On the number of independent strict orders

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1 Motivation and Aim

Let me start with a simple example. In a given structure $\mathcal{M}$, there might be several different definable orders in $\mathcal{M}$. For example if $\leq$ is a definable order in $\mathcal{M}$, its reverse order $\leq^*$ is also definable, and of course $\leq$ and $\leq^*$ are different. However, they are not independent in the sense of the present article. Now we consider the two (pre)orders $\leq_i$ $(i = 1, 2)$ on $\mathbb{N}$: For $M = 2^{m_0}3^{m_1}\dots$ and $N = 2^{n_0}3^{n_1}\dots$ (prime factorizations), let

1. $M \leq_1 N$ iff $m_0 \leq n_0$,
2. $M \leq_2 N$ iff $m_1 \leq n_1$

Then we have the following:

- Both $\leq_1$ and $\leq_2$ are definable in $\mathbb{N} = (\mathbb{N}, +, \cdot)$;
- By letting $a_i = 2^{2i}$, $b_i = 3^{2i}$ $(i \in \omega)$, we have:
  - $(a_i)_{i\in\omega}$ is strictly increasing in the sense of $\leq_1$;
  - $(b_i)_{i\in\omega}$ is strictly increasing in the sense of $\leq_2$;
  - Two intervals(a_i, a_{i+1}) and (b_j, b_{j+1}) always have a nonempty intersection. $2^{2i+1}3^{3j+1}$ belongs to the intersection.

Such two orders will be called independent in this article. We want to discuss how many independent orders exist in $\mathcal{M}$.

Shelah [2] defined two invariants $\kappa^m_{srd}(T)$ and $\kappa^m_{ird}(T)$, both are concerning the number of independent orders. $\kappa^m_{ird}(T)$ measures the number of possible independent orders by m-formulas, and $\kappa^m_{srd}(T)$ measures the number of independent strict orders. In [2], it was shown that if T is NIP, then the two invariants are equal. In [1], $\kappa^m_{srd}(T)$ was studied from the point of view of indiscernibility, also under the assumption of NIP.

2 Definitions

T is a complete countable theory. We work in a very big saturated model $\mathcal{M}$ of T. $a, b, \ldots$ are finite tuples in $\mathcal{M}$, and $x, y, \ldots$ are finite tuples of variables. We want to investigate $\kappa^m_{srd}(T)$ when m varies.

Now we recall the following basic definition:

Definition 1.

- A formula $\varphi(x, y)$ has the strict order property if there is a formula $\varphi(x, y)$ and a sequence $(a_i)_i$ such that

$$\varphi(\mathcal{M}, a_0) \subsetneq \varphi(\mathcal{M}, a_1) \subsetneq \ldots,$$

  where $\varphi(\mathcal{M}, a)$ denotes the set of all elements (in $\mathcal{M}$) satisfying the formula $\varphi(x, a)$.

- T has the strict order property if some formula has the strict order property.

The invariant $\kappa^m_{srd}(T)$ is defined as follows:

Definition 2. Let $m \in \omega$. $\kappa^m_{\mathrm{srd}}(T)$ is the minimum cardinal κ such that there is no set $\{\varphi_i(x, y_i) : i < \kappa\}$ of formulas with $|x| = m$ and a set $B = \{b_{ij} : (i, j) \in \kappa \times \omega\}$ of parameters satisfying:

1. For each $i < \kappa$, $\{\varphi_i(\mathcal{M}, b_{ij}) : j \in \omega\}$ forms an increasing sequence of definable sets;
2. $\{\varphi_i(x, b_{i,j})^{\text{if } (j \geq \eta(i))} : i \in \omega\}$ is consistent, for all $\eta \in \omega^\kappa$. ($\varphi^{\text{ if } (*)} = \varphi$ if $(*)$ holds, otherwise $\varphi^{\text{ if } (*)} = \neg\varphi$)

We write $\kappa^m_{\mathrm{srd}}(T) = \infty$, if there is no such κ.

3 Results

Our main results are the following:

Theorem 3 (A)**.** *Suppose* $\kappa^m_{\mathrm{srd}}(T) = \infty$*. Then* $\kappa^1_{\mathrm{srd}}(T) = \infty$*.*

Theorem 4 (B). *Suppose $\kappa^m_{\mathrm{srd}}(T) = \omega$. Then $\kappa^1_{\mathrm{srd}}(T) = \omega$.*

We do not prove the statements here. The details of proofs will be given in our forthcoming paper. But, the important facts to be used are summarized in the following remark.

Remark 5. 1. For $\kappa < \kappa^m_{\mathrm{srd}}(T)$, there are formulas φ_i $(i < \kappa)$ and a set B witnessing the conditions 1 and 2 in the definition. By replacing B with a new one, the following additional conditions can be assumed.

(a) The column size of B can be chosen arbitrarily large. Moreover, B can be assumed to have the form $B = (b_{ij})_{(i,j)\in\kappa\times J}$, where J is an arbitrary infinite order.

(b) By letting $B_i = (b_{ij})_{j\in J}$ $(i \in \kappa)$, B_i is assumed to be indiscernible over $\bigcup_{k\neq i} B_k$.

Both can be justified by an easy compactness argument.

Remark 6. 1. Suppose $\kappa^m_{\mathrm{srd}}(T) = \infty$. There is an uncountable set $\{\varphi_i(x, y_i) : i < \omega_1\}$ and $B = (b_{ij})_{(i,j)\in\omega_1\times I}$ witnessing the conditions 1 and 2 in the definition. So, by choosing an uncountable subset of ω_1, we can assume $\varphi_i = \varphi$ for all $i < \omega_1$ (fixed). By compactness, in addition to conditions (a) and (b) above, it can be further assumed that $(B_i)_{i\in I}$ is an indiscernible sequence.

References

[1] Vincent Guingona, Cameron Donnay Hill, Lynn Scow, Characterizing Model-Theoretic Dividing Lines via Collapse of Generalized Indiscernibles, Volume 168, Issue 5, May 2017, Pages 1091-1111

[2] S. Shelah, 'Classification Theory,' North Holland; 2 edition (December 20, 1990)