Local energy decay estimate for the hyperbolic type Stokes equations

Kenji Nakamura
Graduate School of Pure and Applied Sciences,
University of Tsukuba,
Ibaraki 305-8571, Japan

1 Introduction

This article is a survey of the work [6], which is a joint work with Takayuki Kobayashi, Professor of Osaka University and Takayuki Kubo, Professor of University of Tsukuba.

We consider the local energy decay properties of solutions to the initial-boundary value problem of the hyperbolic type Stokes equations:

$$\begin{cases} \tau\partial_t^2 u - \Delta u + \partial_t u + (1+\tau\partial_t)\nabla\pi = 0 & \text{in } \Omega\times(0,\infty), \\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,\infty), \\ u|_{\partial\Omega} = 0, \quad (u,\partial_t u)|_{t=0} = (u_0,u_1) \end{cases} \tag{HS}$$

with unknown velocity field $u = (u_1(x,t),\ldots,u_n(x,t))$, unknown pressure $\pi = \pi(x,t)$ and given vector function (u_0,u_1). Here, $\tau < 1$ is a positive constant describing the relaxation parameter and Ω is a domain of $\mathbb{R}^n\,(n \geq 2)$ with smooth boundary. We consider the following cases:

(i) Ω is an exterior domain of the n-dimensional Euclidean space $\mathbb{R}^n$, that is, there exists a number $r > 0$ such that $\Omega\setminus B_r = \mathbb{R}^n\setminus B_r$, where $B_r = \{x\in\mathbb{R}^n \mid |x| < r\}$.

(ii) Ω is a perturbed half-space, that is, there exists a number $r > 0$ such that $\Omega\setminus B_r = \mathbb{R}^n_+\setminus B_r$, where $\mathbb{R}^n_+ = \{(x',x_n) \mid x'\in\mathbb{R}^{n-1},\ x_n > 0\}$.

This model arises from a time delayed version for the deformation tensor in the parabolic type Stokes equations (see [9]).

The investigation of the local energy decay properties are essential step to prove the global-in-time unique existence theorem corresponding to the nonlinear problem in an exterior domain and a perturbed half-space.

In the case of the parabolic type Stokes equations, when Ω is an exterior domain of $\mathbb{R}^n\,(n\geq 3)$, Iwashita [5] investigated the local energy decay properties based on the resolvent expansion near the origin. Lator on Iwashita's work, Dan, Kobayashi and Shibata [3] and Dan and Shibata [2] also proved the local energy decay estimate. They improved Iwashita's results and extended to 2D case. Here, the decay rate of the local energy decay estimate is $t^{-1}(\log t)^{-2}\,(n=2)$, $t^{-n/2}\,(n\geq 3)$ as $t\to\infty$.

When Ω is a perturbed half-space, Kubo and Shibata [8] proved the local energy decay estimate based on the resolvent expansion near the origin obtained in [7]. Here, the decay rate of the local energy decay estimate is $t^{-(n+1)/2}$ $(n \geq 2)$ as $t \to \infty$.

In the case of the hyperbolic type Stokes equations (HS), when Ω is an exterior domain or a perturbed half-space, as far as we know, there are no results. The idea of our proof is based on Dan and Shibata [1]. They proved in [1] the local energy decay estimate to the dissipative wave equations in 2D exterior domain by use of the resolvent expansion for the Laplace operator. Therefore, we prove the local energy decay estimete to (HS) by use of the resolvent expansion for the Stokes operator.

2 Main result

To state our results more precisely, we outline our notaion. Let r_0 be a fixed constant satisfying (i) or (ii) in Section 1. We set $\Omega_r = \Omega \cap B_r$ for $r > r_0$ and

$$\begin{aligned} C^\infty_{0,\sigma}(\Omega) &= \{u \in C^\infty_0(\Omega) \mid \nabla \cdot u = 0 \text{ in } \Omega\}, \\ L^2_\sigma(\Omega) &= \text{the completion of } C^\infty_{0,\sigma}(\Omega) \text{ with respect to } \|\cdot\|_{L^2(\Omega)}, \\ H^1_{0,\sigma}(\Omega) &= \text{the completion of } C^\infty_{0,\sigma}(\Omega) \text{ with respect to } \|\cdot\|_{H^1(\Omega)}, \\ \hat{H}^1_{0,\sigma}(\Omega) &= \text{the completion of } C^\infty_{0,\sigma}(\Omega) \text{ with respect to } \|\nabla \cdot\|_{L^2(\Omega)}, \\ G(\Omega) &= \{\nabla \pi \in L^2(\Omega) \mid \pi \in L^2_{loc}(\Omega)\}. \end{aligned}$$

We set $v = \partial_t u$ and $\mathbb{U} = {}^T(u, v)$ and define a Hilbert space $\mathscr{H}(\Omega)$ by

$$\mathscr{H}(\Omega) = \left\{ \mathbb{U} = {}^T(u, v) \,|\, u \in \hat{H}^1_{0,\sigma}(\Omega), v \in L^2_\sigma(\Omega) \right\}$$

with inner product

$$(\mathbb{U}, \mathbb{W})_{\mathscr{H}(\Omega)} = (u, w)_D + \tau(v, z),$$

where $(u, w)_D = (\nabla u, \nabla w)$. Moreover, we set $L^2_r(\Omega) = \{f \in L^2(\Omega) \mid \operatorname{supp} f \subset \Omega_r\}$ and

$$\mathscr{H}_r(\Omega) = \left\{ \mathbb{U} = {}^T(u, v) \in \mathscr{H}(\Omega) \mid \operatorname{supp} u \cup \operatorname{supp} v \subset \Omega_r \right\}.$$

We treat (HS) as the semigroup theoretical framework. To do this, we use the Helmholtz decomposition: $L^2(\Omega) = L^2_\sigma(\Omega) \oplus G(\Omega)$, where $\oplus$ denotes the direct sum. Let P be a continuous projection from $L^2(\Omega)$ to $L^2_\sigma(\Omega)$. The Stokes operator A is defined by $A = -P\Delta$ with domain $\mathscr{D}(A) = H^1_{0,\sigma}(\Omega) \cap H^2(\Omega)$. We define an operator $\mathbb{L}$ by

$$\mathbb{L} = \frac{1}{\tau} \begin{pmatrix} 0 & -\tau \\ A & 1 \end{pmatrix}$$

with domain

$$\mathscr{D}(\mathbb{L}) = \left\{ \mathbb{U} = {}^T(u, v) \in \mathscr{H}(\Omega) \,|\, u \in \mathscr{D}(A), v \in \hat{H}^1_{0,\sigma}(\Omega) \right\}.$$

Applying the projection P to (HS), the problem is written in the following form:

$$\frac{d}{dt}\mathbb{U}(t) = -\mathbb{L}\mathbb{U}(t) \quad \text{for } t > 0, \quad \mathbb{U}|_{t=0} = \mathbb{U}_0,$$

where $\mathbb{U}_0 = {}^T(u_0, u_1)$. Then, we obtain the following two theorems.

Theorem 2.1. $-\mathbb{L}$ *generates a* C_0 *contraction semigroup* $\{T(t)\}_{t\geq 0}$ *on* $\mathscr{H}(\Omega)$.

Theorem 2.2. *Let $n \geq 2$ be an integer and let $r > r_0$. Suppose that the initial data $u_0 \in H^1_{0,\sigma}(\Omega)$ and $u_1 \in L^2_\sigma(\Omega)$ and* $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega_r$*. Then, the solution u of* (HS) *holds the following properties:*

(i) *When Ω is an exterior domain, it holds that*

$$\|u(t)\|_{H^1(\Omega_r)} + \sqrt{\tau}\|\partial_t u(t)\|_{L^2(\Omega_r)} \leq C_{n,r}(1+t)^{-\frac{n}{2}}(\|u_0\|_{H^1(\Omega)} + \sqrt{\tau}\|u_1\|_{L^2(\Omega)})$$

for any $t \geq 0$.

(ii) *When Ω is a perturbed half-space, it holds that*

$$\|u(t)\|_{H^1(\Omega_r)} + \sqrt{\tau}\|\partial_t u(t)\|_{L^2(\Omega_r)} \leq C_{n,r}(1+t)^{-\frac{n+1}{2}}(\|u_0\|_{H^1(\Omega)} + \sqrt{\tau}\|u_1\|_{L^2(\Omega)})$$

for any $t \geq 0$.

Remark 2.1. The decay rate of (ii) in Theorem 2.2 is one half better compared with the case (i) because the order of asymptotic behavior of the Stokes resolvent near the origin in Proposition 3.1 is one half better compared with the exterior domain case due to the reflection principle on the boundary in the half-space unlike the whole space.

3 Key lemma

We consider the resolvent problem:

$$\lambda u + Au = f \quad \text{in}\, \Omega, \tag{3.1}$$

where $\lambda \in \Sigma_{\ell,\epsilon} = \{\lambda \in \mathbf{C} \setminus \{0\} \mid |\lambda| < \ell, |\arg \lambda| < \pi - \epsilon\}$, $0 < \ell < 1$, $0 < \epsilon < \pi/2$ and A is the Stokes operator. Let $S(\lambda)f$ be defined as a solution to (3.1), $W^{m,2}_s(\Omega)$ be a weighted Sobolev space defined by

$$W^{m,2}_s(\Omega) = \left\{ f \mid \left(1 + |\cdot|^2\right)^{\frac{s}{2}} \partial^k_j f \in L^2(\Omega),\, k \leq m \right\}$$

for any non-negative integer m and real number s. Moreover, for $s > n/2$ and $s' < -n/2$, we set

$$\mathcal{B}_1 = \mathcal{L}\left(L^2_\sigma(\Omega) \cap W^{0,2}_s(\Omega)^n, L^2_\sigma(\Omega) \cap W^{2,2}_{s'}(\Omega)^n\right),$$
$$\mathcal{B}_2 = \mathcal{L}\left(L^2_\sigma(\Omega) \cap L^2_r(\Omega), L^2_\sigma(\Omega) \cap H^2(\Omega_r)\right).$$

Then, by [2, Proposition 3.6], [5, Theorem 3.1, Corollary 3.2] and [8, Theorem 3.1], it holds that the following proposition.

Proposition 3.1. *There exist an $\ell > 0$ and an $S(\lambda) \in \mathrm{Hol}\,(\Sigma_{\ell,\epsilon}, \mathcal{B}_1)$ which has the following expansion formula:*

(i) *When Ω is an exterior domain,*

$$S(\lambda) = \begin{cases} G_1\lambda^{\frac{n}{2}-1}\log\lambda + G_2(\lambda) + G_3(\lambda)\lambda^{\frac{n}{2}-1} & \text{where } n \text{ is even,} \\ G_1\lambda^{\frac{n}{2}-1} + G_2(\lambda) + G_3(\lambda)\lambda^{\frac{n}{2}-1} & \text{where } n \text{ is odd,} \end{cases}$$

where $G_1 \in \mathcal{B}_1$, $G_2(\lambda)$ is a polynomial of λ of $\deg G_2(\lambda) \le [n/2] - 1$ and $G_3(\lambda) \to 0$ as $\lambda \to 0$. Particularly in case of $n = 2$, the following holds:

$$S(\lambda) = V_1 + V_2(\log\lambda)^{-1} + O\left((\log\lambda)^{-2}\right),$$

where $V_1, V_2 \in \mathcal{B}_2$.

(ii) *When Ω is a perturbed half-space,*

$$S(\lambda) = \begin{cases} H_1(\lambda)\lambda^{\frac{n-1}{2}} + H_2(\lambda)\lambda^{\frac{n}{2}}\log\lambda + H_3(\lambda) & \text{where } n \text{ is even,} \\ H_1(\lambda)\lambda^{\frac{n}{2}} + H_2(\lambda)\lambda^{\frac{n-1}{2}}\log\lambda + H_3(\lambda) & \text{where } n \text{ is odd,} \end{cases}$$

where $H_1, H_2 \in \mathrm{Hol}\,(\Sigma_{\ell,\epsilon}, \mathcal{B}_2)$ and $H_3 \in \mathrm{Hol}\,(\Sigma_{\ell,\epsilon} \cup \{0\}, \mathcal{B}_2)$.

In Section 4, we investigate the stability of $(\lambda\mathbb{I} + \mathbb{L})^{-1}$ near the origin. To do this, we use the class $\mathcal{C}^k$ defined as follows and the properties of the class.

Definition 3.2 ([1])**.** Let X be a Banach space with norm $|\cdot|_X$. Let $N \ge 0$ be an integer and $k = N + \sigma$ with $0 < \sigma \le 1$. Set

$$\mathcal{C}^k(\mathbf{R}, X) = \left\{ f \in C^\infty(\mathbf{R} \setminus \{0\}, X) \,|\, \langle\!\langle f \rangle\!\rangle_{k,X} < \infty \right\},$$

where

$$\langle\!\langle f \rangle\!\rangle_{k,X} = \sum_{j=0}^{N} \int_{-\infty}^{\infty} \left| \left(\frac{d}{ds}\right)^j f(s) \right|_X ds + \sup_{h \neq 0} \frac{1}{|h|^\sigma} \int_{-\infty}^{\infty} \left| \Delta_h \left(\frac{d}{ds}\right)^N f(s) \right|_X ds \ (0 < \sigma < 1),$$

$$\langle\!\langle f \rangle\!\rangle_{k,X} = \sum_{j=0}^{N} \int_{-\infty}^{\infty} \left| \left(\frac{d}{ds}\right)^j f(s) \right|_X ds + \sup_{h \neq 0} \frac{1}{|h|} \int_{-\infty}^{\infty} \left| \Delta_h^2 \left(\frac{d}{ds}\right)^N f(s) \right|_X ds \ \ (\sigma = 1).$$

Here, we have set

$$\Delta_h f(s) = f(s+h) - f(s), \quad \Delta_h^2 f(s) = f(s+h) - 2f(s) + f(s-h).$$

Proposition 3.3 ([10])**.** *Let N be a positive integer and X be a Banach space with norm $|\cdot|_X$. Assume that $f \in C^\infty(\mathbf{R} \setminus \{0\}, X)$, $f(s) = 0$ if $|s| \ge 2$ and set $I = (-2, 2)$.*

(i) *Let $k = N + \sigma$ with $0 < \sigma < 1$ and f satisfy the following condition* (a).

(a) *For any* $s \in I \setminus \{0\}$,

$$\left| \left(\frac{d}{ds} \right)^j f(s) \right|_X \le C_f \quad \text{for any integer } j \in [0, N-1],$$

$$\left| \left(\frac{d}{ds} \right)^N f(s) \right|_X \le C_f |s|^{\sigma-1}, \quad \left| \left(\frac{d}{ds} \right)^{N+1} f(s) \right|_X \le C_f |s|^{\sigma-2}.$$

Then, $f \in \mathcal{C}^k(\mathbf{R}, X)$ *satisfies*

$$\langle\!\langle f \rangle\!\rangle_{k,X} \le C_{\sigma,N} C_f.$$

(ii) *Let* $k = N+1$ *and* f *satisfy the following conditions* (a) *and* (b).

(a) *There exist* $f_0 \in X$ *and a* X*-valued function* $f_1(s)$ *defined on* I *such that*

$$\left(\frac{d}{ds} \right)^N f(s) = f_0 \log |s| + f_1(s) \quad \text{for } s \in I \setminus \{0\}.$$

(b) *For any* $s \in I \setminus \{0\}$,

$$\left| \left(\frac{d}{ds} \right)^j f(s) \right|_X \le C_f \quad \text{for any integer } j \in [0, N-1],$$

$$|f_0|_X \le C_f, \quad |f_1(s)|_X \le C_f, \quad \left| \left(\frac{d}{ds} \right)^{N+1} f(s) \right|_X \le C_f |s|^{-1},$$

$$\left| \left(\frac{d}{ds} \right)^{N+2} f(s) \right|_X \le C_f |s|^{-2}.$$

Then, $f \in \mathcal{C}^k(\mathbf{R}, X)$ *satisfies*

$$\langle\!\langle f \rangle\!\rangle_{k,X} \le C_{\sigma,N} C_f.$$

Proposition 3.4 ([1]). *Let* X *be a Banach space with norm* $|\cdot|_X$. *Let* $f(s) \in C^2(\mathbf{R} \setminus \{0\}, X)$. *If* $\left| \left(\frac{d}{ds} \right)^j f(s) \right|_X \le C_f |s|^{-j}$ *for any* $s \in \mathbf{R} \setminus \{0\}$ *and* $j = 0, 1, 2$. *Then, it holds that*

$$\frac{1}{|h|} \int_{-\infty}^{\infty} \left| \Delta_h^2 f(s) \right|_X ds \le C_f.$$

Proposition 3.5 ([10]). *Let* X *be a Banach space with norm* $|\cdot|_X$. *Let* $N \ge 0$ *be an integer and* $0 < \sigma \le 1$. *Assume that* $f \in \mathcal{C}^{N+\sigma}(\mathbf{R}, X)$. *Set*

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{its} ds.$$

Then, the following estimate holds:

$$|F(t)|_X \le C(1+|t|)^{-(N+\sigma)} \langle\!\langle f \rangle\!\rangle_{N+\sigma,X}.$$

4 Outline of the proof

Theorem 2.1 follows from the Lumer-Phillips theorem. Hereafter, We concentrate the proof of Theorem 2.2.

Lemma 4.1. *Set*

$$b(a) = \frac{a}{2a\sqrt{\tau} + 2(3a\tau + 1)\sqrt{\tau + 1}}.$$

Then for any $a > 0$, there exists an $M_a > 0$ such that

$$\left\|(\lambda\mathbb{I} + \mathbb{L})^{-1}\right\|_{\mathcal{L}(\mathscr{H}(\Omega))} \leq M_a$$

for $\lambda \in D_{a,b(a)} = \{\lambda \in \mathbf{C} \,|\, |\mathrm{Re}\,\lambda| \leq b(a), |\mathrm{Im}\,\lambda| \geq a\} \cup \{\lambda \in \mathbf{C} \,|\, \mathrm{Re}\,\lambda \geq b(a)\}$.

In what follows, $\ell > 0$ denotes the same positive number in Proposition 3.1. Moreover, let φ_r be a function in $C_0^\infty(\mathbf{R}^n)$ such that $\varphi_r(x) = 1$ if $|x| \leq r$ and $\varphi_r(x) = 0$ if $|x| \geq r+1$ and let ρ_d be a function in $C_0^\infty(\mathbf{R})$ such that $\rho_d(s) = 1$ if $|s| < d/2$ and $\rho_d(s) = 0$ if $|s| > d$.

Lemma 4.2. *Let $Q_d = \{\lambda \in \mathbf{C} \,|\, 0 < \mathrm{Re}\,\lambda < d, |\mathrm{Im}\,\lambda| < d\}$. Then the following assertions hold.*

(i) *There exist a $d > 0$ and an $\mathbb{R}(\lambda) \in \mathrm{Hol}\,(Q_d, \mathcal{L}(\mathscr{H}_r(\Omega), \mathcal{H}(\Omega_r)))$ such that*

$$\mathbb{R}(\lambda)\mathbb{X} = (\lambda\mathbb{I} + \mathbb{L})^{-1}\mathbb{X} \quad \text{for} \quad \mathbb{X} \in \mathscr{H}_r(\Omega) \quad \text{and} \quad \lambda \in Q_d,$$

where we have set

$$\mathcal{H}(\Omega_r) = \left\{{}^T(f, g) \,|\, f \in H^1(\Omega_r) \cap L^2_\sigma(\Omega),\, g \in L^2(\Omega_r) \cap L^2_\sigma(\Omega)\right\}.$$

(ii) *For any $\mathbb{X} \in \mathscr{H}_r(\Omega)$, $\mathbb{Y} \in \mathscr{H}(\Omega)$ and $\alpha < d$, there exists a positive constant $C = C_{n,r,\rho_d,\varphi_r}$ such that the following assertions hold.*

  (a) *When Ω is an exterior domain, the following estimate holds.*

$$\left\langle\!\left\langle \rho_d(\cdot)(\varphi_r\mathbb{R}(\alpha + i\cdot)\mathbb{X}, \mathbb{Y})_{\mathscr{H}(\Omega)} \right\rangle\!\right\rangle_{\frac{n}{2},\mathbf{R}} \leq C\|\mathbb{X}\|_{\mathscr{H}(\Omega)}\|\mathbb{Y}\|_{\mathscr{H}(\Omega)}.$$

  (b) *When Ω is a perturbed half-space, the following estimate holds.*

$$\left\langle\!\left\langle \rho_d(\cdot)(\varphi_r\mathbb{R}(\alpha + i\cdot)\mathbb{X}, \mathbb{Y})_{\mathscr{H}(\Omega)} \right\rangle\!\right\rangle_{\frac{n+1}{2},\mathbf{R}} \leq C\|\mathbb{X}\|_{\mathscr{H}(\Omega)}\|\mathbb{Y}\|_{\mathscr{H}(\Omega)}.$$

Proof. (i) In terms of $S(\lambda)$, we shall represent $(\lambda\mathbb{I} + \mathbb{L})^{-1}$. If we set $\mathbb{X} = {}^T(f, g)$ and

$$(\lambda\mathbb{I} + \mathbb{L})\mathbb{U} = \mathbb{X} \quad \text{for} \quad \mathbb{U} \in \mathscr{D}(\mathbb{L}),$$

we have

$$v = \lambda u - f \quad \text{and} \quad \{\lambda(\tau\lambda + 1) + A\}u = (\tau\lambda + 1)f + \tau g.$$

We take $\ell' < \ell$ so small that there exists an $\epsilon' < \pi/2$ such that $\lambda(\tau\lambda+1) \in \Sigma_{\ell,\epsilon}$ if $\lambda \in \Sigma_{\ell',\epsilon'}$. If we set

$$\mathbb{R}(\lambda) = \begin{bmatrix} (\tau\lambda+1)S(\lambda(\tau\lambda+1)) & \tau S(\lambda(\tau\lambda+1)) \\ \lambda(\tau\lambda+1)S(\lambda(\tau\lambda+1)) - 1 & \tau\lambda S(\lambda(\tau\lambda+1)) \end{bmatrix}, \tag{4.1}$$

we obtain

$$\mathbb{R}(\lambda)\mathbb{X} = (\lambda\mathbb{I} + \mathbb{L})^{-1}\mathbb{X} \quad \text{for} \quad \mathbb{X} \in \mathscr{H}_r(\Omega) \quad \text{and} \quad \lambda \in \Sigma_{\ell',\epsilon'},$$

because $\mathbb{R}(\lambda)\mathbb{X} \in \mathscr{D}(\mathbb{L})$ as it follows from the fact that $S(\lambda(\tau\lambda+1)) \in \mathcal{L}(L^2(\Omega), H^2(\Omega))$. Therefore, $\mathbb{R}(\lambda)$ satisfies the property of (i) with $d = 2\ell'/3$.

(ii) follows from Proposition 3.1, 3.3 and 3.4. □

By Theorem 2.1, the following estimate holds:

$$\|T(t)\|_{\mathcal{L}(\mathscr{H}(\Omega))} \leq 1, \quad \forall t \geq 0. \tag{4.2}$$

Then, by a lemma due to Huang [4, Lemma 1], we have the following lemma.

Lemma 4.3. *For any $\alpha > 0$ and $\mathbb{X} \in \mathscr{H}(\Omega)$, set*

$$g(\omega) = \left\|((\alpha + i\omega)\mathbb{I} + \mathbb{L})^{-1}\mathbb{X}\right\|_{\mathscr{H}(\Omega)}.$$

Then, $g(\omega) \in L^2(\mathbf{R})$ and

$$\lim_{|\omega|\to\infty} g(\omega) = 0,$$
$$\int_{-\infty}^{\infty} g(\omega)^2 d\omega \leq \frac{\pi}{\alpha}\|\mathbb{X}\|^2_{\mathscr{H}(\Omega)}.$$

Now, we shall give a proof of Theorem 2.2. Since the proofs of an exterior domain case and a perturbed half-space case are essentially the same, we only prove an exterior domain case. To do this, it is sufficient to prove the following proposition.

Proposition 4.4. *Let Ω be an exterior domain, φ_r be the same in Lemma 4.2. Then, it holds that*

$$\|\varphi_r T(t)\mathbb{X}\|_{\mathscr{H}(\Omega)} \leq C(1+t)^{-\frac{n}{2}}\|\mathbb{X}\|_{\mathscr{H}(\Omega)} \tag{4.3}$$

for any $t \geq 0$ and $\mathbb{X} \in \mathscr{H}_r(\Omega)$, where $C = C_{M_a,n,\varphi_r}$, and M_a denotes the constant arising from Lemma 4.1.

Proof. Since (4.2) holds, we have the following expression formula:

$$T(t)\mathbb{X} = \lim_{\omega\to\infty} \frac{1}{2\pi i}\int_{\alpha-i\omega}^{\alpha+i\omega} e^{\lambda t}(\lambda\mathbb{I} + \mathbb{L})^{-1}\mathbb{X}d\lambda, \quad \alpha > 0.$$

Hereafter, we set $\rho(s) = \rho_d(s)$. Let us take $\alpha < d$, $\mathbb{X} \in \mathscr{H}_r(\Omega)$ and $\mathbb{Y} \in \mathscr{H}(\Omega)$. Then, we see

$$(\varphi_r T(t)\mathbb{X}, \mathbb{Y})_{\mathscr{H}(\Omega)} = J_0(t) + J_\infty(t),$$

where

$$J_0(t) = \frac{1}{2\pi} e^{\alpha t} \int_{-\infty}^{\infty} e^{ist} \rho(s) \left(\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1}\mathbb{X}, \mathbb{Y} \right)_{\mathscr{H}(\Omega)} ds,$$
$$J_\infty(t) = \frac{1}{2\pi} e^{\alpha t} \lim_{\omega\to\infty} \int_{-\omega}^{\omega} e^{ist}(1 - \rho(s)) \left(\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1}\mathbb{X}, \mathbb{Y} \right)_{\mathscr{H}(\Omega)} ds.$$

By Lemma 4.2 and Proposition 3.5, we obtain

$$|J_0(t)| \le Ce^{\alpha t}(1+t)^{-\frac{n}{2}} \|\mathbb{X}\|_{\mathscr{H}(\Omega)} \|\mathbb{Y}\|_{\mathscr{H}(\Omega)}. \tag{4.4}$$

We set

$$J_\infty(t) = \frac{1}{2\pi} e^{\alpha t} \lim_{\omega\to\infty} L_\omega(t),$$

where

$$L_\omega(t) = \int_{-\omega}^{\omega} e^{ist}(1 - \rho(s)) \left(\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1}\mathbb{X}, \mathbb{Y} \right)_{\mathscr{H}(\Omega)} ds.$$

By the relation $(it)^{-1} de^{ist}/ds = e^{ist}$ and integration by parts, we have

$$L_\omega(t) = \sum_{k=1}^{\ell} \frac{(-1)^{k-1}}{(it)^k} L_\omega^k(t) + \frac{(-1)^l}{(it)^l} M_\omega^l(t),$$

where

$$L_\omega^k(t) = \left[e^{ist} \frac{d^{k-1}}{ds^{k-1}} \left\{ (1 - \rho(s)) \left(\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1}\mathbb{X}, \mathbb{Y} \right)_{\mathscr{H}(\Omega)} \right\} \right]_{s=-\omega}^{s=\omega},$$
$$M_\omega^l(t) = \int_{-\omega}^{\omega} e^{ist} \frac{d^l}{ds^l} \left\{ (1 - \rho(s)) \left(\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1}\mathbb{X}, \mathbb{Y} \right)_{\mathscr{H}(\Omega)} \right\} ds.$$

Since we have by Lemma 4.1

$$\left\| \frac{d^j}{ds^j} ((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \right\|_{\mathcal{L}(\mathscr{H}(\Omega))} \le j! M_a^j \left\| ((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \right\|_{\mathcal{L}(\mathscr{H}(\Omega))} \quad \text{for } |s| \ge a,$$

it follows from Lemma 4.3 that

$$\left| L_\omega^k(t) \right| \to 0, \quad \omega \to \infty. \tag{4.5}$$

Using the Leibniz rule and the adjoint operator $\mathbb{L}^*$ of $\mathbb{L}$, we obtain

$$\begin{aligned} \left| M_\omega^l(t) \right| &\le l! \int_{\frac{d}{2} \le |s| \le \omega} (1 - \rho(s)) \left| \left(((\alpha + is)\mathbb{I} + \mathbb{L})^{-l}\mathbb{X}, ((\alpha - is)\mathbb{I} + \mathbb{L}^*)^{-1} \varphi_r \mathbb{Y} \right)_{\mathscr{H}(\Omega)} \right| ds \\ &\quad + \sum_{k=0}^{l-1} \binom{l}{k} k! \int_{\frac{d}{2} \le |s| \le d} \left| \frac{d^{l-k}}{ds^{l-k}} \rho(s) \right| \left| \left(\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-k-1}\mathbb{X}, \mathbb{Y} \right)_{\mathscr{H}(\Omega)} \right| ds \\ &= K_1 + K_2. \end{aligned}$$

If we take $a < d/2$, by Lemma 4.1 and Lemma 4.3, we have

$$\begin{aligned} K_1 &\le Cl!M_a^{l-1}\left(\int_{\frac{d}{2}\le|s|}\left\|((\alpha+is)\mathbb{I}+\mathbb{L})^{-1}\mathbb{X}\right\|_{\mathscr{H}(\Omega)}^2 ds\right)^{\frac12} \\ &\quad\times\left(\int_{\frac{d}{2}\le|s|}\left\|((\alpha-is)\mathbb{I}+\mathbb{L}^*)^{-1}\varphi_r\mathbb{Y}\right\|_{\mathscr{H}(\Omega)}^2 ds\right)^{\frac12} \\ &\le C_{l,M_a}\|\mathbb{X}\|_{\mathscr{H}(\Omega)}\|\mathbb{Y}\|_{\mathscr{H}(\Omega)}. \end{aligned} \tag{4.6}$$

Moreover, by Lemma 4.1, we see

$$K_2 \le C_{l,M_a}\|\mathbb{X}\|_{\mathscr{H}(\Omega)}\|\mathbb{Y}\|_{\mathscr{H}(\Omega)}. \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), we obtain

$$|J_\infty(t)| \le \frac{e^{\alpha t}}{2\pi}t^{-l}C_{\ell,M_a}\|\mathbb{X}\|_{\mathscr{H}(\Omega)}\|\mathbb{Y}\|_{\mathscr{H}(\Omega)} \tag{4.8}$$

for any $l \ge 1$. Letting $\alpha \to 0$ in (4.4) and (4.8), we obtain (4.3) for any $\mathbb{X} \in \mathscr{H}_r(\Omega)$. □

References

[1] W. Dan, Y. Shibata, On a local energy decay of solutions of a dissipative wave equation, Funkcial. Ekvac. 38 (1995), 545–568.

[2] W. Dan, Y. Shibata, On the L_q-L_r estimates of the Stokes semigroup in a two-dimensional exterior domain, J. Math. Soc. Japan. 51 (1999), 181–207.

[3] W. Dan, T. Kobayashi, Y. Shibata, On the local energy decay approach to some fluid flow in an exterior domain. Recent topics on mathematical theory of viscous incompressible fluid (Tsukuba, 1996), Lecture Notes Numer. Appl. Anal. vol. 16, Kinokuniya, Tokyo, 1998, pp. 1–51.

[4] F. Huang, Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces, Ann. of Differential. Equations, 1 (1985), 43–56.

[5] H. Iwashita, L_q-L_r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q, Math. Ann. 285 (1989), 265–288.

[6] T. Kobayashi, T. Kubo, K. Nakamura On a local energy decay estimate of solutions to the hyperbolic type Stokes equations, J. Differential Equations, 264 (2018), 6061–6081.

[7] T. Kubo, Y. Shibata, On some properties of solutions to the Stokes equation in the half-space and perturbed half-space. Dispersive nonlinear problems in mathematical physics, Quad. Mat. vol. 15 Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 149–220.

[8] T. Kubo, Y. Shibata, On the Stokes and Navier-Stokes equations in a perturbed half-space, Adv. Differential Equations 10 (2005), 695–720.

[9] R. Racke, J. Saal, Hyperbolic Navier-Stokes equations I: Local well-posedness, Evol. Equ. Control Theory 1 (2012), 195–215.

[10] Y. Shibata, On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain, Tsukuba J. Math. 7 (1983), 1–68.