Ideas of proving symmetry of Kim-independence

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1 Introduction and Preliminaries

In [1], the notion of Kim-independence was introduced, and it was shown that $NSOP_1$-theories are characterized as those theories for which Kim-independence has the symmetric property over models. In the proof of this characterization, the authors of [1] used Erdös-Rado theorem, which is a combinatorial set theoretic result on uncountable cardinals. In this article, we try to present a new proof of this fact only using Compactness theorem and Ramsey's theorem. We give an outline of the idea of the proof.

In this article, L is a language and T is a complete L-theory having an infinite model. For simplicity, we assume L is countable. We fix a big saturated model M^* of T and we work in M^*. Small subsets of M^* are denoted by $A, B, C, \dots$. Finite tuples in M^* are denoted by $a, b, c, \dots$. Variables are denoted by $x, y, z, \dots$. Formulas are deoted by $\varphi, \psi, \dots$. Types are denoted by $p, q, r, \dots$ and $S(A)$ is the set of all complete types over A. We say a and b have the same type over A (in symbol $a \equiv_A b$) if there is a type $p \in S(A)$ for which $a, b \models p$. For any $A \subset B$ and $p \in S(B)$, $p|_A = \{\varphi(x) \in p \mid \varphi(x) : L(A)\text{-formula}\}$. Let $\mathrm{Aut}(M^*/A) = \{\sigma : M^* \to M^* \mid \sigma \text{ is an automorphism over } A\}$. A sequence $\langle a_i \mid i < \alpha \rangle$, where α is an ordinal, is called an indiscernible sequence over A, if for any strictly increasing partial function $f : \alpha \to \alpha$, there is an $\sigma \in \mathrm{Aut}(M^*/A)$ with $\sigma \supset \{(a_i, a_{f(i)}) \mid i < \alpha\}$.

2 Kim-independence

A complete type p over the domain M^* will be called a global type. The following definitions are from [1].

Definition 1 (A-invariant global type)**.** We say a global type $p(x) \in S(M^*)$ is A-invariant, if

$$\varphi(x, a) \in q \iff \varphi(x, b) \in q.$$

holds, for any $a, b \in M^*$ with $a \equiv_A b$ and any L-formula $\varphi(x, y)$.

Definition 2 (Morley sequence)**.** Let q be an A-invariant global type. $\langle b_i \mid i < \omega \rangle$ will be called an A-Morley sequence (defined by q) if $b_i \models q|_{Ab_{<i}}$, for all $i < \omega$.

Remark 3. For any set A, an A-Morley sequence is an indiscernible sequence over A. This can be shown by an induction on the length of the sequence.

Example 4 ($T = \mathrm{Th}(\mathbb{Q}, <)$)**.** Let $q(x)$ be an M-invariant global type extending $\{x > a \mid a \in M\}$.

1. Suppose that all formulas $x < a$ with $a > M$ belong to q. Then any decreasing sequence $a_0 > a_1 > \cdots > a_i > \cdots > M$ becomes an M-Morley sequence defined by q.

2. Suppose that all formulas $x > a$ with $a > M$ belong to q. Then any increasing sequence $M < a_0 < a_1 < \cdots < a_i < \ldots$ becomes an M-Morley sequence defined by q.

Definition 5 (Kim-divide)**.** We say that a formula $\varphi(x, b)$ Kim-divides over A if there are an A-invariant global type q and an A-Morley sequence $I = \langle b_i \mid i < \omega \rangle$ defined by q such that

1. $b_0 = b$,

2. $\{\varphi(x, b_i) \mid i < \omega\}$ is inconsistent.

A type $p \in S(B)$ Kim-divides over A if there is a formula $\varphi(x, b) \in p$ that Kim-divides over A.

Example 6 (T=Th($\mathbb{Q}, <$))**.** Let M be a model of T. Let us consider the formula $a_0 < x < b_0$.

1. Suppose that there is an element $m \in M$ with $a_0 < m < b_0$. Then the formula $a_0 < x < b_0$ does not Kim-divide over M.

2. Suppose that $M < a_0 < b_0$. Let $q(y, z)$ be the global type $\{a < y < z : a \in M^*\}$. Then, the formula $a_0 < x < b_0$ Kim-divides over M by this q.

Definition 7 (Kim-fork)**.** $\varphi(x, b)$ Kim-forks over A if there are $n < \omega$ and $\psi_0(x, c), \ldots, \psi_n(x, c)$ such that

1. $\psi_i(x, c)$: Kim-divides over A,

2. $M^* \models \forall x \left[\varphi(x, b) \to \bigvee_{i \le n} \psi_i(x, c)\right]$.

$p \in S(B)$ Kim-forks over A if there is $\varphi(x, b) \in p$ Kim-forks over A.

By definition, If $\varphi(x, a)$ Kim-divides over A, then $\varphi(x, a)$ Kim-forks over A(but not the converse).

3 NSOP_1 theories

Definition 8 (NSOP_1)**.** T has SOP_1 if there exist $\varphi(x, y) \in L$ and a binary tree of tuples $(c_\eta)_{\eta \in 2^{<\omega}}$ suth that

1. For all $\beta \in 2^\omega$, $\{\varphi(x, c_{\beta \lceil m}) \mid m < \omega\}$ is consistent,
2. For all $\gamma \in 2^{<\omega}$ and $\gamma \trianglerighteq \eta^\frown\langle 0\rangle$, $\{\varphi(x, c_{\alpha^\frown\langle 1\rangle}), \varphi(x, c_\gamma)\}$ is inconsistent.

T is NSOP_1 if T does not have SOP_1.

"T is NSOP_1" characterizes an L-formula and two infinite sequence of tuples.

Fact 9 ([1])**.** Let T be a complete theory. T.F.A.E.

1. T has SOP_1.
2. There are $\langle a_i b_i \mid i < \omega\rangle$ and $\varphi(x, y) \in L$ suth that
   - $a_i \equiv_{(ab)_{<i}} b_i$ $(\forall i < \omega)$,
   - $\{\varphi(x, a_i) \mid i < \omega\}$ is consistent,
   - $\{\varphi(x, b_i) \mid i < \omega\}$ is 2-inconsistent.

By Fact 9,

Fact 10 ([1])**.** Let T be a complete theory. T.F.A.E.

1. T is NSOP_1,
2. For all $M \models T$, $\varphi(x, b)$ and M-invariant global type $q \supset \mathrm{tp}(b/M)$, if $\varphi(x, b)$ Kim-divides over M by q, then for all M-invariant global type r satisfies $r|_M = q|_M$, $\varphi(x, b)$ Kim-divides over M by r.

By Fact 10,

Fact 11 ([1], T : NSOP_1)**.** If $\varphi(x, b)$ Kim-forks over M, $\varphi(x, b)$ Kim-divides over M.

By Fact 11,

Fact 12 (T : NSOP_1)**.** For any B and $p \in S(B)$,

$$p \text{ Kim-divides over } M \iff p \text{ Kim-forks over } M.$$

Notation. $a \mathop{\downarrow}^K_A b \iff \mathrm{tp}(a/Ab)$ does not Kim-fork over A.

Fact 13 ([1], T : NSOP_1)**.** $\mathop{\downarrow}^K$ satisfies the following conditions :

1. (Extension over models) If $a \mathop{\downarrow}^K_M b$, then for all c, there exists $a' \equiv_{Mb} a$ satisfies $a' \mathop{\downarrow}^K_M bc$.

2. (Chain condition) If $a \mathop{\downarrow}^K_M b$ and M-Morley sequence $I = \langle b_i \mid i < \omega \rangle$ starts with b, there exists $a' \equiv_{Mb} a$ suth that
   - $a' \mathop{\downarrow}^K_M I$
   - I : an indiscernible sequence over Ma'

In [1], Kaplan and Ramsey proved

Fact 14 ([1])**.** T.F.A.E.

1. T is NSOP_1.
2. Symmetry : $a \mathop{\downarrow}^K_M b \iff b \mathop{\downarrow}^K_M a$.

4 Ideas proving Symmetry of Kim-independence

I want to prove

Theorem 15 ([1])**.** If T is NSOP_1, $\mathop{\downarrow}^K$ satisfies symmetry over models, i.e. $a \mathop{\downarrow}^K_M b \Rightarrow b \mathop{\downarrow}^K_M a$.

by only using Compactness theorem and Ramsey's theorem. We introduce the notion of finitely satisfiability of types.

Notation. We denote $\mathrm{tp}(a/A) = \{\varphi(x) : L(A)\text{-formula}, M^* \models \varphi(a)\}$.

Definition 16. $p(x) \in S(A)$ is finitely satisfiable in B if for any $n < \omega$ and $\varphi_0(x), \ldots, \varphi_n(x) \in p$, there is $b \in B$ satisfies

$$M^* \models \bigwedge_{i \le n} \varphi(b).$$

Let α be an ordinal. $I = \langle a_i \mid i < \alpha \rangle$ is cohheir sequence over A if for any $i < \alpha$, $\mathrm{tp}(a_i/Aa_{<i})$ is finitely satisfiable in A and I is an indiscernible sequence over A.

My main idea proving Theorem 15 is using Fact 9. First, I proved

Lemma 17 (T : NSOP_1)**.** We put $p(x, a) = \mathrm{tp}(b/Ma)$. If $a \mathop{\downarrow}^K_M b$, then for all $n < \omega$, there is a sequence $(a_i a'_i)_{i<n}$ satisfies the following conditions :

1. $a_i \equiv_M a'_i \equiv_M a$ $(\forall i < n)$,
2. $a_i \equiv_{M(aa')_{>i}} a'_i$ $(\forall i < n)$,
3. $\bigcup_{i<n} p(x, a_i)$: consistent,
4. $(a'_i)_{i<n}$: For all $i < n$, $\mathrm{tp}(a'_i/Ma'_{<i})$ is finitely satisfiable in M.

Proof. We confirm only $n = 2$. But same method is applicable for all $n < \omega$. Let κ be a sufficiently large cardinal. Since $a \mathop{\downarrow}^K_M b$, there is $I_0 = (b_i)_{i<\kappa}$ starts with b satisfies the following conditions,

- $a \downarrow^K_M I_0$,
- I_0 : coheir sequence over M and Ma-indsicernible sequence.

Since $a \downarrow^K_M I_0$, there is a'' and $I_1 = (c_i I'_i)_{i<\kappa}$ starts with aI_0 satisfies the following conditions,

- $a' \equiv_{MI_0} a$,
- $a' \downarrow^K_M I_1$,
- I'_1 : coheir sequence over M and Ma'-indsicernible sequence.

Let $I_2 = (c'_i I''_i)_{i<\kappa}$ be an coheir sequence over M starts with $a'I_1$. Let $a_1 = c'_0$ and $a'_1 = c'_1$. Since $(c_i)_{i<\kappa}$ is sufficiently long, there is $i < j < \kappa$ suth that $c_i \equiv_{Ma_1a'_1} c_j$. Let $a_0 = c_i$ and $a'_0 = c_j$. □

Question 18. For all $n < \omega$, Can we take $(a_i a'_i)_{i<n}$ satisfies the following condition ? :

1. For all $m \leq n$ and $(b_i b'_i)_{i<m}$ are taken by Lemma 17, $(a_i a'_i)_{i<m} \equiv_M (b_i b'_i)_{i<m}$.

   $\Longrightarrow$ If $m < 2$, $(a_i a'_i)_{i<m} \equiv_M (b_i b'_i)_{i<m}$ but the other case can't satisfy this condition.

2. $\mathrm{tp}(a'_i/Ma'_{<i}) \subset \mathrm{tp}(a'_{i+1}/Ma'_{<i+1})$ for all $i < n-1$?

   $\Longrightarrow$ For all $n < \omega$ and $(a_i a'_i)_{i<n}$, $\mathrm{tp}(a'_0/M) \subset \mathrm{tp}(a'_i/Ma'_{<i})$, but the other case can't satisfy this condition.

We explain another idea.

Lemma 19. Let T be a complete theory. Let $M \subset A$, where $M \models T$ and $p(x) \in S(A)$ be a type finitely satisfiable in M. Let $q(X) \in S(M)$, where X is a set of variables with $x \in X$. Suppose that $p(x) \cup q(X)$ is consistent, in other words, $p|_M \subset q(X)$. Then, there is a type $q^*(X) \in S(A)$ such that

1. $q^*(X)$ is finitely satisfiable in M , and
2. $q^*(X) \supset p(x) \cup q(X)$.

Proof. Let $\Pi(X) = \{\neg\theta(X) \mid \theta(X) : L(A)\text{-formula}, \theta \text{ isn't satisfiable in } M\}$ and $\Gamma(X)$ be $p(x) \cup q(X) \cup \Pi(X)$. We claim that $\Gamma(X)$ is consistent. Suppose otherwise, We can find $\varphi_p(x) \in p$, $\varphi_q(x) \in q$, $n < \omega$ and $\psi_0(X), \ldots, \psi_n(X) \in \Pi(X)$ suth that

$$\varphi_p(x) \wedge \varphi_q(X) \models \bigvee_{i \leq n} \neg\psi_i(X).$$

But this is contradiction since $\exists X \setminus x[\varphi_p(x) \wedge \varphi_q(X)] \in p$ and p is finitely satisfiable in M. □

Proposition 20. Let T be a NSOP$_1$ theory and $M \models T$. Let $r(x, y) = \mathrm{tp}(ab/M)$, where $a \downarrow^K_M b$. Then for any $n < \omega$, there is a tree $(a_\eta)_{\eta \in 2^{\leq n}}$ suth that

1. $a_{\eta\restriction m} a_\eta \models \gamma$, for any $\eta \in \omega^n$ and $m < n$;

2. For any $i < \omega$ and $i \frown \eta \in \omega^{<n}$, $I_{i\frown\eta} = \langle a_{j\frown i\frown\eta} \mid j < \omega\rangle$ is indiscernible sequence over $M \cup \{a_{i\frown\eta}\} \cup \{a_{\nu\frown k\frown\eta} \mid k < i, \nu \in \omega^{<n-1}\}$.

3. $a_{0^{n-2}\frown 1}, a_{0^{n-3}\frown 1}, \ldots, a_1$ forms a coheir sequence over M.

Proof. Suppose we already defined a desired tree $(a_\eta)_{\eta\in 2^{\leq n}}$ for $n < \omega$. We rename each a_η to $b_{0\frown\eta}$. By assumption, $b_{0^{n-1}\frown 1}, b_{0^{n-2}\frown 1}, \ldots, b_{0\frown 1}$ forms a coheir sequence over M. Let $B = \{b_{0^2\frown\eta} \mid \eta \in \omega^{<n-1}\}$. Since the type $p(x) = \mathrm{tp}(b_{0\frown 1}/MB)$ is finitely satisfiable in M, there is a coheir extension $p'(x) \in S(M(a_\eta)_{\eta\in\omega^{\leq n}})$ of p. Let $q(X) = \mathrm{tp}((a_\eta)_{\eta\in\omega^{\leq n}}/M)$, where $X = (x_\eta)_{\eta\in\omega^{\leq n}}$ and the variable corresponds to a_η. By Lemma 19, there is a type $p^*(X) \in S(M(a_\eta)_{\eta\in 2^{\leq n}})$ which is finitely satisfiable in M and extends $p'(x_\emptyset) \cup q(X)$. Choose a realization $B_1 = (b_{1\frown\eta})_{\eta\in\omega^{\leq n}}$ of p^*. Notice that $b_{1\frown\eta}$ corresponds to x_η. Then we choose $B_i (i \geq 2)$ suth that $B_0, B_1, B_2, \ldots$ forms a coheir sequence over M. Since $\mathrm{tp}(b_0/M(b_{0\frown\eta})_\eta)$ is does not Kim-fork over M and Fact 13, there is $r(x) \in S(M(B_i)_{i<\omega})$ satisfies $r(x) \supset \mathrm{tp}(b_0/M(b_{0\frown\eta})_\eta)$ and does not Kim-fork over M. we choose a realisation $b_\emptyset$ of $r(x)$. Then $(b_\eta)_{\eta\in\omega^{\leq n+1}}$ satisfies the condition 1-3. $\square$

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