On a compressible fluid model of Korteweg type in a maximal regularity class

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1 Introduction

This article shows local and global existence theorems in a maximal regularity class for a compressible fluid model of Korteweg type as follows:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0 & \text{in } \Omega\times(0,T_0), \\ \rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}) = \operatorname{Div}(\mathbf{S}(\mathbf{u})+\mathbf{K}(\rho)-P(\rho)\mathbf{I}) & \text{in } \Omega\times(0,T_0), \\ \mathbf{n}\cdot\nabla\rho = 0, \quad \mathbf{u}=0 & \text{on } \Gamma\times(0,T_0), \\ (\rho,\mathbf{u})|_{t=0} = (\rho_0+\rho_\infty, \mathbf{u}_0) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a domain in $\mathbf{R}^N$, $N\geq 2$, with boundary Γ and T_0 is a positive number.

Here $\rho=\rho(x,t)$ and $\mathbf{u}=(u_1(x,t),\dots,u_N(x,t))^{\mathsf{T}}$[1] are respectively the fluid density and the fluid velocity at $x=(x_1,\dots,x_N)\in\Omega$ and $t\in(0,T_0)$; P is a given function describing the pressure and $\mathbf{I}$ is the $N\times N$ identity matrix; $\mathbf{S}(\mathbf{u})$ is the viscous stress tensor given by $\mathbf{S}(\mathbf{u})=\mu\mathbf{D}(\mathbf{u})+(\nu-\mu)\operatorname{div}\mathbf{u}\mathbf{I}$, where $\mu,\nu>0$ are viscosity coefficients and $\mathbf{D}(\mathbf{u})$ is the doubled deformation tensor, i.e. $\mathbf{D}(\mathbf{u})$ is an $N\times N$ matrix whose (i,j) component is given by $\partial_iu_j+\partial_ju_i$ for $\partial_i=\partial/\partial x_i$; $\mathbf{K}(\rho)$ is the so-called Korteweg tensor, i.e.

$$\begin{aligned} \mathbf{K}(\rho) &= \frac{\kappa}{2}\left(\Delta\rho^2-|\nabla\rho|^2\right)\mathbf{I}-\kappa\nabla\rho\otimes\nabla\rho \\ &= \kappa\left(\rho\Delta\rho+\frac{|\nabla\rho|^2}{2}\right)\mathbf{I}-\kappa\nabla\rho\otimes\nabla\rho, \end{aligned} \tag{1.2}$$

where $\kappa>0$ is a capillarity coefficient and $\nabla\rho\otimes\nabla\rho$ is an $N\times N$ matrix whose (i,j) component is given by $(\partial_i\rho)(\partial_j\rho)$; $\mathbf{n}$ is the unit outward normal vector to Γ and $\mathbf{a}\cdot\mathbf{b}=\sum_{i=1}^N a_ib_i$ for N-vectors $\mathbf{a}=(a_1,\dots,a_N)^{\mathsf{T}}$ and $\mathbf{b}=(b_1,\dots,b_N)^{\mathsf{T}}$; $\rho_0=\rho_0(x)$ and $\mathbf{u}_0=(u_{01}(x),\dots,u_{0N}(x))^{\mathsf{T}}$ are given initial data, while ρ_∞ is a positive constant.

Throughout this article, we assume

[1] $\mathbf{M}^{\mathsf{T}}$ denotes the transpose of $\mathbf{M}$.

Assumption 1.1. (1) *The coefficients μ, ν, and κ are positive constants.*

(2) *The pressure $P : (\rho_\infty/8, 8\rho_\infty) \to \mathbf{R}$ is smooth enough.*

Korteweg formulated in 1901 some tensor that included gradients of density in order to model fluid capillarity effects, and Dunn and Serrin [2] derived (1.2) in view of rational mechanics by introducing the thermodynamics of interstitial working. Concerning the mathematical analysis of Korteweg-type model, we refer e.g. to [5, 1, 4, 15] for the whole space problem and to [6, 7, 8, 9] for boundary value problems. On the other hand, [10] employs the Korteweg-type model in order to analyze the structure of liquid-vapor phase transition in numerical analysis.

2 Preliminaries

Let $p \in (1,\infty)$, and let $q \in (1,\infty)$ or $q = \infty$. We here introduce function spaces used throughout this article as follows:

- $L_q(G)$ and $H_q^m(G)$, $m \in \mathbf{N}$, are respectively the usual Lebesgue spaces and the Sobolev spaces, where G is a domain in $\mathbf{R}^N$. The norm of $L_q(G)$ is denoted by $\|\cdot\|_{L_q(G)}$, while the norm of $H_q^m(G)$ is denoted by $\|\cdot\|_{H_q^m(G)}$.
- Let $(\cdot,\cdot)_{\theta,p}$ be the real interpolation functor for $\theta \in (0,1)$. Then the Besov spaces $B_{q,p}^{3-2/p}(G)$ and $B_{q,p}^{2-2/p}(G)$ are defined as
$$B_{q,p}^{3-2/p}(G) = (H_q^1(G), H_q^3(G))_{1-1/p,p}, \quad B_{q,p}^{2-2/p}(G) = (L_q(G), H_q^2(G))_{1-1/p,p}.$$
- Let X be a Banach space and I be an interval in $\mathbf{R}$. Then $L_p(I,X)$ and $H_p^1(I,X)$ are respectively the X-valued Lebesgue spaces and the X-valued Sobolev spaces. The norm of $L_p(I,X)$ is denoted by $\|\cdot\|_{L_p(I,X)}$, while the norm of $H_p^1(I,X)$ is denoted by $\|\cdot\|_{H_p^1(I,X)}$.
- Let $T \in (0,\infty)$ or $T = \infty$. Then ${}_0H_p^1((0,T),X)$ is given by
$${}_0H_p^1((0,T),X) = \{f \in H_p^1((0,T),X) : f|_{t=0} = 0 \text{ in } X\}.$$

Next, we introduce the definition of uniform C^3 domains.

Definition 2.1 ([3, 14]). *Let D be a domain in $\mathbf{R}^N$ with boundary ∂D. Then D is called a uniform C^3 domain, if there exist positive constants α, β, and K such that the following assertion holds: for any $x_0 = (x_{01}, \dots, x_{0N}) \in \partial D$, there are coordinate* *number j and a C^3 function $h(x')$ $(x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N))$ on $B'_\alpha(x'_0)$, with $x'_0 = (x_{01}, \ldots, x_{0j-1}, x_{0j+1}, \ldots, x_{0N})$,*

$$B'_\alpha(x'_0) = \{x \in \mathbf{R}^N \mid |x' - x'_0| < \alpha\}, \quad \|h\|_{H^3_\infty(B'_\alpha(x'_0))} \le K,$$

such that

$$D \cap B_\beta(x_0) = \{x \in \mathbf{R}^N : x_j > h(x'), x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0),$$
$$\partial D \cap B_\beta(x_0) = \{x \in \mathbf{R}^N : x_j = h(x'), x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0).$$

Here $B_\beta(x_0) = \{x \in \mathbf{R}^N : |x - x_0| < \beta\}$.

Remark 2.2. Typical examples of uniform C^3 domains are as follows: bounded domains; exterior domains; half-spaces, layers, tubes, and their perturbed domains.

3 Local solvability

3.1 Linearization

Let us start with the linearization of (1.1). Replace ρ by $\rho + \rho_\infty$ in (1.1). Then the first equation becomes

$$\partial_t \rho + \rho_\infty \operatorname{div} \mathbf{u} = -\rho \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla \rho; \tag{3.1}$$

the second equation becomes

$$\begin{aligned} \rho_\infty \partial_t \mathbf{u} - \operatorname{Div}(\mathbf{S}(\mathbf{u}) + \kappa \rho_\infty \Delta \rho \mathbf{I}) &= -\rho \partial_t \mathbf{u} - (\rho + \rho_\infty) \mathbf{u} \cdot \nabla \mathbf{u} \\ &\quad + \operatorname{Div}(\mathbf{K}(\rho + \rho_\infty) - \kappa \rho_\infty \Delta \rho \mathbf{I}) \\ &\quad - P'(\rho + \rho_\infty) \nabla \rho, \end{aligned} \tag{3.2}$$

where $P'(s) = (dP/ds)(s)$. To prove the local solvabiliity, we further rewrite (3.1) and (3.2) as follows:

$$\begin{aligned} \partial_t \rho + (\rho_0 + \rho_\infty) \operatorname{div} \mathbf{u} &= -(\rho - \rho_0) \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla \rho \\ &=: F(\rho, \mathbf{u}); \end{aligned}$$

for $\widetilde{\mu} = \mu/\kappa$ and $\widetilde{\nu} = \nu/\kappa$,

$$\begin{aligned} &(\rho_0 + \rho_\infty) \partial_t \mathbf{u} - \kappa \operatorname{Div}(\widetilde{\mu} \mathbf{D}(\mathbf{u}) + (\widetilde{\nu} - \widetilde{\mu}) \operatorname{div} \mathbf{u} \mathbf{I} + (\rho_0 + \rho_\infty) \Delta \rho \mathbf{I}) \\ &= -(\rho - \rho_0) \partial_t \mathbf{u} - (\rho + \rho_\infty) \mathbf{u} \cdot \nabla \mathbf{u} \\ &+ \operatorname{Div}(\mathbf{K}(\rho + \rho_\infty) - \kappa (\rho_0 + \rho_\infty) \Delta \rho \mathbf{I}) - P'(\rho + \rho_\infty) \nabla \rho \\ &=: G(\rho, \mathbf{u}). \end{aligned}$$

Setting $\gamma = \rho_0 + \rho_\infty$, we have achieved the following equivalent system of (1.1):

$$\begin{cases} \partial_t \rho + \gamma \operatorname{div} \mathbf{u} = F(\rho, \mathbf{u}) & \text{in } \Omega \times (0, T_0), \\ \partial_t \mathbf{u} - \gamma^{-1}\kappa \operatorname{Div}(\widetilde{\mu}\mathbf{D}(\mathbf{u}) + (\widetilde{\nu} - \widetilde{\mu}) \operatorname{div} \mathbf{u}\mathbf{I} + \gamma\Delta\rho\mathbf{I}) = \gamma^{-1}G(\rho, \mathbf{u}) & \text{in } \Omega \times (0, T_0), \\ \mathbf{n} \cdot \nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma \times (0, T_0), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{3.3}$$

3.2 Linearized problem

For a positive number S, we consider a linearized problem associated with (3.3) as follows:

$$\begin{cases} \partial_t \rho + \gamma_1 \operatorname{div} \mathbf{u} = f & \text{in } \Omega \times (0, S), \\ \partial_t \mathbf{u} - \gamma_4^{-1} \operatorname{Div}(\gamma_2\mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I}) = \mathbf{g} & \text{in } \Omega \times (0, S), \\ \mathbf{n} \cdot \nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma \times (0, S), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega, \end{cases} \tag{3.4}$$

where Ω and $\gamma_i = \gamma_i(x)$ $(i = 1, 2, 3, 4)$ satisfy the following assumption.

Assumption 3.1. (1) *The domain Ω is a uniform C^3 domain in $\mathbf{R}^N$, $N \geq 2$, and its boundary is denoted by Γ.*

(2) *The coefficients $\gamma_i = \gamma_i(x)$ $(i = 1, 2, 3, 4)$ are uniformly Lipschitz continuous functions on $\overline{\Omega}$, i.e. there exists a positive constant γ_L such that $|\gamma_i(x) - \gamma_i(y)| \leq \gamma_L|x - y|$ for any $x, y \in \overline{\Omega}$ and for $i = 1, 2, 3, 4$. In addition, there exist positive constants γ_*, γ^* such that $\gamma_* \leq \gamma_i(x) \leq \gamma^*$ for any $x \in \overline{\Omega}$ and for $i = 1, 2, 3, 4$.*

Let $q \in (1, \infty)$ and $X_q = H^1_q(\Omega) \times L_q(\Omega)^N$. We define an operator A_q by

$$A_q(\rho, \mathbf{u}) = (-\gamma_1 \operatorname{div} \mathbf{u}, \gamma_4^{-1} \operatorname{Div}(\gamma_2\mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I})),$$

with the domain $D(A_q)$:

$$D(A_q) = \{(\rho, \mathbf{u}) \in \times H^3_q(\Omega) \times H^2_q(\Omega)^N \mid \mathbf{n} \cdot \nabla\rho = 0,\ \mathbf{u} = 0 \text{ on } \Gamma\}.$$

Note that $D(A_q) \subset X_q$ and $A_q : D(A_q) \to X_q$. One then has

Lemma 3.2 ([12])**.** *Let $q \in (1, \infty)$ and suppose that Assumption* 3.1 *holds. Then A_q generates an analytic C_0-semigroup $\{e^{A_q t}\}_{t \geq 0}$ on X_q. In addition, there exist constants $\delta_1 \geq 1$ and $C_{N,q,\delta_1} > 0$ such that for any $t > 0$*

$$\|e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} \leq C_{N,q,\delta_1} e^{(\delta_1/2)t} \|(\rho_0, \mathbf{u}_0)\|_{X_q} \qquad ((\rho_0, \mathbf{u}_0) \in X_q),$$

$$\|\partial_t e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} \le C_{N,q,\delta_1} e^{(\delta_1/2)t} t^{-1} \|(\rho_0, \mathbf{u}_0)\|_{X_q} \quad ((\rho_0, \mathbf{u}_0) \in X_q),$$
$$\|\partial_t e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} \le C_{N,q,\delta_1} e^{(\delta_1/2)t} \|(\rho_0, \mathbf{u}_0)\|_{D(A_q)} \quad ((\rho_0, \mathbf{u}_0) \in D(A_q)),$$

where $\|\cdot\|_{D(A_q)}$ *denotes the graph norm of* A_q*.*

Setting $D_{q,p}(\Omega) = (X_q, D(A_q))_{1-1/p,p}$ for $p, q \in (1, \infty)$, we have

Lemma 3.3 ([12])**.** *Let* $(p, q) \in (1, \infty)$ *and suppose that Assumption* 3.1 *holds. Then, for any* $(\rho_0, \mathbf{u}_0) \in D_{q,p}(\Omega)$*,* $(\rho, \mathbf{u}) = e^{A_q t}(\rho_0, \mathbf{u}_0)$ *is a unique solution to the system* (3.4) *under the condition of* $(f, \mathbf{g}) = (0, 0)$*. In addition,*

$$\begin{aligned} &\|\partial_t \rho\|_{L_p((0,S),H^1_q(\Omega))} + \|\rho\|_{L_p((0,S),H^3_q(\Omega))} \\ &+ \|\partial_t \mathbf{u}\|_{L_p((0,S),L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0,S),H^2_q(\Omega)^N)} \\ &\le C_{N,p,q,\delta_1} e^{\delta_1 S} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)} \end{aligned}$$

for some positive constant C_{N,p,q,δ_1} *independent of* S*, where* δ_1 *is the same constant as in Lemma* 3.2*.*

The following lemma is also proved in [12].

Lemma 3.4 (Maximal regularity)**.** *Let* $p, q \in (1, \infty)$ *and suppose that Assumption* 3.1 *holds. Then, for* $(\rho_0, \mathbf{u}_0) = (0, 0)$ *and for any* f *and* $\mathbf{g}$ *with*

$$f \in L_p((0, S), H^1_q(\Omega)), \quad \mathbf{g} \in L_p((0, S), L_q(\Omega)^N),$$

the system (3.4) *admits a unique solution* $(\rho, \mathbf{u})$ *with*

$$\begin{aligned} \rho &\in {}_0H^1_p((0, S), H^1_q(\Omega)) \cap L_p((0, S), H^3_q(\Omega)), \\ \mathbf{u} &\in {}_0H^1_p((0, S), L_q(\Omega)^N) \cap L_p((0, S), H^2_q(\Omega)^N). \end{aligned}$$

In addition, the solution $(\rho, \mathbf{u})$ *satisfies the estimate:*

$$\begin{aligned} &\|\partial_t \rho\|_{L_p((0,S),H^1_q(\Omega))} + \|\rho\|_{L_p((0,S),H^3_q(\Omega))} \\ &+ \|\partial_t \mathbf{u}\|_{L_p((0,S),L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0,S),H^2_q(\Omega)^N)} \\ &\le C_{N,p,q,\delta_2} e^{\delta_2 S} \left(\|f\|_{L_p((0,S),H^1_q(\Omega))} + \|\mathbf{g}\|_{L_p((0,S),L_q(\Omega)^N)}\right) \end{aligned}$$

for positive constants δ_2 *and* C_{N,p,q,δ_2} *independent of* S*.*

3.3 Local existence theorem

Let $(p, q) \in (2, \infty) \times (N, \infty)$ and $(\rho_0, \mathbf{u}_0) \in B_{q,p}(\Omega)^{3-2/p}(\Omega) \times B^{2-2/p}_{q,p}(\Omega)^N$ with the following conditions:

$$\mathbf{n} \cdot \nabla \rho_0 = 0, \quad \mathbf{u}_0 = 0 \quad \text{on } \Gamma, \tag{3.5}$$

$$\frac{\rho_\infty}{2} \le \rho_0(x) + \rho_\infty \le 2\rho_\infty \quad (x \in \overline{\Omega}), \tag{3.6}$$

and let $(\rho_*, \mathbf{u}_*) = e^{A_q t}(\rho_0, \mathbf{u}_0)$ for $\gamma_1 = \gamma$, $\gamma_2 = \widetilde{\mu}$, $\gamma_3 = \widetilde{\nu}$, and $\gamma_4 = \gamma\kappa^{-1}$. Then, setting $\rho = \sigma + \rho_*$ and $\mathbf{u} = \mathbf{v} + \mathbf{u}_*$ in (3.3), we observe that

$$\begin{cases} \partial_t \sigma + \gamma \operatorname{div} \mathbf{v} = F(\sigma + \rho_*, \mathbf{v} + \mathbf{u}_*) & \text{in } \Omega \times (0, T_0), \\ \partial_t \mathbf{v} - \gamma^{-1}\kappa \operatorname{Div}(\widetilde{\mu}\mathbf{D}(\mathbf{v}) + (\widetilde{\nu} - \widetilde{\mu}) \operatorname{div} \mathbf{v}\mathbf{I} + \gamma \Delta \sigma \mathbf{I}) & \\ \qquad = \gamma^{-1} G(\sigma + \rho_*, \mathbf{v} + \mathbf{u}_*) & \text{in } \Omega \times (0, T_0), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma \times (0, T_0), \\ (\sigma, \mathbf{v})|_{t=0} = (0, 0) & \text{in } \Omega. \end{cases} \tag{3.7}$$

To use the contraction mapping principle, we introduce the following notation:

- For $T > 0$, ${}_0Z_T := {}_0Z_T^1 \times {}_0Z_T^2$ with

$$\begin{aligned} {}_0Z_T^1 &= {}_0H_p^1((0,T), H_q^1(\Omega)) \cap L_p((0,T), H_q^3(\Omega)), \\ {}_0Z_T^2 &= {}_0H_p^1((0,T), L_q(\Omega)^N) \cap L_p((0,T), H_q^2(\Omega)^N). \end{aligned}$$

  Here the norm $\|\cdot\|_{{}_0Z_T}$ of ${}_0Z_T$ is given by

$$\begin{aligned} \|(\rho, \mathbf{u})\|_{{}_0Z_T} &= \|\rho\|_{H_p^1((0,T),H_q^1(\Omega))} + \|\rho\|_{L_p((0,T),H_q^3(\Omega))} \\ &\quad + \|\mathbf{u}\|_{H_p^1((0,T),L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0,T),H_q^2(\Omega)^N)}. \end{aligned}$$

- For $T > 0$ and $r > 0$,

$$\begin{aligned} {}_0Z_T(r) := \Big\{ &(\tau, \mathbf{w}) \in {}_0Z_T : \|(\tau, \mathbf{w})\|_{{}_0Z_T} \le r, \\ &\frac{\rho_\infty}{4} \le \tau(x,t) + \rho_*(x,t) + \rho_\infty \le 4\rho_\infty \text{ for any } (x,t) \in \overline{\Omega} \times [0,T] \Big\}. \end{aligned}$$

Let $(\tau, \mathbf{w}) \in {}_0Z_T(L)$ for a suitable positive number L and for $T > 0$, and replace $(\sigma, \mathbf{v})$ by $(\tau, \mathbf{w})$ in the right-hand sides of (3.7). Then, by the maximal regularity stated in Lemma 3.4, we can define a contraction mapping $\Phi : {}_0Z_T(L) \ni (\tau, \mathbf{w}) \mapsto (\sigma, \mathbf{v}) \in {}_0Z_T(L)$ for a sufficiently small $T \in (0, T_0)$. We thus obtain by the contraction mapping principle a local existence theorem in the maximal regularity class as follows:

Theorem 3.5. *Assume that Ω is a uniform C^3 domain in $\mathbf{R}^N$, $N \ge 2$, with boundary Γ. Let $(p,q) \in (2,\infty) \times (N,\infty)$, and let R be an arbitrary positive number. Then there exist positive constants L and $T \in (0,T_0)$ such that, for any $(\rho_0, \mathbf{u}_0) \in B_{q,p}^{3-2/p}(\Omega) \times B_{q,p}^{2-2/p}(\Omega)^N$ satisfying $\|(\rho_0, \mathbf{u}_0)\|_{B_{q,p}(\Omega)^{3-2/p}(\Omega) \times B_{q,p}^{2-2/p}(\Omega)^N} \le R$ with (3.5) and (3.6), the system (3.7) admits a unique solution $(\sigma, \mathbf{v})$ on $(0,T)$ in ${}_0Z_T(L)$.*

4 Global solvability

Throughout this section, we assume

Assumption 4.1. (1) *The domain* Ω *is a bounded domain in* $\mathbf{R}^N$, $N \geq 2$, *with* C^3 *boundary* Γ.

(2) $P'(\rho_\infty) > 0$.

4.1 Linearization

Let $\mu_\infty = \mu/\rho_\infty$, $\nu_\infty = \nu/\rho_\infty$, and $\gamma_\infty = \rho_\infty^{-1}P'(\rho_\infty)$. Then we rewrite (3.1) and (3.2) as follows:

$$\partial_t\rho + \rho_\infty \operatorname{div}\mathbf{u} = -\operatorname{div}(\rho\mathbf{u}) =: \mathsf{F}(\rho, \mathbf{u})$$

and

$$\begin{aligned}
&\partial_t\mathbf{u} - \operatorname{Div}(\mu_\infty\mathbf{D}(\mathbf{u}) + (\nu_\infty - \mu_\infty)\operatorname{div}\mathbf{u}\mathbf{I} + \kappa\Delta\rho\mathbf{I}) + \gamma_\infty\nabla\rho \\
&= \rho_\infty^{-1}\Big\{ -\rho\partial_t\mathbf{u} - (\rho+\rho_\infty)\mathbf{u}\cdot\nabla\mathbf{u} \\
&+ \operatorname{Div}(\mathbf{K}(\rho+\rho_\infty) - \kappa\rho_\infty\Delta\rho\mathbf{I}) - (P'(\rho+\rho_\infty) - P'(\rho_\infty))\,\nabla\rho\Big\} \\
&=: \mathsf{G}(\rho, \mathbf{u}).
\end{aligned}$$

Thus we have achieved the following equivalent system of (1.1) with $T_0 = \infty$:

$$\begin{cases}
\partial_t\rho + \rho_\infty \operatorname{div}\mathbf{u} = \mathsf{F}(\rho, \mathbf{u}) & \text{in } \Omega\times(0,\infty), \\
\partial_t\mathbf{u} - \operatorname{Div}(\mu_\infty\mathbf{D}(\mathbf{u}) + (\nu_\infty - \mu_\infty)\operatorname{div}\mathbf{u}\mathbf{I} + \kappa\Delta\rho\mathbf{I}) + \gamma_\infty\nabla\rho = \mathsf{G}(\rho, \mathbf{u}) & \text{in } \Omega\times(0,\infty), \\
\mathbf{n}\cdot\nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma\times(0,\infty), \\
(\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega.
\end{cases} \tag{4.1}$$

4.2 Linearized problem

We consider a linearized problem associated with (4.1) as follows:

$$\begin{cases}
\partial_t\rho + \rho_\infty \operatorname{div}\mathbf{u} = f & \text{in } \Omega\times(0,\infty), \\
\partial_t\mathbf{u} - \operatorname{Div}(\mu_\infty\mathbf{D}(\mathbf{u}) + (\nu_\infty - \mu_\infty)\operatorname{div}\mathbf{u}\mathbf{I} + \kappa\Delta\rho\mathbf{I}) + \gamma_\infty\nabla\rho = \mathbf{g} & \text{in } \Omega\times(0,\infty), \\
\mathbf{n}\cdot\nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma\times(0,\infty), \\
(\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega.
\end{cases} \tag{4.2}$$

To construct an analytic C_0-semigroups associated with (4.2), we set for $q \in (1, \infty)$

$$\mathsf{H}^1_q(\Omega) = \left\{\rho \in H^1_q(\Omega) : \int_\Omega \rho \, dx = 0\right\}, \quad \mathsf{X}_q = \mathsf{H}^1_q(\Omega) \times L_q(\Omega)^N,$$

and also

$$\mathsf{H}^3_q(\Omega) = H^3_q(\Omega) \cap \mathsf{H}^1_q(\Omega).$$

Their norms are given by

$$\begin{aligned} &\|\rho\|_{\mathsf{H}^1_q(\Omega)} = \|\rho\|_{H^1_q(\Omega)}, \quad \|(\rho, \mathbf{u})\|_{\mathsf{H}^1_q(\Omega)\times L_q(\Omega)^N} = \|\rho\|_{H^1_q(\Omega)} + \|\mathbf{u}\|_{L_q(\Omega)^N}, \\ &\|\rho\|_{\mathsf{H}^3_q(\Omega)} = \|\rho\|_{H^3_q(\Omega)}. \end{aligned}$$

In addition, an operator A_q is defined by

$$\mathsf{A}_q(\rho, \mathbf{u}) = (-\rho_\infty \operatorname{div} \mathbf{u}, \operatorname{Div}(\mu_\infty \mathbf{D}(\mathbf{u}) + (\nu_\infty - \mu_\infty) \operatorname{div} \mathbf{u}\mathbf{I} + \kappa\Delta\rho\mathbf{I}) + \gamma_\infty \nabla\rho),$$

with the domain $\mathsf{D}(\mathsf{A}_q)$:

$$\mathsf{D}(\mathsf{A}_q) = \{(\rho, \mathbf{u}) \in \mathsf{H}^3_q(\Omega) \times H^2_q(\Omega)^N : \mathbf{n} \cdot \nabla\rho = 0, \ \mathbf{u} = 0 \text{ on } \Gamma\}.$$

Note that $\mathsf{D}(\mathsf{A}_q) \subset \mathsf{X}_q$ and $\mathsf{A}_q : \mathsf{D}(\mathsf{A}_q) \to \mathsf{X}_q$.

Now we introduce the generation of an analytic C_0-semigroup $\{e^{\mathsf{A}_q t}\}_{t\geq 0}$ on X_q and its exponential stability. To this end, we consider the following resolvent problem:

$$\begin{cases} \lambda\rho + \rho_\infty \operatorname{div} \mathbf{u} = f & \text{in } \Omega, \\ \lambda\mathbf{u} - \operatorname{Div}(\mu_\infty \mathbf{D}(\mathbf{u}) + (\nu_\infty - \mu_\infty) \operatorname{div} \mathbf{u}\mathbf{I} + \kappa\Delta\rho\mathbf{I}) + \gamma_\infty\nabla\rho = \mathbf{g} & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma, \end{cases} \tag{4.3}$$

where λ is the resolvent parameter varying in $\mathbf{C}_{+,\delta} = \{z \in \mathbf{C} : \Re z > \delta\}$ for $\delta \in \mathbf{R}$. By [12] and a small perturbation method, we have

Lemma 4.2. *Let $q \in (1, \infty)$ and suppose that Assumption 4.1 holds. Then there exists a positive number δ_3 such that, for any $\lambda \in \mathbf{C}_{+,\delta_3}$ and $(f, \mathbf{g}) \in \mathsf{X}_q$, the system (4.3) admits a unique solution $(\rho, \mathbf{u}) \in \mathsf{H}^3_q(\Omega) \times H^2_q(\Omega)^N$. In addition, the solution $(\rho, \mathbf{u})$ satisfies the estimate:*

$$|\lambda|\|(\rho, \mathbf{u})\|_{\mathsf{X}_q} + \|(\rho, \mathbf{u})\|_{\mathsf{H}^3_q(\Omega)\times H^2_q(\Omega)^N} \leq C_{N,q,\delta_3}\|(f, \mathbf{g})\|_{\mathsf{X}_q}$$

for some positive constant C_{N,q,δ_3} independent of $\lambda \in \mathbf{C}_{+,\delta_3}$.

Let $\overline{\mathbf{C}_+} = \overline{\mathbf{C}_{+,0}} = \{z \in \mathbf{C} : \Re z \geq 0\}$. Combining Lemma 4.2 with a homotopic argument[2] and the closed graph theorem then yields

[2]We refer e.g. to [3, Section 7].

Lemma 4.3. *Let $q \in (1,\infty)$ and suppose that Assumption 4.1 holds. Then, for any $\lambda \in \overline{\mathbf{C}_+}$ and $(f,\mathbf{g}) \in \mathsf{X}_q$, the system (4.3) admits a unique solution $(\rho,\mathbf{u}) \in \mathsf{H}^3_q(\Omega) \times H^2_q(\Omega)^N$. In addition, the solution satisfies the estimate:*

$$|\lambda|\|(\rho,\mathbf{u})\|_{\mathsf{X}_q} + \|(\rho,\mathbf{u})\|_{\mathsf{H}^3_q(\Omega)\times H^2_q(\Omega)^N} \leq C_{N,q}\|(f,\mathbf{g})\|_{\mathsf{X}_q}$$

for some positive constant $C_{N,q}$ independent of $\lambda \in \overline{\mathbf{C}_+}$.

By Lemma 4.3 and the standard theory of analytic C_0-semigroups, we have

Lemma 4.4. *Let $q \in (1,\infty)$ and suppose that Assumption 4.1 holds. Then A_q generates an analytic C_0-semigroup $\{e^{\mathsf{A}_q t}\}_{t\geq 0}$ on X_q. In addition, there there exist constants $\delta_4 \in (0,1)$ and $C_{N,q,\delta_4} > 0$ such that for any $t > 0$*

$$\begin{aligned}
\|e^{\mathsf{A}_q t}(\rho_0,\mathbf{u}_0)\|_{\mathsf{X}_q} &\leq C_{N,q,\delta_4}e^{-2\delta_4 t}\|(\rho_0,\mathbf{u}_0)\|_{\mathsf{X}_q} && ((\rho_0,\mathbf{u}_0) \in \mathsf{X}_q),\\
\|\partial_t e^{\mathsf{A}_q t}(\rho_0,\mathbf{u}_0)\|_{\mathsf{X}_q} &\leq C_{N,q,\delta_4}e^{-2\delta_4 t}t^{-1}\|(\rho_0,\mathbf{u}_0)\|_{\mathsf{X}_q} && ((\rho_0,\mathbf{u}_0) \in \mathsf{X}_q),\\
\|\partial_t e^{\mathsf{A}_q t}(\rho_0,\mathbf{u}_0)\|_{\mathsf{X}_q} &\leq C_{N,q,\delta_4}e^{-2\delta_4 t}\|(\rho_0,\mathbf{u}_0)\|_{\mathsf{D}(\mathsf{A}_q)} && ((\rho_0,\mathbf{u}_0) \in \mathsf{D}(\mathsf{A}_q)),
\end{aligned}$$

where $\|\cdot\|_{\mathsf{D}(\mathsf{A}_q)}$ denotes the graph norm of A_q.

Similarly to [13], we have by setting $\mathsf{D}_{q,p}(\Omega) = (\mathsf{X}_q, \mathsf{D}(\mathsf{A}_q))_{1-1/p,p}$

Lemma 4.5. *Let $p,q \in (1,\infty)$ and suppose that Assumption 4.1 holds. Then, for any $(\rho,\mathbf{u}) \in \mathsf{D}_{q,p}(\Omega)$, $(\rho,\mathbf{u}) = e^{\mathsf{A}_q t}(\rho_0,\mathbf{u}_0)$ is a unique solution to the system (4.2) under the condition of $(f,\mathbf{g}) = (0,0)$. In addition,*

$$\begin{aligned}
&\|e^{\delta_4 t}\partial_t\rho\|_{L_p((0,\infty),\mathsf{H}^1_q(\Omega))} + \|e^{\delta_4 t}\rho\|_{L_p((0,\infty),\mathsf{H}^3_q(\Omega))}\\
&+ \|e^{\delta_4 t}\partial_t\mathbf{u}\|_{L_p((0,\infty),L_q(\Omega)^N)} + \|e^{\delta_4 t}\mathbf{u}\|_{L_p((0,\infty),H^2_q(\Omega)^N)}\\
&\leq C_{N,p,q,\delta_4}\|(\rho_0,\mathbf{u}_0)\|_{\mathsf{D}_{q,p}(\Omega)}
\end{aligned}$$

for some positive constant C_{N,p,q,δ_4}, where δ_4 is the same constant as in Lemma 4.4.

Next, we introduce a maximal regularity with exponential stability for (4.2). To this end, we start with the standard maximal regularity following from [12] with a small perturbation method as follows:

Lemma 4.6. *Let $p,q \in (1,\infty)$ and suppose that Assumption 4.1 holds. Then there exists a positive number δ_5 such that, for $(\rho_0,\mathbf{u}_0) = (0,0)$ and for any f and $\mathbf{g}$ with*

$$e^{-\delta_5 t}f \in L_p((0,\infty),\mathsf{H}^1_q(\Omega)), \quad e^{-\delta_5 t}\mathbf{g} \in L_p((0,\infty),L_q(\Omega)^N),$$

the system (4.2) admits a unique solution $(\rho,\mathbf{u})$ with

$$\rho \in H^1_{p,\mathrm{loc}}((0,\infty),\mathsf{H}^1_q(\Omega)) \cap L_{p,\mathrm{loc}}((0,\infty),\mathsf{H}^3_q(\Omega)),$$

$$\mathbf{u} \in H^1_{p,\mathrm{loc}}((0,\infty), L_q(\Omega)^N) \cap L_{p,\mathrm{loc}}((0,\infty), H^2_q(\Omega)^N).$$

In addition, the solution $(\rho, \mathbf{u})$ *satisfies the estimate:*

$$\begin{aligned}
&\|e^{-\delta_5 t}\partial_t\rho\|_{L_p((0,\infty),\mathsf{H}^1_q(\Omega))} + \|e^{-\delta_5 t}\rho\|_{L_p((0,\infty),\mathsf{H}^3_q(\Omega))} \\
&+ \|e^{-\delta_5 t}\partial_t\mathbf{u}\|_{L_p((0,\infty),L_q(\Omega)^N)} + \|e^{-\delta_5 t}\mathbf{u}\|_{L_p((0,\infty),H^2_q(\Omega)^N)} \\
&\le C_{N,p,q,\delta_5}\left(\|e^{-\delta_5 t}f\|_{L_p((0,\infty),\mathsf{H}^1_q(\Omega))} + \|e^{-\delta_5 t}\mathbf{g}\|_{L_p((0,\infty),L_q(\Omega)^N)}\right)
\end{aligned}$$

for some positive constant C_{N,p,q,δ_5}.

Similarly to [11], we can prove by Lemmas 4.4 and 4.6

Lemma 4.7 (Maximal regularity with exponential stability)**.** *Let* $p, q \in (1,\infty)$ *and suppose that Assumption* 4.1 *holds. Then there exists a positive number* $\delta_6 \in (0,1)$ *such that, for* $(\rho_0, \mathbf{u}_0) = (0,0)$ *and for any* f *and* $\mathbf{g}$ *with*

$$e^{\delta_6 t}f \in L_p((0,\infty), \mathsf{H}^1_q(\Omega)), \quad e^{\delta_6 t}\mathbf{g} \in L_p((0,\infty), L_q(\Omega)^N),$$

the system (4.2) *admits a unique solution* $(\rho, \mathbf{u})$ *with*

$$\begin{aligned}
\rho &\in {}_0H^1_p((0,\infty), \mathsf{H}^1_q(\Omega)) \cap L_p((0,\infty), \mathsf{H}^3_q(\Omega)), \\
\mathbf{u} &\in {}_0H^1_p((0,\infty), L_q(\Omega)^N) \cap L_p((0,\infty), H^2_q(\Omega)^N).
\end{aligned}$$

In addition, the solution $(\rho, \mathbf{u})$ *satisfies the estimate:*

$$\begin{aligned}
&\|e^{\delta_6 t}\partial_t\rho\|_{L_p((0,\infty),\mathsf{H}^1_q(\Omega))} + \|e^{\delta_6 t}\rho\|_{L_p((0,\infty),\mathsf{H}^3_q(\Omega))} \\
&+ \|e^{\delta_6 t}\partial_t\mathbf{u}\|_{L_p((0,\infty),L_q(\Omega)^N)} + \|e^{\delta_6 t}\mathbf{u}\|_{L_p((0,\infty),H^2_q(\Omega)^N)} \\
&\le C_{N,p,q,\delta_6}\left(\|e^{\delta_6 t}f\|_{L_p((0,\infty),\mathsf{H}^1_q(\Omega))} + \|e^{\delta_6 t}\mathbf{g}\|_{L_p((0,\infty),L_q(\Omega)^N)}\right)
\end{aligned}$$

for some positive constant C_{N,p,q,δ_6}.

4.3 Global existence theorem

Let $p, q \in (2,\infty) \times (N,\infty)$ and $(\rho_0, \mathbf{u}_0) \in \mathsf{D}_{q,p}(\Omega)$ with $\|(\rho_0, \mathbf{u}_0)\|_{\mathsf{D}_{q,p}(\Omega)} \le \varepsilon_1$ for some positive number $\varepsilon_1 \in (0,1)$ determined below, and let $(\rho_*, \mathbf{u}_*) = e^{\mathsf{A}_t t}(\rho_0, \mathbf{u}_0)$. Then, setting $\rho = \sigma + \rho_*$ and $\mathbf{u} = \mathbf{v} + \mathbf{u}_*$ in (4.1), we observe that

$$\begin{cases}
\partial_t\sigma + \rho_\infty \operatorname{div}\mathbf{v} = \mathsf{F}(\sigma+\rho_*, \mathbf{v}+\mathbf{u}_*) & \text{in } \Omega\times(0,\infty), \\
\partial_t\mathbf{v} - \operatorname{Div}(\mu_\infty\mathbf{D}(\mathbf{v}) + (\nu_\infty - \mu_\infty)\operatorname{div}\mathbf{v}\mathbf{I} + \kappa\Delta\sigma\mathbf{I}) + \gamma_\infty\nabla\sigma & \\
\qquad = \mathsf{G}(\sigma+\rho_*, \mathbf{v}+\mathbf{u}_*) & \text{in } \Omega\times(0,\infty), \\
\mathbf{n}\cdot\nabla\sigma = 0, \quad \mathbf{v} = 0 & \text{on } \Gamma\times(0,\infty), \\
(\sigma, \mathbf{v})|_{t=0} = (0,0) & \text{in } \Omega.
\end{cases} \tag{4.4}$$

To use the contraction mapping principle, we introduce the following notation:

- ${}_0Z_\infty := {}_0Z^1_\infty \times {}_0Z^2_\infty$ with

$$
\begin{aligned}
{}_0Z^1_\infty &= {}_0H^1_p((0,\infty), H^1_p(\Omega)) \cap L_p((0,\infty), H^3_q(\Omega)),\\
{}_0Z^2_\infty &= {}_0H^1_p((0,\infty), L_q(\Omega)^N) \cap L_p((0,\infty), H^2_q(\Omega)^N).
\end{aligned}
$$

In addition, for $\delta > 0$,

$$
\begin{aligned}
\|(\rho,\mathbf{u})\|_{{}_0Z^\delta_\infty} := &\|e^{\delta t}\partial_t\rho\|_{L_p((0,\infty),H^1_q(\Omega))} + \|e^{\delta t}\rho\|_{L_p((0,\infty),H^3_q(\Omega))}\\
&+ \|e^{\delta t}\partial_t\mathbf{u}\|_{L_p((0,\infty),L_q(\Omega)^N)} + \|e^{\delta t}\rho\|_{L_p((0,\infty),H^2_q(\Omega)^N)}.
\end{aligned}
$$

- For $\delta > 0$ and $r > 0$,

$$
\begin{aligned}
{}_0Z^\delta_\infty(r) := \Big\{&(\tau,\mathbf{w}) \in {}_0Z_\infty : \|(\tau,\mathbf{w})\|_{{}_0Z^\delta_\infty} \le r,\ \ \mathbf{w} = 0 \text{ on } \Gamma,\\
&\frac{\rho_\infty}{4} \le \tau(x,t) + \rho_*(x,t) + \rho_\infty \le 4\rho_\infty \text{ for any } (x,t) \in \overline{\Omega} \times [0,\infty)\Big\}.
\end{aligned}
$$

Let $(\tau,\mathbf{w}) \in {}_0Z^\delta_\infty(\varepsilon_2)$ for a suitable positive number $\delta \in (0,1)$ and for $\varepsilon_2 \in (0,1)$, and replace $(\sigma,\mathbf{v})$ by $(\tau,\mathbf{w})$ in the right-hand sides of (4.4). Then, by the maximal regularity with exponential stability stated in Lemma 4.7, we can define a contraction mapping $\Phi : {}_0Z^\delta_\infty(\varepsilon_2) \ni (\tau,\mathbf{w}) \mapsto (\sigma,\mathbf{v}) \in {}_0Z^\delta_\infty(\varepsilon_2)$ for sufficiently small positive numbers ε_1 and ε_2. We thus obtain by the contraction mapping principle a global existence theorem in the maximal regularity class as follows:

Theorem 4.8. *Let* $(p,q) \in (2,\infty) \times (N,\infty)$ *and suppose that Assumption* 4.1 *holds. Then there exist positive numbers* δ, ε_1, *and* ε_2 *such that, for any* $(\rho_0,\mathbf{u}_0) \in \mathsf{D}_{q,p}(\Omega)$ *with* $\|(\rho_0,\mathbf{u}_0)\|_{\mathsf{D}_{q,p}(\Omega)} \le \varepsilon_1$, *the system* (4.4) *admits a unique global solution* $(\sigma,\mathbf{v})$ *in* ${}_0Z^\delta_\infty(\varepsilon_2)$.

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