Viscous compressible fluids with only bounded density

Raphaël Danchin[1], Francesco Fanelli[2] and Marius Paicu[3]

Here we survey our paper [4] and give an example of application, Corollary 2.1, that is not contained in [4].

We are concerned with the following compressible Navier-Stokes equations:

$$
(CNS):\begin{cases}(\rho u)_t+\mathrm{div}\,(\rho u\otimes u)-\mu\Delta u-\mu'\nabla\mathrm{div}\,u+\nabla P=0 & \text{in } \mathbb{R}_+\times\mathbb{R}^d,\\ \rho_t+\mathrm{div}\,(\rho u)=0 & \text{in } \mathbb{R}_+\times\mathbb{R}^d,\\ (\rho,u)|_{t=0}=(\rho_0,u_0) & \text{in } \mathbb{R}^d.\end{cases}
$$

The unknowns are the velocity field $u=u(t,x)\in\mathbb{R}^d$ and the density $\rho=\rho(t,x)\geq 0$ with $t\geq 0$ and $x\in\mathbb{R}^d$ $(d\geq 1)$.

The pressure P is a given locally Lipschitz function of the density, and it is assumed that the (constant) viscosity coefficients μ and μ' fulfill

$$
\mu>0 \quad\text{and}\quad \mu+\mu'>0. \tag{1}
$$

The above system is supplemented with initial data ρ_0 and u_0 at time $t=0$. We have in mind the singular situation where the density is *discontinuous* along an interface, like for instance:

$$
\rho_0=\rho_0^1 1_{D_0}+\rho_0^2 1_{{}^cD_0} \quad\text{with}\quad \rho_0^1,\,\rho_0^2>0 \quad\text{and}\quad D_0\subset\subset\mathbb{R}^d, \tag{2}
$$

where ρ_0^1, ρ_0^2 and ∂D_0 are reasonably smooth.

For such an initial density, we would like to find out suitable conditions on u_0 ensuring that (CNS) has a unique local-in-time solution, and that the structure (2) is propagated. Let us underline that, in contrast with the incompressible situation, there is no chance that the characteristic function structure is preserved *stricto* sensu, since the density need not be conserved along the flow of the velocity field.

Before going into more details, let us shortly review some classical results for (CNS). Thanks to the pioneering works by P.-L. Lions [13] in 1996 and E. Feireisl [6] in 2001, the weak solution theory in the case of the isentropic pressure law $P(\rho)=a\rho^\gamma$ with $\gamma>d/2$ is by now well understood. It is based on the following (formal) energy balance

$$
\int_{\mathbb{R}^d}\Big(\frac12\rho|u|^2+e(\rho)\Big)(t)\,dx+\int_0^t\int_{\mathbb{R}^d}(\mu|\nabla u|^2+\mu'(\mathrm{div}\,u)^2)\,dx=\int_{\mathbb{R}^d}\Big(\frac12\rho_0|u_0|^2+e(\rho_0)\Big)\,dx
$$

[1]LAMA, UMR 8050, Université Paris-Est Créteil, France
[2]Institut Camille Jordan, UMR 5208, Université de Lyon 1, France
[3]Institut de Mathématiques de Bordeaux, UMR 5251, Université de Bordeaux, France

where the internal energy e satisfies $ze''(z) = P'(z)$, and on rather subtle compactness arguments implemented on the solutions to a family of suitable approximate systems. It goes without saying that uniqueness in the class of finite energy solutions is a widely open question.

At the exact opposite, one can find the more ancient theory of local-in-time classical solutions for smooth data with no vacuum (J. Serrin [16] in 1959 and J. Nash [15] in 1962), local strong solutions with Sobolev regularity (see the works of A. Tani [18] and V. Solonnikov [17]), global solutions for small perturbations of a constant state $(\bar\rho, 0)$ with $P'(\bar\rho) > 0$ (A. Matsumura and T. Nishida [14] in 1983), and works dedicated to solutions with critical regularity (see [3] and more recent papers in the same spirit).

Unfortunately, none of those works fit in our goal since the weak solution theory does not give much insight on the propagation of density discontinuities, and the strong solution theory does not allow for density discontinuity.

In the 90ies, D. Hoff in [8] came up with an 'intermediate solutions' theory that corresponds to data (ρ_0, u_0) such that ρ_0 is close to some constant $\bar\rho > 0$ in L^∞ and $u_0 \in L^q \cap H^s$ small with $s \geq \frac{d}{2} - 1$ and $q > d$ $(d = 2, 3)$. This enabled him in [9] to prove that if the pressure law is linear then, for any bounded domain D_0 with a $C^{1,\alpha}$ boundary and ρ_0^1, ρ_0^2 in $C^{0,\alpha}$, if $\|\rho_0^2 - \rho_0^1\|_{L^\infty} \ll 1$ then the structure (2) is propagated for all time and $\|(\rho^2 - \rho^1)(t)\|_{L^\infty} \leq Ce^{-ct}$ (see also [11] for a more general result).

However, to the best of our knowledge, it is not known whether Hoff's solutions are unique among a class of functions where the structure (2) is not prescribed. In this direction, one can mention the work by D. Hoff in [10] where it is proved that, if the pressure function is given by $P(\rho) = K\rho$, then one has weak-strong uniqueness on the time interval $[0, T]$ if, essentially, one of the two solutions has velocity in $L^1_{loc}(0, T; \mathrm{Lip})$.

Let us emphasize that Hoff's results (as well as the weak solution theory) strongly rely on the fundamental observation that the *viscous effective flux*

$$F := \operatorname{div} u - \nu^{-1} P(\rho) \quad \text{with} \quad \nu := \mu + \mu', \tag{3}$$

is more regular than $\operatorname{div} u$ or P taken separately.

The main aims of this note are:

– to address the existence issue for a class of initial densities containing the particular case of (2);

– to prove the propagation of related geometrical structures,

– to supplement Hoff's result with a uniqueness statement,

– to present a unified method that works for all dimensions and pressure laws.

The rest of the text unfolds as follows. In the next section, by taking advantage of the classical maximal regularity theory for parabolic systems, we establish an existence result for rather general initial densities with no smoothness, then, in Section 2, we prove the local-in-time propagation of geometrical structures that encompass (2). The last section of the paper is devoted to sketching the proof of uniqueness for the solutions constructed in Section 2.

1 An existence result based on the classical maximal regularity

In the present section, we outline the main ideas leading to a local existence statement in the case where the initial density is only bounded, but close enough to a positive constant $\bar\rho$. The overall approach is essentially based on the use of the viscous effective flux defined in (3), and on parabolic maximal regularity estimates.

For notational simplicity, we shall assume throughout that $\bar\rho = 1$. Then, if one denotes $\varrho := \rho - 1$, the velocity equation may be written:

$$u_t - \mu\Delta u - \mu'\nabla \mathrm{div}\, u = g := \varrho\, u_t - \rho u\cdot\nabla u - \nabla P(\rho). \tag{4}$$

Recall that the maximal regularity theory for the heat equation tells us that, if z fulfills

$$\begin{cases} z_t - \Delta z = f & \text{in } \ \mathbb{R}_+\times\mathbb{R}^d,\\ z|_{t=0} = z_0 & \text{on } \ \mathbb{R}^d,\end{cases}$$

then one has, for all $1 < p, r < \infty$ and $t > 0$, the a priori estimate:

$$\|z\|_{E_t^{p,r}} := \|z\|_{L_t^\infty(\dot B^{2-\frac2r}_{p,r})} + \|(z_t,\nabla^2 z)\|_{L_t^r(L^p)} \lesssim \|z_0\|_{\dot B^{2-\frac2r}_{p,r}} + \|f\|_{L_t^r(L^p)},$$

where the *homogeneous Besov norm* in the right-hand side is defined by

$$\|z_0\|_{\dot B^{2-\frac2r}_{p,r}} := \biggl(\int_0^\infty \Bigl((\sqrt t)^{\frac2r}\|\nabla^2 e^{t\Delta}z_0\|_{L^p}\Bigr)^r\frac{dt}{t}\biggr)^{\frac1r}$$

and the notation $\|\cdot\|_{L_t^r(X)}$ is a shortcut for the norm in $L^r([0,t];X)$.

Now, let us observe that the divergence free and potential parts $\mathcal{P}u$ and $\mathcal{Q}u$ of u in (4) fulfill the following heat equations:

$$(\mathcal{P}u)_t - \mu\Delta\mathcal{P}u = \mathcal{P}g \quad\text{and}\quad (\mathcal{Q}u)_t - (\mu+\mu')\Delta\mathcal{Q}u = \mathcal{Q}g.$$

Hence, since (1) has been assumed and $\mathcal{P},\mathcal{Q} : L^p \to L^p$, we get for all $t \geq 0$,

$$\|u\|_{E_t^{p,r}} \lesssim \|u_0\|_{\dot B^{2-\frac2r}_{p,r}} + \|\varrho\, u_t\|_{L_t^r(L^p)} + \|\rho u\cdot\nabla u\|_{L_t^r(L^p)} + \|\nabla P\|_{L_t^r(L^p)}.$$

Obviously, the second term of the right-hand side may be absorbed by the left-hand side if $\|\varrho\|_{L^\infty} \ll 1$, and it is not difficult to check (by combining Sobolev embedding and Hölder inequality) that the term with $u\cdot\nabla u$ tends to 0 when $t\to 0$ whenever the function space $E_t^{p,r}$ is subcritical with respect to (CNS) (that is $2-2/r > d/p-1$). The troublemaker is ∇P (or equivalently $\nabla\rho$) since having it in a Lebesgue space precludes our considering discontinuity across an interface for the density.

To overcome the difficulty, let us introduce (in the spirit of (3)) the *modified velocity* $w := u + \nabla(-\nu\Delta)^{-1}P$. Since $\mathrm{div}\, w = \mathrm{div}\, u - \nu^{-1}P$, one can expect the coupling between ρ and w to be milder than between ρ and u. Indeed, using the fact that

$$P_t = h\,\mathrm{div}\, u - \mathrm{div}\,(Pu) \quad\text{with}\quad h := P - \rho P',$$

we discover that the equation for w reads

$$w_t - \mu\Delta w - \mu'\nabla \mathrm{div}\, w = \varrho w_t - \rho u\cdot\nabla u + (-\nu\Delta)^{-1}\nabla(h\,\mathrm{div}\, u - \mathrm{div}\,(Pu))\cdot$$

The gain is that all the terms involving the density or the pressure are of order 0. Therefore, after combining the standard maximal regularity estimate with Sobolev embedding and Gagliardo-Nirenberg inequality, we get, if $\|\varrho\|_{L^\infty}$ is small enough:

$$\|w\|_{E_t^{p,r}} \lesssim \|w_0\|_{\dot B^{2-\frac{2}{r}}_{p,r}} + \|u\cdot\nabla u\|_{L^r_t(L^p)} + \|Pu\|_{L^r_t(L^p)} + \text{l.o.t.}$$

Now, since ϱ fulfills

$$\varrho_t + u\cdot\nabla\varrho + (1+\varrho)\mathrm{div}\, w + \nu^{-1}(1+\varrho)P = 0,$$

classical estimates for the transport equation enable us to bound ϱ in any Lebesgue space *provided that* $\mathrm{div}\, w$ *is in* $L^1(0,T;L^\infty)$.

Since we expect w to be in the space $E_T^{p,r}$, that latter condition will be achieved if $p>d$. A technical point that we omit here is that the operator $(-\nu\Delta)^{-1}$ in the definition of w is actually too singular (in particular in dimension 2), and has to be replaced with the milder one $(\mathrm{Id}-\nu\Delta)^{-1}$. In the end, one gets the following statement, the complete proof of which may be found in [4]:

Theorem 1.1 *Let* $d\geq 1$ *and* (p,r) *satisfy* $d<p<\infty$ *and* $1<r<\frac{2p}{2p-d}\cdot$ *Assume that*

- $\varrho_0 := \rho_0 - 1$ *belongs to* $L^p\cap L^\infty$*;*
- $w_0 := u_0 - v_0$ *is in* $\dot B^{2-2/r}_{p,r}$*, with* $v_0 := -\nabla(\mathrm{Id}-\nu\Delta)^{-1}(P(\rho_0))$.

There exist $\varepsilon>0$ *and a time* $T>0$ *such that, if*

$$\|\varrho_0\|_{L^\infty} \leq \varepsilon, \tag{5}$$

then there exists a solution (ρ,u) *to* (CNS) *on* $[0,T]\times\mathbb{R}^d$ *with* $\varrho:=\rho-1$ *satisfying*

$$\|\varrho\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq 2\varepsilon \quad \text{and} \quad \varrho\in\mathcal{C}([0,T];L^q) \quad \text{for all } \ p\leq q<\infty.$$

Furthermore, $u = v+w$ *with* $v := -\nabla(\mathrm{Id}-\nu\Delta)^{-1}(P(\rho))$ *is in* $\mathcal{C}\left([0,T];W^{1,q}(\mathbb{R}^d)\right)$ *for all* $p\leq q<\infty$*, and*

$$w\in\mathcal{C}([0,T];\dot B^{2-2/r}_{p,r}),\quad \partial_t w,\ \nabla^2 w\in L^r(0,T;L^p).$$

That solution is unique if $d=1$.

In higher dimension, the uniqueness issue in the above functional framework is an open question. In order to pinpoint where the difficulty lies, consider two solutions (ρ^1,u^1) and (ρ^2,u^2) of (CNS) emanating from the same initial data. Then $(\delta\rho,\delta u) := (\rho^2-\rho^1, u^2-u^1)$ fulfills

$$\begin{cases} \delta\rho_t + \mathrm{div}\,(u^1\delta\rho) = -\mathrm{div}\,(\delta u\,\rho^2),\\ \delta u_t - \mu\Delta\delta u - \mu'\nabla\mathrm{div}\,\delta u = (1-\rho^1)\delta u_t + \delta\rho\,(u_t^2 + u^2\cdot\nabla u^2)\\ \qquad\qquad + \rho^1(u^1\cdot\nabla\delta u + \delta u\cdot\nabla u^2 + \nabla(P(\rho^1)-P(\rho^2)). \end{cases}$$

The (unavoidable) loss of one derivative in the first equation spoils the second one since, for instance, having $\delta\rho$ in a negative regularity space does not allow to get any control on $\delta\rho\,(u_t^2 + u^2\cdot\nabla u^2)$.

Another weakness of Theorem 1.1 is that, even though one may consider initial density like (2), it does not tell us much on the evolution of that structure.

To some extent, the diagnostic in the two cases is the same: we lack the property that $\nabla u \in L^1([0,T];L^\infty)$, as this would ensure the solution to have a Lipschitz flow. Our aim (and this is the object of the next part) is to exhibit additional assumptions on the data, ensuring that $\nabla u \in L^1([0,T];L^\infty)$ but that, nevertheless, the density may have the structure (2).

2 Striated regularity

A natural question is how far from Lipschitz we are, under the hypotheses of Theorem 1.1. From it and Sobolev embedding, we know that both $\operatorname{div} u$ and $\operatorname{curl} u$ are in $L^1(0,T;L^\infty)$, since

$$\operatorname{div} u = \underbrace{\operatorname{div} w}_{W^{1,p}\hookrightarrow L^\infty} + \nu^{-1}\underbrace{P}_{L^\infty} - \nu^{-1}\underbrace{(\mathrm{Id} - \nu\Delta)^{-1}P}_{\text{smooth}} \quad \text{and} \quad \operatorname{curl} u = \operatorname{curl} w \in L^r(0,T;W^{1,p}).$$

However, unless $d=1$, those two conditions together do not quite imply that $\nabla u \in L^1(0,T;L^\infty)$ (only that $\nabla u \in L^1(0,T;\mathrm{BMO})$ actually), so that we do not know whether u has a Lipschitz flow and if the Lipschitz regularity of the domain D_0 in (2) is preserved.

In order to figure out what are the missing ingredients to achieve the Lipschitz regularity, let us consider the 'flat' situation. Then, if both $\operatorname{div} u$ and $\operatorname{curl} u$ are in L^∞ and, in addition, first order derivatives of u in $d-1$ independent directions are bounded, it is obvious that we do have $\nabla u \in L^\infty$.

The way to generalize that property to *nonflat* situations goes back to the work by J.-Y. Chemin in [1], and relies on the notion of *striated regularity* that we introduce now.

Let $\mathcal{X} = (X_\lambda)_{1\leq\lambda\leq m}$ be a family of vector-fields with components in the space

$$\mathbb{L}^{\infty,p} := \{X \in L^\infty,\ \nabla X \in L^p\},$$

that is *non-degenerate* in the following sense:

$$I(\mathcal{X}) := \inf_{x\in\mathbb{R}^d} \sup_{\Lambda\in\Lambda^m_{d-1}} \left| \overset{d-1}{\wedge} X_\Lambda(x)\right|^{\frac{1}{d-1}} > 0\,.$$

Here $\Lambda \in \Lambda^m_{d-1}$ means that $\Lambda = (\lambda_1,\dots,\lambda_{d-1})$ with $1\leq\lambda_1<\cdots<\lambda_{d-1}\leq m$, while $\overset{d-1}{\wedge} X_\Lambda$ stands for the unique element of $\mathbb{R}^d$ such that

$$\forall\, Y \in \mathbb{R}^d\,, \qquad \Big(\overset{d-1}{\wedge} X_\Lambda\Big)\cdot Y = \det\big(X_{\lambda_1}\dots X_{\lambda_{d-1}}, Y\big)\cdot$$

Since for a general function f in L^∞, $\partial_{X_\lambda} f$ need not be defined (in contrast with $\operatorname{div}(X_\lambda f)$ and $f\operatorname{div} X_\lambda$) and, in the smooth case, we have

$$\partial_{X_\lambda} f = \operatorname{div}(X_\lambda f) - f\operatorname{div} X_\lambda,$$

we adopt the following definition of regularity along a non-degenerate family of vector fields.

Definition 2.1 *Let Y be in $\mathbb{L}^{\infty,p}$ for some $p \in]d,\infty]$. A function $f \in L^\infty$ is said to be in $\mathbb{L}^p_Y$ if* $\operatorname{div}(fY) \in L^p(\mathbb{R}^d)$. *If $\mathcal{X} = (X_\lambda)_{1\leq\lambda\leq m}$ is a non-degenerate family of vector-fields in $\mathbb{L}^{\infty,p}$, then we set*

$$\mathbb{L}^p_{\mathcal{X}} := \bigcap_{1\leq\lambda\leq m} \mathbb{L}^p_{X_\lambda} \quad \text{and} \quad \|f\|_{\mathbb{L}^p_{\mathcal{X}}} := \frac{1}{I(\mathcal{X})}\Big(\|f\|_{L^\infty}\|\mathcal{X}\|_{\mathbb{L}^{\infty,p}} + \|\operatorname{div}(f\,\mathcal{X})\|_{L^p}\Big)\cdot$$

The generalization of the above observation to the nonflat situation is a consequence of the following statement.

Proposition 2.1 *Let $d<p<\infty$ and $m\geq d-1$. Take a nondegenerate family $\mathcal{X} = (X_\lambda)_{1\leq\lambda\leq m}$ of vector-fields belonging to $\mathbb{L}^{\infty,p}$. Then, the following inequality holds true for all $\nu>0$:*

$$\big\|\nabla^2(\mathrm{Id}-\nu\Delta)^{-1}P\big\|_{L^\infty} \lesssim \bigg(1+\frac{\|\mathcal{X}\|^{4d-5}_{L^\infty}\|\nabla\mathcal{X}\|_{L^p}}{(I(\mathcal{X}))^{4d-4}}\bigg)\|P\|_{L^\infty} + \frac{\|\mathcal{X}\|^{4d-5}_{L^\infty}}{(I(\mathcal{X}))^{4d-4}}\|\partial_{\mathcal{X}}P\|_{L^p}\cdot$$

Proof. Fix some $\Lambda\in\Lambda^m_{d-1}$ and consider the set U_Λ of those x in $\mathbb{R}^d$ satisfying

$$\Big(\overset{d-1}{\wedge}\ X_\Lambda(x)\Big) \geq (I(\mathcal{X}))^{d-1}.$$

Then, for all $x\in U_\Lambda$ and $\xi\in\mathbb{R}^d$, one has the following algebraic identity (see [2, Lemma 3.2]):

$$\xi_i\,\xi_j = \frac{\Big(\overset{d-1}{\wedge}\ X_\Lambda(x)\Big)^i\Big(\overset{d-1}{\wedge}\ X_\Lambda(x)\Big)^j}{\Big|\overset{d-1}{\wedge}\ X_\Lambda(x)\Big|^2}|\xi|^2 + \frac{1}{\Big|\overset{d-1}{\wedge}\ X_\Lambda(x)\Big|^4}\sum_{k,\ell} b^{k\ell}_{ij}(x)\,\xi_k(X_{\lambda_\ell}(x)\cdot\xi)$$

where the $b^{k\ell}_{ij}$'s are homogeneous of degree $4d-5$ with respect to the X_λ's.

Multiplying by $(1+\nu|\xi|^2)^{-1}\widehat{P}(\xi)$ then taking the inverse Fourier transform yields:

$$(\mathrm{Id}-\nu\Delta)^{-1}\partial_i\partial_j P = \frac{\Big(\overset{d-1}{\wedge}\ X_\Lambda\Big)^i\Big(\overset{d-1}{\wedge}\ X_\Lambda\Big)^j}{\Big|\overset{d-1}{\wedge}\ X_\Lambda\Big|^2}\Delta(\mathrm{Id}-\nu\Delta)^{-1}P$$
$$+\frac{1}{\Big|\overset{d-1}{\wedge}\ X_\Lambda\Big|^4}\sum_{k,\ell} b^{k\ell}_{ij}\partial_{X_{\lambda_\ell}}\big(\partial_k(\mathrm{Id}-\nu\Delta)^{-1}P\big)\cdot$$

We conclude thanks to the following inequality based on commutator estimates (see the appendix of [4]):

$$\|\partial_{X_{\lambda_\ell}}\partial_k(\mathrm{Id}-\nu\Delta)^{-1}P\|_{L^\infty} \lesssim \|P\|_{L^\infty}\|\nabla X_{\lambda_\ell}\|_{L^p} + \|\partial_{X_{\lambda_\ell}}P\|_{L^p} \quad \text{if} \quad p>d.$$

Since the union of all U_Λ's is equal to $\mathbb{R}^d$, one gets the desired inequality. ∎

The above proposition is the key to the following existence *and uniqueness* statement.

Theorem 2.1 *Let the assumptions of Theorem 1.1 be in force and assume in addition that there exists a non-degenerate family* $\mathcal{X}_0 = (X_{0,\lambda})_{1\leq\lambda\leq m}$ *in* $\mathbb{L}^{\infty,p}$ *such that* ρ_0 *is in* $\mathbb{L}^p_{\mathcal{X}_0}$.

Then, there exists a time $T > 0$ *and a unique solution* (ρ, u) *to* (CNS) *on* $[0,T]$ *satisfying the properties of the previous theorem and* $\nabla u \in L^1([0,T]; L^\infty(\mathbb{R}^d))$.

In particular, u *has a unique Lipschitz flow* ψ*, that is the solution of*

$$\psi(t,x) = x + \int_0^t u(\tau, \psi(\tau,x))\, d\tau \qquad \textit{for all} \;\; (t,x) \in [0,T] \times \mathbb{R}^d,$$

and the family $\mathcal{X}_t := (X_{t,\lambda})_{1\leq\lambda\leq m}$ *with* $X_{t,\lambda}(x) := \partial_{X_{0,\lambda}}\psi(t, \psi^{-1}(t,x))$ *is non-degenerate for all* $t \in [0,T]$ *and in* $\mathbb{L}^{\infty,p}$*, and* $\rho(t)$ *belongs to* $\mathbb{L}^p_{\mathcal{X}_t}$.

Proof. Let us report the main steps of the proof of the existence part of Theorem 2.1 (uniqueness will be discussed in the next section). Since Theorem 1.1 ensures the existence of a solution (ρ, u) on $[0,T] \times \mathbb{R}^d$ for some $T > 0$ with the regularity described therein, it is only a matter of checking that striated regularity is preserved.

Now, the evolution equation for transported vector-fields reads:

$$(\partial_t + u \cdot \nabla) X_\lambda = \partial_{X_\lambda} u.$$

We need to estimate X_λ in L^∞ and ∇X_λ in L^p, which requires at a minimum that

$$\nabla u \in L^1(0,T; L^\infty), \quad \partial_{X_\lambda} u \in L^1(0,T; L^\infty) \quad \text{and} \quad \nabla(\partial_{X_\lambda} u) \in L^1(0,T; L^p). \qquad (6)$$

Next, in order to study the propagation of striated regularity for ρ and P, one may use that

$$\partial_{X_\lambda}\rho = \operatorname{div}(X_\lambda \rho) - \rho \operatorname{div} X_\lambda \quad \text{and} \quad (\partial_t + u\cdot\nabla)(\operatorname{div}(X_\lambda\rho)) = -\operatorname{div} u \operatorname{div}(X_\lambda \rho).$$

Hence, to estimate $\operatorname{div}(X_\lambda \rho)$ in L^p, we need $\operatorname{div} u \in L^1(0,T; L^\infty)$.

Finally, since $u = w - \nabla(\mathrm{Id} - \nu\Delta)^{-1}P$, we have

$$\nabla u = \nabla w - \nabla^2(\mathrm{Id} - \nu\Delta)^{-1}P \quad \text{and} \quad \partial_{X_\lambda}\nabla u = \partial_{X_\lambda}\nabla w - \partial_{X_\lambda}\nabla(\mathrm{Id} - \nu\Delta)^{-1}\nabla P.$$

Hence, to achieve (6), it suffices to prove that:

- $\nabla w \in L^1(0,T; L^\infty)$ and $\nabla(\partial_{X_\lambda} w) \in L^1(0,T; L^p)$;
- $\nabla^2(\mathrm{Id} - \nu\Delta)^{-1}P \in L^\infty(0,T; L^\infty)$;
- $\nabla(\partial_{X_\lambda}\nabla(\mathrm{Id} - \nu\Delta)^{-1}P) \in L^1(0,T; L^p)$.

The first item is an easy consequence of the fact that $w \in E_T^{p,r}$ and of the embedding $W^{1,p} \hookrightarrow L^\infty$ while the second one is given by Prop. 1. The last part follows from rather tricky commutator estimates that are omitted here.

Of course, the estimates given by the above arguments depend the one from the others, hence one has to resort to a bootstrap argument in order to eventually close the estimates for small enough time. ■

Corollary 2.1 *Assume that* $\rho_0 = \rho_0^1 1_{D_0} + \rho_0^2 1_{^cD_0}$ *where* ρ_0^1 *and* ρ_0^2 *are in* $L^\infty \cap \dot{W}^{1,p}$ *and* D_0 *is a* $W^{2,p}$ *bounded domain of* $\mathbb{R}^d$ *with* $p > d$*. If* u_0 *is as above and, in addition,* $\|\rho_0^1 - \rho_0^2\|_{L^\infty} \ll 1$ *then* (CNS) *admits a unique solution on* $[0,T]$ *such that*

$$\rho(t) = \rho_t^1 1_{D_t} + \rho_t^2 1_{^cD_t} \quad \textit{for all} \;\; t \in [0,T],$$

with $D_t := \psi(t, D_0)$ *having* $W^{2,p}$ *regularity, and* ρ_t^1 *and* ρ_t^2 *in* $W^{1,p}$.

Proof. Assume that D_0 corresponds (locally) to the level set $\{\phi_0 = 0\}$ of some $W^{2,p}$ function with nondegenerate gradient on ∂D_0. Then, one can take for $(X_{0,\lambda})_{1\leq\lambda\leq m}$ a suitable family constructed from linear combinations of components of $\nabla\phi_0$ (one may use for instance the construction given in Prop. 5.1 of [2]) and check that ρ_0 is in $\mathbb{L}^p_{\mathcal{X}_0}$. Hence Theorem 2.1 applies.

Now, if one defines $\phi_t := \phi_0(\psi(t,\cdot))$, then ∂D_t corresponds to $\{\phi_t = 0\}$ and we have ∂D_t in $W^{2,p}$ (one may argue as in [2]). Furthermore,

$$(\partial_t + u\cdot\nabla)1_{D_t} = 0.$$

Let $F := \operatorname{div} u - \nu^{-1}P(\rho)$. Let us define ρ^1 and ρ^2 to be the solutions of

$$(\partial_t + u\cdot\nabla)\rho^i + \rho^i F + \nu^{-1}\Pi(\rho^i) = 0 \quad \text{with} \qquad \Pi(z) := zP(z)$$

and data ρ_0^1 and ρ_0^2, respectively.

The fact that ρ_0^1 and ρ_0^2 are bounded, and that the function F is in $L^1(0,T;L^\infty)$ ensures that $\rho^i \in L^\infty(0,T;L^\infty)$. Furthermore, $\widetilde{\rho} := 1_D\rho^1 + 1_{^cD}\rho^2$ fulfills

$$(\partial_t + u\cdot\nabla)\widetilde{\rho} + \widetilde{\rho}\,F + \nu^{-1}\Pi(\widetilde{\rho}) = 0$$

while

$$(\partial_t + u\cdot\nabla)\rho + \rho\,F + \nu^{-1}\Pi(\rho) = 0.$$

Since $\widetilde{\rho}|_{t=0} = \rho|_{t=0} = \rho_0$, this ensures that $\widetilde{\rho} \equiv \rho$ on $[0,T]$: to prove it, one can for instance estimate $\delta\rho := \widetilde{\rho} - \rho$ in L^∞ after noticing that

$$(\partial_t + u\cdot\nabla)\delta\rho + \delta\rho\,F + \nu^{-1}\delta\rho\int_0^1 \Pi'(\rho + \tau\delta\rho)\,d\tau = 0.$$

Now, differentiating the equation of ρ^i, we see that

$$(\partial_t + u\cdot\nabla)(\nabla\rho^i) + \nabla u\cdot\nabla\rho^i + (F + \nu^{-1}\Pi'(\rho^i))\nabla\rho^i = -\rho^i\nabla F.$$

Since our assumptions guarantee that the right-hand side of the above equality is in $L^1(0,T;L^p)$, one can deduce that $\nabla\rho^i$ is in $L^\infty(0,T;L^p)$. ■

3 Uniqueness

Since the main responsible for the loss of one derivative in the stability estimates for (CNS) is the density equation (note that the velocity equation is parabolic), it is tempting to use Lagrangian coordinates, since they allow to compute the density from the initial one and the flow, and thus only the velocity equation would have to be considered.

More specifically, we go from Eulerian coordinates (t,x) to Lagrangian coordinates (t,y) by making the change of unknowns:

$$\bar\rho(t,y) := \rho(t,x) \quad \text{and} \quad \bar u(t,y) := u(t,x) \quad \text{with} \quad x := \psi(t,y)$$

where ψ is the unique flow of u defined (according to the Cauchy-Lipschitz theorem) by

$$\psi(t,y) = y + \int_0^t u(\tau, \psi(\tau,y))\, d\tau.$$

Hence

$$\psi(t,y) = y + \int_0^t \bar u(\tau,y)\, d\tau \quad \text{and} \quad D\psi(t,y) = \mathrm{Id} + \int_0^t D\bar u(\tau,y)\, d\tau.$$

We thus have

$$(J\bar\rho)(t) = \rho_0 \quad \text{with} \quad J := \det(D\psi).$$

At the same time, using the following identities:

$$\begin{aligned} \overline{\nabla_x K} &= J_u^{-1}\, \mathrm{div}_y(\mathrm{adj}\, D\psi_u\, \overline{K}) \qquad \text{for} \quad K : \mathbb{R}^d \to \mathbb{R}^d \\ \overline{\mathrm{div}_x H} &= J_u^{-1}\, \mathrm{div}_y(\mathrm{adj}\, D\psi_u\, \overline{H}) \qquad \text{for} \quad H : \mathbb{R}^d \to \mathbb{R} \end{aligned}$$

where $\mathrm{adj}(D\psi)$ stands for the adjugate matrix of $D\psi$, we discover that

$$\rho_0 \bar u_t - \mu \mathrm{div}\,(\mathrm{adj}(D\psi)^T A \nabla u) - \mu' \mathrm{div}\,(\mathrm{adj}(D\psi)^T A : \nabla u) + \mathrm{div}\,(\mathrm{adj}(D\psi) P(J^{-1}\rho_0)) = 0$$

with $A := (D_y\psi)^{-1}$.

Note that if $\int_0^t D\bar u(\tau,\cdot)\, d\tau$ is small enough, then

$$A = \sum_{k=0}^{+\infty} (-1)^k \Big(\int_0^t D\bar u(\tau,\cdot)\, d\tau \Big)^k. \tag{7}$$

Now, consider two solutions (ρ_1,u_1) and (ρ_2,u_2) for the same data (ρ_0,u_0), and perform the Lagrangian change of coordinates for the two solutions, with respect to their own flow:

$$(\rho_i, u_i) \rightsquigarrow (J_i^{-1}\rho_0, \bar u_i), \quad i = 1,2.$$

Then $\delta u := u_2 - u_1$ fulfills

$$\rho_0 \delta u_t - \mathcal{L}_1 \delta u = (\mathcal{L}_2 - \mathcal{L}_1)\bar u_2 + \mathrm{div}\,(\mathrm{adj}(D\psi_1) P(J_1^{-1}\rho_0) - \mathrm{adj}(D\psi_2) P(J_2^{-1}\rho_0))$$

with $\mathcal{L}_j := \mu \mathrm{div}\,(\mathrm{adj}(D\psi_j)^T A_j \nabla u_j) + \mu' \mathrm{div}\,(\mathrm{adj}(D\psi_j)^T A_j : \nabla u_j)$.

Performing an energy method and assuming that $\int_0^T \|D\bar{u}^i\|_{L^\infty}\,dt,\ i=1,2,$ is small enough (so that one may use (7) and similar identities) eventually leads to

$$\frac{d}{dt}\int \rho_0|\delta u|^2\,dx + \int |\nabla \delta u|^2 \leq Ct(1+\|\nabla \bar{u}^2\|_{L^\infty}^2)\int_0^t \|\nabla \delta u\|_{L^2}^2\,d\tau \quad \text{on} \quad [0,T].$$

It is then easy to conclude to uniqueness by Gronwall lemma, if

$$\int_0^T t\|\nabla u^2(t)\|_{L^\infty}^2\,dt < \infty.$$

That latter property which is not utterly obvious stems from the fact that (ρ^2, u^2) is a solution to (CNS), and from the regularity properties that have been exhibited so far. Here one has to first prove *time weighted* maximal regularity estimates (see Proposition 4.1 in [4], and its corollary therein).

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