdash tp(b/M)$ where $tp^{or}(b/acl(Ma)) \in S_1^{or}(acl(Ma))$. So there is a realization c of $q(x)$ such that $c \in acl(Ma)$. Thus there is a definable function $f(x)$ over M such that $f(a) = c$.

Let $p(x) = \{m < x < d\,;\, m \in M,\, m < d\}$ for some fixed $d \in M$ and $a < c$. (The other cases are proved similarly.) We may assume that the set $I \cap$"$x < d$"$= \{n \in I : n < d, n \in N\}$ is the domain of $f(x)$. By the monotonicity theorem of strongly locally o-minimal structures as above, we may assume that $I \cap$"$x < d$" is monotone and continuous. Moreover as $M \prec N$, there is $e \in M$ such that $f(x)$ is monotone and continuous on the interval (e, d).

Intuitively, it is obvious that the function $f(x)$ deduces a contradiction. But we show details.

W.l.o.g, we assume that $f(x)$ is strictly increasing on (e, d). (Another case is proved similarly.) Let $(e, d) \cap M = \{m \in M : e < m < d\}$. As $f(x)$ is monotone and continuous, its image of $(e, d) \cap M$ is an interval $(f(e), f(d))$ in M ($f(e)$ may be $-\infty$ and $f(d)$ may be ∞).

And as $q(x)$ *is cut, for the realization* c *of* $q(x)$*, there is* $g \in M$ *such that* $c < g < f(d)$*. Now* $N \models \forall x\,(f^{-1}(g) < x < d \longrightarrow g < f(x))$*. As* $f^{-1}(g) < a < d$*,* $g < f(a) = c$*. Contradiction.* ▮

Proposition 18 *Let* M *be locally o-minimal and* N *with* $M \prec N$ *be strongly locally o-minimal. And let* $p(x) \in S_1(M)$ *be definable, and* a *be a realization of* $p(x)$ *and* $I \subset N$ *be an interval such that* $a \in I$ *and* I *has* $SLOM-property$*.*

Then for any $b \in I$*, if* $tp^{or}(b/M) \in S_1^{or}(M)$ *is incomplete, then* $tp(b/M) \in S_1(M)$ *is definable.*

Proposition 19 *Let* M *be a locally o-minimal structure and let* $p(x), q(x) \in S_1^{or}(M)$*, and* $q(x)$ *be incomplete over* M*.*

Moreover let $p(x)$ *be realized in* N *and* $q(x)$ *be not realized in* N *for some* N *with* $M \prec N$*.*

Then no realizations of $p(x)$ *and* $q(x)$ *have a common interval* $I\,(\subset N')$ *with* $SLOM-$ *property in any strongly locally o-minimal structure* N' *with* $N \prec N'$*.*

5. Further problems

As is mentioned above, some results about o-minimal structures are generalized to the context of locally o-minimal structures. I will continue this attempt hereafter.

I studied about the independence relation in locally o-minimal structures before. I will investigate whether the difference between two kinds of 1−types has effect on the independence relation.

References

[1] C.Toffalori and K.Vozoris, *Note on local* $o-minimality$, MLQ Math. Log. Quart., 55, pp 617−632, 2009.

[2] T.Kawakami, K.Takeuchi, H.Tanaka and A.Tsuboi, *Locally* $o-minimal$ *structures*, J. Math. Soc. Japan, vol.64, no.3, pp 783−797, 2012.

[3] A.Pillay and C.Steinhorn, *Definable sets in ordered structures. I*, Trans. Amer. Math. Soc, 295, pp 565−592, 1986.

[4] J.Knight, A.Pillay and C.Steinhorn, *Definable sets in ordered structures. II*, Trans. Amer. Math. Soc, 295, pp 593−605, 1986.

[5] D.Marker, *Omitting types in* $o-minimal$ *theories*, J. Symb. Logic, vol.51, pp 63−74, 1986.

[6] D.Marker and C.Steinhorn, *Definable types in* $o-minimal$ *structures*, J. Symb. Logic, vol.59, pp 185−198, 1994.

[7] A.Pillay, *Some remarks on definable equivalence relations in o − minimal structures*, J. Symb. Logic, vol.51, pp 709−714, 1986.

[8] A.Pillay and C.Steinhorn, *On Dedekind complete o−minimal structures*, J. Symb. Logic, vol.52, pp 156−164, 1987.

[9] H.D.Macpherson, D.Marker and C.Steinhorn, *Weakly o − minimal structures and real closed fields*, Trans. Amer. Math. Soc, 352, pp 5435−5482, 2000.

[10] A.Dolich, C.Miller and C.Steinhorn, *Structures having o − minimal open core*, Trans. Amer. Math. Soc, 362, pp 1371−1411, 2010.

[11] A.Berenstein, C.Ealy and A.Gunaydin, *Thorn independence in the field of real numbers with a small multiplicative group*, Ann. Pure Appl. Logic, 150, pp 1−18, 2007.

[12] E.Hrushovski, Y.Peterzil and A.Pillay, *Groups, measures, and the NIP*, J. Amer. Math. Soc, Vol.21, pp 563−596, 2008.

[13] E.Hrushovski and A.Pillay, *On NIP and invariant measures*, J. Eur. Math. Soc, 13, pp 1005−1061, 2011.

[14] A.Conversano and A.Pillay, *Connected components of definable groups and o − minimality I*, Advan. Math, vol.231, pp 605−623, 2012.

[15] L.van den Dries, *Tame topology and o − minimal structures*, London Math. Soc. Lecture Note Ser, 248, Cambridge University Press, 1998.

[16] A.Pillay, *Geometric Stability Theory*, Oxford University Press, 1996.