LIOUVILLE THEOREMS FOR THE STOKES EQUATIONS IN EXTERIOR DOMAINS

K. ABE

Abstract. This is a resume of the paper [1]. We study Liouville theorems for the non-stationary Stokes equations in exterior domains in $\mathbb{R}^n$ under decay conditions for spatial variables. As applications, we show boundedness of the Stokes semigroup on L^∞_σ for all $t > 0$ for $n \geq 3$ and for $n = 2$ with zero net force.

1. Introduction

We consider the Stokes equations:

$$\begin{aligned} \partial_t v - \Delta v + \nabla q = 0, \quad \operatorname{div} v &= 0 \qquad \text{in } \Omega \times (0,\infty), \\ v &= 0 \qquad \text{on } \partial\Omega \times (0,\infty), \\ v &= v_0 \qquad \text{on } \Omega \times \{t = 0\}, \end{aligned} \tag{1.1}$$

for exterior domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$. It is known that a solution operator (called the Stokes semigroup) $S(t) : v_0 \longmapsto v(\cdot, t) = (v_i(\cdot, t))_{1 \leq i \leq n}$ forms an analytic semigroup on L^p_σ for $p \in (1, \infty)$, of angle $\pi/2$ [30], [16], i.e., $S(t)v_0$ is a holomorphic function in the half plane $\{\mathrm{Re}\, t > 0\}$ on L^p_σ. Here, L^p_σ denotes the L^p-closure of $C^\infty_{c,\sigma}$, the space of all smooth solenoidal vector fields with compact support in Ω.

We say that an analytic semigroup is a *bounded* analytic semigroup of angle $\pi/2$ if the semigroup is bounded in the sector $\Sigma_\theta = \{t \in \mathbb{C}\backslash\{0\} \mid |\arg t| < \theta\}$ for each $\theta \in (0, \pi/2)$ [6, Definition 3.7.3]. The boundedness in the sector implies the bounds on the positive real line

$$\|S(t)\| \leq C, \quad \|A_p S(t)\| \leq \frac{C'}{t}, \quad t > 0, \tag{1.2}$$

where $\|\cdot\|$ denotes the operator norm and A_p is the generator. The estimates (1.2) are important to study large time behavior of solutions to (1.1). In terms of the resolvent, the boundedness of $S(t)$ of angle $\pi/2$ is equivalent to the estimate

$$\|(\lambda - A_p)^{-1}\| \leq \frac{C_\theta}{|\lambda|}, \quad \lambda \in \Sigma_{\theta+\pi/2}. \tag{1.3}$$

When Ω is a half space, $S(t)$ is a bounded analytic semigroup on L^p_σ of angle $\pi/2$ [26], [33], [8].

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The problem becomes more difficult when Ω is an exterior domain. For $n \geq 3$, the boundedness of $S(t)$ on L^p_σ is proved in [10] based on the resolvent estimate

$$|\lambda|\|v\|_{L^p} + |\lambda|^{1/2}\|\nabla v\|_{L^p} + \|\nabla^2 v\|_{L^p} \leq C\|f\|_{L^p}, \quad 1 < p < \frac{n}{2}, \tag{1.4}$$

for $v = (\lambda - A_p)^{-1}f$ and $\lambda \in \Sigma_{\theta+\pi/2} \cup \{0\}$. The estimate (1.4) implies (1.3) for $p \in (1, n/2)$ and the case $p \in [n/2, \infty)$ follows from a duality. Due to the restriction on p, the two-dimensional case is more involved. Indeed, the estimate (1.4) is optimal in the sense that

$$\|\nabla^2 v\|_{L^p} \leq C\|A_p v\|_{L^p}, \quad v \in D(A_p),$$

is not valid for any $p \in [n/2, \infty)$ [9]. Here, $D(A_p) = W^{2,p} \cap W^{1,p}_0 \cap L^p_\sigma$ and $W^{1,p}_0$ denotes the space of all $f \in W^{1,p}$ vanishing on $\partial\Omega$. For $n = 2$, the boundedness of the Stokes semigroup on L^p_σ is proved in [11] based on layer potentials for the Stokes resolvent (see also [34]).

We study the case $p = \infty$. When Ω is a half space, $S(t)$ is a bounded analytic semigroup on L^∞_σ of angle $\pi/2$ by explicit solution formulas [15], [31]. Here, L^∞_σ is defined by

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_\Omega f \cdot \nabla\Phi \mathrm{d}x = 0,\ \nabla\Phi \in G^1(\Omega) \right\},$$

and $G^1(\Omega) = \{\nabla\Phi \in L^1(\Omega) \mid \Phi \in L^1_{\mathrm{loc}}(\Omega)\}$. For a half space and domains with compact boundary, L^∞_σ agrees with the space of all $f \in L^\infty$ satisfying $\operatorname{div} f = 0$ in Ω and $f \cdot N = 0$ on $\partial\Omega$. Here, N is the unit outward normal vector field on $\partial\Omega$. Since $S(t)$ is bounded on L^∞, the associated generator A_∞ is defined also for $p = \infty$. For bounded domains [3] and exterior domains [4], analyticity of the semigroup on L^∞_σ follows from the a priori estimate

$$\|v\|_{L^\infty} + t^{1/2}\|\nabla v\|_{L^\infty} + t\|\nabla^2 v\|_{L^\infty} + t\|\partial_t v\|_{L^\infty} + t\|\nabla q\|_{L^\infty} \leq C\|v_0\|_{L^\infty}, \tag{1.5}$$

for $v = S(t)v_0$ and $t \leq T$. The estimate (1.5) implies (1.2) for $t \leq T$ and that $S(t)$ is analytic on L^∞_σ. Moreover, the angle of analyticity is $\pi/2$ by the resolvent estimate on L^∞_σ [5]. When Ω is bounded, the sup-norms in (1.5) exponentially decay as $t \to \infty$ and $S(t)$ is a bounded analytic semigroup on L^∞_σ of angle $\pi/2$. For exterior domains, it is non-trivial whether the Stokes semigroup is a bounded analytic semigroup on L^∞_σ.

For the Laplace operator or general elliptic operators, it is known that corresponding semigroups are analytic on L^∞ of angle $\pi/2$ [25], [32], [22]. Moreover, if the operators are uniformly elliptic, by Gaussian upper bounds for complex time heat kernels, the semigroups are bounded analytic on L^∞ of angle $\pi/2$. See [14, Chapter 3]. In particular, the heat semigroup with the Dirichlet boundary condition in an exterior domain for $n \geq 2$ is a bounded analytic semigroup on L^∞ of angle $\pi/2$. For the Stokes equations, the Gaussian upper bound may not hold. See [15], [31], [28] for a half space.

Large time L^∞-estimates of the Stokes semigroup have been studied for $n \geq 3$. Maremonti [23] proved the estimate

$$\|S(t)v_0\|_{L^\infty} \leq C\|v_0\|_{L^\infty}, \quad t > 0, \tag{1.6}$$

for exterior domains and $n \geq 3$ based on the short time estimate in [3]. Subsequently, Hieber-Maremonti [19] proved the estimate $t\|AS(t)v_0\|_{L^\infty} \leq C\|v_0\|_{L^\infty}$ for $t > 0$ and the results are extended in [7] for complex time $t \in \Sigma_\theta$ and $\theta \in (0, \pi/2)$ based on the approach in [23]. Of these papers, the case $n = 2$ is excluded. We are able to observe the difference between $n \geq 3$ and $n = 2$ from the representation formula of the Stokes flow due to Mizumachi [27]. See below (1.9). We shall study large time behavior of Stokes flows for $n \geq 2$ by a different approach.

Our approach is by a Liouville theorem. A Liouville theorem is a fundamental property to study regularity problems. It rules out non-trivial solutions defined in $\Omega \times (-\infty, 0]$, called ancient solutions. See [21], [29] for Liouville theorems of the Navier-Stokes equations and [20] for the Stokes equations. Liouville theorems are also important to study large time behavior of solutions. We prove non-existence of ancient solutions of (1.1) in exterior domains under spatial decay conditions. We then apply our Liouville theorems and prove the large time estimate (1.6) for complex time $t \in \Sigma_\theta$ and $\theta \in (0, \pi/2)$.

We say that $v \in L^1_{\mathrm{loc}}(\overline{\Omega} \times (-\infty, 0])$ is an ancient solution to the Stokes equations (1.1) if $\operatorname{div} v = 0$ in $\Omega \times (-\infty, 0)$, $v \cdot N = 0$ on $\partial\Omega \times (-\infty, 0)$ and

$$\int_{-\infty}^{0} \int_{\Omega} v \cdot (\partial_t \varphi + \Delta\varphi) \mathrm{d}x\mathrm{d}t = 0, \tag{1.7}$$

for all $\varphi \in C^{2,1}_c(\overline{\Omega} \times (-\infty, 0])$ satisfying $\operatorname{div} \varphi = 0$ in $\Omega \times (-\infty, 0)$ and $\varphi = 0$ on $\partial\Omega \times (-\infty, 0) \cup \Omega \times \{t = 0\}$. The conditions $\operatorname{div} v = 0$ and $v \cdot N = 0$ are understood in the sense that

$$\int_{\Omega} v \cdot \nabla\Phi \mathrm{d}x = 0 \quad \text{a.e. } t \in (-\infty, 0),$$

for all $\Phi \in C^1_c(\overline{\Omega})$.

Theorem 1.1 (Liouville theorem)**.** *Let Ω be an exterior domain with C^3-boundary in $\mathbb{R}^n$, $n \geq 2$. Let v be an ancient solution to the Stokes equations (1.1). Assume that*

$$v \in L^\infty(-\infty, 0; L^p) \quad \text{for } p \in (1, \infty). \tag{1.8}$$

Then, $v \equiv 0$.

If one removes the spatial decay condition (1.8), the assertion of Theorem 1.1 becomes false for $n \geq 3$ due to existence of stationary solutions which are asymptotically constant as $|x| \to \infty$ [9] (cf. [20] for bounded ancient solutions.) For $n = 2$, it is known that bounded stationary solutions do not exist [13]. See [1] for the proof of Theorem 1.1.

Theorem 1.1 is useful to study the large time estimate (1.6) for $t > 0$. We invoke the representation formula of the Stokes flow

$$v(x,t)=\int_{\Omega}\Gamma(x-y,t)v_0(y)\mathrm{d}y+\int_0^t\int_{\partial\Omega}V(x-y,t-s)T(y,s)N(y)\mathrm{d}H(y)\mathrm{d}s. \tag{1.9}$$

Here, $T=\nabla v+\nabla^T v-qI$ is the stress tensor and $V=(V_{ij}(x,t))_{1\le i,j\le n}$ is the Oseen tensor

$$V_{ij}(x,t)=\Gamma(x,t)\delta_{ij}+\partial_i\partial_j\int_{\mathbb{R}^n}E(x-y)\Gamma(y,t)\mathrm{d}y, \tag{1.10}$$

defined by the heat kernel $\Gamma(x,t)=(4\pi t)^{-n/2}e^{-|x|^2/4t}$ and the fundamental solutions of the Laplace equation E, i.e., $E(x)=(n(n-2)\alpha(n))^{-1}|x|^{-(n-2)}$ for $n\ge 3$ and $E(x)=-(2\pi)^{-1}\log|x|$ for $n=2$, where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. The formula (1.9) is obtained by regarding $v=S(t)v_0$ as the Stokes flow in $\mathbb{R}^n$ with a measure as the external force. It describes the asymptotic behavior of bounded Stokes flow as $|x|\to\infty$. We show that if the Stokes flow is bounded for all $t>0$, the stress tensor is also bounded on $\partial\Omega$. Observe that by the pointwise estimate of the Oseen tensor

$$|V(x,t)|\le\frac{C}{(|x|+t^{1/2})^n},\quad x\in\mathbb{R}^n,\ t>0, \tag{1.11}$$

the remainder term is estimated by

$$\left|v(x,t)-\int_{\Omega}\Gamma(x-y,t)v_0(y)\mathrm{d}y\right|\le\frac{C}{|x|^{n-2}}\sup_{0<s\le t}\|T\|_{L^\infty(\partial\Omega)}(s), \tag{1.12}$$

for $|x|\ge 2R_0$ and $t>0$ such that $\Omega^c\subset B_0(R_0)$, where $B_0(R_0)$ denotes the open ball centered at the origin with radius $R_0>0$. The right-hand side is decaying as $|x|\to\infty$ uniformly for $t>0$ if $n\ge 3$. We are able to show that the large time estimate (1.6) is reduced to showing non-existence of ancient solutions by a contradiction argument. Since (1.12) yields a spatial decay condition for ancient solutions as $|x|\to\infty$, we are able to obtain a contradiction by applying the Liouville theorem (Theorem 1.1). We apply a similar argument on the half line $\gamma=\{t\in\mathbb{C}\backslash\{0\}|\ \arg t=\theta\}$ and prove (1.6) for complex time $t\in\Sigma_\theta$ and $\theta\in(0,\pi/2)$.

Theorem 1.2. *When $n\ge 3$, the Stokes semigroup is a bounded analytic semigroup on L^∞_σ of angle $\pi/2$.*

For $n=2$, the remainder term estimate (1.12) is different. By a simple calculation from the formula (1.9), we see an asymptotic profile of the two-dimensional Stokes flow:

$$\left|v(x,t)-\int_{\Omega}\Gamma(x-y,t)v_0(y)\mathrm{d}y-\int_0^t V(x,t-s)F(s)\mathrm{d}s\right|\le\frac{C}{|x|}\sup_{0<s\le t}\|T\|_{L^\infty(\partial\Omega)}(s), \tag{1.14}$$

for $|x|\ge 2R_0$ and $t>0$, with the net force

$$F(s) = \int_{\partial\Omega} T(y,s)N(y)\mathrm{d}H(y).$$

Since $|\int_0^t V(x,s)\mathrm{d}s| \lesssim \log(1+t/|x|^2)$, the decay as $|x| \to \infty$ of the third term in the left-hand side is not uniform for $t > 0$ in contrast to (1.12) for $n \geq 3$. If the net force vanishes, the situation is the same as $n = 3$ and we are able to prove (1.6). For example, if initial data is rotationally symmetric, the net force vanishes. Following [12], we consider initial data invariant under a cyclic group or a dihedral group. For integers $m \geq 2$, we set the matrices

$$R_m = \begin{pmatrix} \cos(2\pi/m) & -\sin(2\pi/m) \\ \sin(2\pi/m) & \cos(2\pi/m) \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let C_m denote the cyclic group of order m generated by the rotation R_m. Let D_m denote the dihedral group of order $2m$ generated by R_m and the reflection J. Any finite subgroup of the orthogonal group $O(2)$ is either a cyclic group or a dihedral group. See [17, Chapter 2]. Let G be a subgroup of $O(2)$ and Ω^c be a disk centered at the origin. We say that a vector field v is G-covariant if $v(x) = {}^tAv(Ax)$ for all $A \in G$ and $x \in \Omega$. It is known that if v_0 is C_m-covariant, so is $v = S(t)v_0$ and the net force vanishes, i.e., $F \equiv 0$ [18]. Thus for C_m-covariant vector fields $v_0 \in L^\infty_\sigma$, the remainder term estimate is the same as $n = 3$.

Theorem 1.3. *For $n = 2$, the estimate (1.6) holds for $t \in \Sigma_\theta$ and $v_0 \in L^\infty_\sigma$, for which the net force vanishes (e.g., C_m-covariant vector fields when Ω^c is a disk.)*

Theorem 1.3 improves the pointwise estimates of the two-dimensional Navier-Stokes flows for rotationally symmetric initial data [18], in which (1.6) is noted as an open question together with the applications to the nonlinear problem. We are able to apply (1.6) to improve the results although initial data is restricted to rotationally symmetric.

We hope it is possible to extend our approach to study the case with net force, for which (1.6) is unknown even if initial data is with finite Dirichlet integral. The estimate (1.6) with non-vanishing net force is important to study large time behavior of asymptotically constant solutions as $|x| \to \infty$. We refer to [2] for asymptotically constant solutions of the two-dimensional Navier-Stokes equations. See also [24].

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(K. ABE) Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku Osaka, 558-8585, Japan

E-mail address: kabe@sci.osaka-cu.ac.jp