# 博　士　論　文

# Algebraic cycles and cohomology with torsion coefficients of algebraic varieties

（代数的サイクルと代数多様体の捩れ係数コホモロジーについて）


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# Algebraic cycles and cohomology with torsion coefficients of algebraic varieties

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To my family.

# Abstract

The Tate conjecture predicts that the Galois invariant subspace of the $\ell$-adic cohomology group of a projective smooth variety over a field finitely generated over its prime subfield is spanned by the classes of algebraic cycles. It is also expected that the assertion of the Tate conjecture holds for the cohomology group with $\ell$-torsion coefficients of such a variety for all but finitely many prime numbers $\ell$.

Madapusi Pera deduced the Tate conjecture for K3 surfaces in full generality from the Tate conjecture for divisors on abelian varieties by using a period map from the moduli space of K3 surfaces to the integral canonical model of an orthogonal Shimura variety, called the Kuga-Satake morphism. The first part of this thesis is devoted to give further applications of the Kuga-Satake morphism to the arithmetic of K3 surfaces. We prove the Tate conjecture for the square of any K3 surface over a finite field of characteristic different from 2 and 3. Moreover, we prove the torsion analogue of the Tate conjecture for K3 surfaces.

The Tate conjecture and its torsion analogue guarantee that a projective smooth variety over a field finitely generated over its prime subfield satisfies the property that the Galois invariant subspace of the cohomology group with $\ell$-torsion coefficients coincides with the reduction modulo $\ell$ of the Galois invariant subspace of the $\ell$-adic cohomology group for all but finitely many prime numbers $\ell$. We can ask whether projective smooth varieties over non-archimedean local fields satisfy the same property or not, although the assertion of the Tate conjecture does not hold for such varieties in general. This leads to a torsion analogue of the weight-monodromy conjecture. In the second part of this thesis, we formulate and study a torsion analogue of the weight-monodromy conjecture for a proper smooth variety over a non-archimedean local field. We prove it for proper smooth varieties over equal characteristic non-archimedean local fields, surfaces, and set-theoretic complete intersections in toric varieties. In the course of the proof for set-theoretic complete intersections in toric varieties, we prove an $\ell$-independence result on étale cohomology of tubular neighborhoods of rigid analytic varieties.

As a related topic, in the third part of this thesis, we study deformations of rational curves on surfaces and their singularities in positive characteristic. One of the initial motivations of this study is to construct infinitely many rational curves on a K3 surface. It is known that rational curves on a K3 surface are important to study the Chow group of it.

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# Contents

CHAPTER 1

# Introduction

## 1.1. Outline of this thesis

The theory of algebraic cycles has a lot of applications in many branches of mathematics, such as algebraic geometry and arithmetic geometry. The Tate conjecture is one of the main problems in the theory of algebraic cycles. In this thesis, we study the Tate conjecture, a torsion analogue of it and related topics. More specifically, we prove the following results.

(1) The Tate conjecture holds for the square of any K3 surface over a finite field of characteristic different from 2 and 3. See Chapter 2 for details. This result is based on the joint work [**65**] with Tetsushi Ito and Teruhisa Koshikawa.
(2) A closed subvariety of a proper variety over an algebraically closed complete non-archimedean field has a small open neighborhood in the analytic topology such that, for every prime number $\ell$ different from the residue characteristic, the closed variety and the open neighborhood have the same étale cohomology with $\mathbb{F}_\ell$-coefficients. See Chapter 3 for details. This result is based on the preprint [**63**].
(3) A torsion analogue of the weight-monodromy conjecture holds for proper smooth varieties over equal characteristic non-archimedean local fields, abelian varieties, surfaces, varieties uniformized by Drinfeld upper half spaces, and set-theoretic complete intersections in toric varieties. See Chapter 4 for details. This result is based on the preprint [**64**].
(4) If a proper smooth surface in positive characteristic $p$ is dominated by a family of rational curves such that one member has all $\delta$-invariants strictly less than $(p-1)/2$, then the surface has negative Kodaira dimension. See Chapter 5 for details. This result is based on the joint work [**66**] with Tetsushi Ito and Christian Liedtke.

In Chapter 2, we study the integral canonical models of orthogonal Shimura varieties and the Kuga-Satake construction for K3 surfaces. The result (1) is deduced from a theorem on CM (complex multiplication) liftings of K3 surfaces. We also prove a torsion analogue of the Tate conjecture for K3 surfaces.

In Chapter 4, inspired by a torsion analogue of the Tate conjecture, we formulate and study a torsion analogue of the weight-monodromy conjecture for a proper smooth variety over a non-archimedean local field. Then we prove the result (3). In the proof for set-theoretic complete intersections in toric varieties, we need the result (2). Chapter 3 is devoted to study local constancy of étale cohomology of rigid analytic varieties and prove the result (2).

Finally, as a related topic, we study deformations of rational curves on surfaces and their singularities in positive characteristic and prove the result (4) in Chapter 5.

In the rest of this chapter, we shall give precise statements of our results. At the beginning of each chapter, a more detailed introduction will be provided.

## 1.2. Kuga-Satake constructions for K3 surfaces

Let $k$ be a field. Let $\overline{k}$ be an algebraic closure of $k$ and $k^{\mathrm{sep}}$ the separable closure of $k$ in $\overline{k}$. The absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ of $k$ is denoted by $G_k$. Let $p \geq 0$ be the characteristic of $k$.

Let $X$ be an algebraic variety over $k$. Let $\ell \neq p$ be a prime number. The absolute Galois group $G_k$ naturally acts on the $\ell$-adic cohomology $H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$, where we put $X_{\overline{k}} := X \otimes_k \overline{k}$.

We recall the following conjecture due to Tate [**127**, Section 1], called the *Tate conjecture.*

**Conjecture 1.2.1 (Tate).** *Let $k$ be a field which is finitely generated over its prime subfield. Let $p \geq 0$ be the characteristic of $k$. Let $X$ be a projective smooth variety over $k$. Then, for every $i$, the $\ell$-adic cycle class map*

$$\mathrm{cl}^i_\ell \colon Z^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^{2i}_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(i))^{G_k}$$

*is surjective for every prime number $\ell \neq p$. Here $Z^i(X)$ denotes the group of algebraic cycles of codimension $i$ on $X$ and $(i)$ denotes the Tate twist.*

Conjecture 1.2.1 is known to be true in the following cases. Let $k$ be a field which is finitely generated over its prime subfield. Let $p \geq 0$ be the characteristic of $k$.

- Let $A$ and $B$ be abelian varieties over $k$. Then the natural morphism

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{Hom}_{G_k}(H^1_{\text{ét}}(B_{\overline{k}}, \mathbb{Q}_\ell), H^1_{\text{ét}}(A_{\overline{k}}, \mathbb{Q}_\ell))$$

  is an isomorphism for every $\ell \neq p$. If $k$ is a finite field, it was proved by Tate [**125**]. If $p = 0$, it was proved by Faltings [**40, 41**]. For the case where $p > 0$ and $k$ is infinite, it was proved by Zarhin [**136**] if $p > 2$ and by Mori [**92**, Chapitre XII, Théorème 2.5] if $p = 2$. This result implies the Tate conjecture for divisors on abelian varieties over $k$.
- Let $X$ be a K3 surface over $k$. The Tate conjecture for $X$ is known to be true. If $p = 0$, this follows from the work of Deligne [**31**] (see also André's paper [**1**]). If $p > 0$, many authors contributed to this conjecture [**96, 97, 23, 88**], and it was proved in full generality by Madapusi Pera [**85, 72**].

Recall that a *K3 surface* $X$ over a field is a projective smooth surface with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$.

The integral canonical models of orthogonal Shimura varieties and the Kuga-Satake morphism have applications to the arithmetic of K3 surfaces. For example, Madapusi Pera used it in his proof of the Tate conjecture for K3 surfaces. As a further application, we will prove the following theorem in Chapter 2.

**Theorem 1.2.2 (Theorem 2.1.4).** *Let $\mathbb{F}_q$ denote a finite field of cardinality $q$. Assume that $\mathbb{F}_q$ is of characteristic $p \geq 5$. Let $X$ be a K3 surface over $\mathbb{F}_q$. Then the Tate conjecture (Conjecture 1.2.1) for $X \times_{\mathrm{Spec}\,\mathbb{F}_q} X$ is true.*

Here, just for simplicity, we assume that $p \geq 5$. Our results are valid over finite fields of any characteristics; see [**65**] for details. Theorem 1.2.2 is deduced from a theorem on *CM liftings* of K3 surfaces over finite fields; see Theorem 2.1.1.

As another application of the Kuga-Satake morphism, we will prove a torsion analogue of the Tate conjecture for K3 surfaces in Chapter 2. More precisely, we will prove the following theorem.

**Theorem 1.2.3 (Theorem 2.9.1).** *Let $k$ be a field which is finitely generated over its prime subfield, and $X$ a K3 surface over $k$. Let $p \geq 0$ be the characteristic of $k$. Then the Chern class map for $\ell$-torsion coefficients*

$$\mathrm{Pic}(X) \to H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_\ell(1))^{G_k}$$

*is surjective for all but finitely many $\ell \neq p$. Here* (1) *denotes the Tate twist.*

We will also give an application to the finiteness of the Brauer group of a K3 surface over a field finitely generated over its prime field in Chapter 2. If the characteristic of the base field $k$ is 0, Theorem 1.2.3 was proved by Skorobogatov-Zarhin [**118**]. For the proof, we need the Kuga-Satake morphism. Even when $k$ is of characteristic $p > 0$, we can use the same methods to prove the result, as shown in [**119**] when $p \geq 3$ and in [**62**] when $p = 2$.

**Remark 1.2.4.** In characteristic $p > 0$, Theorem 1.2.3 also follows from a more general result due to Cadoret-Hui-Tamagawa [**21**], which says that the Tate conjecture for divisors implies the Tate conjecture with torsion coefficients for divisors, without using the Kuga-Satake morphism. A key ingredient of the proof is the following result [**20**, Theorem 4.5], which is also due to Cadoret-Hui-Tamagawa:

- Let $k$ be a field which is finitely generated over $\mathbb{F}_p$ and $X$ a proper smooth variety over $k$. Then the natural map $H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_\ell) \to H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_\ell)$ gives an isomorphism
$$H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_\ell)^{\mathrm{Gal}(k^{\mathrm{sep}}/k.\overline{\mathbb{F}}_p)} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \cong H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_\ell)^{\mathrm{Gal}(k^{\mathrm{sep}}/k.\overline{\mathbb{F}}_p)}$$
for all but finitely many $\ell \neq p$ and for every $i$.

In Chapter 4, we investigate an analogue of [**20**, Theorem 4.5] for local fields. It leads to a torsion analogue of the weight-monodromy conjecture, which we explain in the next section.

## 1.3. A torsion analogue of the weight-monodromy conjecture

Let $K$ be a non-archimedean local field, i.e. the field of fractions of a complete discrete valuation ring whose residue field is a finite field $\mathbb{F}_q$. Let $\overline{K}$ be an algebraic closure of $K$. Let $p > 0$ be the characteristic of the residue field $\mathbb{F}_q$.

**1.3.1. The statement.** Let $X$ be a proper smooth variety over $K$ and $w$ an integer. By Grothendieck's quasi-unipotence theorem, the action of an open subgroup of the inertia group $I_K$ of $K$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ defines the monodromy filtration

$$\{M_{i,\mathbb{Q}_\ell}\}_i$$

on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. It is an increasing filtration stable by the action of $G_K$. The *weight-monodromy conjecture* due to Deligne states that the $i$-th graded piece

$$\mathrm{Gr}^M_{i,\mathbb{Q}_\ell} := M_{i,\mathbb{Q}_\ell}/M_{i-1,\mathbb{Q}_\ell}$$

of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is of weight $w + i$, i.e. every eigenvalue of a lift of the geometric Frobenius element $\mathrm{Frob}_q \in G_{\mathbb{F}_q}$ is an algebraic integer such that the complex absolute values of its conjugates are $q^{(w+i)/2}$. When $X$ has good reduction over the ring of integers $\mathcal{O}_K$ of $K$, it is nothing more than the Weil conjecture [**32, 34**]. In general, the weight-monodromy conjecture is still open. We shall propose a torsion analogue of the weight-monodromy conjecture and prove it in some cases.

By the work of Rapoport-Zink [**106**] and de Jong's alteration [**28**], we can take an open subgroup $J \subset I_K$ so that the action of $J$ on the étale cohomology group with $\mathbb{F}_\ell$-coefficients $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ is unipotent for every $\ell \neq p$. By the same construction as in the $\ell$-adic case, we can define the monodromy filtration

$$\{M_{i,\mathbb{F}_\ell}\}_i$$

on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$, which is stable by the action of $G_K$; see Section 4.3 for details.

We propose the following conjecture.

**Conjecture 1.3.1 (A torsion analogue of the weight-monodromy conjecture, Conjecture 4.3.4).** *Let $X$ be a proper smooth variety over $K$ and $w$ an integer. For every $i$, there exists a non-zero monic polynomial $P_i(T) \in \mathbb{Z}[T]$ satisfying the following conditions:*

- *The roots of $P_i(T)$ have complex absolute values $q^{(w+i)/2}$.*
- *We have $P_i(\mathrm{Frob}) = 0$ on the $i$-th graded piece*

$$\mathrm{Gr}^M_{i,\mathbb{F}_\ell} := M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}$$

  *for all but finitely many $\ell \neq p$ and for every lift $\mathrm{Frob} \in G_K$ of the geometric Frobenius element.*

In Chapter 4, we will prove the following theorem.

**Theorem 1.3.2 (Theorem 4.3.6).** *Let $X$ be a proper smooth variety over $K$ and $w$ an integer. Assume that one of the following conditions holds:*

*(A) $K$ is of equal characteristic, i.e. the characteristic of $K$ is $p$.*
*(B) $X$ is an abelian variety.*
*(C) $w \leq 2$ or $w \geq 2\dim X - 2$.*
*(D) $X$ is uniformized by a Drinfeld upper half space.*
*(E) $X$ is a set-theoretic complete intersection in a projective smooth toric variety.*

*Then the assertion of Conjecture 1.3.1 for $(X, w)$ is true.*

The weight-monodromy conjecture for $\mathbb{Q}_\ell$-coefficients is known to be true for $(X, w)$ if one of the above conditions (A)–(E) holds. However, it seems that the weight-monodromy conjecture for $\mathbb{Q}_\ell$-coefficients does not automatically imply Conjecture 4.3.4; see Section 4.3.3 for details. For the case (A), we will use an ultraproduct variant of Weil II established by Cadoret [**19**].

As applications, we will show some finiteness properties of the Brauer group and the codimension two Chow group of a proper smooth variety over $K$. See Section 4.10 for details.

**1.3.2. On $\ell$-independence for Huber's tubular neighborhoods.** We shall explain our next result, which is used in the proof of Theorem 1.3.2 in the case (E).

As in [**114**], by using the tilting equivalence of Scholze, we will deduce the case (E) from the case (A). In his proof of the weight-monodromy conjecture for the case (E), Scholze used a theorem of Huber on étale cohomology of tubular neighborhoods of rigid analytic varieties, which we recall below.

We assume that $K$ is of characteristic 0. Let $\mathbb{C}_p$ be the completion of the algebraic closure $\overline{K}$ of $K$, which is a complete non-archimedean field. Let $L \subset \mathbb{C}_p$ be a complete non-archimedean subfield. For a variety $X$ over $L$, the rigid analytic variety associated with $X$ is denoted by $X^{\mathrm{an}}$. (Here we consider rigid analytic varieties as adic spaces.) For a rigid analytic variety $Y$ over $L$, let $Y_{\mathbb{C}_p}$ denote the base change of $Y$ to $\mathbb{C}_p$.

The theory of étale cohomology for rigid analytic varieties was developed in [**52**]. Let us recall the following theorem of Huber:

- Let $Y$ be a proper variety over $L$ and $X \hookrightarrow Y$ a closed immersion. We have a closed immersion $X^{\mathrm{an}} \hookrightarrow Y^{\mathrm{an}}$ of rigid analytic varieties over $L$. We fix a prime number $\ell \neq p$. Then, there is an open subset $V$ of $Y^{\mathrm{an}}$ containing $X^{\mathrm{an}}$, such that the pull-back map
$$H^w_{\text{ét}}(V_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(X^{\mathrm{an}}_{\mathbb{C}_p}, \mathbb{F}_\ell)$$
of étale cohomology groups is an isomorphism for every $w$.

The open subset $V$ as above can be taken as a *tubular neighborhood* of $X^{\mathrm{an}}$. (See Section 3.4 for the definition of tubular neighborhoods.)

In our case, we need the following uniform variant of Huber's theorem:

**Theorem 1.3.3 (Corollary 3.4.11).** *Let $Y$ be a proper variety over $L$ and $X \hookrightarrow Y$ a closed immersion. We have a closed immersion $X^{\mathrm{an}} \hookrightarrow Y^{\mathrm{an}}$ of rigid analytic varieties over $L$. Then, there is an open subset $V$ of $Y^{\mathrm{an}}$ containing $X^{\mathrm{an}}$, such that, for every prime number $\ell \neq p$, the pull-back map*
$$H^w_{\text{ét}}(V_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(X^{\mathrm{an}}_{\mathbb{C}_p}, \mathbb{F}_\ell)$$
*is an isomorphism for every $w$.*

The same result holds for a complete non-archimedean field of positive characteristic. We will prove Theorem 1.3.3 in Chapter 3.

## 1.4. Rational curves on surfaces

As a related topic, we will study deformations of rational curves on surfaces and their singularities in positive characteristic.

Before stating our result, let us recall some classical invariants of singularities. Let $C$ be an integral curve over an algebraically closed field $k$ of characteristic $p > 0$. For each closed point $x \in C$, the $\delta$-invariant and the Jacobian number of $C$ at $x$ are defined as follows:

- The *$\delta$-invariant* is defined by
$$\delta(C, x) := \dim_k(\pi_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C)_x,$$
where $\pi\colon \widetilde{C} \to C$ is the normalization morphism.
- The *Jacobian number* is defined by
$$\mathrm{jac}(C, x) := \dim_k \left(\mathcal{O}_C / \operatorname{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k})\right)_x,$$
where $\operatorname{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k}) \subset \mathcal{O}_C$ is the first Fitting ideal of the sheaf $\Omega^1_{C/k}$ of Kähler differentials on $C$.

Let $X$ be a proper smooth surface over $k$. A *family of rational curves on $X$* means a closed subvariety $\mathscr{C} \subset U \times X$ with projections $\pi\colon \mathscr{C} \to U$ and $\varphi\colon \mathscr{C} \to X$ where $U$ is an integral variety over $k$, $\pi$ is proper flat, and every geometric fiber of $\pi$ is an integral rational curve. We say that a rational curve $C \subset X$ is *topologically non-rigid* if there exists a family of rational curves $(\pi, \varphi)$ on $X$ such that $\varphi\colon \mathscr{C} \to X$ is dominant and such that $\varphi(\mathscr{C}_u) = C$ for some closed point $u \in U$. Otherwise, we say that $C$ is *topologically rigid*.

In Chapter 5, we will prove the following theorem.

**Theorem 1.4.1 (Theorem 5.1.1).** *Let $X$ be a proper smooth surface over an algebraically closed field $k$ of characteristic $p > 0$. Assume that $X$ contains a topologically non-rigid rational curve $C \subset X$ satisfying one of the following conditions:*

*(1) The $\delta$-invariants of $C$ are strictly less than $(p-1)/2$ at every closed point.*
*(2) The Jacobian numbers of $C$ are strictly less than $p$ at every closed point.*

*Then, $X$ is separably uniruled and thus, has negative Kodaira dimension.*

Our results are optimal in some sense; see Proposition 5.7.7.

**Remark 1.4.2.** It is conjectured that there are infinitely many rational curves on a K3 surface over an algebraically closed field; see [**77**, Conjecture in Introduction]. The initial motivation of this work is to construct infinitely many rational curves on a K3 surface $X$ defined over a number field by lifting rational curves on the reduction of $X$ modulo $p$ following methods given in [**10, 77**]. It is known that rational curves on a K3 surface are important to study the Chow group of 0-cycles on it; see [**57**] for details.

## 1.5. Notation

The following notation will be used throughout this thesis.

(1) Let $k$ be a field. An algebraic closure of $k$ is denoted by $\overline{k}$ and a separable closure of $k$ is denoted by $k^{\mathrm{sep}}$ (except for Chapter 3). We often take $k^{\mathrm{sep}}$ as the separable closure of $k$ in $\overline{k}$. The absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ of $k$ is denoted by $G_k$.
(2) Let $\mathbb{F}_q$ denote a finite field of cardinality $q$. For a prime number $\ell$, let $\mathbb{Z}_\ell$ be the ring of $\ell$-adic integers and let $\mathbb{Q}_\ell$ be the field of fractions of $\mathbb{Z}_\ell$.
(3) A non-archimedean local field is the field of fractions of a complete discrete valuation ring whose residue field is a finite field.
(4) Let $f\colon X \to S$ be a morphism of schemes. For a morphism $T \to S$ of schemes, the base change $X \times_S T$ of $X$ is denoted by $X_T$ and $f_T\colon X_T \to T$ denotes the base change of $f$. If $S = \operatorname{Spec} R$ and $T = \operatorname{Spec} R'$ are affine schemes, the base change $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ is also denoted by $X_{R'}$ or $X \otimes_R R'$. We use similar notation for the base change of group schemes, $p$-divisible groups, line bundles, étale sheaves, morphisms between them, etc.
(5) Let $k$ be a field. A *variety* $X$ over $k$ means a separated scheme $X$ of finite type over $k$ such that $X_{\overline{k}}$ is connected. A *surface* over $k$ means a variety of pure dimension 2 over $k$. Moreover, a *curve* over $k$ means a variety of pure dimension 1 over $k$. For a variety $X$ (or a scheme $X$ of finite type) over $k$, let $\dim X$ denote the dimension of $X$.
(6) Let $G$ be a group and let $M$ be an abelian group equipped with an action of $G$. Let $M^G$ denote the $G$-fixed part of $M$. Let $M_G$ denote be the group of $G$-coinvariants of $M$.
(7) For a perfect field $k$ of characteristic $p$, the ring of Witt vectors of $k$ is denoted by $W(k)$. The Frobenius automorphism of $W(k)$ is denoted by $\sigma\colon W(k) \to W(k)$.

CHAPTER 2

# Kuga-Satake constructions for K3 surfaces

## 2.1. Introduction

The integral canonical models of orthogonal Shimura varieties and the Kuga-Satake morphism have applications to the arithmetic of K3 surfaces. For example, Madapusi Pera used it to prove the Tate conjecture for divisors on K3 surfaces over finitely generated fields [**85, 72**].

The aim of this chapter is to give further applications. The major part of this chapter is devoted to prove the following results.

(1) (see Theorem 2.1.1) Every K3 surface $X$ of finite height over a finite field $\mathbb{F}_q$ with $q$ elements admits a CM lifting after replacing $\mathbb{F}_q$ by its finite extension (i.e. it admits a characteristic 0 lifting whose generic fiber has complex multiplication).

(2) (see Theorem 2.1.4) The Tate conjecture holds for algebraic cycles of codimension 2 on the square $X \times X$ of any K3 surface $X$ (of any height) over $\mathbb{F}_q$.

These results are consequences of our results on characteristic 0 liftings of K3 surfaces; see Theorem 2.1.6. These results were obtained in the joint work with Tetsushi Ito and Teruhisa Koshikawa [**65**]. For simplicity, we will assume that the characteristic of the base field is greater than or equal to 5. However, all of our results are valid over finite fields of any characteristics; see [**65**] for details.

In the last section of this chapter, we prove the Tate conjecture with torsion coefficients for a K3 surface over a field finitely generated over its prime field; see Theorem 2.9.1. If the characteristic of the base field is 0, this was proved by Skorobogatov-Zarhin [**118**]. For the proof, we need the Kuga-Satake morphism. Even when the base field is of characteristic $p > 0$, we can use the same methods to prove the result, as shown in [**119**] when $p \geq 3$ and in [**62**] when $p = 2$. However, in characteristic $p > 0$, the result also follows from a more general result due to Cadoret-Hui-Tamagawa [**21**] without using the Kuga-Satake morphism. See Section 2.9 for details. We will also give an application to the finiteness of the Brauer group of a K3 surface over a field finitely generated over its prime field.

In the rest of this section, we shall first give precise statements of our results on CM liftings and the Tate conjecture; see Theorem 2.1.1 and Theorem 2.1.4. Then we explain our results on characteristic 0 liftings (see Theorem 2.1.6), and how to obtain the results (1) and (2) from them.

**2.1.1. CM liftings of K3 surfaces of finite height over finite fields.** First we state our results on CM liftings.

Recall that a projective smooth surface $X$ over a field is called a *K3 surface* if its canonical bundle is trivial and it satisfies $H^1(X, \mathcal{O}_X) = 0$. More generally, an algebraic space $\mathscr{X}$ over a scheme $S$ is a K3 surface over $S$ if $\mathscr{X} \to S$ is proper, smooth, and every geometric fiber is a K3 surface.

We say that a projective K3 surface $Y$ over $\mathbb{C}$ has *complex multiplication* (*CM*) if the Mumford-Tate group associated with the singular cohomology $H^2_B(Y, \mathbb{Q})$ is commutative;

see Section 2.7.1. We say that a K3 surface $Y$ over a number field $F$ has CM if $Y_{\mathbb{C}}$ has CM for every embedding $F \hookrightarrow \mathbb{C}$.

Throughout this section, we fix a prime number $p \geq 5$. Let $q$ be a power of $p$. Let $X$ be a K3 surface over $\mathbb{F}_q$. We say that $X$ admits a *CM lifting* if there exist a number field $F$, a finite place $v$ of $F$ with residue field $\mathbb{F}_q$, and a K3 surface $\mathscr{X}$ over the localization $\mathcal{O}_{F,(v)}$ of the ring of integers $\mathcal{O}_F$ of $F$ at $v$ such that the special fiber $\mathscr{X}_{\mathbb{F}_q}$ is isomorphic to $X$, and the generic fiber $\mathscr{X}_F$ is a K3 surface with CM.

The height $h$ of the formal Brauer group of $X$ is called the *height* of $X$; it satisfies $1 \leq h \leq 10$ or $h = \infty$. When $1 \leq h \leq 10$ (resp. $h = \infty$), we say that $X$ is *of finite height* (resp. *supersingular*).

Here is the first main theorem of this chapter.

**Theorem 2.1.1 (Corollary 2.7.10).** *Let $X$ be a K3 surface over $\mathbb{F}_q$. If $X$ is of finite height, then there is a positive integer $m \geq 1$ such that $X_{\mathbb{F}_{q^m}} := X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}_{q^m}$ admits a CM lifting.*

**Remark 2.1.2.** Yang also proved the above theorem under the additional conditions that $p \geq 5$ and $X$ admits a quasi-polarization whose degree is not divisible by $p$; see [**134**, Theorem 1.6]. Our method and Yang's method share several ingredients but there is one difference; Yang used Kisin's result [**74**, Theorem 0.4] on the CM liftings, up to isogeny, of closed points of the special fiber of the integral canonical model of a Shimura variety of Hodge type, while we give a refinement of Kisin's result (or argument) itself; see Theorem 2.1.6 for details.

**Remark 2.1.3.** Deuring proved that every elliptic curve over a finite field admits a characteristic 0 lifting whose generic fiber is an elliptic curve with CM; see [**22**, Theorem 1.7.4.6]. Theorem 2.1.1 is an analogue of this result for K3 surfaces of finite height. It is an interesting question to ask whether Theorem 2.1.1 holds also for supersingular K3 surfaces over finite fields. Our methods in this chapter cannot be applied to supersingular K3 surfaces.

**2.1.2. The Tate conjecture for the squares of K3 surfaces over finite fields.** As the second main theorem of this chapter, we shall prove the Tate conjecture (Conjecture 1.2.1) for the square of a K3 surface over a finite field of characteristic $p \geq 5$.

**Theorem 2.1.4.** *Let $X$ be a K3 surface (of any height) over $\mathbb{F}_q$. We put $X \times X := X \times_{\operatorname{Spec} \mathbb{F}_q} X$ and $X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q} := X_{\overline{\mathbb{F}}_q} \times_{\operatorname{Spec} \overline{\mathbb{F}}_q} X_{\overline{\mathbb{F}}_q}$. Then, for every $i$, the $\ell$-adic cycle class map*

$$\mathrm{cl}^i_\ell \colon Z^i(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^{2i}_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$$

*is surjective for every prime number $\ell \neq p$.*

Here $Z^i(X \times X)$ denotes the group of algebraic cycles of codimension $i$ on $X \times X$.

**Remark 2.1.5.** Theorem 2.1.4 was previously known to hold for some K3 surfaces.

(1) Theorem 2.1.4 obviously holds for any $i \notin \{1, 2, 3\}$.
(2) The surjectivity of $\mathrm{cl}^1_\ell$ and $\mathrm{cl}^3_\ell$ follow from the Tate conjecture for $X$ [**85, 72**]; see also Lemma 2.8.5.
(3) Theorem 2.1.4 holds when $X$ is supersingular. In fact, the Tate conjecture for $X$ implies the Picard number of $X_{\overline{\mathbb{F}}_q}$ is 22; see Lemma 2.8.2. Then the Tate conjecture for the square $X \times X$ follows by the Künneth formula; see Lemma 2.8.3 and Remark 2.8.4.

(4) Zarhin proved the Tate conjecture for $X \times X$ when $X$ is an ordinary K3 surface; see [**138**, Corollary 6.1.2]. Here a K3 surface $X$ is called *ordinary* if it is of height 1. (More generally, Zarhin proved the Tate conjecture for any power $X \times \cdots \times X$ of an ordinary K3 surface $X$.)

(5) Yu-Yui proved the Tate conjecture for $X \times X$ when $X$ satisfies some conditions on the characteristic polynomial of the Frobenius morphism; see [**135**, Lemma 3.5, Corollary 3.6].

In the cases studied by Zarhin and Yu-Yui, it turns out that all the Tate cycles of codimension 2 on $X \times X$ are spanned by the classes of the cycles of the form $X \times \{x_0\}$, $\{x_0\} \times X$, and $D_1 \times D_2$, and the classes of the graphs of powers of the Frobenius morphism on $X$. Here $x_0$ is a closed point on $X$, and $D_1$ and $D_2$ are divisors on $X$. In general, there are Tate classes on $X \times X$ which are not spanned by these classes. Therefore, in order to prove Theorem 2.1.4 in full generality, we shall prove the algebraicity of Tate cycles on $X \times X$ which are not spanned by Tate cycles considered by Zarhin and Yu-Yui. We shall prove it by constructing characteristic 0 liftings, and applying the results of Mukai and Buskin on the Hodge conjecture.

**2.1.3. Construction of characteristic 0 liftings preserving the action of tori.** Here we explain our results on the construction of characteristic 0 liftings of K3 surfaces.

Let $X$ be a K3 surface over $\mathbb{F}_q$, and $\mathscr{L}$ a line bundle on $X$ defined over $\mathbb{F}_q$ which gives a primitive quasi-polarization. Assume that $X$ is of finite height. After replacing $\mathbb{F}_q$ by a finite extension of it, the *Kuga-Satake abelian variety* $A$ associated with $(X, \mathscr{L})$ is defined over $\mathbb{F}_q$. (Precisely, we shall use the Kuga-Satake abelian variety introduced by Madapusi Pera in [**86, 85**], which has dimension $2^{21}$; it is larger than the dimension of the classical Kuga-Satake abelian variety. See Section 2.3.3.)

We have an action of a general spin group, denoted by $\mathrm{GSpin}(L_{\mathbb{Q}})$ in this chapter, on the cohomology of $X$ and $A$. We put $G := \mathrm{GSpin}(L_{\mathbb{Q}})$ in this section. We do not recall the precise definition of $G$ here. Instead, we give some of its properties:

- For every prime number $\ell \neq p$, the group of $\mathbb{Q}_\ell$-valued points $G(\mathbb{Q}_\ell)$ acts on the primitive part
$$P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)) := \mathrm{ch}_\ell(\mathscr{L})^\perp \subset H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$$
of the $\ell$-adic cohomology of $X$ and the $\ell$-adic cohomology $H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ of $A$.
- There is a $G(\mathbb{Q}_\ell)$-equivariant $\mathbb{Q}_\ell$-linear map
$$P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)) \to \mathrm{End}_{\mathbb{Q}_\ell}(H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^\vee),$$
where $()^\vee$ denotes the $\mathbb{Q}_\ell$-linear dual.
- There is an element $\mathrm{Frob}_q \in G(\mathbb{Q}_\ell)$ such that its action on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ (resp. $H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$) coincides with the action of the geometric Frobenius morphism on the $\ell$-adic cohomology of $X$ (resp. $A$).

Following Kisin [**74**], we attach an algebraic group $I$ over $\mathbb{Q}$ to the quasi-polarized K3 surface $(X, \mathscr{L})$; see Definition 2.6.1. Instead of giving the precise definition here, we give its properties:

- The group of $\mathbb{Q}$-valued points $I(\mathbb{Q})$ is considered as a subgroup of the multiplicative group of the endomorphism algebra of $A_{\overline{\mathbb{F}}_q}$ tensored with $\mathbb{Q}$:
$$I(\mathbb{Q}) \subset (\mathrm{End}_{\overline{\mathbb{F}}_q}(A_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} \mathbb{Q})^\times.$$

- For every prime number $\ell \neq p$, there is an embedding $I_{\mathbb{Q}_\ell} \hookrightarrow G_{\mathbb{Q}_\ell}$, and an element of $G(\mathbb{Q}_\ell)$ is in $I(\mathbb{Q}_\ell)$ if and only if it commutes with $\mathrm{Frob}_q^m$ for a sufficiently divisible $m \geq 1$.
- The algebraic groups $G$ and $I$ have the same rank.

The existence of an algebraic group $I$ over $\mathbb{Q}$ which satisfies these properties is not obvious; it is considered as Kisin's group-theoretic interpretation and generalization of Tate's original proof of the Tate conjecture for endomorphisms of abelian varieties over finite fields.

As the third main theorem of this chapter, we shall construct a characteristic 0 lifting of a quasi-polarized K3 surface of finite height preserving the action of a maximal torus of the algebraic group $I$.

**Theorem 2.1.6 (Theorem 2.7.7).** *Let $T \subset I$ be a maximal torus over $\mathbb{Q}$. Then there exist a finite extension $K$ of $W(\overline{\mathbb{F}}_q)[1/p]$ and a quasi-polarized K3 surface $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_K$ such that the special fiber $(\mathcal{X}_{\overline{\mathbb{F}}_q}, \mathcal{L}_{\overline{\mathbb{F}}_q})$ is isomorphic to $(X_{\overline{\mathbb{F}}_q}, \mathscr{L}_{\overline{\mathbb{F}}_q})$, and, for every embedding $K \hookrightarrow \mathbb{C}$, the quasi-polarized K3 surface $(\mathcal{X}_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}})$ satisfies the following properties:*

1. *The K3 surface $\mathcal{X}_{\mathbb{C}}$ has CM.*
2. *There is a homomorphism of algebraic groups over $\mathbb{Q}$*
$$T \to \mathrm{SO}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))).$$
*Here $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ is the primitive part of the Betti cohomology of $\mathcal{X}_{\mathbb{C}}$.*
3. *For every $\ell \neq p$, the action of $T(\mathbb{Q}_\ell)$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is identified with the action of $T(\mathbb{Q}_\ell)$ on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ via the canonical isomorphisms*
$$P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong P^2_{\text{ét}}(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}_\ell(1)) \cong P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$$
*(using the embedding $K \hookrightarrow \mathbb{C}$, we consider $K$ as a subfield of $\mathbb{C}$).*
4. *The action of every element of $T(\mathbb{Q})$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure on it.*

Our construction of characteristic 0 liftings relies on the theory of integral canonical models of Shimura varieties of Hodge type developed by Milne, Vasiu, Kisin, and Kim-Madapusi Pera.

**Remark 2.1.7.** It is known that every K3 surface with CM is defined over a number field; see Proposition 2.7.1 and Remark 2.7.2. Therefore, Theorem 2.1.6 implies Theorem 2.1.1.

**Remark 2.1.8.** When $X$ is ordinary, Theorem 2.1.6 was essentially proved by Nygaard in [**96**] although the algebraic group $I$ did not appear there. When $X$ is ordinary, the canonical lifting of $X$ is a CM lifting.

**2.1.4. Outline of the proofs of the main theorems.** We shall prove Theorem 2.1.1 and Theorem 2.1.6 at the same time. Then, combined with the results of Mukai and Buskin, we shall prove Theorem 2.1.4.

*Proofs of Theorem 2.1.1 and Theorem 2.1.6.* In the following argument, we replace $\mathbb{F}_q$ by a sufficiently large finite extension of it. We write $W := W(\mathbb{F}_q)$.

Let $\widehat{\mathrm{Br}} := \widehat{\mathrm{Br}}(X)$ be the formal Brauer group associated with $X$. First we shall show that $I_{\mathbb{Q}_p}$ acts on $\widehat{\mathrm{Br}}$, up to isogeny. Then we take a finite totally ramified extension $E$ of $W[1/p]$, and a one-dimensional smooth formal group $\mathcal{G}$ over $\mathcal{O}_E$ lifting $\widehat{\mathrm{Br}}$ such that the action of $I_{\mathbb{Q}_p}$ on $\widehat{\mathrm{Br}}$ lifts to an action of $I_{\mathbb{Q}_p}$ on $\mathcal{G}$, up to isogeny.

The lifting $\mathcal{G}$ defines filtrations on $P^2_{\mathrm{cris}}(X/W) \otimes_W E$ and $H^1_{\mathrm{cris}}(A/W) \otimes_W E$ as follows. Here $P^2_{\mathrm{cris}}(X/W)$ is the primitive part of the crystalline cohomology of $X$. The Kuga-Satake construction gives embeddings which are homomorphisms of $F$-isocrystals after inverting $p$:

$$\mathbb{D}(\widehat{\mathrm{Br}})(1) \subset P^2_{\mathrm{cris}}(X/W)(1) \subset \widetilde{L}_{\mathrm{cris}} \subset \mathrm{End}_W(H^1_{\mathrm{cris}}(A/W)^\vee).$$

(Here $\mathbb{D}(\widehat{\mathrm{Br}})$ is the Dieudonné module of $\widehat{\mathrm{Br}}$ considered as a connected $p$-divisible group. For the $W$-module $\widetilde{L}_{\mathrm{cris}}$, see Section 2.5.) The lifting $\mathcal{G}$ defines a filtration on $\mathbb{D}(\widehat{\mathrm{Br}})(1) \otimes_W E$:

$$\mathrm{Fil}^1(\mathcal{G}) \subset \mathbb{D}(\widehat{\mathrm{Br}})(1) \otimes_W E.$$

Thus, it gives the filtration on $P^2_{\mathrm{cris}}(X/W)(1) \otimes_W E$:

$$\mathrm{Fil}^1(\mathcal{G}) \subset \mathrm{Fil}^1(\mathcal{G})^\perp \subset P^2_{\mathrm{cris}}(X/W)(1) \otimes_W E.$$

Take a generator $e \in \mathrm{Fil}^1(\mathcal{G})$, and denote the image of the action of $e$ by

$$\mathrm{Fil}^1 := \mathrm{Im}(e) \subset H^1_{\mathrm{cris}}(A/W) \otimes_W E.$$

It gives a filtration on $H^1_{\mathrm{cris}}(A/W) \otimes_W E$, which does not depend on the choice of $e$.

When $p \geq 5$, the results of Nygaard-Ogus [**97**] imply the existence of a lifting $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_E$ corresponding to the filtration defined as above.

We shall show that, for every embedding $E \hookrightarrow \mathbb{C}$, the action of an element of $T(\mathbb{Q})$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure. To show this, we note that each element of $T(\mathbb{Q})$ can be considered as an element of $(\mathrm{End}_{\mathbb{F}_q}(A) \otimes_{\mathbb{Z}} \mathbb{Q})^\times$. Since its action preserves the filtration on $H^1_{\mathrm{cris}}(A/W) \otimes_W E$, it lifts to an element of $(\mathrm{End}_{\mathbb{C}}(\mathcal{A}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q})^\times$, where $\mathcal{A}_{\mathbb{C}}$ is the Kuga-Satake abelian variety over $\mathbb{C}$ associated with $(\mathcal{X}_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}})$. In particular, it preserves the Hodge structure on the singular cohomology $H^1_B(\mathcal{A}_{\mathbb{C}}, \mathbb{Q})$. Since we have a $T(\mathbb{Q})$-equivariant embedding respecting the $\mathbb{Q}$-Hodge structures

$$P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \hookrightarrow \mathrm{End}_{\mathbb{C}}(H^1_B(\mathcal{A}_{\mathbb{C}}, \mathbb{Q})^\vee),$$

the action of each element of $T(\mathbb{Q})$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure on it.

As the algebraic groups $G$ and $I$ have the same rank, we conclude that the Mumford-Tate group of $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ is commutative. Thus $\mathcal{X}_{\mathbb{C}}$ is a K3 surface with CM. Consequently, the quasi-polarized K3 surface $(\mathcal{X}_E, \mathcal{L}_E)$ is defined over a number field, and Theorem 2.1.1 and Theorem 2.1.6 are proved.

*Proof of Theorem 2.1.4.* We may assume that $X$ is of finite height. Fix a prime number $\ell \neq p$. By the Künneth formula, we have

$$\begin{aligned} &H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2)) \\ \cong &\bigoplus_{(i,j)=(0,4),(2,2)(4,0)} H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^j_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(2). \end{aligned}$$

It is enough to show that every element fixed by $\mathrm{Frob}_q$ in the component of type $(2,2)$ is spanned by the classes of algebraic cycles of codimension 2 on $X \times X$. By the Poincaré duality, such an element can be considered as an endomorphism of $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ commuting with $\mathrm{Frob}_q$.

Thus, we consider the action of $I(\mathbb{Q}_\ell)$ on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$. It can be shown that, after replacing $\mathbb{F}_q$ by a finite extension of it, there exist maximal tori $T_1, \ldots, T_n \subset I$ over $\mathbb{Q}$ such that the $\mathbb{Q}_\ell$-vector space of endomorphisms on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ commuting with $\mathrm{Frob}_q$ is spanned by the images of $T_1(\mathbb{Q}), \ldots, T_n(\mathbb{Q})$.

Therefore, it is enough to show that, for each $i$, the action of every element of $T_i(\mathbb{Q})$ on $P^2_{\mathrm{\acute{e}t}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ comes from an algebraic cycle of codimension 2 on $X \times X$. It can be proved by combining Theorem 2.1.6 with the results of Mukai and Buskin on the Hodge conjecture for certain Hodge cycles on the product of two K3 surfaces over $\mathbb{C}$.

**2.1.5. Outline of this chapter.** In Section 2.2, we recall basic results on Clifford algebras and general spin groups. Then, in Section 2.3 and Section 2.4, we fix notation and recall necessary results on the integral canonical models of orthogonal Shimura varieties and the Kuga-Satake morphism used in this chapter. In Section 2.5, we compare $F$-crystals on Shimura varieties and the crystalline cohomology of K3 surfaces. In Section 2.6, we define and study an analogue of Kisin's algebraic group associated with a quasi-polarized K3 surface of finite height over a finite field. We also study its action on the formal Brauer group. In Section 2.7, we prove Theorem 2.1.1 and Theorem 2.1.6. In Section 2.8, combined with the results of Mukai and Buskin, we prove Theorem 2.1.4. In Section 2.9, we prove a torsion analogue of the Tate conjecture for K3 surfaces. We also give an application to the finiteness of the Brauer group of a K3 surface over a field finitely generated over its prime field.

**2.1.6. Notation.** Throughout this chapter, we fix a prime number $p > 0$, and $q$ denotes a power of $p$. We will assume that $p \geq 5$ in several places in this chapter to avoid technical issues.

A *quadratic space* over a commutative ring $R$ means a free $R$-module $M$ of finite rank equipped with a quadratic form $Q$. We equip $M$ with a symmetric bilinear pairing $(\ ,\ )$ defined by $(x, y) = Q(x + y) - Q(x) - Q(y)$ for $x, y \in M$. For a module $M$ over a commutative ring $R$ equipped with a symmetric bilinear form $(\ ,\ )$, we say that $M$ (or the bilinear form $(\ ,\ )$) is *even* if, for every $x \in M$, we have $(x, x) = 2a$ for some $a \in R$.

The base change of a module or a scheme is denoted by a subscript. For example, for a module $M$ over a commutative ring $R$ and an $R$-algebra $R'$, the tensor product $M \otimes_R R'$ is denoted by $M_{R'}$. For a scheme (or an algebraic space) $X$ over $R$, the base change $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ is denoted by $X_{R'}$. We use similar notation for the base change of group schemes, $p$-divisible groups, line bundles, morphisms between them, etc. For a homomorphism $f \colon M \to N$ of $R$-modules, the base change $f_{R'} \colon M_{R'} \to N_{R'}$ is also denoted by the same notation $f$ if there is no possibility of confusion. For an element $x \in M$, the $R$-submodule of $M$ generated by $x$ is denoted by $\langle x \rangle$. The dual of $M$ as an $R$-module is denoted by $M^\vee := \operatorname{Hom}_R(M, R)$.

## 2.2. Clifford algebras and general spin groups

In this section, we introduce notation on quadratic spaces and Clifford algebras which will be used in this chapter. Our basic references are [**4**] and [**86**, Section 1].

**2.2.1. Embeddings of lattices.** A quadratic space $U := \mathbb{Z}x \oplus \mathbb{Z}y$ whose associated bilinear form is given by $(x, x) = (y, y) = 0$ and $(x, y) = 1$ is called the *hyperbolic plane*. The *K3 lattice* $\Lambda_{K3}$ is defined by

$$\Lambda_{K3} := {E_8}^{\oplus 2} \oplus U^{\oplus 3},$$

which is a quadratic space over $\mathbb{Z}$. It is unimodular and the signature of it is $(19, 3)$.

We fix a positive integer $d > 0$. Let $L$ be an orthogonal complement of $x - dy$ in $\Lambda_{K3}$, where $x - dy$ is considered as an element in the third $U$. Hence $L$ is equal to

$${E_8}^{\oplus 2} \oplus U^{\oplus 2} \oplus \langle x + dy \rangle,$$

and the signature of it is $(19, 2)$.

The following result is well known.

**Lemma 2.2.1.** *Let $p$ be a prime number. There is a quadratic space $\widetilde{L}$ of rank 22 over $\mathbb{Z}$ satisfying the following properties:*

*(1) The signature of it is $(20, 2)$.*
*(2) $\widetilde{L}$ is self-dual at $p$ (i.e. the discriminant of $\widetilde{L}$ is not divisible by $p$).*
*(3) There is an embedding $L \hookrightarrow \widetilde{L}$ as quadratic spaces which sends $L$ onto a direct summand of $\widetilde{L}$ as a $\mathbb{Z}$-module.*

Proof. This result was proved in [**86**, Lemma 6.8] when $p > 2$. Here we briefly give a proof which is valid for every prime number $p$. We consider a quadratic space $L' := \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ such that the associated bilinear form is given by $(v_1, v_1) = 2d$, $(v_2, v_2) = 2p$, and $(v_1, v_2) = 1$. We put

$$\widetilde{L} := {E_8}^{\oplus 2} \oplus U^{\oplus 2} \oplus L'.$$

This is of signature $(20, 2)$ and self-dual at $p$. We have an embedding of quadratic spaces

$$L = {E_8}^{\oplus 2} \oplus U^{\oplus 2} \oplus \langle x + dy \rangle \hookrightarrow \widetilde{L}$$

which is the identity on ${E_8}^{\oplus 2} \oplus U^{\oplus 2}$ and sends $x + dy$ to $v_1$. □

**2.2.2. Clifford algebras and general spin groups.** In the rest of this chapter, we fix an embedding of quadratic spaces $L \subset \widetilde{L}$ as in Lemma 2.2.1.

Let $\mathrm{Cl} := \mathrm{Cl}(\widetilde{L})$ be the *Clifford algebra* over $\mathbb{Z}$ associated with the quadratic space $(\widetilde{L}, q_{\widetilde{L}})$. There is an embedding of $\mathbb{Z}$-modules $\widetilde{L} \hookrightarrow \mathrm{Cl}$ which is universal for morphisms $f\colon \widetilde{L} \to R$ of $\mathbb{Z}$-modules into an associative $\mathbb{Z}$-algebra $R$ such that $f(v)^2 = q_{\widetilde{L}}(v)$ for every $v \in \widetilde{L}$. The algebra Cl has a $\mathbb{Z}/2\mathbb{Z}$-grading structure $\mathrm{Cl} := \mathrm{Cl}^+ \oplus \mathrm{Cl}^-$, where $\mathrm{Cl}^+$ is a subalgebra of Cl. The quadratic space $\widetilde{L}$ is naturally embedded into $\mathrm{Cl}^-$.

Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal $(p)$. We define the *general spin group* $\widetilde{G} := \mathrm{GSpin}(\widetilde{L}_{\mathbb{Z}_{(p)}})$ over $\mathbb{Z}_{(p)}$ by

$$\widetilde{G}(R) := \{\, g \in (\mathrm{Cl}_R^+)^\times \mid g\widetilde{L}_R g^{-1} = \widetilde{L}_R \text{ in } \mathrm{Cl}_R^- \,\}$$

for every $\mathbb{Z}_{(p)}$-algebra $R$. Since $\widetilde{L}$ is self-dual at $p$, the group scheme $\widetilde{G}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$.

The *special orthogonal group* $\widetilde{G}_0 := \mathrm{SO}(\widetilde{L}_{\mathbb{Z}_{(p)}})$ is a reductive group scheme over $\mathbb{Z}_{(p)}$, whose generic fiber $\widetilde{G}_{0,\mathbb{Q}} := \widetilde{G}_0 \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is $\mathrm{SO}(\widetilde{L}_{\mathbb{Q}})$.

We have the canonical morphism $\widetilde{G} \to \widetilde{G}_0$ defined by $g \mapsto (v \mapsto gvg^{-1})$, whose kernel is the multiplicative group $\mathbb{G}_{m,\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. We have the following exact sequence of group schemes over $\mathbb{Z}_{(p)}$:

$$1 \to \mathbb{G}_{m,\mathbb{Z}_{(p)}} \to \widetilde{G} = \mathrm{GSpin}(\widetilde{L}_{\mathbb{Z}_{(p)}}) \to \widetilde{G}_0 = \mathrm{SO}(\widetilde{L}_{\mathbb{Z}_{(p)}}) \to 1.$$

**2.2.3. Representations of general spin groups and Hodge tensors.** We define a $\mathbb{Z}$-module $H$ by $H := \mathrm{Cl}$. We consider $H_{\mathbb{Z}_{(p)}}$ as a $\widetilde{G}$-representation over $\mathbb{Z}_{(p)}$ by the left multiplication. We have a closed embedding of group schemes over $\mathbb{Z}_{(p)}$:

$$\widetilde{G} \hookrightarrow \mathrm{GL}(H_{\mathbb{Z}_{(p)}}).$$

As in [**73**, (1.3.1)], let $H^{\otimes}_{\mathbb{Z}_{(p)}}$ be the direct sum of all $\mathbb{Z}_{(p)}$-modules obtained from $H_{\mathbb{Z}_{(p)}}$ by taking tensor products, duals, symmetric powers, and exterior powers. (In fact, symmetric powers and exterior powers are unnecessary; see [**37**].) By [**73**, Proposition 1.3.2], the group scheme $\widetilde{G}$ over $\mathbb{Z}_{(p)}$ is the stabilizer of a finite collection of tensors

$$\{s_\alpha\} \subset H^{\otimes}_{\mathbb{Z}_{(p)}}.$$

(See also [**72**, Lemma 4.7].) In the rest of this chapter, we fix such tensors $\{s_\alpha\}$.

We regard $\widetilde{L}_{\mathbb{Z}_{(p)}}$ as a $\widetilde{G}$-representation via the canonical homomorphism $\widetilde{G} \to \widetilde{G}_0$ as above. Then the injective homomorphism

$$i\colon \widetilde{L}_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{End}_{\mathbb{Z}_{(p)}}(H_{\mathbb{Z}_{(p)}})$$

defined by $v \mapsto (h \mapsto vh)$ is $\widetilde{G}$-equivariant. The cokernel of this homomorphism $i$ is torsion-free as a $\mathbb{Z}_{(p)}$-module.

We end this section by describing filtrations on Clifford algebras defined by isotropic elements. Let $F$ be a field of characteristic 0. Take a non-zero element $e \in \widetilde{L}_F$ satisfying $(e, e) = 0$. We consider an endomorphism

$$i(e) := (i \otimes_{\mathbb{Z}_{(p)}} F)(e) \in \mathrm{End}_F(H_F)$$

which is the image of $e$ under the embedding $i \otimes_{\mathbb{Z}_{(p)}} F\colon \widetilde{L}_F \hookrightarrow \mathrm{End}_F(H_F)$. Let $i(e)(H_F)$ be the image of the endomorphism $i(e)\colon H_F \to H_F$. Then we have the following proposition:

**Proposition 2.2.2.** *The dimension of $i(e)(H_F)$ as an $F$-vector space is $2^{21}$.*

Proof. We have a decomposition $\widetilde{L}_F = \langle e\rangle \oplus \langle f\rangle \oplus (\langle e\rangle \oplus \langle f\rangle)^\perp$ such that $(e, f) = 1$ and $(f, f) = 0$. Let $v_3, \ldots, v_{22}$ be an orthogonal basis for $(\langle e\rangle \oplus \langle f\rangle)^\perp$. We have

$$H_F = \bigoplus_{a_j \in \{0,1\}} \langle e^{a_1} f^{a_2} v_3^{a_3} \ldots v_{22}^{a_{22}} \rangle$$

by [**14**, §9.3, Théorème 1]. Since $e^2 = 2^{-1}(e, e) = 0$ in the Clifford algebra $H_F = \mathrm{Cl}_F$, we have

$$i(e)(H_F) = \bigoplus_{a_j \in \{0,1\}} \langle e f^{a_2} v_3^{a_3} \ldots v_{22}^{a_{22}} \rangle.$$

Hence the dimension of $i(e)(H_F)$ as an $F$-vector space is $2^{21}$. □

## 2.3. Shimura varieties

In this section, we recall basic results on Shimura varieties and their integral models associated with general spin groups and special orthogonal groups. We follow Madapusi Pera's paper [**86**] for orthogonal Shimura varieties. (For integral models of more general Shimura varieties of abelian type, see Kisin's paper [**73**]. For the construction of 2-adic integral canonical models, see also [**72**].)

We retain the notation of Section 2.2. Recall that we fix an embedding of quadratic spaces $L \subset \widetilde{L}$ as in Lemma 2.2.1.

**2.3.1. Orthogonal Shimura varieties over $\mathbb{Q}$.** Let $X_{\widetilde{L}}$ be the symmetric domain of oriented negative definite planes in $\widetilde{L}_{\mathbb{R}}$. We have Shimura data $(\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$ and $(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}})$. Each of them has the reflex field $\mathbb{Q}$; see [**1**, Appendix 1, Lemma] for example. The canonical homomorphism $\widetilde{G} \to \widetilde{G}_0$ induces a morphism of Shimura data $(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}}) \to (\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$.

We put $\widetilde{\mathrm{K}}_{0,p} := \widetilde{G}_0(\mathbb{Z}_p)$ (resp. $\widetilde{\mathrm{K}}_p := \widetilde{G}(\mathbb{Z}_p)$), which is a hyperspecial subgroup. Let $\widetilde{\mathrm{K}}_0^p \subset \widetilde{G}_0(\mathbb{A}_f^p)$ (resp. $\widetilde{\mathrm{K}}^p \subset \widetilde{G}(\mathbb{A}_f^p)$) be an open compact subgroup and $\widetilde{\mathrm{K}}_0 := \widetilde{\mathrm{K}}_{0,p}\widetilde{\mathrm{K}}_0^p \subset \widetilde{G}_0(\mathbb{A}_f)$ (resp. $\widetilde{\mathrm{K}} := \widetilde{\mathrm{K}}_p\widetilde{\mathrm{K}}^p \subset \widetilde{G}(\mathbb{A}_f)$). We have the Shimura varieties $\mathrm{Sh}_{\widetilde{\mathrm{K}}_0} := \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}(\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$, $\mathrm{Sh}_{\widetilde{\mathrm{K}}} := \mathrm{Sh}_{\widetilde{\mathrm{K}}}(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}})$ associated with the Shimura data $(\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$, $(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}})$. We assume $\widetilde{\mathrm{K}}_0^p$ and $\widetilde{\mathrm{K}}^p$ are small enough such that $\mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ and $\mathrm{Sh}_{\widetilde{\mathrm{K}}}$ are smooth quasi-projective schemes over $\mathbb{Q}$. Moreover, we assume the image of $\widetilde{\mathrm{K}}^p$ under the homomorphism $\widetilde{G} \to \widetilde{G}_0$ is $\widetilde{\mathrm{K}}_0^p$. Then we have a finite étale morphism over $\mathbb{Q}$:

$$\mathrm{Sh}_{\widetilde{\mathrm{K}}} \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}.$$

We also consider the reductive group $\mathrm{SO}(L_{\mathbb{Q}})$ over $\mathbb{Q}$. Let $X_L$ be the symmetric domain of oriented negative definite planes in $L_{\mathbb{R}}$. We have a Shimura datum $(\mathrm{SO}(L_{\mathbb{Q}}), X_L)$ and a morphism of Shimura data: $(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \to (\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$. Let $\mathrm{K}_{0,p} \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{Q}_p)$ be the maximal subgroup which stabilizes $L_{\mathbb{Z}_p}$ and acts on $L^{\vee}_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}$ trivially. Let $\mathrm{K}_0^p \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{A}_f^p)$ be an open compact subgroup which stabilizes $L_{\widehat{\mathbb{Z}}^p}$ and acts on $L^{\vee}_{\widehat{\mathbb{Z}}^p}/L_{\widehat{\mathbb{Z}}^p}$ trivially. We assume $\mathrm{K}_0^p$ is small enough such that it is contained in $\widetilde{\mathrm{K}}_0^p$ and the associated Shimura variety $\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L)$ is a smooth quasi-projective variety over $\mathbb{Q}$, where $\mathrm{K}_0 := \mathrm{K}_{0,p}\,\mathrm{K}_0^p$. Note that we have $\mathrm{K}_{0,p} \subset \widetilde{\mathrm{K}}_{0,p}$; see the proof of [**86**, Lemma 2.6]. Hence we have a morphism of Shimura varieties over $\mathbb{Q}$:

$$\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}.$$

**2.3.2. Symplectic embeddings of general spin groups.** By [**86**, Lemma 3.6], there is a non-degenerate alternating bilinear form

$$\psi\colon H \times H \to \mathbb{Z}$$

satisfying the following properties:

- The left multiplication induces a closed embedding of algebraic groups over $\mathbb{Q}$
$$\widetilde{G}_{\mathbb{Q}} \hookrightarrow \mathrm{GSp} := \mathrm{GSp}(H_{\mathbb{Q}}, \psi_{\mathbb{Q}}).$$
- The left multiplication induces a morphism of Shimura data
$$(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}}) \to (\mathrm{GSp}, S^{\pm}),$$
where $S^{\pm}$ denotes the Siegel double spaces associated with the symplectic space $(H_{\mathbb{Q}}, \psi)$.

Let $\mathrm{K}'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ be an open compact subgroup containing the image of $\widetilde{\mathrm{K}}^p$. Let $\mathrm{K}'_p \subset \mathrm{GSp}(\mathbb{Q}_p)$ be the stabilizer of $H_{\mathbb{Z}_p}$. We put $\mathrm{K}' := \mathrm{K}'_p\mathrm{K}'^p$. After replacing $\mathrm{K}'^p$ and $\widetilde{\mathrm{K}}^p$ by their open compact subgroups, we may assume that the associated Shimura variety $\mathrm{Sh}_{\mathrm{K}'}(\mathrm{GSp}, S^{\pm})$ is a smooth quasi-projective scheme over $\mathbb{Q}$ and the morphism of Shimura data $(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}}) \to (\mathrm{GSp}, S^{\pm})$ induces the following morphism of Shimura varieties over $\mathbb{Q}$:

$$\mathrm{Sh}_{\widetilde{\mathrm{K}}} \to \mathrm{Sh}_{\mathrm{K}'}(\mathrm{GSp}, S^{\pm}).$$

Let us summarize our situation by the following commutative diagram of algebraic groups over $\mathbb{Q}$:

$$\begin{array}{ccccc} \mathrm{GSpin}(L_{\mathbb{Q}}) & \hookrightarrow & \widetilde{G}_{\mathbb{Q}} & \hookrightarrow & \mathrm{GSp} \\ \downarrow & & \downarrow & & \\ \mathrm{SO}(L_{\mathbb{Q}}) & \hookrightarrow & \widetilde{G}_{0,\mathbb{Q}}. & & \end{array}$$

We also have the corresponding diagram of Shimura varieties over $\mathbb{Q}$:

$$\begin{array}{ccc} & \mathrm{Sh}_{\widetilde{\mathrm{K}}} = \mathrm{Sh}_{\widetilde{\mathrm{K}}}(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}}) & \longrightarrow \mathrm{Sh}_{\mathrm{K}'}(\mathrm{GSp}, S^{\pm}) \\ & \downarrow & \\ \mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \longrightarrow & \mathrm{Sh}_{\widetilde{\mathrm{K}}_0} = \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}(\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}}). & \end{array}$$

**2.3.3. Integral canonical models and the Kuga-Satake abelian scheme.** Since $\widetilde{\mathrm{K}}_{0,p} \subset \widetilde{G}_0(\mathbb{Q}_p)$ and $\widetilde{\mathrm{K}}_p \subset \widetilde{G}(\mathbb{Q}_p)$ are hyperspecial subgroups, the Shimura varieties $\mathrm{Sh}_{\widetilde{\mathrm{K}}}$, $\mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ admit the integral canonical models $\mathscr{S}_{\widetilde{\mathrm{K}}}$, $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$ over $\mathbb{Z}_{(p)}$, respectively. (This result is proved by Kisin when $p \neq 2$ [**73**], and by Kim-Madapusi Pera when $p = 2$ [**72**]. The integral canonical models are characterized by the extension properties. See [**73**] for details.)

By the construction of $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$, the morphism $\mathrm{Sh}_{\widetilde{\mathrm{K}}} \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ extends to a finite étale morphism $\mathscr{S}_{\widetilde{\mathrm{K}}} \to \mathscr{S}_{\widetilde{\mathrm{K}}_0}$ over $\mathbb{Z}_{(p)}$.

Let $m := |H_{\mathbb{Z}}^{\vee}/H_{\mathbb{Z}}|$ be the discriminant of $H_{\mathbb{Z}}$. We put $g := (\dim_{\mathbb{Q}} H_{\mathbb{Q}})/2 = 2^{21}$. Let $\mathscr{A} := \mathscr{A}_{g,m,\mathrm{K}'}$ be the moduli space over $\mathbb{Z}_{(p)}$ of triples $(A, \lambda, \epsilon'^p)$ consisting an abelian scheme $A$ of dimension $g$, a polarization $\lambda\colon A \to A^*$ of degree $m$, and a $\mathrm{K}'^p$-level structure $\epsilon'^p$. For a sufficiently small $\mathrm{K}'^p$, this is represented by a quasi-projective scheme over $\mathbb{Z}_{(p)}$. We have a canonical open and closed immersion

$$\mathrm{Sh}_{\mathrm{K}'}(\mathrm{GSp}, S^{\pm}) \hookrightarrow \mathscr{A}_{\mathbb{Q}}$$

over $\mathbb{Q}$. Hence, we have a morphism $\mathrm{Sh}_{\widetilde{\mathrm{K}}} \to \mathscr{A}_{\mathbb{Q}}$. over $\mathbb{Q}$. By the construction of $\mathscr{S}_{\widetilde{\mathrm{K}}}$, this morphism extends to a morphism $\mathscr{S}_{\widetilde{\mathrm{K}}} \to \mathscr{A}$ over $\mathbb{Z}_{(p)}$; see [**73**, (2.3.3)], [**72**, Section 4.4]. In summary, we have the following diagram of schemes over $\mathbb{Z}_{(p)}$:

$$\begin{array}{ccc} \mathscr{S}_{\widetilde{\mathrm{K}}} & \longrightarrow & \mathscr{A} \\ \downarrow & & \\ \mathscr{S}_{\widetilde{\mathrm{K}}_0}. & & \end{array}$$

Let $\mathcal{A}_{\mathscr{S}_{\widetilde{\mathrm{K}}}} \to \mathscr{S}_{\widetilde{\mathrm{K}}}$ be the abelian scheme corresponding to the morphism $\mathscr{S}_{\widetilde{\mathrm{K}}} \to \mathscr{A}$. The abelian scheme $\mathcal{A}_{\mathscr{S}_{\widetilde{\mathrm{K}}}}$ is called the *Kuga-Satake abelian scheme*. We often drop the subscript $\mathscr{S}_{\widetilde{\mathrm{K}}}$ in the notation. For every $\mathscr{S}_{\widetilde{\mathrm{K}}}$-scheme $S$, we denote the pullback of $\mathcal{A}_{\mathscr{S}_{\widetilde{\mathrm{K}}}}$ to $S$ by $\mathcal{A}_S$.

**Remark 2.3.1.** If the discriminant of the quadratic space $L$ is divisible by $p$, we do not yet have a satisfactory theory of integral canonical models of the Shimura varieties $\mathrm{Sh}_{\mathrm{K}}(\mathrm{GSpin}(L_{\mathbb{Q}}), X_L)$ and $\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L)$ associated with $L$. (The open compact subgroup $\mathrm{K}_{0,p} \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{Q}_p)$ may not be hyperspecial.) Following Madapusi Pera [**86, 85, 72**], we embed $L$ into $\widetilde{L}$ whose discriminant is not divisible by $p$ as in Lemma 2.2.1, and use the integral canonical models $\mathscr{S}_{\widetilde{\mathrm{K}}}$ and $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$ associated with $\widetilde{L}$.

**2.3.4. Local systems on Shimura varieties.** We recall basic results on (complex analytic, $\ell$-adic, and $p$-adic) local systems on orthogonal Shimura varieties. (For details, see [**86, 72**].)

The $\widetilde{G}$-representation $\widetilde{L}_{\mathbb{Z}_{(p)}}$ and the $\widetilde{G}$-equivariant embedding $i\colon \widetilde{L}_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{End}_{\mathbb{Z}_{(p)}}(H_{\mathbb{Z}_{(p)}})$ induce the following objects:

- A $\mathbb{Q}$-local system $\widetilde{\mathbb{V}}_B$ over the complex analytic space $\mathrm{Sh}^{\mathrm{an}}_{\widetilde{\mathrm{K}},\mathbb{C}}$ and an embedding of $\mathbb{Q}$-local systems:
$$i_B\colon \widetilde{\mathbb{V}}_B \hookrightarrow \underline{\mathrm{End}}(H_B^\vee).$$
Here $H_B$ is the relative first singular cohomology with coefficients in $\mathbb{Q}$ of $\mathcal{A}^{\mathrm{an}}_{\mathrm{Sh}_{\widetilde{\mathrm{K}},\mathbb{C}}}$ over $\mathrm{Sh}^{\mathrm{an}}_{\widetilde{\mathrm{K}},\mathbb{C}}$, and $H_B^\vee$ is its dual.
- An $\mathbb{A}^p_f$-local system $\widetilde{\mathbb{V}}^p$ over the integral canonical model $\mathscr{S}_{\widetilde{\mathrm{K}}}$ and an embedding of $\mathbb{A}^p_f$-local systems:
$$i^p\colon \widetilde{\mathbb{V}}^p \hookrightarrow \underline{\mathrm{End}}(V^p\mathcal{A}).$$
Here we put
$$V^p\mathcal{A} := (\varprojlim_{p\nmid n} \mathcal{A}[n]) \otimes_{\mathbb{Z}} \mathbb{Q}$$
and consider it as an $\mathbb{A}^p_f$-local system over $\mathscr{S}_{\widetilde{\mathrm{K}}}$.
- A $\mathbb{Z}_p$-local system $\widetilde{\mathbb{L}}_p$ over the Shimura variety $\mathscr{S}_{\widetilde{\mathrm{K}},\mathbb{Q}} = \mathrm{Sh}_{\widetilde{\mathrm{K}}}$ and an embedding of $\mathbb{Z}_p$-local systems:
$$i_p\colon \widetilde{\mathbb{L}}_p \hookrightarrow \underline{\mathrm{End}}(T_p\mathcal{A}_{\mathrm{Sh}_{\widetilde{\mathrm{K}}}}).$$
Here $T_p\mathcal{A}_{\mathrm{Sh}_{\widetilde{\mathrm{K}}}}$ is the $p$-adic Tate module of $\mathcal{A}_{\mathrm{Sh}_{\widetilde{\mathrm{K}}}}$ over $\mathrm{Sh}_{\widetilde{\mathrm{K}}}$.

**2.3.5. Hodge tensors.** Recall that we fix tensors $\{s_\alpha\}$ of $H^{\otimes}_{\mathbb{Z}_{(p)}}$ defining the closed embedding $\widetilde{G} \hookrightarrow \mathrm{GL}(H_{\mathbb{Z}_{(p)}})$ over $\mathbb{Z}_{(p)}$; see Section 2.2.3. The tensors $\{s_\alpha\}$ of $H^{\otimes}_{\mathbb{Z}_{(p)}}$ give rise to global sections $\{s_{\alpha,B}\}$ of $H_B^{\otimes}$, global sections $\{s^p_\alpha\}$ of $(V^p\mathcal{A})^{\otimes}$, and global sections $\{s_{\alpha,p}\}$ of $(T_p\mathcal{A})^{\otimes}$.

We recall properties of these tensors. (See [**74**, (1.3.6)], [**72**, Proposition 4.10] for details.)

- Let $k$ be a field of characteristic 0. For every $x \in \mathscr{S}_{\widetilde{\mathrm{K}},\mathbb{Q}}(k)$ and a geometric point $\overline{x} \in \mathscr{S}_{\widetilde{\mathrm{K}},\mathbb{Q}}(\overline{k})$ above $x$, the stalk $\widetilde{\mathbb{L}}_{p,\overline{x}}$ at $\overline{x}$ is equipped with an even perfect bilinear form $(\ ,\ )$ over $\mathbb{Z}_p$. The bilinear form is $\mathrm{Gal}(\overline{k}/k)$-invariant, i.e. we have
$$(gy_1, gy_2) = (y_1, y_2)$$
for every $y_1, y_2 \in \widetilde{\mathbb{L}}_{p,\overline{x}}$ and every $g \in \mathrm{Gal}(\overline{k}/k)$.

  We identify $T_p(\mathcal{A}_{\overline{x}})$ with $H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Z}_p)^\vee$. The $\mathrm{Gal}(\overline{k}/k)$-module $\widetilde{\mathbb{L}}_{p,\overline{x}}$ and the homomorphism $i_{p,\overline{x}}$ are characterized by the property that there is an isomorphism of $\mathbb{Z}_p$-modules
$$H_{\mathbb{Z}_p} \cong H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Z}_p)^\vee$$

which carries $\{s_\alpha\}$ to $\{s_{\alpha,p,\overline{x}}\}$ and induces the commutative diagram

$$\begin{array}{ccc} \widetilde{L}_{\mathbb{Z}_p} & \xrightarrow{i} & \mathrm{End}_{\mathbb{Z}_p}(H_{\mathbb{Z}_p}) \\ \downarrow \cong & & \downarrow \cong \\ \widetilde{\mathbb{L}}_{p,\overline{x}} & \xrightarrow{i_{p,\overline{x}}} & \mathrm{End}_{\mathbb{Z}_p}(H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Z}_p)^\vee), \end{array}$$

where $\widetilde{L}_{\mathbb{Z}_p} \cong \widetilde{\mathbb{L}}_{p,\overline{x}}$ is an isometry over $\mathbb{Z}_p$. (We will often drop the subscript $\overline{x}$ of $i_{p,\overline{x}}$.)

- Let $k$ a field of characteristic 0 or $p$. For every $x \in \mathscr{S}_{\widetilde{\mathrm{K}}}(k)$ and a geometric point $\overline{x} \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\overline{k})$ above $x$, the stalk $\widetilde{\mathbb{V}}^p_{\overline{x}}$ at $\overline{x}$ has a bilinear form $(\ ,\ )$ over $\mathbb{A}^p_f$ satisfying the same property as above with $\mathbb{Z}_p$ replaced by $\mathbb{A}^p_f$.
- For every $x \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbb{C})$, the stalk $\widetilde{\mathbb{V}}_{B,x}$ at $x$ has a bilinear form $(\ ,\ )$ over $\mathbb{Q}$ satisfying the same property as above with $\mathbb{Z}_p$ replaced by $\mathbb{Q}$.

**2.3.6.** ***F*-crystals.** Let $k$ be a finite field $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$. Let $x \in \mathscr{S}_{\widetilde{\mathrm{K}},\mathbb{F}_p}(k)$ be an element. The $\widetilde{G}$-representation $\widetilde{L}_{\mathbb{Z}_{(p)}}$ and the $\widetilde{G}$-equivariant embedding $i\colon \widetilde{L}_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{End}_{\mathbb{Z}_{(p)}}(H_{\mathbb{Z}_{(p)}})$ induces a free $W$-module $\widetilde{L}_{\mathrm{cris},x}$ of finite rank and an embedding

$$i_{\mathrm{cris}}\colon \widetilde{L}_{\mathrm{cris},x} \hookrightarrow \mathrm{End}_W(H^1_{\mathrm{cris}}(\mathcal{A}_x/W)^\vee).$$

The $W[1/p]$-vector space $\widetilde{L}_{\mathrm{cris},x}[1/p]$ has the structure of an $F$-isocrystal. Namely, it is equipped with a Frobenius automorphism $\varphi$. The embedding $i_{\mathrm{cris}}$ induces an embedding of $F$-isocrystals

$$i_{\mathrm{cris}}[1/p]\colon \widetilde{L}_{\mathrm{cris},x}[1/p] \hookrightarrow \mathrm{End}_{W[1/p]}(H^1_{\mathrm{cris}}(\mathcal{A}_x/W[1/p])^\vee).$$

There is an even perfect bilinear form on $\widetilde{L}_{\mathrm{cris},x}$. When $p$ is inverted, we have

$$(\varphi(y_1), \varphi(y_2)) = \sigma(y_1, y_2)$$

for every $y_1, y_2 \in \widetilde{L}_{\mathrm{cris},x}[1/p]$. The tensors $\{s_\alpha\}$ of $H^{\otimes}_{\mathbb{Z}_{(p)}}$ give rise to Frobenius invariant tensors $\{s_{\alpha,\mathrm{cris},x}\}$ of $H^1_{\mathrm{cris}}(\mathcal{A}_x/W)^{\otimes}$. There is an isomorphism of $W$-modules

$$H_W \cong H^1_{\mathrm{cris}}(\mathcal{A}_x/W)^\vee$$

which carries $\{s_\alpha\}$ to $\{s_{\alpha,\mathrm{cris},x}\}$ and induces the following commutative diagram:

$$\begin{array}{ccc} \widetilde{L}_W & \xrightarrow{i} & \mathrm{End}_W(H_W) \\ \downarrow \cong & & \downarrow \cong \\ \widetilde{L}_{\mathrm{cris},x} & \xrightarrow{i_{\mathrm{cris}}} & \mathrm{End}_W(H^1_{\mathrm{cris}}(\mathcal{A}_x/W)^\vee), \end{array}$$

where $\widetilde{L}_W \cong \widetilde{L}_{\mathrm{cris},x}$ is an isometry over $W$. (For details, see [**73, 74, 86, 72**]. See also [**65**, Section 4.6].)

**2.3.7. Λ-structures for integral canonical models.** Recall that we have fixed an embedding of quadratic spaces $L \hookrightarrow \widetilde{L}$. Let $\Lambda := L^\perp \subset \widetilde{L}$ be the orthogonal complement of $L$ in $\widetilde{L}$, and $\iota\colon \Lambda \hookrightarrow \widetilde{L}$ the natural inclusion.

We recall the definition of Λ-structures from [**86**].

**Definition 2.3.2 (see [86, Definition 6.11]).** A $\Lambda$-*structure* for an $\mathscr{S}_{\widetilde{\mathrm{K}}}$-scheme $S$ is a homomorphism of $\mathbb{Z}_{(p)}$-modules

$$\iota_S \colon \Lambda_{\mathbb{Z}_{(p)}} \to \mathrm{End}_S(\mathcal{A}_S)_{\mathbb{Z}_{(p)}}$$

satisfying the following properties:

- For any algebraically closed field $\overline{K}$ of characteristic 0 and $x \in S(\overline{K})$, there is an isometry $\iota_p \colon \Lambda_{\mathbb{Z}_p} \to \widetilde{\mathbb{L}}_{p,x}$ over $\mathbb{Z}_p$ such that the homomorphism induced by $\iota_S$
$$\Lambda_{\mathbb{Z}_p} \to \mathrm{End}_{\mathbb{Z}_p}(T_p\mathcal{A}_x)$$
factors as
$$\Lambda_{\mathbb{Z}_p} \xrightarrow{\iota_p} \widetilde{\mathbb{L}}_{p,x} \xrightarrow{i_p} \mathrm{End}_{\mathbb{Z}_p}(T_p\mathcal{A}_x).$$
- For any perfect field $k$ of characteristic $p$ and $x \in S(k)$, there is an isometry $\iota_{\mathrm{cris}} \colon \Lambda_W \to \widetilde{L}_{\mathrm{cris},x}$ over $W$ such that the homomorphism induced by $\iota_S$
$$\Lambda_W \to \mathrm{End}_W(H^1_{\mathrm{cris}}(\mathcal{A}_x/W)^\vee)$$
factors as
$$\Lambda_W \xrightarrow{\iota_{\mathrm{cris}}} \widetilde{L}_{\mathrm{cris},x} \xrightarrow{i_{\mathrm{cris}}} \mathrm{End}_W(H^1_{\mathrm{cris}}(\mathcal{A}_x/W)^\vee) \ .$$

It turns out that these conditions imply the following:

- By [86, Corollary 5.22], for every geometric point $x \to S$, there is an isometry $\iota^p \colon \Lambda_{\mathbb{A}_f^p} \to \widetilde{\mathbb{V}}_x^p$ over $\mathbb{A}_f^p$ such that the homomorphism induced by $\iota_S$
$$\Lambda_{\mathbb{A}_f^p} \to \mathrm{End}_{\mathbb{A}_f^p}(V^p(\mathcal{A}_x))$$
factors as
$$\Lambda_{\mathbb{A}_f^p} \xrightarrow{\iota^p} \widetilde{\mathbb{V}}_x^p \xrightarrow{i^p} \mathrm{End}_{\mathbb{A}_f^p}(V^p(\mathcal{A}_x)).$$
- For every $\mathbb{C}$-valued point $x \in S(\mathbb{C})$, there is an isometry $\iota_B \colon \Lambda_{\mathbb{Q}} \to \widetilde{\mathbb{V}}_{B,x}$ over $\mathbb{Q}$ such that the homomorphism induced by $\iota_S$
$$\Lambda_{\mathbb{Q}} \to \mathrm{End}_{\mathbb{Q}}(H^1_B(\mathcal{A}_x, \mathbb{Q})^\vee)$$
factors as
$$\Lambda_{\mathbb{Q}} \xrightarrow{\iota_B} \widetilde{\mathbb{V}}_{B,x} \xrightarrow{i_B} \mathrm{End}_{\mathbb{Q}}(H^1_B(\mathcal{A}_x, \mathbb{Q})^\vee).$$

We recall the definition of a $\mathrm{K}^p$-level structure. Here $\mathrm{K}^p \subset \mathrm{GSpin}(L_{\mathbb{Q}})(\mathbb{A}_f^p)$ is an open compact subgroup whose image under the homomorphism $\mathrm{GSpin}(L_{\mathbb{Q}})(\mathbb{A}_f^p) \to \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{A}_f^p)$ is $\mathrm{K}_0^p$.

Let $S$ be an $\mathscr{S}_{\widetilde{\mathrm{K}}}$-scheme. For simplicity, we assume $S$ is locally Noetherian and connected. Let $\epsilon'$ be the corresponding $\mathrm{K}'^p$-level structure on $\mathcal{A}_S$; as in [73, (3.2.4)], for a geometric point $s \to S$, the $\mathrm{K}'^p$-level structure $\epsilon'$ is induced by a $\widetilde{\mathrm{K}}^p$-orbit $\widetilde{\epsilon}$ of an isometry $H_{\mathbb{A}_f^p} \cong V^p(\mathcal{A}_s)$ over $\mathbb{A}_f^p$ which carries $\{s_\alpha\}$ to $\{s^p_{\alpha,s}\}$ and carries $\widetilde{L}_{\mathbb{A}_f^p}$ to $\widetilde{\mathbb{V}}_s^p$ such that $\widetilde{\epsilon}$ is $\pi_1(S,s)$-invariant. Here $\pi_1(S,s)$ denotes the étale fundamental group of $S$, and the Tate module $V^p(\mathcal{A}_s)$ over $\mathbb{A}_f^p$ has a natural action of $\pi_1(S,s)$.

**Definition 2.3.3.** Let $S$ be a locally Noetherian connected scheme over $\mathscr{S}_{\widetilde{\mathrm{K}}}$. Let $s \to S$ be a geometric point. A $\mathrm{K}^p$-*level structure* on $(S, \iota_S)$ is a $\pi_1(S,s)$-invariant $\mathrm{K}^p$-orbit $\epsilon_\iota$ of an isometry of $\mathbb{A}_f^p$-modules

$$H_{\mathbb{A}_f^p} \cong V^p(\mathcal{A}_s)$$

satisfying the following properties:

- It carries $\{s_\alpha\}$ to $\{s^p_{\alpha,s}\}$.
- The following diagram is commutative:

$$\begin{array}{ccccc} \Lambda_{\mathbb{A}^p_f} & \xrightarrow{\iota} & \widetilde{L}_{\mathbb{A}^p_f} & \xrightarrow{i} & \mathrm{End}_{\mathbb{A}^p_f}(H_{\mathbb{A}^p_f}) \\ & \searrow^{\iota^p} & \downarrow \cong & & \downarrow \cong \\ & & \widetilde{\mathbb{V}}^p_s & \xrightarrow{i^p} & \mathrm{End}_{\mathbb{A}^p_f}(V^p(\mathcal{A}_s)). \end{array}$$

- The $\mathrm{K}^p$-orbit $\epsilon_\iota$ induces the $\widetilde{\mathrm{K}}^p$-orbit $\widetilde{\epsilon}$ on $S$.

**Definition 2.3.4.** Let $Z_{\mathrm{K}^p}(\Lambda)$ be the functor on $\mathscr{S}_{\widetilde{\mathrm{K}}}$-schemes defined by

$$Z_{\mathrm{K}^p}(\Lambda)(S) := \{\, (\iota_S, \epsilon_\iota) \mid \iota_S \text{ is a } \Lambda\text{-structure and } \epsilon_\iota \text{ is a } \mathrm{K}^p\text{-level structure on } (S, \iota_S) \,\}$$

for an $\mathscr{S}_{\widetilde{\mathrm{K}}}$-scheme $S$.

Similarly, we can define a $\Lambda$-structure $\iota_{0,S}$ for an $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$-scheme $S$, a $\mathrm{K}^p_0$-level structure on $(S, \iota_{0,S})$ and a functor $Z_{\mathrm{K}^p_0}(\Lambda)$. (See [**86**, Definition 6.11] for details.)

The following result was proved by Madapusi Pera.

**Proposition 2.3.5 (Madapusi Pera [86]).** *The functor $Z_{\mathrm{K}^p}(\Lambda)$ (resp. $Z_{\mathrm{K}^p_0}(\Lambda)$) is represented by a scheme which is finite and unramified over $\mathscr{S}_{\widetilde{\mathrm{K}}}$ (resp. $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$). Moreover there is a natural morphism*

$$Z_{\mathrm{K}^p}(\Lambda) \to Z_{\mathrm{K}^p_0}(\Lambda),$$

*which is finite and étale.*

PROOF. See [**86**, Proposition 6.13]. □

## 2.4. Moduli spaces of K3 surfaces and the Kuga-Satake morphism

In this section, we recall definitions and basic properties of the moduli space of K3 surfaces and the level structure. Then we recall definitions and basic results on the Kuga-Satake morphism over $\mathbb{Z}_{(p)}$ introduced by Madapusi Pera [**85, 72**].

**2.4.1. Moduli spaces of K3 surfaces.** We say that $f\colon \mathscr{X} \to S$ is a *K3 surface over* $S$ if $S$ is a scheme, $\mathscr{X}$ is an algebraic space, and $f$ is a proper smooth morphism whose geometric fibers are K3 surfaces.

A *quasi-polarization* of $f\colon \mathscr{X} \to S$ is a section $\xi \in \underline{\mathrm{Pic}}(\mathscr{X}/S)(S)$ of the relative Picard functor whose fiber $\xi(s)$ at every geometric point $s \to S$ is a line bundle on the K3 surface $\mathscr{X}_s$ which is nef and big. We say that $\xi \in \underline{\mathrm{Pic}}(\mathscr{X}/S)(S)$ is *primitive* if, for every geometric point $s \to S$, the cokernel of the inclusion $\langle \xi(s) \rangle \hookrightarrow \mathrm{Pic}(\mathscr{X}_s)$ is torsion-free. We say that $\xi$ has degree $2d$ if, for every geometric point $s \to S$, we have $(\xi(s), \xi(s)) = 2d$, where $(\ ,\ )$ denotes the intersection pairing on $\mathscr{X}_s$. We say that a pair $(f\colon \mathscr{X} \to S, \xi)$ is a *quasi-polarized K3 surface over $S$ of degree* $2d$ if $f\colon \mathscr{X} \to S$ is a K3 surface over $S$ and $\xi \in \underline{\mathrm{Pic}}(\mathscr{X}/S)(S)$ is a primitive quasi-polarization of degree $2d$.

Let $M_{2d}$ be the moduli functor that sends a $\mathbb{Z}$-scheme $S$ to the groupoid consists of quasi-polarized K3 surfaces over $S$ of degree $2d$. The moduli functor $M_{2d}$ is a Deligne-Mumford stack of finite type over $\mathbb{Z}$; see [**110**, Theorem 4.3.4] and [**88**, Proposition 2.1].

We put $M_{2d,\mathbb{Z}_{(p)}} := M_{2d} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let $S$ be an $M_{2d,\mathbb{Z}_{(p)}}$-scheme. For the quasi-polarized K3 surface $(f\colon \mathscr{X} \to S, \xi)$ associated with the structure morphism $S \to M_{2d,\mathbb{Z}_{(p)}}$ and a prime number $\ell \neq p$, we equipped $R^2 f_* \mathbb{Z}_\ell(1)$ with the *negative* of the cup product pairing. Let

$$P^2 f_* \mathbb{Z}_\ell(1) := \mathrm{ch}_\ell(\xi)^\perp \subset R^2 f_* \mathbb{Z}_\ell(1)$$

be the orthogonal complement of the $\ell$-adic Chern class $\mathrm{ch}_\ell(\xi) \in R^2 f_* \mathbb{Z}_\ell(1)(S)$ with respect to the pairing. We set

$$P^2 f_* \widehat{\mathbb{Z}}^p(1) := \prod_{\ell \neq p} P^2 f_* \mathbb{Z}_\ell(1).$$

The stalk of $P^2 f_* \mathbb{Z}_\ell(1)$ (resp. $P^2 f_* \widehat{\mathbb{Z}}^p(1)$) at a geometric point $s \to S$ will be denoted by $P^2_{\text{ét}}(\mathscr{X}_s, \mathbb{Z}_\ell(1))$ (resp. $P^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1))$).

Let $M^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ be the smooth locus of $M_{2d,\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. Madapusi Pera constructed a twofold finite étale cover $\widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}} \to M^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ parameterizing orientations of $P^2 f_* \widehat{\mathbb{Z}}^p(1)$, which satisfies the following property. For every morphism $S \to \widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$, there is a natural isometry of $\widehat{\mathbb{Z}}^p$-local systems on $S$

$$\nu \colon \underline{\det L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p} \cong \det P^2 f_* \widehat{\mathbb{Z}}^p(1)$$

such that, for every $s \in S(\mathbb{C})$, the isometry $\nu$ restricts to an isometry over $\mathbb{Z}$

$$\nu_s \colon \det L \cong \det P^2_B(\mathscr{X}_s, \mathbb{Z}(1))$$

under the canonical isomorphism

$$P^2_B(\mathscr{X}_s, \mathbb{Z}(1)) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \cong P^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1)),$$

where we put $P^2_B(\mathscr{X}_s, \mathbb{Z}(1)) := \mathrm{ch}_B(\xi_s)^\perp \subset H^2_B(\mathscr{X}_s, \mathbb{Z}(1))$. See [**85**, Section 5] for details.

For an open compact subgroup $\mathrm{K}^p_0 \subset \mathrm{SO}(L_\mathbb{Q})(\mathbb{A}^p_f)$ as in Section 2.3.1, we recall the notion of (oriented) $\mathrm{K}^p_0$-level structures from [**85**, Section 3]. For simplicity, we only consider the case $S$ is a locally Noetherian connected $\widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$-scheme. Let $s \to S$ be a geometric point and $\pi_1(S, s)$ the étale fundamental group of $S$. A $\mathrm{K}^p_0$-*level structure* on $(f \colon \mathscr{X} \to S, \xi)$ is a $\pi_1(S, s)$-invariant $\mathrm{K}^p_0$-orbit $\eta$ of an isometry over $\widehat{\mathbb{Z}}^p$

$$\Lambda_{K3} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \cong H^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1))$$

which carries $e - df$ to $\mathrm{ch}_{\widehat{\mathbb{Z}}^p}(\xi(s))$ such that the induced isometry

$$\det L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \cong \det P^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1))$$

coincides with $\nu_s$. Here the étale cohomology $H^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1))$ has a natural action of $\pi_1(S, s)$, and $\Lambda_{K3} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$ has a natural action of $\mathrm{K}^p_0$; see [**86**, Lemma 2.6].

Let $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$ be the moduli functor over $\widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ which sends an $\widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$-scheme $S$ to the set of (oriented) $\mathrm{K}^p_0$-level structures on the quasi-polarized K3 surface $(f \colon \mathscr{X} \to S, \xi)$.

The following result is well known.

**Proposition 2.4.1.** *If $\mathrm{K}^p_0 \subset \mathrm{SO}(L_\mathbb{Q})(\mathbb{A}^p_f)$ is small enough, $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$ is an algebraic space over $\mathbb{Z}_{(p)}$ which is finite, étale, and faithfully flat over $M^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$.*

Proof. This result was essentially proved by Rizov [**110**, Theorem 6.2.2], Maulik [**88**, Proposition 2.8] and Madapusi Pera [**85**, Proposition 3.11]. Note that their proofs work in every characteristic $p$, including $p = 2$. Their proofs rely on the injectivity of the map

$$\mathrm{Aut}(X) \to \mathrm{GL}(H^2_{\text{ét}}(X, \mathbb{Q}_\ell))$$

for every $\ell \neq p$, where $X$ is a K3 surface over an algebraically closed field of characteristic $p > 0$. The injectivity was proved by Ogus when $p > 2$; see [**99**, Corollary 2.5]. (Precisely, Ogus proved it for the crystalline cohomology. The injectivity for the $\ell$-adic cohomology follows from Ogus' results; see [**110**, Proposition 3.4.2].) Recently, Keum proved that the injectivity holds also when $p = 2$; see [**71**, Theorem 1.4]. $\square$

We will assume an open compact subgroup $\mathrm{K}_0^p \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{A}_f^p)$ as in Section 2.3.1 is small enough so that Proposition 2.4.1 can be applied.

**2.4.2. The Kuga-Satake morphism.** Rizov and Madapusi Pera defined the following étale morphism over $\mathbb{Q}$:

$$M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Q}} \to \mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L).$$

It is called the *Kuga-Satake morphism* over $\mathbb{Q}$. (See [**111**, Theorem 3.9.1], [**85**, Corollary 5.4] for details.)

Since $M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$, the composite of the following morphisms over $\mathbb{Q}$

$$M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Q}} \to \mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$$

extends to a morphism over $\mathbb{Z}_{(p)}$

$$M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}} \to \mathscr{S}_{\widetilde{\mathrm{K}}_0}$$

by the extension properties of the integral canonical model $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$; see [**73**, (2.3.7)], [**85**, Proposition 5.7].

The following results are proved by Madapusi Pera.

**Proposition 2.4.2 (Madapusi Pera).** *There is a natural étale $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$-morphism*

$$\mathrm{KS}\colon M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}} \to Z_{\mathrm{K}_0^p}(\Lambda).$$

PROOF. The morphism $\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ over $\mathbb{Q}$ factors through the generic fiber $Z_{\mathrm{K}_0^p}(\Lambda)_{\mathbb{Q}}$ of $Z_{\mathrm{K}_0^p}(\Lambda)$; see [**86**, 6.15] for details. Hence we have a morphism over $\mathbb{Q}$

$$M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Q}} \to Z_{\mathrm{K}_0^p}(\Lambda)_{\mathbb{Q}}.$$

This morphism extends to a $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$-morphism

$$\mathrm{KS}\colon M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}} \to Z_{\mathrm{K}_0^p}(\Lambda)$$

by [**42**, Chapter I, Proposition 2.7].

For the étaleness of the morphism KS, see the proof of [**72**, Proposition A.12]. For the case where $p = 2$, see also [**87**] and [**65**, Section 6.4]. $\square$

One usually calls the morphism KS in Proposition 2.4.2 a period map, and calls a morphism from the moduli space of K3 surfaces to the moduli space of abelian varieties a Kuga-Satake morphism. In this chapter, following Madapusi Pera, we call the morphism KS a *Kuga-Satake morphism* for convenience.

Summarizing the above, we have the following commutative diagram of algebraic spaces over $\mathbb{Z}_{(p)}$:

$$\begin{array}{ccccc} & & Z_{\mathrm{K}^p}(\Lambda) & \longrightarrow & \mathscr{S}_{\widetilde{\mathrm{K}}} \\ & & \downarrow & & \downarrow \\ M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}} & \xrightarrow{\mathrm{KS}} & Z_{\mathrm{K}_0^p}(\Lambda) & \longrightarrow & \mathscr{S}_{\widetilde{\mathrm{K}}_0}. \end{array}$$

Here $\mathscr{S}_{\widetilde{\mathrm{K}}}$ (resp. $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$) is the integral canonical model of the Shimura variety associated with $\widetilde{G} = \mathrm{GSpin}(\widetilde{L}_{\mathbb{Z}_{(p)}})$ (resp. $\widetilde{G}_0 = \mathrm{SO}(\widetilde{L}_{\mathbb{Z}_{(p)}})$), and $Z_{\mathrm{K}^p}(\Lambda)$ (resp. $Z_{\mathrm{K}_0^p}(\Lambda)$) is the scheme over $\mathscr{S}_{\widetilde{\mathrm{K}}}$ (resp. $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$) as in Definition 2.3.4 and Proposition 2.3.5.

## 2.5. $F$-crystals on Shimura varieties

In this section, we assume $k$ is a finite field $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$. We shall study the $W$-module $\widetilde{L}_{\mathrm{cris},s}$ associated with a $k$-valued point $s \in \mathscr{S}_{\widetilde{\mathrm{K}}}(k)$ of the integral canonical model $\mathscr{S}_{\widetilde{\mathrm{K}}}$. We write $W := W(k)$.

**2.5.1. $F$-crystals on Shimura varieties and the cohomology of K3 surfaces.** We consider the following situation.

(1) Let $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}(k)$ be a $k$-valued point on the smooth locus and $(X, \mathscr{L})$ a quasi-polarized K3 surface over $k$ of degree $2d$ associated with $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}(k)$.
(2) The image of $s$ under the Kuga-Satake morphism KS is denoted by the same notation $s \in Z_{\mathrm{K}_0^p}(\Lambda)(k)$. After replacing $k$ by a finite extension of it, there is a $k$-valued point of $Z_{\mathrm{K}^p}(\Lambda)(k)$ mapped to $s$. We fix such a point, and denote it also by $s \in Z_{\mathrm{K}^p}(\Lambda)(k)$. Let $\mathcal{A}_s$ be the Kuga-Satake abelian variety over $k$ associated with the point $s \in Z_{\mathrm{K}^p}(\Lambda)(k)$.
(3) We take an $\mathcal{O}_K$-valued point $t \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}(\mathcal{O}_K)$ which is a lift of $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}(k)$. The morphism
$$Z_{\mathrm{K}^p}(\Lambda) \to Z_{\mathrm{K}_0^p}(\Lambda)$$
is étale by Proposition 2.3.5. Hence the image of the $\mathcal{O}_K$-valued point $t$ under the Kuga-Satake morphism lifts a unique $\mathcal{O}_K$-valued point on $Z_{\mathrm{K}^p}(\Lambda)$ which is a lift of $s \in Z_{\mathrm{K}^p}(\Lambda)$. We also denote it by $t \in Z_{\mathrm{K}^p}(\Lambda)(\mathcal{O}_K)$.
(4) Let $(\mathscr{Y}, \xi)$ be a quasi-polarized K3 surface over $\mathcal{O}_K$ of degree $2d$ associated with $t \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}(\mathcal{O}_K)$. We assume $\xi$ is a line bundle whose restriction to every geometric fiber is big and nef.
(5) Let $\bar{t}$ be a geometric point of $Z_{\mathrm{K}^p}(\Lambda)$ above the generic point of $t \in Z_{\mathrm{K}^p}(\Lambda)(\mathcal{O}_K)$.

We denote the orthogonal complements with respect to the cup product of the first Chern class of $\xi$ in the de Rham cohomology and in the crystalline cohomology by
$$P^2_{\mathrm{dR}}(\mathscr{Y}/\mathcal{O}_K) := \mathrm{ch}_{\mathrm{dR}}(\xi)^{\perp} \subset H^2_{\mathrm{dR}}(\mathscr{Y}/\mathcal{O}_K),$$
$$P^2_{\mathrm{cris}}(X/W) := \mathrm{ch}_{\mathrm{cris}}(\xi)^{\perp} = \mathrm{ch}_{\mathrm{cris}}(\mathscr{L})^{\perp} \subset H^2_{\mathrm{\acute{e}t}}(X/W).$$

We use the following notation on twists of $\varphi$-modules. Let $(N, \varphi)$ be a pair of a free $W$-module of finite rank and a $\sigma$-linear map $\varphi$ on $N[1/p]$. We denote the pair $(N, p^{-i}\varphi)$ by $N(i)$. We put
$$\widetilde{L}_{\mathrm{cris}} := \widetilde{L}_{\mathrm{cris},s}.$$
The Frobenius automorphism of $\widetilde{L}_{\mathrm{cris}}(-1)[1/p]$ maps the $W$-module $\widetilde{L}_{\mathrm{cris}}(-1)$ into itself. Therefore $\widetilde{L}_{\mathrm{cris}}(-1)$ is an $F$-crystal. Let us recall the following proposition, which is important for the proof of the étaleness of the Kuga-Satake morphism and for the proofs of our main results.

**Proposition 2.5.1.** *We have an isomorphism of $F$-crystals*
$$P^2_{\mathrm{cris}}(X/W) \cong \iota_{\mathrm{cris}}(\Lambda_W)^{\perp}(-1) \subset \widetilde{L}_{\mathrm{cris}}(-1).$$

Proof. This proposition was proved in [**85**, Corollary 5.14] when $p \neq 2$. A proof using the integral comparison theorem of Bhatt-Morrow-Scholze [**9**] was given in [**65**, Proposition 6.4], which is valid for any characteristic $p$. □

In particular, we have the following composition

$$P^2_{\mathrm{cris}}(X/W)(1) \hookrightarrow \widetilde{L}_{\mathrm{cris}} \xrightarrow{i_{\mathrm{cris}}} \mathrm{End}_W(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee),$$

which is also denoted by $i_{\mathrm{cris}}$. Similarly, we have the following map

$$i_{\mathrm{dR}} \colon P^2_{\mathrm{dR}}(\mathscr{Y}/\mathcal{O}_K)(1) \to \mathrm{End}_{\mathcal{O}_K}(H^1_{\mathrm{dR}}(\mathcal{A}_t/\mathcal{O}_K)^\vee)$$

which preserves Hodge filtrations. Via the isomorphisms of Berthelot-Ogus [**8**, Theorem 2.4], we have the following commutative diagram:

$$\begin{array}{ccc} P^2_{\mathrm{cris}}(X/W)(1) \otimes_W K & \xrightarrow{i_{\mathrm{cris}}} & \mathrm{End}_K((H^1_{\mathrm{cris}}(\mathcal{A}_s/W) \otimes_W K)^\vee) \\ \downarrow \cong & & \downarrow \cong \\ P^2_{\mathrm{dR}}(\mathscr{Y}_K/K)(1) & \xrightarrow{i_{\mathrm{dR}}} & \mathrm{End}_K(H^1_{\mathrm{dR}}(\mathcal{A}_t/K)^\vee). \end{array}$$

The commutativity of the diagram follows from the constructions of $i_{\mathrm{cris}}$ and $i_{\mathrm{dR}}$; see also [**65**, Section 6 and Section 11].

**2.5.2. Formal Brauer groups.** We consider the situation as in Section 2.5.1.

Let $\widehat{\mathrm{Br}} := \widehat{\mathrm{Br}}(X)$ be the *formal Brauer group* associated with the K3 surface $X$. Recall that $\widehat{\mathrm{Br}}$ is a one-dimensional smooth formal group scheme pro-representing the functor

$$\Phi^2_X \colon \mathrm{Art}_k \to (\text{Abelian groups})$$

defined by

$$R \mapsto \mathrm{Ker}(H^2_{\text{ét}}(X_R, \mathbb{G}_m) \to H^2_{\text{ét}}(X, \mathbb{G}_m)),$$

where $\mathrm{Art}_k$ is the category of local Artinian $k$-algebras with residue field $k$, and (Abelian groups) is the category of abelian groups; see [**2**, Chapter II, Corollary 2.12]. (For basic properties of the formal Brauer group, see also [**79**, Section 6].) The *height* $h$ of the K3 surface $X$ is defined to be the height of $\widehat{\mathrm{Br}}$. We have $1 \leq h \leq 10$ or $h = \infty$.

There is a natural equivalence from the category of one-dimensional smooth formal group schemes of finite height over $k$ to the category of one-dimensional connected $p$-divisible groups over $k$. If the height of $X$ is finite, we identify the formal Brauer group $\widehat{\mathrm{Br}}$ with the corresponding connected $p$-divisible group over $k$, and let $\widehat{\mathrm{Br}}^*$ be the Cartier dual of $\widehat{\mathrm{Br}}$.

Let $\mathrm{CRIS}(k/\mathbb{Z}_p)$ be the (absolute) crystalline site of $k$. For a $p$-divisible group $\mathscr{G}$ over $k$, we have a (contravariant) crystal $\mathbb{D}(\mathscr{G})$ over $\mathrm{CRIS}(k/\mathbb{Z}_p)$; see [**7**, Définition 3.3.6]. Its value

$$\mathbb{D}(\mathscr{G})(W) := \mathbb{D}(\mathscr{G})_{W \twoheadrightarrow k}$$

in $(\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, W)$ is an $F$-crystal. For a crystal $\mathscr{E}$ over $\mathrm{CRIS}(k/\mathbb{Z}_p)$, we will denote simply by the same letter $\mathscr{E}$ the value $\mathscr{E}_{W \twoheadrightarrow k}$ in $(\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, W)$. By [**7**, (5.3.3.1)], we have a canonical perfect bilinear form

$$\mathbb{D}(\widehat{\mathrm{Br}}^*) \times \mathbb{D}(\widehat{\mathrm{Br}})(-1) \to W(-2).$$

**Proposition 2.5.2.** *Assume the height of $X$ is finite. The following assertions hold:*

(1) *There is an isomorphism of $F$-crystals*

$$\widetilde{L}_{\mathrm{cris}}(-1) \cong \mathbb{D}(\widehat{\mathrm{Br}}^*) \oplus \mathbb{D}(D)(-1) \oplus \mathbb{D}(\widehat{\mathrm{Br}})(-1),$$

*where $D$ is an étale $p$-divisible group over $k$.*

(2) *Under this isomorphism, the bilinear form on $\widetilde{L}_{\mathrm{cris}}(-1)$ is the direct sum of a perfect bilinear form*

$$\mathbb{D}(D)(-1) \times \mathbb{D}(D)(-1) \to W(-2)$$

*and the canonical perfect bilinear form*

$$\mathbb{D}(\widehat{\mathrm{Br}}^*) \times \mathbb{D}(\widehat{\mathrm{Br}})(-1) \to W(-2).$$

Proof. The breaking points of the Newton polygon of $\widetilde{L}_{\mathrm{cris}}(-1)$ lie on the Hodge polygon of it; see [**65**, Lemma 6.8] and its proof.

By the Hodge-Newton decomposition [**70**, Theorem 1.6.1], there is a decomposition as an $F$-crystal over $W$

$$\widetilde{L}_{\mathrm{cris}}(-1) \cong \widetilde{L}_{1-1/h} \oplus \widetilde{L}_1 \oplus \widetilde{L}_{1+1/h},$$

where $\widetilde{L}_\lambda$ is an $F$-crystal over $W$ has a single slope $\lambda$ for each $\lambda \in \{1-1/h,\, 1,\, 1+1/h\}$.

Via this decomposition, the bilinear form $(\ ,\ )$ is the direct sum of a perfect bilinear form

$$\widetilde{L}_1 \times \widetilde{L}_1 \to W(-2)$$

and a perfect bilinear form

$$\widetilde{L}_{1-1/h} \times \widetilde{L}_{1+1/h} \to W(-2).$$

Similarly, we have a decomposition

$$P^2_{\mathrm{cris}}(X/W) \cong L_{1-1/h} \oplus L_1 \oplus L_{1+1/h}.$$

By Proposition 2.5.1, we have $P^2_{\mathrm{cris}}(X/W) = \iota_{\mathrm{cris}}(\Lambda_W)(-1)^\perp$. Since $\iota_{\mathrm{cris}}(\Lambda_W)(-1)$ is contained in $\widetilde{L}_1$, we have

$$L_{1-1/h} = \widetilde{L}_{1-1/h} \quad \text{and} \quad L_{1+1/h} = \widetilde{L}_{1+1/h}.$$

We have a natural isomorphism of $F$-crystals over $W$

$$\mathbb{D}(\widehat{\mathrm{Br}}^*) \cong L_{1-1/h}.$$

(See [**122**, Proposition 7] for example. See also Remark 2.5.3.) Using the perfect bilinear form

$$\widetilde{L}_{1-1/h} \times \widetilde{L}_{1+1/h} \to W(-2),$$

we identify $L_{1+1/h}$ with $L^\vee_{1-1/h}(-2) \cong \mathbb{D}(\widehat{\mathrm{Br}})(-1)$.

Since we have $p\widetilde{L}_1 = \varphi(\widetilde{L}_1)$, there is an étale $p$-divisible group $D$ over $k$ such that $\mathbb{D}(D)(-1) \cong \widetilde{L}_1$. □

**Remark 2.5.3.** We assume the height of $X$ is finite. The proof of [**122**, Proposition 7] shows that there is a natural isomorphism of $F$-crystals over $W$

$$\mathrm{TC}(\widehat{\mathrm{Br}}) \cong L_{1-1/h}.$$

Here $\mathrm{TC}(\widehat{\mathrm{Br}})$ is the Cartier-Dieudonné module of typical curves of $\widehat{\mathrm{Br}}$; see [**2**, I, Section 3] for example. The $F$-crystal $\mathrm{TC}(\widehat{\mathrm{Br}})$ is naturally isomorphic to the $F$-crystal $\mathbb{D}(\widehat{\mathrm{Br}}^*)$ by [**15**, (5.8)], [**7**, Théorème 4.2.14, (5.3.3.1)].

## 2.6. Kisin's algebraic groups

In this section, we attach an algebraic group $I$ over $\mathbb{Q}$ to a quasi-polarized K3 surface of finite height over $\overline{\mathbb{F}}_q$. It is a subgroup of the multiplicative group of the endomorphism algebra of the Kuga-Satake abelian variety. Then we study its action on the formal Brauer group of the K3 surface.

**2.6.1. Kisin's algebraic groups.** Let $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$ be an $\mathbb{F}_q$-valued point and $\overline{s} \in Z_{\mathrm{K}^p}(\Lambda)(\overline{\mathbb{F}}_q)$ a geometric point above $s$. Let $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{\overline{s}})$ be the algebraic group over $\mathbb{Q}$ defined by

$$\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{\overline{s}})(R) := (\mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{\overline{s}}) \otimes_{\mathbb{Z}} R)^{\times}$$

for every $\mathbb{Q}$-algebra $R$.

After replacing $\mathbb{F}_q$ by a finite extension of it, we may assume that all endomorphisms of $\mathcal{A}_{\overline{s}}$ are defined over $\mathbb{F}_q$. Namely we have

$$\mathrm{End}_{\mathbb{F}_q}(\mathcal{A}_s) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{\overline{s}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The global sections $\{s_\alpha^p\}$ of $V^p(\mathcal{A})^{\otimes}$ give rise to global sections $\{s_{\alpha,\ell}\}$ of $V_\ell(\mathcal{A})^{\otimes}$. The $\mathbb{A}_f^p$-local system $\widetilde{\mathbb{V}}^p$ and the embedding $i^p$ determine a $\mathbb{Q}_\ell$-local system $\widetilde{\mathbb{V}}_\ell$ and an embedding $\widetilde{\mathbb{V}}_\ell \hookrightarrow \underline{\mathrm{End}}(V_\ell(\mathcal{A}))$ of $\mathbb{Q}_\ell$-local systems for every $\ell \neq p$. The homomorphism $\iota^p$ induced by the $\Lambda$-structure for $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$ determines a homomorphism $\iota_\ell \colon \Lambda_{\mathbb{Q}_\ell} \to \widetilde{\mathbb{V}}_{\ell,\overline{s}}$.

We fix an isomorphism $H_{\mathbb{Q}_\ell} \cong V_\ell(\mathcal{A}_{\overline{s}})$ of $\mathbb{Q}_\ell$-vector spaces which carries $\{s_\alpha\}$ to $\{s_{\alpha,\ell,\overline{s}}\}$ and induces the following commutative diagram:

$$\begin{array}{ccccc}
\Lambda_{\mathbb{Q}_\ell} & \longrightarrow & \widetilde{L}_{\mathbb{Q}_\ell} & \xrightarrow{\ i\ } & \mathrm{End}_{\mathbb{Q}_\ell}(H_{\mathbb{Q}_\ell}) \\
& {\scriptstyle \iota_\ell}\searrow & \downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow \\
& & \widetilde{\mathbb{V}}_{\ell,\overline{s}} & \xrightarrow{\ i_\ell\ } & \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(\mathcal{A}_{\overline{s}})),
\end{array}$$

where $\widetilde{L}_{\mathbb{Q}_\ell} \cong \widetilde{\mathbb{V}}_{\ell,\overline{s}}$ is an isometry over $\mathbb{Q}_\ell$.

Let $H_W \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee$ be an isomorphism as in Section 2.3.6. By [**86**, Lemma 2.8] and the fact that every $\mathrm{GSpin}(L_{W[1/p]})$-torsor over $W[1/p]$ is trivial (see [**104**, Theorem 6.4]), after inverting $p$ and composing an element of $\widetilde{G}(W[1/p])$, we can find an isomorphism

$$H_{W[1/p]} \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p]$$

which carries $\{s_\alpha\}$ to $\{s_{\alpha,\mathrm{cris},s}\}$ and induces the same diagram as above. See also the arguments in [**65**, Section 7.3]. We fix such an isomorphism.

Kisin introduced an algebraic group $\widetilde{I}_\ell$ over $\mathbb{Q}_\ell$ for every prime number $\ell$ (including $\ell = p$), and an algebraic group $\widetilde{I}$ over $\mathbb{Q}$ as follows; see [**74**, (2.1.2)] for details.

(1) For a prime number $\ell \neq p$, let $\mathrm{Frob}_q \in \mathrm{End}_{\mathbb{Q}_\ell}(H_{\mathbb{Q}_\ell})$ be the $\mathbb{Q}_\ell$-endomorphism of $H_{\mathbb{Q}_\ell}$ induced by the $q$-th power Frobenius of $\mathcal{A}_{\overline{s}}$. Since $\mathrm{Frob}_q$ fixes the tensors $\{s_{\alpha,\ell,\overline{s}}\}$, we have $\mathrm{Frob}_q \in \widetilde{G}(\mathbb{Q}_\ell)$. For every integer $m \geq 1$, we define an algebraic $\mathbb{Q}_\ell$-subgroup $\widetilde{I}_{\ell,m}$ of $\widetilde{G}_{\mathbb{Q}_\ell}$ by

$$\widetilde{I}_{\ell,m}(R) := \{\, g \in \widetilde{G}(R) \mid g\,\mathrm{Frob}_q^m = \mathrm{Frob}_q^m\, g \,\}$$

for every $\mathbb{Q}_\ell$-algebra $R$. For sufficiently divisible $m \geq 1$, the algebraic group $\widetilde{I}_{\ell,m}$ does not depend on $m$, and we denote it by $\widetilde{I}_\ell$.

(2) For $\ell = p$, we define an algebraic group $\widetilde{I}_{p,m}$ over $\mathbb{Q}_p$ by

$$\widetilde{I}_{p,m}(R) := \{ g \in \widetilde{G}(R \otimes_{\mathbb{Q}_p} W(\mathbb{F}_{q^m})[1/p]) \mid g\varphi = \varphi g \}.$$

For sufficiently divisible $m \geq 1$, the algebraic group $\widetilde{I}_{p,m}$ does not depend on $m$, and we denote it by $\widetilde{I}_p$.

(3) Let $\widetilde{I} \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{\overline{s}})$ be the largest closed $\mathbb{Q}$-subgroup of $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{\overline{s}})$ mapped into $\widetilde{I}_\ell$ for every $\ell$ (including $\ell = p$).

Replacing $\mathbb{F}_q$ be a finite extension of it, we may assume $\widetilde{I}_{\ell,1} = \widetilde{I}_\ell$ and $\widetilde{I}_{p,1} = \widetilde{I}_p$.

Kisin proved that the natural map

$$\widetilde{I}_{\mathbb{Q}_\ell} \to \widetilde{I}_\ell$$

is an isomorphism of algebraic groups over $\mathbb{Q}_\ell$ for every $\ell$ (including $\ell = p$) [**74**, Corollary 2.3.2].

For our purpose, we need an algebraic subgroup $I \subset \widetilde{I}$ over $\mathbb{Q}$ defined using the $\Lambda$-structure; see Definition 2.3.2. If $L$ is self-dual at $p$, it coincides with Kisin's algebraic group associated with an $\mathbb{F}_q$-valued point of the integral canonical model of $\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L)$ taken in a similar way as in Section 2.5.1.

**Definition 2.6.1.**

(1) Let $I \subset \widetilde{I}$ be the algebraic subgroup over $\mathbb{Q}$ defined by

$$I(R) := \{ g \in \widetilde{I}(R) \mid ghg^{-1} = h \text{ in } \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{\overline{s}})_R \text{ for every } h \in \iota(\Lambda_{\mathbb{Q}}) \}$$

for every $\mathbb{Q}$-algebra $R$.

(2) For a prime number $\ell \neq p$, let $I_\ell \subset \widetilde{I}_\ell$ be the algebraic subgroup over $\mathbb{Q}_\ell$ defined by

$$I_\ell(R) := \{ g \in \widetilde{I}_\ell(R) \mid ghg^{-1} = h \text{ in } \mathrm{End}_{\mathrm{Frob}_q}(H_{\mathbb{Q}_\ell})_R \text{ for every } h \in i(\Lambda_{\mathbb{Q}_\ell}) \}$$

for every $\mathbb{Q}_\ell$-algebra $R$.

(3) For $\ell = p$, let $I_p \subset \widetilde{I}_p$ be the algebraic subgroup over $\mathbb{Q}_p$ defined in a similar way as above.

As in Kisin's paper [**74**], we shall prove that the natural map

$$I_{\mathbb{Q}_\ell} \to I_\ell$$

is an isomorphism of algebraic groups over $\mathbb{Q}_\ell$ for every $\ell$ (including $\ell = p$). Here we prove it for some $\ell \neq p$. The case of general $\ell$ will be proved later; see Corollary 2.7.9.

**Proposition 2.6.2.**

*(1) For some prime number $\ell \neq p$, the natural map $I_{\mathbb{Q}_\ell} \to I_\ell$ is an isomorphism of algebraic groups over $\mathbb{Q}_\ell$.*

*(2) The algebraic groups $I$ and $\mathrm{GSpin}(L_{\mathbb{Q}})$ over $\mathbb{Q}$ have the same rank. (Recall that the rank of an algebraic group over a field $k$ is the dimension of a maximal $k$-torus of it.)*

Proof. (1) We fix a prime number $\ell \neq p$ such that $\mathrm{GSpin}(L_{\mathbb{Q}})$ and $\widetilde{G}_{\mathbb{Q}}$ are split at $\ell$, and all the eigenvalues of $\mathrm{Frob}_q$ acting on $H_{\mathbb{Q}_\ell}$ are contained in $\mathbb{Q}_\ell$. We shall show that the assertion (1) holds for such $\ell$. By the proof of [**74**, Corollary 2.1.7], the homomorphism

$$\widetilde{I}_{\mathbb{Q}_\ell} \to \widetilde{I}_\ell$$

is an isomorphism. (Precisely, Kisin proved it in [**74**] assuming $p \geq 3$ and the restriction of $\psi$ to $H_{\mathbb{Z}_{(p)}}$ is perfect. These assumptions are unnecessary; see the proof of [**72**, Theorem A.8].) By Tate's theorem, we have

$$\mathrm{End}_{\mathbb{F}_q}(\mathcal{A}_s)_{\mathbb{Q}_\ell} \cong \mathrm{End}_{\mathrm{Frob}_q}(H_{\mathbb{Q}_\ell}).$$

Now, the assertion (1) follows from the definitions of $I$ and $I_\ell$.

(2) We follow Kisin's proof of [**74**, Corollary 2.1.7]. Since $\mathrm{Frob}_q$ and $I_\ell$ act trivially on $i(\Lambda_{\mathbb{Q}_\ell})$, we have $\mathrm{Frob}_q \in \mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ and $I_\ell \subset \mathrm{GSpin}(L_{\mathbb{Q}_\ell})$; see [**86**, (2.6.1)]. The element $\mathrm{Frob}_q \in \mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ is semisimple since the action of $\mathrm{Frob}_q$ on $V_\ell(\mathcal{A}_{\overline{s}})$ is semisimple. Thus the connected component $S$ of the Zariski closure of the group $\langle \mathrm{Frob}_q \rangle$ generated by $\mathrm{Frob}_q$ is a split torus in $\mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ by the hypotheses on $\ell$. Since $I_\ell$ is the same as the centralizer of $\mathrm{Frob}_q^m$ in $\mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ for a sufficiently divisible $m$, it follows that $I_\ell$ coincides with the centralizer of $S$. Therefore, $I_\ell$ contains a split maximal torus of $\mathrm{GSpin}(L_{\mathbb{Q}_\ell})$. (Hence $I_\ell$ is a connected split reductive group over $\mathbb{Q}_\ell$.) In particular, the rank of $I_\ell$ as an algebraic group over $\mathbb{Q}_\ell$ is equal to the rank of $\mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ as an algebraic group over $\mathbb{Q}_\ell$. Since we have $I_{\mathbb{Q}_\ell} \cong I_\ell$, the ranks of the algebraic groups $I$ and $\mathrm{GSpin}(L_{\mathbb{Q}})$ over $\mathbb{Q}$ are equal. □

**2.6.2. The action of Kisin's groups on the formal Brauer groups of K3 surfaces.** We consider the situation as in Section 2.5.1 and keep the notation. We assume the height $h$ of $X$ is finite.

As in the proof of Proposition 2.5.2, we have a decomposition

$$P^2_{\mathrm{cris}}(X/W) \cong L_{1-1/h} \oplus L_1 \oplus L_{1+1/h}.$$

Here, $L_\lambda$ is an $F$-crystal over $W$ which has a single slope $\lambda$ for each $\lambda \in \{1-1/h,\ 1,\ 1+1/h\}$. Moreover, there is an isomorphism of $F$-crystals over $W$:

$$\mathbb{D}(\widehat{\mathrm{Br}}) \cong L_{1+1/h}(1).$$

By Proposition 2.5.1, the algebraic group $I$ acts on $P^2_{\mathrm{cris}}(X/W)(1)[1/p]$. Hence we have the following homomorphism of algebraic groups over $\mathbb{Q}_p$:

$$I_{\mathbb{Q}_p} \to \mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}(P^2_{\mathrm{cris}}(X/W)(1)[1/p]).$$

For a one-dimensional smooth formal group $\mathscr{G}$ over a ring $A$, let $\mathrm{Aut}_{\mathbb{Q}_p}(\mathscr{G})$ be the $\mathbb{Q}_p$-group such that

$$\mathrm{Aut}_{\mathbb{Q}_p}(\mathscr{G})(R) := (\mathrm{End}_A(\mathscr{G}) \otimes_{\mathbb{Z}_p} R)^\times$$

for every $\mathbb{Q}_p$-algebra $R$.

**Lemma 2.6.3.** *There is a homomorphism*

$$I_{\mathbb{Q}_p} \to (\mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}))^{\mathrm{op}}$$

*which is compatible with*

$$I_{\mathbb{Q}_p} \to \mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}(P^2_{\mathrm{cris}}(X/W)(1)[1/p])$$

*via the composite* $P^2_{\mathrm{cris}}(X/W)(1) \to L_{1+1/h}(1) \cong \mathbb{D}(\widehat{\mathrm{Br}})$, *where the first map is the projection.*

Proof. We put $V_\lambda := L_{1+\lambda}(1)[1/p]$ for each $\lambda \in \{-1/h,\ 0,\ 1/h\}$. For an $F$-isocrystal $M$ over $K_0$, let $\mathrm{GL}_\varphi(M)$ be the algebraic group over $\mathbb{Q}_p$ defined by

$$\mathrm{GL}_\varphi(M)(R) := \{\, g \in \mathrm{GL}_{K_0 \otimes_{\mathbb{Q}_p} R}(M \otimes_{\mathbb{Q}_p} R) \mid g\varphi = \varphi g \,\}$$

for every $\mathbb{Q}_p$-algebra $R$, where $\varphi$ is the Frobenius of $M$. We have an isomorphism of algebraic groups over $\mathbb{Q}_p$:

$$\mathrm{GL}_\varphi(P^2_{\mathrm{cris}}(X/W)(1)[1/p]) \cong \mathrm{GL}_\varphi(V_{-1/h}) \times \mathrm{GL}_\varphi(V_0) \times \mathrm{GL}_\varphi(V_{1/h}).$$

Let $I_{\mathbb{Q}_p} \to \mathrm{GL}_\varphi(V_{1/h})$. be the composite of the map $I_{\mathbb{Q}_p} \to \mathrm{GL}_\varphi(P^2_{\mathrm{cris}}(X/W)(1)[1/p])$ with the projection $\mathrm{GL}_\varphi(P^2_{\mathrm{cris}}(X/W)(1)[1/p]) \to \mathrm{GL}_\varphi(V_{1/h})$. We have an isomorphism

$$(\mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}))^{\mathrm{op}} \cong \mathrm{GL}_\varphi(V_{1/h}).$$

Hence we have a homomorphism of algebraic groups $I_{\mathbb{Q}_p} \to (\mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}))^{\mathrm{op}}$ over $\mathbb{Q}_p$. □

## 2.7. Lifting of K3 surfaces over finite fields with actions of tori

**2.7.1. K3 surfaces with complex multiplication.** In this subsection, we recall the definition and basic properties of K3 surfaces with complex multiplication over $\mathbb{C}$.

Let $Y$ be a projective K3 surface over $\mathbb{C}$. Let

$$T_Y := \mathrm{Pic}(Y)^{\perp}_{\mathbb{Q}} \subset H^2_B(Y, \mathbb{Q}(1))$$

be the transcendental part of the singular cohomology, which has the $\mathbb{Q}$-Hodge structure coming from $H^2(Y, \mathbb{Q}(1))$.

Let

$$E_Y := \mathrm{End}_{\mathrm{Hdg}}(T_Y)$$

be the $\mathbb{Q}$-algebra of $\mathbb{Q}$-linear endomorphisms on $T_Y$ preserving the $\mathbb{Q}$-Hodge structure on it. We say that $Y$ has *complex multiplication (CM)* if $E_Y$ is a CM field and $\dim_{E_Y}(T_Y) = 1$. Here a number field is called CM if it is a purely imaginary quadratic extension of a totally real number field.

Let $\mathrm{MT}(T_Y)$ be the *Mumford-Tate group* of $T_Y$. By the definition, it is the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{SO}(T_Y)$ such that $h_Y(\mathbb{S}(\mathbb{R})) \subset \mathrm{MT}(T_Y)(\mathbb{R})$, where

$$h_Y : \mathbb{S} \to \mathrm{SO}(T_Y)_{\mathbb{R}}$$

is the homomorphism over $\mathbb{R}$ corresponding to the $\mathbb{Q}$-Hodge structure of $T_Y$. By the results of Zarhin [**137**, Section 2], the K3 surface $Y$ has CM if and only if the Mumford-Tate group $\mathrm{MT}(T_Y)$ is commutative; see [**122**, Proposition 8] for example.

In the rest of this subsection, we fix a $\mathbb{C}$-valued point $t \in M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Q}}(\mathbb{C})$. Let $(Y, \xi)$ be a quasi-polarized K3 surface over $\mathbb{C}$ associated with $t$. The image of $t$ under the Kuga-Satake morphism KS is also denoted by $t \in Z_{\mathrm{K}^p_0}(\Lambda)(\mathbb{C})$.

**Proposition 2.7.1.** *Assume $Y$ is a K3 surface with CM over $\mathbb{C}$. Then there exist a number field $F \subset \mathbb{C}$ and an $F$-valued point $t_0 \in M_{2d,\mathrm{K}^p_0,\mathbb{Q}}(F)$ such that the morphism $t\colon \mathrm{Spec}\,\mathbb{C} \to Z_{\mathrm{K}^p_0}(\Lambda)$ factors through the image of $t_0$ under* KS.

Proof. This proposition follows from Rizov's result [**111**, Corollary 3.9.4] as follows. The image of $t \in Z_{\mathrm{K}^p_0}(\Lambda)(\mathbb{C})$ under the morphism $Z_{\mathrm{K}^p_0}(\Lambda) \to \mathscr{S}_{\widetilde{\mathrm{K}}_0}$ is denoted by the same symbol $t$. If $Y$ has CM, then the residue field of the image $t \in \mathscr{S}_{\widetilde{\mathrm{K}}_0}(\mathbb{C})$ is a number field by the definition of the canonical model $\mathscr{S}_{\widetilde{\mathrm{K}}_0,\mathbb{Q}} = \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ over $\mathbb{Q}$. Since the morphism $Z_{\mathrm{K}^p_0}(\Lambda) \to \mathscr{S}_{\widetilde{\mathrm{K}}_0}$ is finite by Proposition 2.3.5, the residue field of $t \in Z_{\mathrm{K}^p_0}(\Lambda)(\mathbb{C})$ is a number field. Now, the assertion follows from the étaleness of the Kuga-Satake morphism KS (in characteristic 0). □

**Remark 2.7.2.** Pjateckiĭ-Šapiro and Šafarevič also showed every K3 surface with CM is defined over a number field; see [**103**, Theorem 4].

For the quasi-polarized K3 surface $(Y, \xi)$ over $\mathbb{C}$, the primitive singular cohomology is defined by

$$P^2_B(Y, \mathbb{Q}(1)) := \mathrm{ch}_B(\xi)^\perp \subset H^2_B(Y, \mathbb{Q}(1)).$$

We fix a $\mathbb{C}$-valued point of $Z_{\mathrm{K}^p}(\Lambda)$ mapped to $t$, and also denote it by $t \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{C})$. We have the Kuga-Satake abelian variety $\mathcal{A}_t$ over $\mathbb{C}$ corresponding to $t \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{C})$. The stalk

$$\widetilde{V}_t := \widetilde{\mathbb{V}}_{B,t}$$

satisfies the following properties:

- $\widetilde{V}_t$ admits a perfect bilinear form $(\ ,\ )$ over $\mathbb{Q}$ which is a quasi-polarization.
- There is a homomorphism $\iota_B \colon \Lambda_{\mathbb{Q}} \to \widetilde{V}_t$ preserving the bilinear forms and the $\mathbb{Q}$-Hodge structures.
- There is an isometry over $\mathbb{Q}$

  $$P^2_B(Y, \mathbb{Q}(1)) \cong \iota_B(\Lambda_{\mathbb{Q}})^\perp$$

  preserving the $\mathbb{Q}$-Hodge structures.
- The following diagram commutes:

  $$\begin{array}{ccccc} & & \mathrm{GSpin}(\widetilde{V}_t)_{\mathbb{R}} & \longrightarrow & \mathrm{GL}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)_{\mathbb{R}} \\ & \overset{h}{\nearrow} & \downarrow & & \\ \mathbb{S} & \xrightarrow[h_0]{} & \mathrm{SO}(\widetilde{V}_t)_{\mathbb{R}}, & & \end{array}$$

  where $h_0$ is the homomorphism of algebraic groups over $\mathbb{R}$ corresponding to the $\mathbb{Q}$-Hodge structure on $\widetilde{V}_t$ and the composite of the following homomorphisms

  $$\mathbb{S} \xrightarrow{h} \mathrm{GSpin}(\widetilde{V}_t)_{\mathbb{R}} \to \mathrm{GL}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)_{\mathbb{R}}$$

  is the homomorphism corresponding to the $\mathbb{Q}$-Hodge structure on $H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee$.

**Proposition 2.7.3.** *The K3 surface $Y$ has CM if and only if the Kuga-Satake abelian variety $\mathcal{A}_t$ has CM.*

PROOF. This proposition was essentially proved by Tretkoff; see [**132**, Corollary 3.2]. We give an argument from the point of view of algebraic groups. Since $h_0$ is the composite of $h_Y$ with the following inclusions

$$\mathrm{SO}(T_Y)_{\mathbb{R}} \hookrightarrow \mathrm{SO}(P^2_B(Y, \mathbb{Q}(1)))_{\mathbb{R}} \hookrightarrow \mathrm{SO}(\widetilde{V}_t)_{\mathbb{R}},$$

we have $\mathrm{MT}(T_Y) \cong \mathrm{MT}(\widetilde{V}_t)$. We shall show that $\mathrm{MT}(\widetilde{V}_t)$ is commutative if and only if $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$ is commutative. It suffices to show that $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$ is commutative if $\mathrm{MT}(\widetilde{V}_t)$ is so since $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$ is contained in $\mathrm{GSpin}(\widetilde{V}_t)$ and $\mathrm{MT}(\widetilde{V}_t)$ is contained in the image of $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$.

We assume $\mathrm{MT}(\widetilde{V}_t)$ is commutative. Then the inverse image of $\mathrm{MT}(\widetilde{V}_t)$ under the homomorphism $\mathrm{GSpin}(\widetilde{V}_t) \to \mathrm{SO}(\widetilde{V}_t)$ is a solvable algebraic group and contains $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$. Since $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$ is a reductive group (see [**36**, Proposition 3.6]), it is commutative. □

**Corollary 2.7.4.** *Let $F$ be a field which can be embedded in $\mathbb{C}$. Let $Z$ be a K3 surface over $F$. If $Z \otimes_{F,j} \mathbb{C}$ has CM for an embedding $j \colon F \hookrightarrow \mathbb{C}$, then $Z \otimes_{F,j'} \mathbb{C}$ has CM for every embedding $j' \colon F \hookrightarrow \mathbb{C}$.*

Proof. The assertion follows from Proposition 2.7.3 and the fact that, for an abelian variety $A$ over $\mathbb{C}$ with CM and every automorphism $f\colon \mathbb{C} \cong \mathbb{C}$, the abelian variety $A \otimes_{\mathbb{C},f} \mathbb{C}$ has CM. □

**Remark 2.7.5.** Let $F$ be a field of characteristic 0 which can be embedded into $\mathbb{C}$, and $Z$ a K3 surface over $F$. We say that $Z$ has CM if $Z \otimes_{F,j} \mathbb{C}$ has CM for some embedding $j\colon F \hookrightarrow \mathbb{C}$, in which case $Z \otimes_{F,j'} \mathbb{C}$ has CM for *every* embedding $j'\colon F \hookrightarrow \mathbb{C}$ by Corollary 2.7.4.

### 2.7.2. A lemma on liftings of formal groups with action of tori.

The following result on characteristic 0 liftings of one-dimensional smooth formal groups is presumably well known. (For the definition of $\mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0)$ and $\mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G})$, see Section 2.6.2.)

**Lemma 2.7.6.** *Let $\mathcal{G}_0$ be a one-dimensional smooth formal group over $\mathbb{F}_q$. Let $T_p$ be an algebraic torus over $\mathbb{Q}_p$ and*

$$\rho\colon T_p \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0)$$

*a homomorphism of algebraic groups over $\mathbb{Q}_p$. Assume that the height of $\mathcal{G}_0$ is finite, and the Frobenius $\Phi$ of $\mathcal{G}_0$ over $\mathbb{F}_q$ is contained in $\rho(T_p(\mathbb{Q}_p))$. Then, there exist a finite totally ramified extension $E$ of $K_0$ and a smooth formal group $\mathcal{G}$ over $\mathcal{O}_E$ satisfying the following properties:*

*(1) The special fiber of $\mathcal{G}$ is isomorphic to $\mathcal{G}_0$.*
*(2) The homomorphism $\rho$ factors as*

$$T_p \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}) \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0).$$

Proof. We fix an isomorphism $\mathcal{G}_0 \cong \mathrm{Spf}\,\mathbb{F}_q[[x]]$ and consider $\mathcal{G}_0$ as a formal group law in $\mathbb{F}_q[[x,y]]$. The composite of the following homomorphism

$$T_p \xrightarrow{\rho} \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0) \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_{0,\overline{\mathbb{F}}_q})$$

is also denoted by $\rho$. Take a maximal $\mathbb{Q}_p$-torus $T'_p$ of $\mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_{0,\overline{\mathbb{F}}_q})$ containing $\rho(T_p)$.

It is well known that $\mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{G}_{0,\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a central division algebra over $\mathbb{Q}_p$ and $\mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{G}_{0,\overline{\mathbb{F}}_q})$ is the maximal order of it; see [**49**, Corollary 20.2.14]. Hence, there is a maximal commutative $\mathbb{Q}_p$-subalgebra

$$K' \subset \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{G}_{0,\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

such that $K'^{\times} = T'_p$ as algebraic groups over $\mathbb{Q}_p$, and we have $\mathcal{O}_{K'} \subset \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{G}_{0,\overline{\mathbb{F}}_q})$. Moreover, since $T'_p(\mathbb{Q}_p)$ contains the Frobenius $\Phi$ of $\mathcal{G}_0$ over $\mathbb{F}_q$, the endomorphisms in $K'$ commute with $\Phi$. Hence, we have $\mathcal{O}_{K'} \subset \mathrm{End}_{\mathbb{F}_q}(\mathcal{G}_0)$.

We regard the formal group law $\mathcal{G}_0$ as a formal $\mathcal{O}_{K'}$-module over $\mathbb{F}_q$ in the sense of [**49**, (18.6.1)]. The universal formal $\mathcal{O}_{K'}$-module $\mathcal{G}^{\mathrm{univ}}$ exists and it is a formal $\mathcal{O}_{K'}$-group over a polynomial ring $\mathcal{O}_{K'}[(S_i)_{i\in\mathbb{N}}]$ with infinitely many variables over $\mathcal{O}_{K'}$; see [**49**, (21.4.8)]. (Note that $\mathcal{G}^{\mathrm{univ}}$ does not classify isomorphism classes of formal $\mathcal{O}_{K'}$-modules, but formal $\mathcal{O}_{K'}$-modules.)

We take a finite totally ramified extension $E$ of $K_0$ such that $E$ is a $K'$-algebra. Then there is a formal $\mathcal{O}_{K'}$-module $\mathcal{G} \in \mathcal{O}_E[[x,y]]$ over $\mathcal{O}_E$ whose reduction modulo the maximal ideal of $\mathcal{O}_E$ is equal to $\mathcal{G}_0$ such that the homomorphism $\mathcal{O}_{K'} \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{G}_0)$ factors as

$$\mathcal{O}_{K'} \to \mathrm{End}_{\mathcal{O}_E}(\mathcal{G}) \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{G}_0).$$

It follows that $\rho$ factors as $T_p \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}) \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0)$. □

**2.7.3. Liftings of K3 surfaces over finite fields with actions of tori.** Let $(Y, \xi)$ be a quasi-polarized K3 surface over a field $k$ of characteristic 0 or $p$. Let $\overline{k}$ be an algebraic closure of $k$. For a prime number $\ell \neq p$, the primitive part of the $\ell$-adic cohomology is denoted by

$$P^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) := \mathrm{ch}_\ell(\xi)^\perp \subset H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1)).$$

It is equipped with a canonical action of $\mathrm{Gal}(\overline{k}/k)$. When $\overline{k}$ is a subfield of $\mathbb{C}$, we have a canonical isomorphism

$$P^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) \cong P^2_B(Y_\mathbb{C}, \mathbb{Q}(1)) \otimes_\mathbb{Q} \mathbb{Q}_\ell,$$

where $P^2_B(Y_\mathbb{C}, \mathbb{Q}(1))$ denotes the primitive part of the singular cohomology of $Y_\mathbb{C}$.

We consider the situation as in Section 2.5.1 and keep the notation. In particular, $(X, \mathscr{L})$ is a quasi-polarized K3 surface over $\mathbb{F}_q$. We assume that the height $h$ of $X$ is finite. We attach the algebraic group $I$ over $\mathbb{Q}$ to the $\mathbb{F}_q$-valued point $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$; see Definition 2.6.1.

The following theorem concerns characteristic 0 liftings of K3 surfaces of finite height over finite extensions of $W(\overline{\mathbb{F}}_q)[1/p]$. Since every K3 surface with CM is defined over a number field (see Proposition 2.7.1 and Remark 2.7.2), Theorem 2.7.7 implies Theorem 2.1.1; see Corollary 2.7.10.

**Theorem 2.7.7.** *We assume that $p \geq 5$. Let $T \subset I$ be a maximal torus over $\mathbb{Q}$. Then there exist a finite extension $K$ of $W(\overline{\mathbb{F}}_q)[1/p]$ and a quasi-polarized K3 surface $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_K$ such that the special fiber $(\mathcal{X}_{\overline{\mathbb{F}}_q}, \mathcal{L}_{\overline{\mathbb{F}}_q})$ is isomorphic to $(X_{\overline{\mathbb{F}}_q}, \mathscr{L}_{\overline{\mathbb{F}}_q})$, and, for every embedding $K \hookrightarrow \mathbb{C}$, the quasi-polarized K3 surface $(\mathcal{X}_\mathbb{C}, \mathcal{L}_\mathbb{C})$ satisfies the following properties:*

*(1) The K3 surface $\mathcal{X}_\mathbb{C}$ has CM.*

*(2) There is a homomorphism of algebraic groups over $\mathbb{Q}$*

$$T \to \mathrm{SO}(P^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1))).$$

*(3) For every $\ell \neq p$, the action of $T(\mathbb{Q}_\ell)$ on $P^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1)) \otimes_\mathbb{Q} \mathbb{Q}_\ell$ is identified with the action of $T(\mathbb{Q}_\ell)$ on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ via the canonical isomorphisms*

$$P^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1)) \otimes_\mathbb{Q} \mathbb{Q}_\ell \cong P^2_{\text{ét}}(\mathcal{X}_\mathbb{C}, \mathbb{Q}_\ell(1)) \cong P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)))$$

*(using the embedding $K \hookrightarrow \mathbb{C}$, we consider $K$ as a subfield of $\mathbb{C}$).*

*(4) The action of every element of $T(\mathbb{Q})$ on $P^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure on it.*

Proof. Recall $\widehat{\mathrm{Br}} = \widehat{\mathrm{Br}}(X)$ is the formal Brauer group associated with $X$. Since the height of $X$ is $h < \infty$, there is a natural homomorphism $I_{\mathbb{Q}_p} \to (\mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}))^{\mathrm{op}}$ by Lemma 2.6.3. Hence we have a homomorphism

$$T_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}})$$

of algebraic groups over $\mathbb{Q}_p$.

By Lemma 2.7.6, there exist a finite totally ramified extension $E$ of $K_0$ and a one-dimensional smooth formal group $\mathcal{G}$ over $\mathcal{O}_E$ whose special fiber is isomorphic to $\widehat{\mathrm{Br}}$ such that the homomorphism $\rho$ factors as

$$T_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}) \to \mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}).$$

We have the filtration associated with $\mathcal{G}$

$$\mathrm{Fil}^1(\mathcal{G}) \hookrightarrow \mathbb{D}(\widehat{\mathrm{Br}}) \otimes_W E \hookrightarrow P^2_{\mathrm{cris}}(X/W)(1) \otimes_W E.$$

This is an isotropic line on $P^2_{\mathrm{cris}}(X/W)(1) \otimes_W E$. Let $K$ be the composition of $E$ and $W(\overline{\mathbb{F}}_q)[1/p]$. Since $p \geq 5$, by the proof of [**97**, Proposition 5.5], there exists a quasi-polarized K3 surface $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_K$ such that the special fiber $(\mathcal{X}_{\overline{\mathbb{F}}_q}, \mathcal{L}_{\overline{\mathbb{F}}_q})$ is isomorphic to $(X_{\overline{\mathbb{F}}_q}, \mathscr{L}_{\overline{\mathbb{F}}_q})$ and $\mathrm{Fil}^1_{\mathrm{Hdg}} \subset P^2_{\mathrm{Hdg}}(\mathcal{X}_{\overline{K}}/K)(1)$ coincides with $\mathrm{Fil}^1(\mathcal{G}) \otimes_E K$ under the isomorphism of Berthelot-Ogus

$$P^2_{\mathrm{cris}}(X/W)(1) \otimes_W K \cong P^2_{\mathrm{dR}}(\mathcal{X}_K/K)(1).$$

Let $\widetilde{s} \in M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}(\mathcal{O}_K)$ be the $\mathcal{O}_K$-valued point corresponding to $(\mathcal{X}, \mathcal{L})$. Since $Z_{\mathrm{K}^p}(\Lambda) \to Z_{\mathrm{K}^p_0}(\Lambda)$ is étale, the image of $\widetilde{s}$ in $Z_{\mathrm{K}^p_0}(\Lambda)(\mathcal{O}_K)$ under the Kuga-Satake morphism KS canonically lifts to an element $\widetilde{s} \in Z_{\mathrm{K}^p}(\Lambda)(\mathcal{O}_K)$ which is a lift of $\overline{s} \in Z_{\mathrm{K}^p}(\Lambda)(\overline{\mathbb{F}}_q)$.

We have the following commutative diagram:

$$\begin{array}{ccc}
P^2_{\mathrm{cris}}(X/W)(1) \otimes_W K & \xrightarrow{i_{\mathrm{cris}}} & \mathrm{End}_K((H^1_{\mathrm{cris}}(\mathcal{A}_s/W) \otimes_W K)^\vee) \\
\downarrow \cong & & \downarrow \cong \\
P^2_{\mathrm{dR}}(\mathcal{X}_K/K)(1) & \xrightarrow{i_{\mathrm{dR}}} & \mathrm{End}_K(H^1_{\mathrm{dR}}(\mathcal{A}_{\widetilde{s}}/K)^\vee),
\end{array}$$

Take a generator $e$ of $\mathrm{Fil}^1(\mathcal{G})$, and consider the endomorphism

$$i_{\mathrm{cris}}(e)\colon (H^1_{\mathrm{cris}}(\mathcal{A}_s/W) \otimes_W K)^\vee \to (H^1_{\mathrm{cris}}(\mathcal{A}_s/W) \otimes_W K)^\vee.$$

Since $i_{\mathrm{dR}}$ preserves filtrations, it follows from Proposition 2.2.2 that the 0-th piece

$$\mathrm{Fil}^0_{\widetilde{s}} \subset (H^1_{\mathrm{cris}}(\mathcal{A}_s/W) \otimes_W K)^\vee \cong H^1_{\mathrm{dR}}(\mathcal{A}_{\widetilde{s}}/K)^\vee$$

of the Hodge filtration coincides with the image $i_{\mathrm{cris}}(e)(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W K)$.

Since the embedding $i_{\mathrm{cris}}$ is $I_{\mathbb{Q}_p}$-equivariant and the action of $T_{\mathbb{Q}_p}$ on $P^2_{\mathrm{cris}}(X/W)(1) \otimes_W E$ preserves $\mathrm{Fil}^1(\mathcal{G})$, we see that the action of $T$ on $H^1_{\mathrm{dR}}(\mathcal{A}_{\widetilde{s}}/K)^\vee$ preserves $\mathrm{Fil}^0_{\widetilde{s}}$. Therefore the $\mathbb{Q}$-torus $T$ can be considered as a $\mathbb{Q}$-torus in $(\mathrm{End}_{\mathcal{O}_K}(\mathcal{A}_{\widetilde{s}}) \otimes_{\mathbb{Z}} \mathbb{Q})^\times$.

We choose an embedding $K \hookrightarrow \mathbb{C}$. We shall show that $\mathcal{X} \otimes_{\mathcal{O}_K} \mathbb{C}$ satisfies the conditions of Theorem 2.7.7. We denote by $x$ the $\mathbb{C}$-valued point comes from $\widetilde{s}$ and the embedding $K \hookrightarrow \mathbb{C}$. The algebraic group $T$ over $\mathbb{Q}$ can be considered as a subgroup of $\mathrm{GL}(H^1_B(\mathcal{A}_x, \mathbb{Q})^\vee)$. We fix an isomorphism of $\mathbb{Q}$-vector spaces $H_{\mathbb{Q}} \cong H^1_B(\mathcal{A}_x, \mathbb{Q})^\vee$ which carries $\{s_\alpha\}$ to $\{s_{\alpha,B,x}\}$ and induces the following commutative diagram:

$$\begin{array}{ccccc}
\Lambda_{\mathbb{Q}} & \longrightarrow & \widetilde{L}_{\mathbb{Q}} & \xrightarrow{i} & \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}}) \\
& \searrow^{\iota_B} & \downarrow \cong & & \downarrow \cong \\
& & \widetilde{\mathbb{V}}_{B,x} & \xrightarrow{i_B} & \mathrm{End}_{\mathbb{Q}}(H^1_B(\mathcal{A}_x, \mathbb{Q})^\vee).
\end{array}$$

This isomorphism identifies $\mathrm{GSpin}(\widetilde{V})$ with $\mathrm{GSpin}(\widetilde{V}_{x,B})$ and identifies $\mathrm{GSpin}(\widetilde{V}_{x,B})$ with the subgroup of $\mathrm{GL}(H^1_B(\mathcal{A}_x, \mathbb{Q}))^\vee$ defined by $\{s_{\alpha,B,x}\}$, where $\widetilde{V} := \widetilde{L}_{\mathbb{Q}}$. Since $T$ is contained in $\widetilde{I}$, we can consider $T$ as a subgroup of $\mathrm{GSpin}(\widetilde{V}_{x,B})$. Moreover, since $T$ is contained in $I$, we see that $T$ is compatible with $\iota(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$. Hence we have

$$T \hookrightarrow \mathrm{GSpin}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))).$$

Composing this homomorphism with

$$\mathrm{GSpin}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))) \to \mathrm{SO}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))),$$

we have a homomorphism of algebraic groups over $\mathbb{Q}$

$$T \to \mathrm{SO}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))).$$

The base change of this homomorphism to $\mathbb{Q}_\ell$ is identified with the homomorphism

$$T_{\mathbb{Q}_\ell} \to \mathrm{SO}(P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)))$$

via the canonical isomorphism $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$.

Finally, we shall prove the K3 surface $\mathcal{X}_{\mathbb{C}}$ has CM. It is enough to show that the Mumford-Tate group of $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ is commutative; see Section 2.7.1. It suffices to prove the image of

$$\mathbb{S} \to \mathrm{SO}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{R}(1)))$$

is contained in the image of $T_{\mathbb{R}}$. To prove this, it suffices to show that the image of

$$\mathbb{S} \to \mathrm{GSpin}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{R}(1)))$$

is contained in $T_{\mathbb{R}}$. By Proposition 2.6.2, it follows that $T$ is a maximal $\mathbb{Q}$-torus of $\mathrm{GSpin}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)))$. Therefore, it suffices to show that the image of $\mathbb{S}$ is contained in the centralizer of $T_{\mathbb{R}}$. Since $T(\mathbb{Q})$ is Zariski dense in $T_{\mathbb{R}}$, this follows from the fact that every element of $T(\mathbb{Q})$ comes from an element of $\mathrm{End}_{\mathbb{C}}(\mathcal{A}_x) \otimes_{\mathbb{Z}} \mathbb{Q}$. □

**Remark 2.7.8.** Theorem 2.7.7 is also valid for $p = 2, 3$; see [**65**, Theorem 9.7]. Since we are assuming that $p \geq 5$, a result stronger than Theorem 2.7.7 can be obtained. In fact, the quasi-polarized K3 surface $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_K$ in the proof above can be defined over $\mathcal{O}_E$ by the results of Nygaard-Ogus [**97**]. Currently, we do not know how to obtain a quasi-polarized K3 surface over $\mathcal{O}_E$ as in Theorem 2.7.7 for $p = 2, 3$.

Using Theorem 2.7.7, we can show that the assertion of Proposition 2.6.2 (1) holds for every $\ell$ (including $\ell = p$).

**Corollary 2.7.9.** *For every prime number $\ell$ (including $\ell = p$), the canonical homomorphism $I_{\mathbb{Q}_\ell} \to I_\ell$ is an isomorphism.*

Proof. The assertion follows from Proposition 2.6.2 and Theorem 2.7.7. (See the proof of [**74**, Corollary 2.3.2] for details.) □

We shall give an application of Theorem 2.7.7 to CM liftings of K3 surfaces of finite height over a finite field. For the definition of CM liftings used in this chapter, see Section 2.1.1.

**Corollary 2.7.10.** *Let $X$ be a K3 surface of finite height over $\mathbb{F}_q$. Then there is a positive integer $m \geq 1$ such that $X_{\mathbb{F}_{q^m}}$ admits a CM lifting.*

Proof. After replacing $\mathbb{F}_q$ by its finite extension, we may assume $X$ comes from an $\mathbb{F}_q$-valued point $s \in M^{\mathrm{sm}}_{2d, \mathrm{K}^p_0, \mathbb{Z}_{(p)}}(\mathbb{F}_q)$ satisfying the conditions as in Section 2.5.1. After replacing $\mathbb{F}_q$ by its finite extension again, there exist a number field $F$, a finite place $v$ of $F$ with residue field $\mathbb{F}_q$, and a K3 surface $\mathscr{X}$ over $\mathcal{O}_{F,(v)}$ whose special fiber $\mathscr{X}_{\mathbb{F}_q}$ is isomorphic to $X$, and generic fiber $\mathscr{X}_F$ is a K3 surface with CM over $F$ by Theorem 2.7.7 and Proposition 2.7.1. □

## 2.8. The Tate conjecture for the square of a K3 surfaces over finite fields

In this section, we will prove Theorem 2.1.4.

**2.8.1. Previous results on the Tate conjecture.** In this subsection, we recall previously known results on the Tate conjecture which will be used to prove Theorem 2.1.4.

**Lemma 2.8.1.** *Let $V$ be a projective smooth variety over $\mathbb{F}_q$. Let $\ell$ be a prime number different from $p$. Let $i$ be an integer, and $m \geq 1$ a positive integer. If the Tate conjecture holds for algebraic cycles of codimension $i$ on the variety $V_{\mathbb{F}_{q^m}}$ over $\mathbb{F}_{q^m}$, then the Tate conjecture holds for algebraic cycles of codimension $i$ on the variety $V$ over $\mathbb{F}_q$.*

PROOF. See [**131**, Section 2, p.6] for example. □

The followings results on supersingular K3 surfaces are well known.

**Lemma 2.8.2.** *Let $X$ be a K3 surface over an algebraically closed field $k$ of characteristic $p > 0$. Then $X$ is supersingular (i.e. the height of $X$ is $\infty$) if and only if the rank of the Picard group $\mathrm{Pic}(X)$ is 22.*

PROOF. See [**79**, Theorem 4.8] for example. (Precisely, the characteristic $p$ is assumed to be odd in [**79**, Theorem 4.8]. But the same proof works in the case $p = 2$ because the Tate conjecture for K3 surfaces in characteristic 2 is now proved by [**72**, Theorem A.1].) □

**Lemma 2.8.3.** *Let $X$ be a supersingular K3 surface over $\mathbb{F}_q$. Then the Tate conjecture for $X \times X$ holds for the $\ell$-adic cohomology for every prime number $\ell \neq p$, and for the crystalline cohomology.*

PROOF. Fix a prime number $\ell \neq p$. After replacing $\mathbb{F}_q$ by a finite extension of it, we may assume $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ is spanned by classes of divisors on $X$ defined over $\mathbb{F}_q$ by Lemma 2.8.2. We have an isomorphism

$$\begin{aligned} & H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2)) \\ \cong & \bigoplus_{(i,j)=(0,4),(2,2),(4,0)} H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^j_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(2) \end{aligned}$$

by the Künneth formula. Hence $H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2))$ is spanned by classes of algebraic cycles of codimension 2 on $X \times X$. Thus the Tate conjecture holds for $X \times X$. The same proof works for the crystalline cohomology. □

**Remark 2.8.4.** By the same argument, we can prove the Tate conjecture holds for any power $X \times \cdots \times X$ for a supersingular K3 surface $X$ over $\mathbb{F}_q$.

**2.8.2. Endomorphisms of the cohomology of a K3 surface over a finite field.** Let $X$ be a K3 surface of finite height over $\mathbb{F}_q$. After replacing $\mathbb{F}_q$ by its finite extension, we may assume $X$ comes from an $\mathbb{F}_q$-valued point $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}(\mathbb{F}_q)$ satisfying the conditions as in Section 2.5.1. Let $I$ be the algebraic group over $\mathbb{Q}$ associated with $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$; see Definition 2.6.1.

In this subsection, we fix a prime number $\ell \neq p$. Let

$$V_\ell := \mathrm{ch}_\ell(\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}))^\perp \subset H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$$

be the transcendental part of the $\ell$-adic cohomology. By the Tate conjecture for $X$, every eigenvalue of $\mathrm{Frob}_q$ is not a root of unity.

**Lemma 2.8.5.**

*(1) There is a* $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$*-equivariant isomorphism*

$$\begin{aligned} &H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2)) \\ &\cong \mathbb{Q}_\ell \oplus (\mathrm{Pic}(X_{\overline{\mathbb{F}}_q})^{\otimes 2} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \oplus (\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} V_\ell)^{\oplus 2} \oplus \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell). \end{aligned}$$

*(2) The Tate conjecture holds for* $X \times X$ *if and only if the* $\mathbb{Q}_\ell$*-vector subspace*

$$\mathrm{End}_{\mathrm{Frob}_q}(V_\ell) = \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$$

*is spanned by classes of algebraic cycles of codimension* 2 *on* $X \times X$*.*

Proof. (1) We have

$$H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell & i = 0 \\ (\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(-1)) \oplus V_\ell(-1) & i = 2 \\ \mathbb{Q}_\ell(-2) & i = 4 \\ 0 & i \neq 0, 2, 4. \end{cases}$$

By the Poincaré duality, we have isomorphisms

$$V_\ell \otimes_{\mathbb{Q}_\ell} V_\ell \cong V_\ell^\vee \otimes_{\mathbb{Q}_\ell} V_\ell \cong \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell).$$

Hence the assertion (1) follows by the Künneth formula.

(2) The $\mathbb{Q}_\ell$-vector space $\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} V_\ell$ has no non-zero $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-invariants. Hence the assertion (2) follows. □

Since the action of $I(\mathbb{Q}_\ell)$ on the primitive part $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ commutes with $\mathrm{Frob}_q$, it also acts on $V_\ell$. For a sufficiently divisible $m \geq 1$, the following conditions are satisfied:

- $I_\ell = I_{\ell,m} = I_{\mathbb{Q}_\ell}$ (for the definition of $I_\ell, I_{\ell,m}$, see Section 2.6.1).
- The image of $I_{\ell,m}$ under the homomorphism $\mathrm{GSpin}(V_\ell) \to \mathrm{SO}(V_\ell)$ is equal to the centralizer $\mathrm{SO}_{\mathrm{Frob}_q^m}(V_\ell)$ of $\mathrm{Frob}_q^m$ in $\mathrm{SO}(V_\ell)$.

In the rest of this subsection, we fix an integer $m \geq 1$ satisfying the above conditions.

Let $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell)$ be the set of $\mathbb{Q}_\ell$-linear endomorphisms of $V_\ell$ commuting with $\mathrm{Frob}_q^m$. Similarly, let $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ be the set of $\overline{\mathbb{Q}}_\ell$-linear endomorphisms of $V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ commuting with $\mathrm{Frob}_q^m$. We have a map

$$I(\mathbb{Q}_\ell) \to \mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell).$$

Similarly, we also have a map

$$I(\overline{\mathbb{Q}}_\ell) \to \mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell).$$

**Lemma 2.8.6.** *The* $\overline{\mathbb{Q}}_\ell$*-vector space* $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ *is spanned by the image of* $I(\overline{\mathbb{Q}}_\ell)$*.*

Proof. Let $R \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ be the $\overline{\mathbb{Q}}_\ell$-vector subspace generated by the image of $I(\overline{\mathbb{Q}}_\ell)$. Since the action of $I(\overline{\mathbb{Q}}_\ell)$ on $V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ is semisimple, $R$ is a semisimple $\overline{\mathbb{Q}}_\ell$-subalgebra. Hence it suffices to prove every element of $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ commutes with every element in the commutant of $R$.

Since the action of $\mathrm{Frob}_q^m$ preserves the bilinear form on $V_\ell$, if $\alpha$ is an eigenvalue of $\mathrm{Frob}_q^m$, then $\alpha^{-1}$ is also an eigenvalue of $\mathrm{Frob}_q^m$. Since all the eigenvalues of $\mathrm{Frob}_q^m$ on $V_\ell$ are not roots of unity, we denote the distinct eigenvalues of $\mathrm{Frob}_q^m$ by $\alpha_1, \alpha_1^{-1}, \ldots, \alpha_r, \alpha_r^{-1} \in$

$\overline{\mathbb{Q}}_\ell$. Let $W_1, W_1^-, \dots, W_r, W_r^-$ be the eigenspaces of the eigenvalues $\alpha_1, \alpha_1^{-1}, \dots, \alpha_r, \alpha_r^{-1}$, respectively. Since $\mathrm{Frob}_q^m$ acts semisimply on $V_\ell$, we have

$$V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \cong \bigoplus_{i=1}^r (W_i \oplus W_i^-).$$

Hence we have

$$\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{i=1}^r (\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(W_i) \oplus \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(W_i^-)),$$
$$\mathrm{SO}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell) \cong \prod_{i=1}^r \mathrm{GL}(W_i).$$

By Schur's lemma, every element $g \in \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ in the commutant of $R$ is written as

$$g = \bigoplus_{i=1}^r (g_i \oplus g_i^-),$$

where $g_1, g_1^-, \dots, g_r, g_r^-$ are multiplication by scalars. Hence $g$ commutes with every element of $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$. □

**Lemma 2.8.7.** *Let $G$ be an algebraic group over an algebraically closed field $k$ of characteristic 0. Let $V$ be a finite dimensional $k$-vector space, and $\rho\colon G \to \mathrm{GL}(V)$ a morphism of algebraic groups over $k$. For any Zariski dense subset $Z \subset G(k)$, we have $\langle \rho(Z) \rangle = \langle \rho(G(k)) \rangle$, where $\langle \rho(Z) \rangle$ (resp. $\langle \rho(G(k)) \rangle$) is the $k$-vector subspace of $\mathrm{End}_k(V)$ spanned by $\rho(Z)$ (resp. $\rho(G(k))$).*

PROOF. We put $d := \dim_k \langle \rho(G(k)) \rangle$. Let $\psi$ be the composite of the following maps

$$G(k)^d := \prod_{i=1}^d G(k) \to \prod_{i=1}^d \mathrm{End}_k(V) \to \wedge^d \mathrm{End}_k(V).$$

If $\dim_k \langle \langle \rho(Z) \rangle < d$, we have $\psi(Z^d) = \{0\}$. Since $Z^d \subset G(k)^d$ is Zariski dense, we have $\psi(G(k)^d) = \{0\}$, which is absurd. The contradiction shows $\dim_k \langle \rho(Z) \rangle = d$. □

**Lemma 2.8.8.**

*(1) As a $\mathbb{Q}_\ell$-vector space, $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell)$ is spanned by the image of semisimple elements in $I(\mathbb{Q})$.*
*(2) There exist maximal tori $T_1, \dots, T_n \subset I$ over $\mathbb{Q}$ such that the $\mathbb{Q}_\ell$-vector space $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell)$ is spanned by the image of $T_1(\mathbb{Q}), \dots, T_n(\mathbb{Q})$.*

PROOF. (1) Since $I$ is a connected reductive algebraic group over $\mathbb{Q}$, the set of semisimple elements in $I(\mathbb{Q})$ is Zariski dense in $I(\overline{\mathbb{Q}}_\ell)$; see [**SGA 3**, Expose XIV, Corollaire 6.4]. By Lemma 2.8.6 and Lemma 2.8.7, the $\overline{\mathbb{Q}}_\ell$-vector space $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ is spanned by the image of semisimple elements in $I(\mathbb{Q})$. Since

$$\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell) = \mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell,$$

the $\mathbb{Q}_\ell$-vector space $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell)$ is spanned by the image of semisimple elements in $I(\mathbb{Q})$.

(2) The assertion follows from the fact that every semisimple element of $I(\mathbb{Q})$ is contained in a maximal torus of $I$ over $\mathbb{Q}$. □

**2.8.3. The results of Mukai and Buskin.** The following theorem will be used in our proof of Theorem 2.1.4.

**Theorem 2.8.9 (Mukai, Buskin).** *Let $T$ and $S$ be projective K3 surfaces over $\mathbb{C}$. Let $\psi\colon H^2_B(S,\mathbb{Q}) \cong H^2_B(T,\mathbb{Q})$ be an isomorphism of $\mathbb{Q}$-vector spaces which preserves the cup product pairings and the $\mathbb{Q}$-Hodge structure. Let $[\psi] \in H^4_B(S \times T, \mathbb{Q}(2))$ be the class corresponding to $\psi$ by the Poincaré duality and the Künneth formula. Then $[\psi]$ is the class of an algebraic cycle of codimension 2 on $S \times T$.*

Proof. See [**18**, Theorem 1.1], [**93**, Theorem 2]. (See also [**58**, Corollary 0.4].) □

**2.8.4. Proof of Theorem 2.1.4.** In this subsection, we shall prove Theorem 2.1.4.

By Lemma 2.8.3, it is enough to prove Theorem 2.1.4 for K3 surfaces of finite height. Let $X$ be a K3 surface of finite height over $\mathbb{F}_q$. After replacing $\mathbb{F}_q$ by its finite extension, we may assume $X$ comes from an $\mathbb{F}_q$-valued point $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}(\mathbb{F}_q)$ satisfying the conditions as in Section 2.5.1. Let $I$ be the algebraic group over $\mathbb{Q}$ associated with $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$; see Definition 2.6.1.

We fix a prime number $\ell \neq p$. We take a sufficiently divisible integer $m \geq 1$ as in Section 2.8.2. Replacing $\mathbb{F}_q$ by a finite extension of it (see Lemma 2.8.1), we may assume $m = 1$.

By Lemma 2.8.8, there exist maximal tori $T_1, \ldots, T_n \subset I$ over $\mathbb{Q}$ such that $\mathrm{End}_{\mathrm{Frob}_q}(V_\ell)$ is spanned by the image of $T_1(\mathbb{Q}), \ldots, T_n(\mathbb{Q})$. By Lemma 2.8.5, it is enough to show that, for every $i$ with $1 \leq i \leq n$, the image of $T_i(\mathbb{Q})$ in $\mathrm{End}_{\mathrm{Frob}_q}(V_\ell)$ is spanned by classes of algebraic cycle of codimension 2 on $X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}$.

Fix an integer $i$ with $1 \leq i \leq n$. By Theorem 2.7.7, there exist a finite extension $K$ of $W(\overline{\mathbb{F}}_q)[1/p]$ and a quasi-polarized K3 surface $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_K$ whose special fiber is isomorphic to $(X_{\overline{\mathbb{F}}_q}, \mathscr{L}_{\overline{\mathbb{F}}_q})$ such that, for any embedding $K \hookrightarrow \mathbb{C}$, there is a homomorphism of algebraic groups over $\mathbb{Q}$

$$T_i \to \mathrm{SO}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))),$$

and the action of every element of $T_i(\mathbb{Q})$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure on it. We extend the action of $T_i(\mathbb{Q})$ on the primitive part $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ to the full cohomology $H^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ so that every element of $T_i(\mathbb{Q})$ acts trivially on the first Chern class $\mathrm{ch}_B(\mathcal{L}_{\mathbb{C}})$. Hence we have a homomorphism of algebraic groups over $\mathbb{Q}$

$$T_i \to \mathrm{SO}(H^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)))$$

whose image preserves the $\mathbb{Q}$-Hodge structure on $H^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$.

By the results of Mukai and Buskin (see Theorem 2.8.9), the image of every element of $T_i(\mathbb{Q})$ in $\mathrm{SO}(H^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)))$ is a class of an algebraic cycle of codimension 2 on $\mathcal{X}_{\mathbb{C}} \times \mathcal{X}_{\mathbb{C}}$.

Taking the specialization of algebraic cycles, we conclude that the image of every element of $T_i(\mathbb{Q})$ in $\mathrm{End}_{\mathrm{Frob}_q}(V_\ell)$ is a class of an algebraic cycle of codimension 2 on $X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}$.

The proof of Theorem 2.1.4 is complete. □

## 2.9. The Tate conjecture with torsion coefficients for K3 surfaces

In this section, by using the methods of Skorobogatov-Zarhin [**118, 119, 62**], we will prove the following theorem. Let $\mathrm{char}(F)$ denote the characteristic of a field of $F$.

**Theorem 2.9.1.** *Let $k$ be a field which is finitely generated over its prime subfield, and $X$ a K3 surface over $k$. Then the Chern class map for $\ell$-torsion coefficients*

$$\mathrm{Pic}(X) \to H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_\ell(1))^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}$$

*is surjective for all but finitely many* $\ell \neq \mathrm{char}(k)$.

Before proving Theorem 2.9.1, we recall well known results on the Chern class maps for divisors. Let $Z$ be a proper smooth scheme over a field $F$. Let $\mathrm{NS}(Z_{\overline{F}})$ be the Néron-Severi group of $Z_{\overline{F}}$, which is a finitely generated $\mathbb{Z}$-module. The absolute Galois group $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ of $F$ acts on $\mathrm{NS}(Z_{\overline{F}})$ via the isomorphism $\mathrm{Aut}(\overline{F}/F) \cong G_F$. Let $\Lambda_\ell$ be either $\mathbb{Q}_\ell$ or $\mathbb{Z}_\ell$. We put $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} := \mathrm{NS}(Z_{\overline{F}}) \otimes_{\mathbb{Z}} \Lambda_\ell$. The Chern class map with $\Lambda_\ell$-coefficients gives an injection

$$\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \hookrightarrow H^2_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(1))$$

for every $\ell \neq \mathrm{char}(F)$.

**Lemma 2.9.2.** *Let the notation be as above. Let* $\Lambda_\ell = \mathbb{Q}_\ell$ *(resp.* $\Lambda_\ell = \mathbb{Z}_\ell$*). Then there exists a* $\Lambda_\ell$*-submodule* $M_\ell \subset H^2_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(1))$ *stable by the action of* $G_F$ *such that the injection* $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \hookrightarrow H^2_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(1))$ *gives a* $G_F$*-equivariant isomorphism*

$$H^2_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(1)) \cong \mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \oplus M_\ell$$

*for every* $\ell \neq \mathrm{char}(F)$ *(resp. all but finitely many* $\ell \neq \mathrm{char}(F)$*).*

Proof. We may assume that $Z$ is connected. We first assume that $Z$ is projective. Let $d := \dim Z$. If $d = 1$, then $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \to H^2_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(1))$ is an isomorphism for every $\ell \neq \mathrm{char}(F)$ and the assertion is trivial. So we assume that $d \geq 2$. Let $D$ be an ample divisor on $Z$. The cohomology class of $D$ in $H^2_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(1))$ is also denoted by $D$. Let $D^{d-2} \in H^{2d-4}_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(d-2))$ be the $(d-2)$-times self-intersection of $D$ with respect to the cup product. We have the following $G_F$-equivariant map:

$$\begin{aligned} f_D \colon H^2_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(1)) &\to \mathrm{Hom}_{\Lambda_\ell}(\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell}, \Lambda_\ell) \\ x &\mapsto (y \mapsto \mathrm{tr}(D^{d-2} \cup x \cup y)), \end{aligned}$$

where $D^{d-2} \cup x \cup y \in H^{2d}_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(d))$ is the cup product of the triple $(D^{d-2}, x, y)$, and $\mathrm{tr} \colon H^{2d}_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(d)) \to \Lambda_\ell$ is the trace map. For every $\ell \neq \mathrm{char}(F)$ (resp. all but finitely many $\ell \neq \mathrm{char}(F)$), the restriction of the map $f_D$ to $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell}$ is an isomorphism, and hence $f_D$ gives a $G_F$-equivariant splitting of $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \hookrightarrow H^2_{\mathrm{\acute{e}t}}(Z_{\overline{F}}, \Lambda_\ell(1))$. This proves our claim.

The general case can be reduced to the case where $Z$ is projective as follows. We may assume that $F$ is perfect after replacing $F$ by the perfect closure of it. By [**28**, Theorem 4.1], there exists an alteration $Z' \to Z$ such that $Z'$ is a projective smooth connected scheme over $F$. Since we have already proved the assertion for $Z'$, it suffices to prove that the pull-back map $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \to \mathrm{NS}(Z'_{\overline{F}})_{\Lambda_\ell}$ gives a decomposition

$$\mathrm{NS}(Z'_{\overline{F}})_{\Lambda_\ell} \cong \mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \oplus N_\ell$$

as a $G_F$-module for every $\ell \neq \mathrm{char}(F)$ (resp. all but finitely many $\ell \neq \mathrm{char}(F)$). The pull-back map $\mathrm{NS}(Z_{\overline{F}})_{\mathbb{Q}} \to \mathrm{NS}(Z'_{\overline{F}})_{\mathbb{Q}}$ is a $G_F$-equivariant injection. Since both $\mathrm{NS}(Z_{\overline{F}})$ and $\mathrm{NS}(Z'_{\overline{F}})$ are finitely generated $\mathbb{Z}$-modules and the action of $G_F$ on $\mathrm{NS}(Z'_{\overline{F}})$ factors through a finite quotient of $G_F$, the claim follows. □

**Proof of Theorem 2.9.1.** Let $X$ be a K3 surface over a field $k$ which is finitely generated over its prime subfield. We claim that there exists a finite separable extension $k'$ of $k$ and an ample line bundle $\mathscr{L}$ on $X_{k'}$ such that the pair $(X_{k'}, \mathcal{L})$ lies on the smooth locus $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$ of the moduli space of K3 surfaces as in Section 2.4. Indeed, if $k$ is of characteristic 0, then the assertion is trivial since the moduli space of K3 surfaces of characteristic

0 is smooth. If $\mathrm{char}(k) = p > 0$, then the claim follows from [**38**, Proposition 4.2] and [**99**, Proposition 2.2]. (We note that the assertion of [**38**, Proposition 4.2] is also valid if $p = 2$ and $X$ is supersingular since the Tate conjecture for $X$ is true; see [**62**, Lemma 2.5].)

By replacing $k'$ by its separable finite extension, we may assume further that $(X_{k'}, \mathcal{L})$ satisfies the conditions as in Section 2.5.1. In particular, we have the Kuga-Satake abelian variety $A$ over $k'$. After possibly replacing $\mathrm{K}_0^p$ by its open compact subgroup, we have a natural $G_{k'}$-equivariant homomorphism

$$i_\ell \colon P^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_\ell(1)) \to \mathrm{End}_{\mathbb{Z}_\ell}(H^1_{\text{ét}}(A_{\overline{k}}, \mathbb{Z}_\ell)^\vee)$$

for every $\ell \neq \mathrm{char}(k)$ (see also Section 2.3.4).

By Lemma 2.9.2, for all but finitely many $\ell \neq \mathrm{char}(k)$, we have a decomposition

$$H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_\ell(1)) \cong \mathrm{NS}(X_{\overline{k}})_{\mathbb{Z}_\ell} \oplus M_\ell$$

as a $G_k$-module over $\mathbb{Z}_\ell$. By the Tate conjecture for $X$, we have $(M_\ell)^{G_k} = 0$ for all but finitely many $\ell \neq \mathrm{char}(k)$. Then, by [**119**, Proposition 4.2], we obtain that $(M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell)^{G_k} = 0$ for all but finitely many $\ell \neq \mathrm{char}(k)$. (We remark that [**119**, Proposition 4.2] is deduced from the Tate conjecture for torsion coefficients for endomorphisms of the abelian variety $A$.) Now the assertion of Theorem 2.9.1 follows from the fact that the natural map $\mathrm{Pic}(X) \to (\mathrm{NS}(X_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{F}_\ell)^{G_k}$ is surjective for all but finitely many $\ell \neq \mathrm{char}(k)$. □

**Remark 2.9.3.** Let $k$ be a field which is finitely generated over $\mathbb{F}_p$, and $X$ a projective smooth variety over $k$. In [**21**], Cadoret-Hui-Tamagawa proved that the Tate conjecture for divisors on $X$ implies that the Chern class map for $\ell$-torsion coefficients

$$\mathrm{Pic}(X) \to H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_\ell(1))^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}$$

is surjective for all but finitely many $\ell \neq p$. A key ingredient of the proof is the following theorem which is also due to Cadoret-Hui-Tamagawa: Let $Z$ be a proper smooth scheme over a finitely generated field $k$ over $\mathbb{F}_p$. Then the natural map $H^i_{\text{ét}}(Z_{\overline{k}}, \mathbb{Z}_\ell) \to H^i_{\text{ét}}(Z_{\overline{k}}, \mathbb{F}_\ell)$ gives an isomorphism

$$H^i_{\text{ét}}(Z_{\overline{k}}, \mathbb{Z}_\ell)^{\mathrm{Gal}(k^{\mathrm{sep}}/k.\overline{\mathbb{F}}_p)} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \cong H^i_{\text{ét}}(Z_{\overline{k}}, \mathbb{F}_\ell)^{\mathrm{Gal}(k^{\mathrm{sep}}/k.\overline{\mathbb{F}}_p)}$$

for all but finitely many $\ell \neq p$ and for every $i$; see [**20**, Theorem 4.5].

Finally, as a corollary of Theorem 2.9.1, we give an application to the finiteness of the Brauer group of a K3 surface over a field which is finitely generated over its prime subfield.

For a scheme $Z$, let $\mathrm{Br}(Z) := H^2_{\text{ét}}(Z, \mathbb{G}_m)$ be the cohomological Brauer group. Recall that $\mathrm{Br}(Z)$ is a torsion abelian group if $Z$ is a Noetherian regular scheme; see [**47**, Corollaire 1.8]. For an integer $n$, let $\mathrm{Br}(Z)[n]$ be the set of elements killed by $n$. For a prime number $p > 0$, let $\mathrm{Br}(Z)[p']$ be the prime-to-$p$ torsion part, i.e. the set of elements $x \in \mathrm{Br}(Z)$ such that we have $nx = 0$ for some non-zero integer $n$ which is not divisible by $p$.

**Proposition 2.9.4.** *Let $k$ be a field which is finitely generated over its prime field. Assume that* $\mathrm{char}(k) = 0$ *(resp.* $\mathrm{char}(k) = p$*). Let $X$ be a projective smooth variety over $k$. We assume that the Tate conjecture for divisors on $Z$ is true, i.e. the $\ell$-adic Chern class map*

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(1))^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}$$

*is surjective for every $\ell \neq$* $\mathrm{char}(k)$*. We assume further that the Chern class map for $\ell$-torsion coefficients*

$$\mathrm{Pic}(X) \to H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_\ell(1))^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}$$

*is surjective for all but finitely many* $\ell \neq \mathrm{char}(k)$*. Then* $\mathrm{Br}(X_{\overline{k}})^{G_k}$ *(resp.* $\mathrm{Br}(X_{\overline{k}})[p']^{G_k}$*) is finite.*

PROOF. We first claim that the $G_k$-fixed part $\mathrm{Br}(X_{\overline{k}})[\ell]^{G_k}$ is zero for all but finitely many $\ell \neq \mathrm{char}(k)$. Indeed, the Kummer sequence gives a short exact sequence

$$0 \to \mathrm{NS}(X_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{F}_\ell \to H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_\ell(1)) \to \mathrm{Br}(X_{\overline{k}})[\ell] \to 0$$

for every $\ell \neq \mathrm{char}(k)$. Thus, by Lemma 2.9.2, there is a decomposition

$$H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_\ell(1)) \cong (\mathrm{NS}(X_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{F}_\ell) \oplus \mathrm{Br}(X_{\overline{k}})[\ell]$$

as a $G_k$-module for all but finitely many $\ell \neq \mathrm{char}(k)$. Hence the claim follows from the assumption on the Chern class map for $\ell$-torsion coefficients.

So it remains to prove that the union $\cup_n \mathrm{Br}(X_{\overline{k}})[\ell^n]^{G_k}$ is finite for every $\ell \neq \mathrm{char}(k)$ under the assumptions. We put

$$T_\ell \mathrm{Br}(X_{\overline{k}}) := \varprojlim_n \mathrm{Br}(X_{\overline{k}})[\ell^n]$$

and $V_\ell \mathrm{Br}(X_{\overline{k}}) := T_\ell \mathrm{Br}(X_{\overline{k}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. As above, the Kummer sequence and Lemma 2.9.2 give a decomposition

$$H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(1)) \cong \mathrm{NS}(X_{\overline{k}})_{\mathbb{Q}_\ell} \oplus V_\ell \mathrm{Br}(X_{\overline{k}})$$

as a $G_k$-module for every $\ell \neq \mathrm{char}(k)$. By the assumption, we have $(V_\ell \mathrm{Br}(X_{\overline{k}}))^{G_k} = 0$. Since $T_\ell \mathrm{Br}(X_{\overline{k}})$ is torsion-free, we have $(T_\ell \mathrm{Br}(X_{\overline{k}}))^{G_k} = 0$ for every $\ell \neq \mathrm{char}(k)$. It follows that $\cup_n \mathrm{Br}(X_{\overline{k}})[\ell^n]^{G_k}$ is finite for every $\ell \neq \mathrm{char}(k)$. □

**Corollary 2.9.5 ([118, 119, 62, 21]).** *Let $k$ be a field which is finitely generated over its prime subfield, and $X$ a K3 surface over $k$. Assume that* $\mathrm{char}(k) = 0$ *(resp.* $\mathrm{char}(k) = p$*). Then* $\mathrm{Br}(X_{\overline{k}})^{G_k}$ *(resp.* $\mathrm{Br}(X_{\overline{k}})[p']^{G_k}$*) is finite.*

PROOF. This follows from the Tate conjecture for $X$, Theorem 2.9.1, and Proposition 2.9.4. □

**Remark 2.9.6.** Let $k$ be a field which is finitely generated over $\mathbb{F}_p$. Let $X$ be a projective smooth variety over $k$. Cadoret-Hui-Tamagawa proved that the Tate conjecture for divisors on $X$ implies that $\mathrm{Br}(X_{\overline{k}})[p']^{G_k}$ is finite; see [**21**, Corollary 1.5]. (If $k$ is finite, this result was proved by Tate; see also the references given in [**127**, Section 4].)

CHAPTER 3

# On $\ell$-independence for Huber's tubular neighborhoods

## 3.1. Introduction

We prove some $\ell$-independence results on local constancy of étale cohomology of rigid analytic varieties. As a result, we show that a closed subscheme of a proper scheme over an algebraically closed complete non-archimedean field has a small open neighborhood in the analytic topology such that, for every prime number $\ell$ different from the residue characteristic, the closed subscheme and the open neighborhood have the same étale cohomology with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients. The existence of such an open neighborhood for each $\ell$ was proved by Huber. A key ingredient in the proof is a uniform refinement of a theorem of Orgogozo on the compatibility of the nearby cycles over general bases with base change. This chapter is based on the preprint [**63**].

Let $K$ be an algebraically closed complete non-archimedean field whose topology is given by a valuation $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ of rank 1. Let $\mathcal{O} := \mathcal{O}_K$ be the ring of integers of $K$.

**3.1.1. A main result.** In this chapter, we study local constancy of étale cohomology of rigid analytic varieties over $K$, or more precisely, of adic spaces of finite type over $\mathrm{Spa}(K, \mathcal{O})$. The theory of étale cohomology for adic spaces was developed by Huber; see [**52**]. Huber obtained several finiteness results on étale cohomology of adic spaces in a series of papers [**53, 54, 56**]. Let us recall one of the main results of [**54**]; see [**54**, Theorem 3.6] for a more precise statement.

**Theorem 3.1.1 (Huber [54, Theorem 3.6]).** *We assume that $K$ is of characteristic 0. Let $X$ be a separated adic space of finite type over $\mathrm{Spa}(K, \mathcal{O})$ and $Z$ a closed adic subspace of $X$. Let $n$ be a positive integer invertible in $\mathcal{O}$. Then there exists an open subset $V$ of $X$ containing $Z$ such that the restriction map*

$$H^i(V, \mathbb{Z}/n\mathbb{Z}) \to H^i(Z, \mathbb{Z}/n\mathbb{Z})$$

*on étale cohomology groups is an isomorphism for every integer $i$. Moreover, we can assume that $V$ is quasi-compact. (In this chapter, we drop the subscript and write $H^i(Z, \mathbb{Z}/n\mathbb{Z})$ instead of $H^i_{\text{ét}}(Z, \mathbb{Z}/n\mathbb{Z})$.)*

It is a natural question to ask whether we can take an open subset $V$ as in Theorem 3.1.1 independent of $n$. In the present chapter, we answer this question in the affirmative for adic spaces which are arising from schemes of finite type over $\mathrm{Spec}\, \mathcal{O}$.

More precisely, we will prove the following theorem. For a scheme $\mathcal{X}$ of finite type over $\mathrm{Spec}\, \mathcal{O}$, let $\widehat{\mathcal{X}}$ denote the $\varpi$-adic formal completion of $\mathcal{X}$, where $\varpi \in K^\times$ is an element with $|\varpi| < 1$. The Raynaud generic fiber of $\widehat{\mathcal{X}}$ is denoted by $(\widehat{\mathcal{X}})^{\mathrm{rig}}$ in this section, which is an adic space of finite type over $\mathrm{Spa}(K, \mathcal{O})$. (It is denoted by $d(\widehat{\mathcal{X}})$ in [**52**] and in the main body of this chapter.)

**Theorem 3.1.2 (Theorem 3.4.9).** *Let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed immersion of separated schemes of finite type over $\mathcal{O}$. We have a closed embedding $(\widehat{\mathcal{Z}})^{\mathrm{rig}} \hookrightarrow (\widehat{\mathcal{X}})^{\mathrm{rig}}$. Then there* *exists an open subset $V$ of $(\widehat{\mathcal{X}})^{\mathrm{rig}}$ containing $(\widehat{\mathcal{Z}})^{\mathrm{rig}}$ such that, for every positive integer $n$ invertible in $\mathcal{O}$, the restriction map*

$$H^i(V, \mathbb{Z}/n\mathbb{Z}) \to H^i((\widehat{\mathcal{Z}})^{\mathrm{rig}}, \mathbb{Z}/n\mathbb{Z})$$

*on étale cohomology groups is an isomorphism for every integer $i$. Moreover, we can assume that $V$ is quasi-compact.*

A more precise statement is given in Theorem 3.4.9. We will use de Jong's alterations in several ways. This is the main reason why we restrict ourselves to the case where adic spaces are arising from schemes of finite type over $\operatorname{Spec}\mathcal{O}$. We remark that, in our case, we need not impose any conditions on the characteristic of $K$. We will also prove an analogous statement for étale cohomology with compact support; see Theorem 3.4.8.

**Remark 3.1.3.** In [**114**], Scholze proved the weight-monodromy conjecture for a projective smooth variety $X$ over a non-archimedean local field $L$ of mixed characteristic $(0, p)$ which is a set-theoretic complete intersection in a projective smooth toric variety, by reduction to the function field case proved by Deligne. In the proof, Scholze used Theorem 3.1.1 to construct, for a fixed prime number $\ell \neq p$, a projective smooth variety $Y$ over a function field of characteristic $p$ and an appropriate mapping from étale cohomology with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients of $X$ to that of $Y$. The initial motivation for the present study is, following the method of Scholze, to prove that an analogue of the weight-monodromy conjecture holds for étale cohomology with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients of such a variety $X$ for all but finitely many $\ell \neq p$ by reduction to an ultraproduct variant of Weil II recently established by Cadoret [**19**]. For this, we shall use Theorem 3.1.2 instead of Theorem 3.1.1. See Chapter 4 for details.

**3.1.2. Local constancy of higher direct images with proper support.** For the proof of Theorem 3.1.2, we need to investigate local constancy of higher direct images with proper support for morphisms of adic spaces. Before stating our results on higher direct images with proper support, let us give an outline of the proof of Theorem 3.1.1.

*Sketch of the proof of Theorem 3.1.1.* We assume that $K$ is of characteristic 0. For simplicity, we assume that the closed embedding $Z \hookrightarrow X$ is of the form $(\widehat{\mathcal{Z}})^{\mathrm{rig}} \hookrightarrow (\widehat{\mathcal{X}})^{\mathrm{rig}}$ for a closed immersion of finite presentation $\mathcal{Z} \hookrightarrow \mathcal{X}$ as in Theorem 3.1.2. By considering the blow-up of $\mathcal{X}$ along $\mathcal{Z}$, we may assume further that the closed subscheme $\mathcal{Z}$ is defined by one global function $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. Let

$$f\colon \mathcal{X} \to \operatorname{Spec}\mathcal{O}[T]$$

be the morphism defined by $T \mapsto f$. The Raynaud generic fiber of the $\varpi$-adic formal completion of $\operatorname{Spec}\mathcal{O}[T]$ is the unit disc $\mathbb{B}(1) := \operatorname{Spa}(K\langle T\rangle, \mathcal{O}\langle T\rangle)$. The set of $K$-rational points of $\mathbb{B}(1)$ is identified with the set

$$\mathbb{B}(1)(K) = \{x \in K \mid |x| \leq 1\}.$$

The morphism $f$ induces the following morphism of adic spaces:

$$f^{\mathrm{rig}}\colon (\widehat{\mathcal{X}})^{\mathrm{rig}} \to \mathbb{B}(1).$$

The inverse image $(f^{\mathrm{rig}})^{-1}(0)$ of the origin $0 \in \mathbb{B}(1)$ is the closed subspace $(\widehat{\mathcal{Z}})^{\mathrm{rig}}$.

We fix a positive integer $n$ invertible in $\mathcal{O}$. We want to take an open subset $V$ in Theorem 3.1.1 as the inverse image

$$V = (f^{\mathrm{rig}})^{-1}(\mathbb{B}(\epsilon))$$

of the disc $\mathbb{B}(\epsilon) \subset \mathbb{B}(1)$ of radius $\epsilon$ centered at 0 for a small $\epsilon \in |K^\times|$. Such a subset is called a *tubular neighborhood* of $(\widehat{\mathcal{Z}})^{\mathrm{rig}}$. For this, we have to compute étale cohomology with $\mathbb{Z}/n\mathbb{Z}$-coefficients of $(f^{\mathrm{rig}})^{-1}(\mathbb{B}(\epsilon))$ for a small $\epsilon \in |K^\times|$. By the Leray spectral sequence for $f^{\mathrm{rig}}$, it suffices to compute the cohomology group

$$H^i(\mathbb{B}(\epsilon), R^j f_*^{\mathrm{rig}} \mathbb{Z}/n\mathbb{Z})$$

for all $i, j$. The key steps are as follows.

- By [**54**, Theorem 2.1], the étale sheaf $R^j f_*^{\mathrm{rig}} \mathbb{Z}/n\mathbb{Z}$ is an oc-quasi-constructible étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules in the sense of [**54**, Definition 1.4]. It follows that there exists an element $\epsilon_1 \in |K^\times|$ such that the restriction $(R^j f_*^{\mathrm{rig}} \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon_1)\backslash\{0\}}$ is a locally constant $\mathbb{Z}/n\mathbb{Z}$-sheaf of finite type.
- By the $p$-adic Riemann existence theorem of Lütkebohmert [**82**, Theorem 2.2], there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq \epsilon_1$ such that $(R^j f_*^{\mathrm{rig}} \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon_0)\backslash\{0\}}$ is trivialized by a Kummer covering $\varphi_m \colon \mathbb{B}(\epsilon_0^{1/m})\backslash\{0\} \to \mathbb{B}(\epsilon_0)\backslash\{0\}$ defined by $T \mapsto T^m$.

Then the desired result can be obtained by explicit calculations. □

In our case, the problem is to show that $\epsilon_0$ and $\epsilon_1$ in the above argument can be taken independent of $n$. To overcome this problem, by using de Jong's alterations and cohomological descent, we will reduce to the case where there exists an element $\epsilon \in |K^\times|$ with $\epsilon \leq 1$ such that the restriction

$$(f^{\mathrm{rig}})^{-1}(\mathbb{B}(\epsilon)\backslash\{0\}) \to \mathbb{B}(\epsilon)\backslash\{0\}$$

of $f^{\mathrm{rig}}$ is *smooth.* In this case, we will analyze the higher direct image sheaf with proper support

$$R^j f_!^{\mathrm{rig}} \mathbb{Z}/n\mathbb{Z}$$

on $\mathbb{B}(1)$, which is defined in [**52**, Definition 5.4.4]. An important fact is that, since $f^{\mathrm{rig}}$ is smooth over $\mathbb{B}(\epsilon)\backslash\{0\}$, the restriction $(R^j f_!^{\mathrm{rig}} \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon)\backslash\{0\}}$ is a constructible étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules (in the sense of [**52**, Definition 2.7.2]) for every positive integer $n$ invertible in $\mathcal{O}$ by [**52**, Theorem 6.2.2].

Our main result on local constancy of higher direct images with proper support is as follows. We do not suppose that $K$ is of characteristic zero. For elements $a, b \in |K^\times|$ with $a < b \leq 1$, let $\mathbb{B}(a, b) \subset \mathbb{B}(1)$ be the annulus with inner radius $a$ and outer radius $b$ centered at 0.

**Theorem 3.1.4 (Proposition 3.6.6 and Theorem 3.6.10).** *Let $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}[T]$ be a separated morphism of finite presentation. We assume that there exists an element $\epsilon \in |K^\times|$ with $\epsilon \leq 1$ such that the induced morphism*

$$f^{\mathrm{rig}} \colon (\widehat{\mathcal{X}})^{\mathrm{rig}} \to \mathbb{B}(1)$$

*is smooth over $\mathbb{B}(\epsilon)\backslash\{0\}$. Then there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq \epsilon$ such that, for every positive integer $n$ invertible in $\mathcal{O}$, the following two assertions hold:*

*(1) The restriction $(R^i f_!^{\mathrm{rig}} \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon_0)\backslash\{0\}}$ is a locally constant $\mathbb{Z}/n\mathbb{Z}$-sheaf of finite type for every $i$.*

*(2) For elements $a, b \in |K^\times|$ with $a < b \leq \epsilon_0$, there exists a composition*

$$h\colon \mathbb{B}(c^{1/m}, d^{1/m}) \xrightarrow{\varphi_m} \mathbb{B}(c, d) \xrightarrow{g} \mathbb{B}(a, b)$$

*of a Kummer covering $\varphi_m$ of degree $m$, where $m$ is invertible in $\mathcal{O}$, with a finite Galois étale morphism $g$, such that $(R^i f_!^{\mathrm{rig}} \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(a,b)}$ is trivialized by $h$ for every* *i. If $K$ is of characteristic zero, then we can take $g$ as a Kummer covering. (The morphism $h$ possibly depends on $n$.)*

Under the assumptions of Theorem 3.1.4, the same results hold for the higher direct image sheaf $R^i f_*^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z}$ by Poincaré duality [**52**, Corollary 7.5.5], which will imply Theorem 3.1.2.

A key ingredient in the proof of Theorem 3.1.4 is the following uniform variant of a theorem of Orgogozo [**100**, Théorème 2.1] on the compatibility of *the sliced nearby cycles functors* with base change. We also obtain a result on uniform unipotency of the sliced nearby cycles functors. See Section 3.2.1 for the definition of the sliced nearby cycles functors and see Definition 3.2.2 for the terminology used in the following theorem.

**Theorem 3.1.5 (Corollary 3.3.17).** *Let $S$ be a Noetherian excellent scheme and $g\colon Y \to S$ a separated morphism of finite type. There exists an alteration $S' \to S$ such that, for every positive integer $n$ invertible on $S$, the following assertions hold:*

1. *The sliced nearby cycles complexes for the base change $g_{S'}\colon Y_{S'} \to S'$ of $g$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are compatible with any base change.*
2. *The sliced nearby cycles complexes for $g_{S'}\colon Y_{S'} \to S'$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are unipotent.*

Theorem 3.1.5 is a corollary of a more general result (Theorem 3.2.7), which may be of independent interest. The proof of Theorem 3.1.5 is quite similar to that of [**101**, Théorème 3.1.1]. A key ingredient in the proof is de Jong's alteration.

By using a comparison theorem of Huber [**52**, Theorem 5.7.8], we will deduce Theorem 3.1.4 from Theorem 3.1.5. Roughly speaking, Theorem 3.1.4 (1) can be deduced from Theorem 3.1.5 (1) by considering a specialization map from an adic space of finite type over $\mathrm{Spa}(K, \mathcal{O})$ to its reduction; see Section 3.5.3 and Section 3.6.2 for details. Theorem 3.1.4 (2) can be deduced from Theorem 3.1.5 (2) and a study of the *discriminant function*

$$\delta_h\colon [0,\infty) \to \mathbb{R}_{\geq 0}$$

associated with a finite Galois étale covering $h\colon Y \to \mathbb{B}(1)\backslash\{0\}$ defined in [**82, 105, 83**]. See Section 3.6.1 and Appendix 3.A for details.

**3.1.3. The outline of this chapter.** This chapter is organized as follows. In Section 3.2, we first recall the definition of the sliced nearby cycles functors. Then we formulate our main result (Theorem 3.2.7) on the sliced nearby cycles functors. In Section 3.3, we prove Theorem 3.2.7.

In Section 3.4, we recall the definition of tubular neighborhoods, and then we state our main results (Theorem 3.4.8 and Theorem 3.4.9) on étale cohomology of tubular neighborhoods. In Section 3.5, we recall a comparison theorem of Huber and use it to study the relation between higher direct images with proper support for morphisms of adic spaces and the sliced nearby cycles functors. In Section 3.6, we prove Theorem 3.1.4. Section 3.5 and Section 3.6 are the technical heart of this chapter. In Section 3.7, we prove Theorem 3.4.8 and Theorem 3.4.9 (and hence Theorem 3.1.2) by using the results of Section 3.6.

Finally, in Appendix 3.A, we prove two theorems (Theorem 3.6.2 and Theorem 3.6.3) on finite étale coverings of annuli in the unit disc, which are essentially proved in [**82, 105, 83**].

## 3.2. Nearby cycles over general bases

In this section, we formulate our main results on nearby cycles over general bases. We will use the following notation. Let $f\colon X \to S$ be a morphism of schemes. For a morphism

$T \to S$ of schemes, the base change $X \times_S T$ of $X$ is denoted by $X_T$ and the base change of $f$ is denoted by $f_T \colon X_T \to T$. For a commutative ring $\Lambda$, let $D^+(X, \Lambda)$ be the derived category of bounded below complexes of étale sheaves of $\Lambda$-modules on $X$. For a complex $\mathcal{K} \in D^+(X, \Lambda)$, the pull-back of $\mathcal{K}$ to $X_T$ is denoted by $\mathcal{K}_T$. We often call an étale sheaf on $X$ simply a sheaf on $X$.

**3.2.1. Sliced nearby cycles functor.** In this chapter, a scheme is called a *strictly local scheme* if it is isomorphic to an affine scheme $\operatorname{Spec} R$ where $R$ is a strictly Henselian local ring. Let $f \colon X \to S$ be a morphism of schemes. Let $q \colon U \to S$ be a morphism from a strictly local scheme $U$. The closed point of $U$ is denoted by $u$. Let $\eta \in U$ be a point. Let $\overline{\eta} \to U$ be an algebraic geometric point lying above $\eta$, i.e. it is a geometric point lying above $\eta$ such that the residue field $\kappa(\overline{\eta})$ is a separable closure of the residue field $\kappa(\eta)$ of $\eta$. The strict localization of $U$ at $\overline{\eta} \to U$ is denoted by $U_{(\overline{\eta})}$. We have the following commutative diagram:

$$\begin{array}{ccccc} X_{U_{(\overline{\eta})}} & \xrightarrow{j} & X_U & \xleftarrow{i} & X_u \\ \downarrow & & \downarrow{\scriptstyle f_U} & & \downarrow \\ U_{(\overline{\eta})} & \longrightarrow & U & \longleftarrow & u. \end{array}$$

Let $\Lambda$ be a commutative ring. We have the following functor:

$$R\Psi_{f_U, \overline{\eta}} := i^* R j_* j^* \colon D^+(X_U, \Lambda) \to D^+(X_u, \Lambda).$$

This functor is called the *sliced nearby cycles functor* in [**60**]. Let $\mathcal{K} \in D^+(X_U, \Lambda)$ be a complex. We have an action of $G_{\kappa(\eta)} = \operatorname{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$ on $R\Psi_{f_U, \overline{\eta}}(\mathcal{K})$ via the canonical isomorphism

$$\operatorname{Aut}(U_{(\overline{\eta})}/\operatorname{Spec}(\mathcal{O}_{U,\eta})) \cong \operatorname{Gal}(\kappa(\overline{\eta})/\kappa(\eta)).$$

Let $q \colon V \to U$ be a local morphism of strictly local schemes over $S$, i.e. a morphism over $S$ which sends the closed point $v$ of $V$ to the closed point $u$ of $U$. Let $\xi \in V$ be a point with image $\eta = q(\xi) \in U$. For an algebraic geometric point $\overline{\xi} \to V$ lying above $\xi$, we have an algebraic geometric point $\overline{\eta} \to U$ lying above $\eta$ by taking the separable closure of $\kappa(\eta)$ in $\kappa(\overline{\xi})$. We call $\overline{\eta} \to U$ the image of $\overline{\xi} \to V$ under the morphism $q$. We have the following commutative diagram:

$$\begin{array}{ccccc} X_{V_{(\overline{\xi})}} & \xrightarrow{j'} & X_V & \xleftarrow{i'} & X_v \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ X_{U_{(\overline{\eta})}} & \xrightarrow{j} & X_U & \xleftarrow{i} & X_u, \end{array}$$

where the vertical morphisms are induced by $q$. For a complex $\mathcal{K} \in D^+(X_U, \Lambda)$, we have the following base change map:

$$q^* R\Psi_{f_U, \overline{\eta}}(\mathcal{K}) \to R\Psi_{f_V, \overline{\xi}}(\mathcal{K}_V).$$

We will use the following terminology.

**Definition 3.2.1.** Let $G$ be a group and $X$ a scheme. We say that a sheaf $\mathcal{F}$ of $\Lambda$-modules on $X$ with a $G$-action is *$G$-unipotent* if $\mathcal{F}$ has a finite filtration which is stable by the action of $G$ such that the action of $G$ on each successive quotient is trivial. We say that a complex $\mathcal{K} \in D^+(X, \Lambda)$ with a $G$-action is *$G$-unipotent* if its cohomology sheaves are $G$-unipotent.

**Definition 3.2.2.** Let $f\colon X \to S$ be a morphism of schemes. Let $\Lambda$ be a commutative ring and $\mathcal{K} \in D^+(X, \Lambda)$ a complex.

(1) We say that *the sliced nearby cycles complexes for $f$ and $\mathcal{K}$ are compatible with any base change* (or simply that *the nearby cycles for $f$ and $\mathcal{K}$ are compatible with any base change*) if for every local morphism $q\colon V \to U$ of strictly local schemes over $S$ and every algebraic geometric point $\overline{\xi} \to V$ with image $\overline{\eta} \to U$, the base change map
$$q^* R\Psi_{f_U,\overline{\eta}}(\mathcal{K}_U) \to R\Psi_{f_V,\overline{\xi}}(\mathcal{K}_V).$$
is an isomorphism.

(2) We say that *the sliced nearby cycles complexes for $f$ and $\mathcal{K}$ are unipotent* (or simply that *the nearby cycles for $f$ and $\mathcal{K}$ are unipotent*) if for every morphism $q\colon U \to S$ from a strictly local scheme $U$, a point $\eta \in U$, and an algebraic geometric point $\overline{\eta} \to U$ lying above $\eta$, the complex $R\Psi_{f_U,\overline{\eta}}(\mathcal{K}_U)$ is $G_{\kappa(\eta)}$-unipotent.

**Remark 3.2.3.** We can restate Definition 3.2.2 (1) in terms of vanishing topoi as follows. Let $f\colon X \to S$ be a morphism of schemes. Let
$$X \overleftarrow{\times}_S S$$
be the vanishing topos, where the étale topos of a scheme $X$ is also denoted by $X$ by abuse of notation. See [**61**, Exposé XI] and [**60**] for the definition and basic properties of the vanishing topos $X \overleftarrow{\times}_S S$. Let $\Lambda$ be a commutative ring. We have a morphism of topoi $\Psi_f\colon X \to X \overleftarrow{\times}_S S$. The direct image functor
$$R\Psi_f\colon D^+(X,\Lambda) \to D^+(X \overleftarrow{\times}_S S, \Lambda)$$
defined by $\Psi_f$ is called the *nearby cycles functor*. For a morphism $q\colon T \to S$ of schemes, we have a morphism of topoi $\overleftarrow{q}\colon X_T \overleftarrow{\times}_T T \to X \overleftarrow{\times}_S S$ and a 2-commutative diagram
$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \big\downarrow{\scriptstyle R\Psi_{f_T}} & & \big\downarrow{\scriptstyle R\Psi_f} \\ X_T \overleftarrow{\times}_T T & \xrightarrow{\overleftarrow{q}} & X \overleftarrow{\times}_S S, \end{array}$$
where $X_T \to X$ is the projection. For a complex $\mathcal{K} \in D^+(X,\Lambda)$, we have the base change map
$$c_{f,q}(\mathcal{K})\colon (\overleftarrow{q})^* R\Psi_f(\mathcal{K}) \to R\Psi_{f_T}(\mathcal{K}_T).$$

For a morphism $f\colon X \to S$ of schemes and a complex $\mathcal{K} \in D^+(X,\Lambda)$, the sliced nearby cycles complexes for $f$ and $\mathcal{K}$ are compatible with any base change in the sense of Definition 3.2.2 (1) if and only if, for every morphism $q\colon T \to S$ of schemes, the base change map $c_{f,q}(\mathcal{K})$ is an isomorphism. This follows from the following descriptions of the stalks of the nearby cycles functor and the sliced nearby cycles functors.

Let $x \to X$ be a geometric point of $X$ and let $s \to S$ denote the composition $x \to X \to S$. Let $t \to S$ be a geometric point with a specialization map $\alpha\colon t \to s$, i.e. an $S$-morphism $\alpha\colon S_{(t)} \to S_{(s)}$, where $S_{(s)}$ (resp. $S_{(t)}$) is the strict localization of $S$ at $s \to S$ (resp. $t \to S$). The triple $(x,t,\alpha)$ defines a point of the vanishing topos $X \overleftarrow{\times}_S S$ and every point of $X \overleftarrow{\times}_S S$ is of this form (up to equivalence). The topos $X \overleftarrow{\times}_S S$ has enough points. For the stalk $R\Psi_f(\mathcal{K})_{(x,t,\alpha)}$ of $R\Psi_f(\mathcal{K})$ at $(x,t,\alpha)$, we have an isomorphism
$$R\Psi_f(\mathcal{K})_{(x,t,\alpha)} \cong R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, \mathcal{K});$$

see [**60**, (1.3.2)]. Here the pull-back of $\mathcal{K}$ to $X_{(x)} \times_{S_{(s)}} S_{(t)}$ is also denoted by $\mathcal{K}$ and we will use this notation in this chapter when there is no possibility of confusion.

We have a similar description of the stalks of the sliced nearby cycles functors. More precisely, let $q\colon U \to S$ be a morphism from a strictly local scheme $U$ and $\overline{\eta} \to U$ an algebraic geometric point. Let $x \to X_u$ be a geometric point of the special fiber $X_u$ of $X_U$. Then, since the morphism $X_{U_{(\overline{\eta})}} \to X_U$ is quasi-compact and quasi-separated, we have

$$R\Psi_{f_U,\overline{\eta}}(\mathcal{K}_U)_x \cong R\Gamma((X_U)_{(x)} \times_U U_{(\overline{\eta})}, \mathcal{K}_U). \tag{3.2.1}$$

**3.2.2. Main results on nearby cycles over general bases.** A proper surjective morphism $f\colon X \to Y$ of Noetherian schemes is called an *alteration* if it sends every generic point of $X$ to a generic point of $Y$ and it is generically finite, i.e. there exists a dense open subset $U \subset Y$ such that the restriction $f^{-1}(U) \to U$ is a finite morphism. If furthermore $X$ and $Y$ are integral schemes, then $f$ is called an integral alteration. An alteration $f\colon X \to Y$ is called a *modification* if there exists a dense open subset $U \subset Y$ such that the restriction $f^{-1}(U) \to U$ is an isomorphism.

Let $f\colon X \to S$ be a morphism of finite type of Noetherian schemes. In [**100**], Orgogozo proved the following result:

**Theorem 3.2.4 (Orgogozo [100, Théorème 2.1]).** *For a positive integer $n$ invertible on $S$ and for a constructible sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $X$, there exists a modification $S' \to S$ such that the sliced nearby cycles complexes for $f_{S'}$ and $\mathcal{F}_{S'}$ are compatible with any base change in the sense of Definition 3.2.2 (1).*

Proof. See [**100**, Théorème 2.1] for the proof and for a more general result. (Actually, Orgogozo formulated his results in terms of vanishing topoi. See Remark 3.2.3.) ☐

To prove Theorem 3.1.2, we need a uniform refinement of Theorem 3.2.4. More precisely, we need a modification (or an alteration) $S' \to S$ such that, for every positive integer $n$ invertible on $S$, the sliced nearby cycles complexes for $f_{S'}$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are compatible with any base change, under the additional assumption that $S$ is excellent.

In order to prove the existence of such a modification, we will use the methods developed in a recent paper [**101**] of Orgogozo. In fact, by the same methods, we can also prove that there exists an *alteration* $S' \to S$ such that, for every positive integer $n$ invertible on $S$, the sliced nearby cycles complexes for $f_{S'}$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are unipotent in the sense of Definition 3.2.2 (2). Such an alteration is also needed in the proof of Theorem 3.1.2.

To formulate our results, we need to recall the definition of a *locally unipotent sheaf* on a Noetherian scheme from [**101**]. Let $X$ be a Noetherian scheme. In this chapter, we call a finite set $\mathfrak{X} = \{X_\alpha\}_\alpha$ of locally closed subsets of $X$ a *stratification* if we have $X = \coprod_\alpha X_\alpha$ (set-theoretically). We endow each $X_\alpha$ with the reduced subscheme structure.

**Definition 3.2.5 (Orgogozo [101, 1.2.1]).** Let $X$ be a Noetherian scheme and $\mathfrak{X}$ a stratification of $X$. We say that an abelian sheaf $\mathcal{F}$ on $X$ is *locally unipotent along* $\mathfrak{X}$ if, for every morphism $q\colon U \to X$ from a strictly local scheme $U$ and every $X_\alpha \in \mathfrak{X}$, the pull-back of $\mathcal{F}$ to $U \times_X X_\alpha$ has a finite filtration whose successive quotients are constant sheaves.

**Remark 3.2.6.** If a constructible abelian sheaf $\mathcal{F}$ on a Noetherian scheme $X$ is locally unipotent along a stratification $\mathfrak{X}$, then it is constructible along $\mathfrak{X}$, i.e. for every $X_\alpha \in \mathfrak{X}$, the pull-back of $\mathcal{F}$ to $X_\alpha$ is locally constant. (See [**101**, 1.2.2].)

The main result on nearby cycles over general bases is as follows.

**Theorem 3.2.7.** *Let $S$ be a Noetherian excellent scheme. Let $f\colon X \to S$ be a proper morphism. Let $\mathfrak{X}$ be a stratification of $X$. Then there exists an alteration $S' \to S$ such that, for every positive integer $n$ invertible on $S$ and every complex $\mathcal{K} \in D^+(X, \mathbb{Z}/n\mathbb{Z})$ whose cohomology sheaves are constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules and are locally unipotent along $\mathfrak{X}$, the following two assertions hold.*

1. *The sliced nearby cycles complexes for $f_{S'}\colon X_{S'} \to S'$ and $\mathcal{K}_{S'}$ are compatible with any base change.*
2. *The sliced nearby cycles complexes for $f_{S'}\colon X_{S'} \to S'$ and $\mathcal{K}_{S'}$ are unipotent.*

In fact, as in [**100**], we can show a more precise result for the compatibility of the sliced nearby cycles functors with base change as a corollary of Theorem 3.2.7:

**Corollary 3.2.8.** *Under the assumptions of Theorem 3.2.7, there exists a modification $S' \to S$ such that, for every positive integer $n$ invertible on $S$ and every complex $\mathcal{K} \in D^+(X, \mathbb{Z}/n\mathbb{Z})$ whose cohomology sheaves are constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules and are locally unipotent along $\mathfrak{X}$, the sliced nearby cycles complexes for $f_{S'}\colon X_{S'} \to S'$ and $\mathcal{K}_{S'}$ are compatible with any base change.*

Proof. This follows from Theorem 3.2.7 together with [**100**, Lemme 3.2 and Lemme 3.3]. □

## 3.3. Proof of Theorem 3.2.7

**3.3.1. Nodal curves.** In this subsection, we recall some results on nodal curves from [**29, 101**]. Let $f\colon X \to S$ be a morphism of Noetherian schemes. We say that $f$ is a *nodal curve* if it is a flat projective morphism such that every geometric fiber of $f$ is a connected reduced curve having at most ordinary double points as singularities. We say that $f$ is a *nodal curve adapted to a pair* $(X^\circ, S^\circ)$ of dense open subsets $X^\circ$ and $S^\circ$ of $X$ and $S$, respectively, if the following conditions are satisfied:

- $f$ is a nodal curve which is smooth over $S^\circ$.
- There is a closed subscheme $D$ of $X$ which is étale over $S$ and is contained in the smooth locus of $f$. Moreover we have $f^{-1}(S^\circ) \cap (X \backslash D) = X^\circ$.

The following proposition will be used in the proof of Theorem 3.2.7, which is one of the main reasons why we introduce the notion of locally unipotent sheaves.

**Proposition 3.3.1 (Orgogozo [101, Proposition 2.3.1]).** *Let $S$ be a Noetherian scheme and $f\colon X \to S$ a nodal curve adapted to a pair $(X^\circ, S^\circ)$ of dense open subsets $X^\circ$ and $S^\circ$ of $X$ and $S$, respectively. Let $u\colon X^\circ \hookrightarrow X$ denote the open immersion. Assume that $S^\circ$ is normal. Then, for every positive integer $n$ invertible on $S$ and every locally constant constructible sheaf $\mathcal{L}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $X^\circ$ such that $u_!\mathcal{L}$ is locally unipotent along the stratification $\mathfrak{X} = \{X^\circ, X\backslash X^\circ\}$ of $X$, the sheaf*

$$R^i f_*(u_!\mathcal{L})$$

*is locally unipotent along the stratification $\mathfrak{S} = \{S^\circ, S\backslash S^\circ\}$ of $S$ for every $i$.*

Proof. See [**101**, Proposition 2.3.1]. □

**Remark 3.3.2.** The proof of Theorem 3.2.7 is inspired by that of Proposition 3.3.1. In fact, we can show that, with the notation of Proposition 3.3.1, the nearby cycles for $f$ and $u_!\mathcal{L}$ are compatible with any base change and unipotent. Since we will not use this fact in the proof of Theorem 3.2.7, we omit the proof of it.

We say that a morphism $f\colon X \to S$ of Noetherian integral schemes is a *pluri nodal curve adapted to a dense open subset* $X^\circ \subset X$ if there are an integer $d \geq 0$, a sequence

$$(X = X_d \underset{f_d}{\to} X_{d-1} \to \cdots \to X_1 \underset{f_1}{\to} X_0 = S)$$

of morphisms of Noetherian integral schemes, and dense open subsets $X_i^\circ \subset X_i$ for every $0 \leq i \leq d$ with $X_d^\circ = X^\circ$ such that $f_i\colon X_i \to X_{i-1}$ is a nodal curve adapted to the pair $(X_i^\circ, X_{i-1}^\circ)$ for every $1 \leq i \leq d$. If $d = 0$, by convention, it means that $X = S$ and $f$ is the identity map.

The following theorem of de Jong plays an important role in the proof of Theorem 3.2.7.

**Theorem 3.3.3 (de Jong [29, Theorem 5.9]).** *Let $f\colon X \to S$ be a proper surjective morphism of Noetherian excellent integral schemes. Let $X^\circ \subset X$ be a dense open subset. We assume that the geometric generic fiber of $f$ is irreducible. Then there is the following commutative diagram:*

$$\begin{array}{ccc} X_0 & \xrightarrow{f'} & S' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S, \end{array}$$

*where the vertical maps are integral alterations and $f'$ is a pluri nodal curve adapted to a dense open subset $X_0^{\circ\circ} \subset X_0$ which is contained in the inverse image of $X^\circ \subset X$.*

Proof. See [29, Theorem 5.9] and the proof of [29, Theorem 5.10]. We note that if the dimension of the generic fiber of $f$ is zero, then $f$ is an integral alteration. Hence we can take $S'$ as $X$ and take $f'$ as the identity map on $X$ in this case. □

**3.3.2. Preliminary lemmas.** We shall give two lemmas, which will be used in the proof of Theorem 3.2.7.

We will need the following terminology.

**Definition 3.3.4.** Let $f\colon X \to S$ be a morphism of schemes. Let $\Lambda$ be a commutative ring and $\mathcal{K} \in D^+(X, \Lambda)$ a complex. Let $\rho$ be an integer.

(1) We say that *the sliced nearby cycles complexes for $f$ and $\mathcal{K}$ are $\rho$-compatible with any base change* (or simply that *the nearby cycles for $f$ and $\mathcal{K}$ are $\rho$-compatible with any base change*) if for every local morphism $q\colon V \to U$ of strictly local schemes over $S$ and every algebraic geometric point $\overline{\xi} \to V$ with image $\overline{\eta} \to U$, we have $\tau_{\leq \rho}\Delta = 0$ for the cone $\Delta$ of the base change map:

$$q^* R\Psi_{f_U, \overline{\eta}}(\mathcal{K}_U) \to R\Psi_{f_V, \overline{\xi}}(\mathcal{K}_V) \to \Delta \to .$$

(2) We say that *the sliced nearby cycles complexes for $f$ and $\mathcal{K}$ are $\rho$-unipotent* (or simply that *the nearby cycles for $f$ and $\mathcal{K}$ are $\rho$-unipotent*) if for every morphism $q\colon U \to S$ from a strictly local scheme $U$, a point $\eta \in U$, and an algebraic geometric point $\overline{\eta} \to U$ lying above $\eta$, the complex

$$\tau_{\leq \rho} R\Psi_{f_U, \overline{\eta}}(\mathcal{K}_U)$$

is $G_{\kappa(\eta)}$-unipotent.

**Lemma 3.3.5.** *Let $f\colon X \to Z$ and $g\colon Z \to S$ be morphisms of schemes. Let $h := g \circ f$ denote the composition. Let $\mathcal{K} \in D^+(X, \mathbb{Z}/n\mathbb{Z})$ be a complex.*

(1) *Assume that $g$ is a closed immersion. If the nearby cycles for $f$ and $\mathcal{K}$ are $\rho$-compatible with any base change (resp. are $\rho$-unipotent), then so are the nearby cycles for $h$ and $\mathcal{K}$.*

(2) *Assume that $f$ is a closed immersion. If the nearby cycles for $h$ and $\mathcal{K}$ are $\rho$-compatible with any base change (resp. are $\rho$-unipotent), then so are the nearby cycles for $g$ and $f_*\mathcal{K}$.*

Proof. (1) Let $q\colon U \to S$ be a morphism from a strictly local scheme $U$ and $\overline{\eta} \to U$ an algebraic geometric point. If the image of $\overline{\eta}$ in $S$ is not contained in $Z$, then we have $R\Psi_{h_U,\overline{\eta}}(\mathcal{K}_U) = 0$. If the image of $\overline{\eta}$ in $S$ is contained in $Z$, then $U' := Z \times_S U$ is a strictly local scheme and $\overline{\eta}$ induces an algebraic geometric point $\overline{\eta}' \to U'$. We have $U'_{(\overline{\eta}')} \cong Z \times_S U_{(\overline{\eta})}$, and hence $R\Psi_{h_U,\overline{\eta}}(\mathcal{K}_U) \cong R\Psi_{f_{U'},\overline{\eta}'}(\mathcal{K}_{U'})$. The assertion follows from this description.

(2) Let $q\colon U \to S$ and $\overline{\eta} \to U$ be as above. Let $u \in U$ be the closed point. Then we have $(f_u)_* R\Psi_{h_U,\overline{\eta}}(\mathcal{K}_U) \cong R\Psi_{g_U,\overline{\eta}}((f_*\mathcal{K})_U)$, where $f_u\colon X_u \to Z_u$ is the base change of $f$. Since $(f_u)_*$ is exact, the assertion follows from this isomorphism. □

As in [**100**], we need some results on cohomological descent. See [**SGA 4 II**, Exposé Vbis] and [**33**, Section 5] for the terminology used here. Let $f\colon Y \to X$ be a morphism of schemes. Let

$$\beta\colon Y_\bullet := \operatorname{cosq}_0(Y/X) \to X$$

be the augmented simplicial object in the category of schemes defined as in [**33**, (5.1.4)], so $Y_m$ is the $(m+1)$-times fiber product $Y \times_X \cdots \times_X Y$ for $m \geq 0$. We can associate to the étale topoi of $Y_m$ $(m \geq 0)$ a topos $(Y_\bullet)^\sim$; see [**33**, (5.1.6)–(5.1.8)]. Moreover, as in [**33**, (5.1.11)], we have a morphism of topoi

$$(\beta_*, \beta^*)\colon (Y_\bullet)^\sim \to X^\sim_{\text{ét}}$$

from $(Y_\bullet)^\sim$ to the étale topos $X^\sim_{\text{ét}}$ of $X$.

**Lemma 3.3.6.** *Let $f\colon X \to S$ be a morphism of schemes. Let $\beta_0\colon Y \to X$ be a proper surjective morphism. We put $\beta\colon Y_\bullet := \operatorname{cosq}_0(Y/X) \to X$. Let $\mathcal{F}$ be a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $X$ and $\mathcal{F}_m := \beta_m^*\mathcal{F}$ the pull-back of $\mathcal{F}$ by $\beta_m\colon Y_m \to X$. The composition $f \circ \beta_m$ is denoted by $f_m$. Let $\rho \geq -1$ be an integer.*

(1) *If the nearby cycles for $f_m$ and $\mathcal{F}_m$ are $(\rho - m)$-compatible with any base change for every $0 \leq m \leq \rho+1$, then the nearby cycles for $f$ and $\mathcal{F}$ are $\rho$-compatible with any base change.*

(2) *If the nearby cycles for $f_m$ and $\mathcal{F}_m$ are $(\rho - m)$-unipotent for every $0 \leq m \leq \rho$, then the nearby cycles for $f$ and $\mathcal{F}$ are $\rho$-unipotent.*

Proof. The assertion (1) is [**100**, Lemme 4.1]. (See also Remark 3.2.3.) Although it is stated for constant sheaves, the same proof works for sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules (or more generally, for torsion abelian sheaves).

The assertion (2) can be proved by the same arguments as in the proof of [**100**, Lemme 4.1]. We shall give a sketch here. Let $q\colon U \to S$ be a morphism from a strictly local scheme $U$ and $\overline{\eta} \to U$ an algebraic geometric point with image $\eta \in U$. Let $u \in U$ be the closed point. We have the following diagram:

$$\begin{array}{ccccc}
(Y_\bullet)_{U_{(\overline{\eta})}} & \xrightarrow{j_\bullet} & (Y_\bullet)_U & \xleftarrow{i_\bullet} & (Y_\bullet)_u \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\
X_{U_{(\overline{\eta})}} & \xrightarrow{j} & X_U & \xleftarrow{i} & X_u,
\end{array}$$

where $\beta\colon (Y_\bullet)_U \to X_U$ is the base change of $\beta$, etc. By [**SGA 4 II**, Exposé Vbis, Proposition 4.3.2], the morphism $\beta_0\colon Y \to X$ is universally of cohomological descent, and hence we have $\mathcal{F}_U \cong R\beta_*\beta^*\mathcal{F}_U$. Using this isomorphism and the proper base change theorem, we obtain

$$R\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U) \cong R\beta_*(i_\bullet)^*R(j_\bullet)_*(j_\bullet)^*\beta^*\mathcal{F}_U.$$

The pull-back of the complex

$$(i_\bullet)^*R(j_\bullet)_*(j_\bullet)^*\beta^*\mathcal{F}_U$$

to $(Y_m)_u$ is isomorphic to $R\Psi_{(f_m)_U,\overline{\eta}}((\mathcal{F}_m)_U)$ for every $m \geq 0$. Thus we have the following spectral sequence:

$$E_1^{k,l} = R^l(\beta_k)_*R\Psi_{(f_k)_U,\overline{\eta}}((\mathcal{F}_k)_U) \Rightarrow R^{k+l}\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U).$$

(See [**33**, (5.2.7.1)].) The assertion follows from this spectral sequence since the sheaf

$$R^l(\beta_k)_*R\Psi_{(f_k)_U,\overline{\eta}}((\mathcal{F}_k)_U) \cong R^l(\beta_k)_*\tau_{\leq l}R\Psi_{(f_k)_U,\overline{\eta}}((\mathcal{F}_k)_U)$$

is $G_{\kappa(\eta)}$-unipotent if $k + l \leq \rho$ by our assumption. □

**3.3.3. Proof of Theorem 3.2.7.** In this subsection, we prove Theorem 3.2.7. Let us stress that the proof is heavily inspired by the methods of [**100, 101**].

In this section, we use the following terminology.

**Definition 3.3.7.** Let $S$ be a Noetherian scheme and $f\colon X \to S$ a morphism of finite type. Let $\rho$ be an integer.

(1) Let $\mathfrak{X}$ be a stratification of $X$. We say that an alteration $S' \to S$ is *$\rho$-adapted to the pair* $(f, \mathfrak{X})$ if, for every positive integer $n$ invertible on $S$ and every constructible sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $X$ which is locally unipotent along $\mathfrak{X}$, the nearby cycles for $f_{S'}\colon X_{S'} \to S'$ and $\mathcal{F}_{S'}$ are $\rho$-compatible with any base change and $\rho$-unipotent.
(2) Let $u\colon U \hookrightarrow X$ be an open immersion. We say that an alteration $S' \to S$ is *$\rho$-adapted to the pair* $(f, U)$ if, for every positive integer $n$ invertible on $S$ and every locally constant constructible sheaf $\mathcal{L}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $U$ such that $u_!\mathcal{L}$ is locally unipotent along the stratification $\{U, X\backslash U\}$, the nearby cycles for $f_{S'}\colon X_{S'} \to S'$ and $(u_!\mathcal{L})_{S'}$ are $\rho$-compatible with any base change and $\rho$-unipotent.

Let $S$ be a Noetherian excellent integral scheme. Let $\rho$ and $d$ be two integers. We shall consider the following statement $\mathbf{P}(S, \rho, d)$:

> $\mathbf{P}(S, \rho, d)$: Let $T \to S$ be an integral alteration and $f\colon Y \to T$ a proper morphism such that the dimension of the generic fiber of $f$ is less than or equal to $d$. Let $\mathfrak{Y}$ be a stratification of $Y$. Then there exists an alteration $T' \to T$ which is $\rho$-adapted to $(f, \mathfrak{Y})$ in the sense of Definition 3.3.7 (1).

**Remark 3.3.8.**

(1) $\mathbf{P}(S, -2, d)$ holds trivially for every Noetherian excellent integral scheme $S$ and every integer $d$.
(2) For an integral scheme $T$ and a proper morphism $f\colon Y \to T$, the condition that the dimension of the generic fiber is less than or equal to $-1$ means that $f$ is not surjective. The statement $\mathbf{P}(S, \rho, -1)$ is not trivial.

**Lemma 3.3.9.** *To prove Theorem 3.2.7, it is enough to prove that statement $\boldsymbol{P}(S, \rho, d)$ holds for every triple $(S, \rho, d)$, where $S$ is a Noetherian excellent integral scheme, and $\rho$ and $d$ are integers.*

PROOF. Let $S$ be a Noetherian scheme and $f\colon X \to S$ a morphism of finite type. Let $N$ be the supremum of dimensions of fibers of $f$. Let $q\colon U \to S$ be a morphism from a strictly local scheme $U$ and $\overline{\eta} \to U$ an algebraic geometric point. Then, for every sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $X$, where $n$ is a positive integer, we have for $i > 2N$

$$R^i\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U) = 0$$

by [**100**, Proposition 3.1]; see also Remark 3.2.3. By using this fact, the assertion can be proved by standard arguments. □

We will prove $\mathbf{P}(S,\rho,d)$ by induction on the triples $(S,\rho,d)$. For two Noetherian excellent integral schemes $S$ and $S'$, we denote

$$S' \prec S$$

if $S'$ is isomorphic to a proper closed subscheme of an integral alteration of $S$. For a Noetherian excellent integral scheme $S$ and an integer $\rho$, we also consider the following statements.

- $\mathbf{P}(S,\rho,*)$: The statement $\mathbf{P}(S,\rho,d')$ holds for every integer $d'$.
- $\mathbf{P}(* \prec S,\rho,*)$: The statement $\mathbf{P}(S',\rho,d')$ holds for every Noetherian excellent integral scheme $S'$ with $S' \prec S$ and every integer $d'$.

We begin with the following lemma.

**Lemma 3.3.10.** *Let $S$ be a Noetherian excellent integral scheme and $\rho$ an integer. If $\boldsymbol{P}(* \prec S,\rho,*)$ holds, then $\boldsymbol{P}(S,\rho,-1)$ holds.*

PROOF. This lemma can be proved by the same arguments as in [**100**, Section 4.2] by using [**101**, Proposition 1.6.2] instead of [**100**, Lemme 4.3]. We recall the arguments for the reader's convenience.

We assume that $\mathbf{P}(* \prec S,\rho,*)$ holds. Let $T \to S$ be an integral alteration and let $f\colon Y \to T$ be a proper morphism. We assume that $f$ is not surjective. Let $\mathfrak{Y}$ be a stratification of $Y$. Let $Z := f(Y)$ be the schematic image of $f$. We write $g\colon Y \to Z$ for the induced morphism. By applying $\mathbf{P}(* \prec S,\rho,*)$ to each irreducible component of $Z$, we can find an alteration $Z' \to Z$ which is $\rho$-adapted to $(g,\mathfrak{Y})$. By [**101**, Proposition 1.6.2], there is an alteration $T' \to T$ such that every irreducible component of $Z_{T'} := Z \times_T T'$ endowed with the reduced closed subscheme structure has a $Z$-morphism to $Z'$.

Let $\mathcal{F}$ be a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $Y$ which is locally unipotent along $\mathfrak{Y}$, where $n$ is a positive integer invertible on $T$. We shall show that the nearby cycles for $f_{T'}$ and $\mathcal{F}_{T'}$ are $\rho$-compatible with any base change and $\rho$-unipotent. By Lemma 3.3.5 (1), it suffices to show that the same properties hold for the nearby cycles for $g_{T'}\colon Y_{T'} \to Z_{T'}$ and $\mathcal{F}_{T'}$. We put

$$W_0 := \coprod_{\alpha\in\Theta}(Y_{T'} \times_{Z_{T'}} Z_\alpha),$$

where $\{Z_\alpha\}_{\alpha\in\Theta}$ is the set of the irreducible components of $Z_{T'}$, and put

$$\beta\colon W_\bullet := \operatorname{cosq}_0(W_0/Y_{T'}) \to Y_{T'}.$$

By the constructions of $Z'$ and $T'$, and by Lemma 3.3.5 (1), we see that the nearby cycles for $g_m$ and $\mathcal{F}_m$ are $\rho$-compatible with any base change and $\rho$-unipotent for every $m \geq 0$, where we write $\mathcal{F}_m := \beta_m^*\mathcal{F}_{T'}$ and $g_m := g_{T'} \circ \beta_m$. By Lemma 3.3.6, it follows that the nearby cycles for $g_{T'}\colon Y_{T'} \to Z_{T'}$ and $\mathcal{F}_{T'}$ are $\rho$-compatible with any base change and $\rho$-unipotent. □

Our next task is to show the following lemma.

**Lemma 3.3.11.** *Let $(S,\rho,d)$ be a triple of a Noetherian excellent integral scheme $S$ and two integers $\rho$ and $d$. Assume that $d \geq 0$. If $\boldsymbol{P}(S,\rho,d-1)$, $\boldsymbol{P}(S,\rho-1,*)$, and $\boldsymbol{P}(* \prec S,\rho,*)$ hold, then $\boldsymbol{P}(S,\rho,d)$ holds.*

The proof of Lemma 3.3.11 is divided into two steps. The first step is to prove the following lemma.

**Lemma 3.3.12.** *We assume that $\boldsymbol{P}(S,\rho,d-1)$, $\boldsymbol{P}(S,\rho-1,*)$, and $\boldsymbol{P}(* \prec S,\rho,*)$ hold. Under this assumption, to prove $\boldsymbol{P}(S,\rho,d)$, it suffices to prove the following statement $\boldsymbol{P}_{nd}(S,\rho,d)$:*

> $\boldsymbol{P}_{nd}(S,\rho,d)$*: Let $T \to S$ be an integral alteration and $f\colon Y \to T$ a pluri nodal curve adapted to a dense open subset $Y^\circ \subset Y$ such that the dimension of the generic fiber of $f$ is less than or equal to $d$. Then there is an alteration $T' \to T$ which is $\rho$-adapted to $(f, Y^\circ)$ in the sense of Definition 3.3.7 (2).*

PROOF. We assume that $\mathbf{P}_{\mathrm{nd}}(S,\rho,d)$ holds. Let $T \to S$ be an integral alteration and $f\colon Y \to T$ a proper morphism such that the dimension of the generic fiber of $f$ is less than or equal to $d$. Let $\mathfrak{Y}$ be a stratification of $Y$. We want to prove that there is an alteration $T' \to T$ which is $\rho$-adapted to $(f,\mathfrak{Y})$.

*Step 1.* It suffices to prove the following claim (I):

(I) Let $u\colon Y^\circ \hookrightarrow Y$ be an open immersion. Then there is an alteration $T' \to T$ which is $\rho$-adapted to $(f, Y^\circ)$.

Indeed, by replacing $\mathfrak{Y}$ by a stratification refining it, we may assume that $\mathfrak{Y}$ is a good stratification in the sense of [**101**, Section 1.1] (it is called a bonne stratification in French). Then every sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $Y$ which is constructible along $\mathfrak{Y}$ has a finite filtration such that each successive quotient is of the form $u_!\mathcal{L}$ where $u\colon Y_\alpha \hookrightarrow Y$ is an immersion for some $Y_\alpha \in \mathfrak{Y}$ and $\mathcal{L}$ is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $Y_\alpha$; see [**101**, Proposition 1.1.4]. If furthermore $\mathcal{F}$ is locally unipotent along $\mathfrak{Y}$, then so is every successive quotient of this filtration. Since $\mathfrak{Y}$ consists of finitely many locally closed subsets, by using Lemma 3.3.5 (2), we see that it suffices to prove the claim (I).

*Step 2.* To prove the claim (I), we may assume that $Y$ is integral, the morphism $f$ is surjective, and the geometric generic fiber of $f$ is irreducible.

Indeed, there is a field $L$ which is a finite extension of the function field of $T$ such that, every irreducible component of $Y \times_T \operatorname{Spec} L$ is geometrically irreducible. Let $T' \to T$ be the normalization of $T$ in $L$. We put

$$\beta_0\colon W_0 := \coprod_{\alpha\in\Theta} Y_\alpha \to Y_{T'},$$

where $\{Y_\alpha\}_{\alpha\in\Theta}$ is the set of the irreducible components of $Y_{T'}$. By $\mathbf{P}(S,\rho-1,*)$ and Lemma 3.3.6, it suffices to show that there is an alteration $T'' \to T'$ which is $\rho$-adapted to $(f_0, \beta_0^{-1}(Y^\circ_{T'}))$, where $f_0 := f_{T'} \circ \beta_0$. It is enough to show that the same assertion holds after restricting to each component $Y_\alpha$. By $\mathbf{P}(* \prec S,\rho,*)$ and Lemma 3.3.10, we may assume that $Y_\alpha \to T'$ is surjective. Then, by the construction of $T'$, the geometric generic fiber of $Y_\alpha \to T'$ is irreducible. This completes the proof of our claim.

*Step 3.* We may assume that $Y^\circ$ is non-empty. We claim that we may assume further that $f\colon Y \to T$ is a pluri nodal curve adapted to a dense open subset $Y^{\circ\circ} \subset Y$ with $Y^{\circ\circ} \subset Y^\circ$.

Indeed, by Theorem 3.3.3, there is the following commutative diagram:

$$\begin{array}{ccc} Y_0 & \xrightarrow{f'} & T' \\ \downarrow{\scriptstyle\beta} & & \downarrow \\ Y & \xrightarrow{f} & T, \end{array}$$

where the vertical maps are integral alterations and $f'$ is a pluri nodal curve adapted to a dense open subset $Y_0^{\circ\circ} \subset Y_0$ which is contained in $\beta^{-1}(Y^\circ)$. The generic fiber of $Y_{T'} \to T'$ is irreducible. Let $W_0$ be the disjoint union of $Y_0$ and the irreducible components of $Y_{T'}$ which do not dominate $T'$. Then the natural morphism $W_0 \to Y_{T'}$ is a proper surjective morphism. By the same arguments as in the proof of the previous step, we see that it suffices to prove that there is an alteration $T'' \to T'$ which is $\rho$-adapted to $(f', \beta^{-1}(Y^\circ))$.

*Step 4.* Finally, we complete the proof of the claim (I).

Let $u'\colon Y^{\circ\circ} \hookrightarrow Y$ and $u''\colon Y^{\circ\circ} \hookrightarrow Y^\circ$ denote the open immersions. Let $n$ be a positive integer invertible on $T$ and $\mathcal{L}$ a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $Y^\circ$ such that $u_!\mathcal{L}$ is locally unipotent along the stratification $\mathfrak{Y} = \{Y^\circ, Y\backslash Y^\circ\}$. We have an exact sequence

$$0 \to u'_! u''^* \mathcal{L} \to u_! \mathcal{L} \to \mathcal{G} \to 0.$$

The sheaf $u'_! u''^* \mathcal{L}$ is locally unipotent along the stratification $\{Y^{\circ\circ}, Y\backslash Y^{\circ\circ}\}$. The sheaf $\mathcal{G}$ is supported on $Y\backslash Y^{\circ\circ}$ and the restriction of $\mathcal{G}$ to $Y\backslash Y^{\circ\circ}$ is locally unipotent along the stratification $\{Y^\circ\backslash Y^{\circ\circ}, Y\backslash Y^\circ\}$. By applying $\mathbf{P}(S, \rho, d-1)$ to $Y\backslash Y^{\circ\circ} \to T$ and the stratification $\{Y^\circ\backslash Y^{\circ\circ}, Y\backslash Y^\circ\}$ and using Lemma 3.3.5 (2), we see that $\mathbf{P}_{\mathrm{nd}}(S, \rho, d)$ implies the claim (I) by dévissage.

The proof of Lemma 3.3.12 is now complete. □

Next, we prove $\mathbf{P}_{\mathrm{nd}}(S, \rho, d)$ in Lemma 3.3.12 under the assumptions:

**Lemma 3.3.13.** *We assume that $\boldsymbol{P}(S, \rho, d-1)$, $\boldsymbol{P}(S, \rho-1, *)$, and $\boldsymbol{P}(* \prec S, \rho, *)$ hold. Then the statement $\boldsymbol{P}_{nd}(S, \rho, d)$ in Lemma 3.3.12 is true.*

Proof. Let $u\colon Y^\circ \hookrightarrow Y$ denote the open immersion. Let $T \to S$ be an integral alteration and $f\colon Y \to T$ a pluri nodal curve adapted to a dense open subset $Y^\circ \subset Y$ such that the dimension of the generic fiber of $f$ is less than or equal to $d$. If $f$ is an isomorphism, then there is nothing to prove. Hence we may assume that $f$ is not an isomorphism, and hence there are a factorization

$$Y \xrightarrow{h} X \xrightarrow{g} T, \qquad f = g \circ h$$

and a dense open subset $X^\circ \subset X$ such that $h\colon Y \to X$ is a nodal curve adapted to the pair $(Y^\circ, X^\circ)$ and $g\colon X \to T$ is a pluri nodal curve adapted to $X^\circ$. Since $\mathbf{P}(S, \rho, d-1)$ holds, we may assume that the identity map $T \to T$ is $\rho$-adapted to the following two pairs

$$(Y\backslash Y^\circ \to T, \{Y\backslash Y^\circ\}) \quad \text{and} \quad (g, \{X^\circ, X\backslash X^\circ\}).$$

By replacing $T$ with its normalization, we may assume that $T$ is normal.

We claim that the identity map $T \to T$ is $\rho$-adapted to $(f, Y^\circ)$. The proof is divided into two parts. First, we prove the assertion after restricting to the smooth locus $Y' \subset Y$ of $h$. Then, we prove our claim by using the results on the smooth locus $Y'$.

**Claim 3.3.14.** *Let $a\colon Y' \to T$ denote the restriction of $f$ to $Y'$. Let $n$ be a positive integer invertible on $T$ and $\mathcal{L}$ a locally constant constructible sheaf $\mathcal{L}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $Y^\circ$ such that $u_!\mathcal{L}$ is locally unipotent along the stratification $\mathfrak{Y} := \{Y^\circ, Y\backslash Y^\circ\}$. Let $\mathcal{F}$ be the pull-back of $u_!\mathcal{L}$ to $Y'$. Then the following assertions hold:*

*(1) The nearby cycles for $a\colon Y' \to T$ and $\mathcal{F}$ are $\rho$-compatible with any base change.*
*(2) The nearby cycles for $a\colon Y' \to T$ and $\mathcal{F}$ are $\rho$-unipotent.*

PROOF. (1) We fix a local morphism $q\colon V \to U$ of strictly local schemes over $T$ and an algebraic geometric point $\overline{\xi} \to V$ with image $\overline{\eta} \to U$. In the following, for a morphism $\phi\colon Z \to T$ and a complex $\mathcal{K} \in D^+(Z, \mathbb{Z}/n\mathbb{Z})$, the cone of the base change map

$$q^* R\Psi_{\phi_U,\overline{\eta}}(\mathcal{K}_U) \to R\Psi_{\phi_V,\overline{\xi}}(\mathcal{K}_V)$$

is denoted by $\Delta(\phi, \mathcal{K})$. For a morphism $\phi\colon Z \to W$ of $T$-schemes and a $T$-scheme $T'$, the base change $Z_{T'} \to W_{T'}$ is often denoted by the same letter $\phi$ when there is no possibility of confusion.

We want to show $\tau_{\leq\rho}\Delta(a, \mathcal{F}) = 0$. It suffices to prove that $\tau_{\leq\rho}\Delta(a, \mathcal{F})_x = 0$ at every geometric point $x \to Y'_s$, where $s \in V$ is the closed point. The morphism

$$(q^* R\Psi_{a_U,\overline{\eta}}(\mathcal{F}_U))_x \to R\Psi_{a_V,\overline{\xi}}(\mathcal{F}_V)_x$$

on the stalks induced by the base change map can be identified with the pull-back map

$$R\Gamma((Y'_U)_{(x)} \times_U U_{(\overline{\eta})}, u_!\mathcal{L}) \to R\Gamma((Y'_V)_{(x)} \times_V V_{(\overline{\xi})}, u_!\mathcal{L}).$$

(See also (3.2.1) in Remark 3.2.3.) Since the sheaf $u_!\mathcal{L}$ is locally unipotent along $\mathfrak{Y} = \{Y^\circ, Y\backslash Y^\circ\}$, we may assume that $\mathcal{L} = \Lambda$ is a constant sheaf on $Y^\circ$ by dévissage.

Note that $Y^\circ$ is contained in $Y'$. Since we have the following exact sequence of sheaves on $Y'$

$$0 \to u_!\Lambda \to \Lambda \to v_*\Lambda \to 0,$$

where $v\colon Y'\backslash Y^\circ \hookrightarrow Y'$ is the closed immersion and the open immersion $u\colon Y^\circ \hookrightarrow Y'$ is denoted by the same letter $u$, it suffices to prove that $\tau_{\leq\rho}\Delta(a, \Lambda) = 0$ and $\tau_{\leq\rho}\Delta(a, v_*\Lambda) = 0$.

It follows from the assumption on $T$ that the nearby cycles for $a \circ v$ and the constant sheaf $\Lambda$ are $\rho$-compatible with any base change. Hence we have $\tau_{\leq\rho}\Delta(a, v_*\Lambda) = 0$ by Lemma 3.3.5 (2). By the assumption on $T$ again, the nearby cycles for $g$ and the constant sheaf $\Lambda$ are $\rho$-compatible with any base change. Since the composition $b\colon Y' \hookrightarrow Y \to X$ is smooth, we have $\Delta(a, \Lambda) \cong b^*\Delta(g, \Lambda)$ by the smooth base change theorem. Hence we obtain that

$$\tau_{\leq\rho}\Delta(a, \Lambda) \cong \tau_{\leq\rho} b^*\Delta(g, \Lambda) \cong b^*\tau_{\leq\rho}\Delta(g, \Lambda) = 0.$$

(2) Let $q\colon U \to T$ be a morphism from a strictly local scheme $U$, a point $\eta \in U$, and an algebraic geometric point $\overline{\eta} \to U$ lying above $\eta$. Let $s \in U$ be the closed point. We want to show that the complex

$$\tau_{\leq\rho} R\Psi_{a_U,\overline{\eta}}(\mathcal{F}_U)$$

is $G_{\kappa(\eta)}$-unipotent.

We first claim that, for every $i \leq \rho$, the sheaf $R^i\Psi_{a_U,\overline{\eta}}(\mathcal{F}_U)$ is constructible. Since we have already shown that the nearby cycles for $a$ and $\mathcal{F}$ are $\rho$-compatible with any base change, we may assume that $U$ is the strict localization of $T$ at $s \to T$, in particular, we may assume that $U$ is Noetherian. Then, by using [**EGA II**, Proposition 7.1.9], we may assume that $U$ is the spectrum of strictly Henselian discrete valuation ring, and in this case, the claim follows from [**SGA 4$\frac{1}{2}$**, Th. finitude, Théorème 3.2]. (See also [**100**, Section 8].)

Now, it suffices to prove that, for every geometric point $x \to Y'_s$, the complex

$$\tau_{\leq\rho} R\Psi_{a_U,\overline{\eta}}(\mathcal{F}_U)_x \cong \tau_{\leq\rho} R\Gamma((Y'_U)_{(x)} \times_U U_{(\overline{\eta})}, u_!\mathcal{L})$$

is $G_{\kappa(\eta)}$-unipotent; see [**101**, Lemme 1.2.5]. Since the sheaf $u_!\mathcal{L}$ is locally unipotent along $\mathfrak{Y} = \{Y^\circ, Y\backslash Y^\circ\}$, we reduce to the case where $\mathcal{L} = \Lambda$ is a constant sheaf on $Y^\circ$ by dévissage.

By the exact sequence $0 \to u_!\Lambda \to \Lambda \to v_*\Lambda \to 0$, it suffices to prove that the nearby cycles for $a$ and the sheaf $v_*\Lambda$ (resp. the constant sheaf $\Lambda$) are $\rho$-unipotent. By using the assumption on $T$, we conclude by the same argument as in the proof of (1). □

**Claim 3.3.15.** *Let $n$ be a positive integer invertible on $T$ and $\mathcal{L}$ a locally constant constructible sheaf $\mathcal{L}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $Y^\circ$ such that $\mathcal{F} := u_!\mathcal{L}$ is locally unipotent along the stratification $\mathfrak{Y} = \{Y^\circ, Y\backslash Y^\circ\}$. Then the following assertions hold:*

*(1) The nearby cycles for $f$ and $\mathcal{F}$ are $\rho$-compatible with any base change.*
*(2) The nearby cycles for $f$ and $\mathcal{F}$ are $\rho$-unipotent.*

Proof. (1) We fix a local morphism $q\colon V \to U$ of strictly local schemes over $T$ and an algebraic geometric point $\overline{\xi} \to V$ with image $\overline{\eta} \to U$. We retain the notation of the proof of Claim 3.3.14 (1). We write $\Delta := \Delta(f, \mathcal{F})$. We want to show $\tau_{\leq\rho}\Delta = 0$. Let $c\colon Z \hookrightarrow Y$ be a closed immersion whose complement is the smooth locus $Y'$ of $h$. By Claim 3.3.14 (1), we have

$$\tau_{\leq\rho}\Delta \cong c_*c^*\tau_{\leq\rho}\Delta,$$

and hence it suffices to show that $c^*\tau_{\leq\rho}\Delta = 0$. Since the composition $d\colon Z \to Y \to X$ is a finite morphism, it is enough to prove that

$$d_*c^*\tau_{\leq\rho}\Delta = 0.$$

By using $\tau_{\leq\rho}\Delta \cong c_*c^*\tau_{\leq\rho}\Delta$, we obtain an isomorphism $d_*c^*\tau_{\leq\rho}\Delta \cong \tau_{\leq\rho}Rh_*\Delta$. By the proper base change theorem, we have $Rh_*\Delta \cong \Delta(g, Rh_*\mathcal{F})$. Note that $X^\circ$ is normal since $T$ is normal. Hence the cohomology sheaves of $Rh_*\mathcal{F}$ are locally unipotent along the the stratification $\{X^\circ, X\backslash X^\circ\}$ by Proposition 3.3.1. By the assumption on $T$, we have $\tau_{\leq\rho}\Delta(g, R^ih_*\mathcal{F}) = 0$ for every $i$. It follows that $\tau_{\leq\rho}\Delta(g, Rh_*\mathcal{F}) = 0$. This completes the proof of (1).

(2) Let $q\colon U \to T$ be a morphism from a strictly local scheme $U$, a point $\eta \in U$, and an algebraic geometric point $\overline{\eta} \to U$ lying above $\eta$. We write $\mathcal{K} := R\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U)$. Let $e\colon Y' \to Y$ denote the open immersion. We have the following distinguished triangle:

$$e_!e^*\tau_{\leq\rho}\mathcal{K} \to \tau_{\leq\rho}\mathcal{K} \to c_*c^*\tau_{\leq\rho}\mathcal{K} \to .$$

By Claim 3.3.14 (2), it suffices to prove that $c^*\tau_{\leq\rho}\mathcal{K}$ is $G_{\kappa(\eta)}$-unipotent. Since $d$ is a finite morphism, it suffices to prove that

$$d_*c^*\tau_{\leq\rho}\mathcal{K}$$

is $G_{\kappa(\eta)}$-unipotent. We have the following distinguished triangle:

$$Rh_*e_!e^*\tau_{\leq\rho}\mathcal{K} \to Rh_*\tau_{\leq\rho}\mathcal{K} \to d_*c^*\tau_{\leq\rho}\mathcal{K} \to .$$

Since the complex $Rh_*e_!e^*\tau_{\leq\rho}\mathcal{K}$ is $G_{\kappa(\eta)}$-unipotent by Claim 3.3.14 (2), it is enough to show that $\tau_{\leq\rho}Rh_*\tau_{\leq\rho}\mathcal{K} \cong \tau_{\leq\rho}Rh_*\mathcal{K}$ is $G_{\kappa(\eta)}$-unipotent. By the proper base change theorem, we have

$$Rh_*\mathcal{K} \cong R\Psi_{g_U,\overline{\eta}}((Rh_*\mathcal{F})_U).$$

As above, by Proposition 3.3.1 and the assumption on $T$, it follows that the complex $\tau_{\leq\rho}R\Psi_{g_U,\overline{\eta}}((Rh_*\mathcal{F})_U)$ is $G_{\kappa(\eta)}$-unipotent, whence (2). □

The proof of Lemma 3.3.13 is complete. □

Now Lemma 3.3.11 follows from Lemma 3.3.12 and Lemma 3.3.13. Finally, we prove the following proposition which completes the proof Theorem 3.2.7.

**Proposition 3.3.16.** *For every triple $(S, \rho, d)$ of a Noetherian excellent integral scheme $S$ and two integers $\rho$ and $d$, the statement $\boldsymbol{P}(S, \rho, d)$ holds.*

Proof. We assume that $\mathbf{P}(S, \rho, d)$ does not hold. Then, by Lemma 3.3.10 and Lemma 3.3.11, we can find infinitely many triples $\{(S_n, \rho_n, d_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ with the following properties:

(1) $\mathbf{P}(S_n, \rho_n, d_n)$ does not hold for every $n \in \mathbb{Z}_{\geq 0}$.
(2) $(S_0, \rho_0, d_0) = (S, \rho, d)$.
(3) For every $n \in \mathbb{Z}_{\geq 0}$, we have
  (a) $S_{n+1} \prec S_n$,
  (b) $S_{n+1} = S_n$, $\rho_{n+1} = \rho_n - 1$, and $d_n \geq 0$, or
  (c) $S_{n+1} = S_n$, $\rho_{n+1} = \rho_n$, and $d_{n+1} = d_n - 1 \geq -1$.

By [**101**, Lemme in 3.4.4], there is an integer $N \geq 0$ such that $S_{n+1} = S_n$ for every $n \geq N$. Since $\mathbf{P}(S', -2, d')$ holds trivially for every Noetherian excellent integral scheme $S'$ and every integer $d'$, there is an integer $N' \geq N$ such that $d_{n+1} = d_n - 1 \geq -1$ for every $n \geq N'$. This leads to a contradiction. □

For future reference, we state the following immediate consequence of Theorem 3.2.7 as a corollary.

**Corollary 3.3.17.** *Let $S$ be a Noetherian excellent scheme and $f: X \to S$ a separated morphism of finite type. Then there exists an alteration $S' \to S$ such that, for every positive integer $n$ invertible on $S$, the sliced nearby cycles complexes for $f_{S'}: X_{S'} \to S'$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are compatible with any base change and are unipotent*

Proof. The morphism $f$ has a factorization $f = g \circ u$ where $u: X \hookrightarrow P$ is an open immersion and $g: P \to S$ is a proper morphism. Let $q: U \to S$ be a morphism from a strictly local scheme $U$ and $\overline{\eta} \to U$ an algebraic geometric point. Let $u \in U$ be the closed point. Then the restriction of $R\Psi_{g_U,\overline{\eta}}(\mathbb{Z}/n\mathbb{Z})$ to $X_u$ is isomorphic to $R\Psi_{f_U,\overline{\eta}}(\mathbb{Z}/n\mathbb{Z})$. Thus, by applying Theorem 3.2.7 to $g: P \to S$ and the stratification $\{P\}$ of $P$, we obtain the desired conclusion. □

## 3.4. Tubular neighborhoods and main results

In this section, we will state the main results of Chapter 3.

**3.4.1. Adic spaces and pseudo-adic spaces.** In this chapter, we will freely use the theory of adic spaces and pseudo-adic spaces developed by Huber. Our basis references are [**50, 51, 52**]. We will use the terminology in [**52**, Section 1.1], such as a valuation of a ring, an affinoid ring, a Tate ring, or a strongly Noetherian Tate ring.

An *adic space* is by definition a triple

$$X = (X, \mathcal{O}_X, \{v_x\}_{x\in X})$$

where $X$ is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings on the topological space $X$, and $v_x$ is an equivalence class of valuations of the stalk $\mathcal{O}_{X,x}$ at $x \in X$ which is locally isomorphic to the *affinoid adic space* $\mathrm{Spa}(A, A^+)$ associated with an affinoid ring $(A, A^+)$; see [**52**, Section 1.1] for details. In this chapter, unless stated otherwise, we assume that every adic space is locally isomorphic to the affinoid adic space $\mathrm{Spa}(A, A^+)$ associated with an affinoid ring $(A, A^+)$ such that $A$ is a strongly Noetherian Tate ring. So we can use the results in [**52**]; see [**52**, (1.1.1)]. In particular, we only treat analytic adic spaces; see [**52**, Section 1.1] for the definition of an analytic adic space.

A *pseudo-adic space* is a pair

$$(X, S)$$

where $X$ is an adic space and $S$ is a subset of $X$ satisfying certain conditions; see [**52**, Definition 1.10.3]. If $X$ is an adic space and $S \subset X$ is a locally closed subset, then $(X, S)$ is a pseudo-adic space. Almost all pseudo-adic spaces which appear in this chapter are of this form. A morphism $f\colon (X, S) \to (X', S')$ of pseudo-adic spaces is by definition a morphism $f\colon X \to X'$ of adic spaces with $f(S) \subset S'$.

We have a functor $X \mapsto (X, X)$ from the category of adic spaces to the category of pseudo-adic spaces. We often consider an adic space as a pseudo-adic space via this functor.

A typical example of an adic space is the following. Let $K$ be a non-archimedean field, i.e. it is a topological field whose topology is induced by a valuation $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ of rank 1. We assume that $K$ is complete. Let $\mathcal{O} = K^\circ$ be the valuation ring of $|\cdot|$. We call $\mathcal{O}$ the ring of integers of $K$. Let $\varpi \in K^\times$ be an element with $|\varpi| < 1$. Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}$. The $\varpi$-adic formal completion of $\mathcal{X}$ is denoted by $\widehat{\mathcal{X}}$ or $\mathcal{X}^\wedge$. Following [**52**, Section 1.9], the Raynaud generic fiber of $\widehat{\mathcal{X}}$ is denoted by $d(\widehat{\mathcal{X}})$, which is an adic space of finite type over $\mathrm{Spa}(K, \mathcal{O})$. In particular $d(\widehat{\mathcal{X}})$ is quasi-compact. For example, we have

$$d((\mathrm{Spec}\, \mathcal{O}[T])^\wedge) = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}\langle T\rangle) =: \mathbb{B}(1).$$

We often identify $d((\mathrm{Spec}\, \mathcal{O}[T])^\wedge)$ with $\mathbb{B}(1)$. For a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of schemes of finite type over $\mathcal{O}$, the induced morphism $d(\widehat{\mathcal{Y}}) \to d(\widehat{\mathcal{X}})$ is denoted by $d(f)$ (rather than $d(\widehat{f})$).

Important examples of pseudo-adic spaces for us are tubular neighborhoods of adic spaces. In the next subsection, we will define them in the case where adic spaces are arising from schemes of finite type over $\mathcal{O}$.

**3.4.2. Tubular neighborhoods.** Let $X = (X, \mathcal{O}_X, \{v_x\}_{x \in X})$ be an adic space. Let $U \subset X$ be an open subset and $g \in \mathcal{O}_X(U)$ an element. Following [**52**], for a point $x \in U$, we write $|g(x)| := v_x(g)$. (Strictly speaking, we implicitly choose a valuation from the equivalence class $v_x$.)

As in the previous subsection, let $K$ be a complete non-archimedean field with ring of integers $\mathcal{O}$.

**Proposition 3.4.1.** *Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. Let $\epsilon \in |K^\times|$ be an element.*

*(1) There exist subsets*

$$S(\mathcal{Z}, \epsilon) \subset d(\widehat{\mathcal{X}}) \quad \textit{and} \quad T(\mathcal{Z}, \epsilon) \subset d(\widehat{\mathcal{X}})$$

*satisfying the following properties; for any affine open subset $\mathcal{U} \subset \mathcal{X}$ and any set $\{g_1, \ldots, g_q\} \subset \mathcal{O}_{\mathcal{U}}(\mathcal{U})$ of elements defining the closed subscheme $\mathcal{Z} \cap \mathcal{U}$ of $\mathcal{U}$, we have*

$$\begin{aligned} S(\mathcal{Z}, \epsilon) \cap d(\widehat{\mathcal{U}}) &= \{x \in d(\widehat{\mathcal{U}}) \mid |g_i(x)| < \epsilon \textit{ for every } 1 \leq i \leq q\} \\ &:= \{x \in d(\widehat{\mathcal{U}}) \mid |g_i(x)| < |\varpi(x)| \textit{ for every } 1 \leq i \leq q\} \end{aligned}$$

*and*

$$T(\mathcal{Z},\epsilon)\cap d(\widehat{\mathcal{U}}) = \{x\in d(\widehat{\mathcal{U}})\,|\,|g_i(x)|\leq \epsilon \textit{ for every } 1\leq i\leq q\}$$
$$:= \{x\in d(\widehat{\mathcal{U}})\,|\,|g_i(x)|\leq |\varpi(x)| \textit{ for every } 1\leq i\leq q\},$$

*where* $\varpi\in K^\times$ *is an element with* $\epsilon=|\varpi|$ *and the element in* $\mathcal{O}_{d(\widehat{\mathcal{U}})}(d(\widehat{\mathcal{U}}))$ *arising from* $g_i$ *is denoted by the same letter. Moreover, they are characterized by the above properties.*

(2) *The subset* $T(\mathcal{Z},\epsilon)$ *is a quasi-compact open subset of* $d(\widehat{\mathcal{X}})$*. The subset* $S(\mathcal{Z},\epsilon)$ *is closed and constructible in* $d(\widehat{\mathcal{X}})$*. (See* [**52**, (1.1.13)] *for the definition of a constructible subset.)*

(3) *For a morphism* $f\colon\mathcal{Y}\to\mathcal{X}$ *of finite type, we have*

$$S(\mathcal{Z}\times_{\mathcal{X}}\mathcal{Y},\epsilon)=d(f)^{-1}(S(\mathcal{Z},\epsilon)) \quad \textit{and} \quad T(\mathcal{Z}\times_{\mathcal{X}}\mathcal{Y},\epsilon)=d(f)^{-1}(T(\mathcal{Z},\epsilon)).$$

Proof. (1) Let $\varpi\in K^\times$ be an element with $\epsilon=|\varpi|$. Let $\mathcal{U}\subset\mathcal{X}$ be an affine open subset. It suffices to show that the subsets

$$\{x\in d(\widehat{\mathcal{U}})\,|\,|g_i(x)|<\epsilon \text{ for every } 1\leq i\leq q\}$$

and

$$\{x\in d(\widehat{\mathcal{U}})\,|\,|g_i(x)|\leq\epsilon \text{ for every } 1\leq i\leq q\}$$

are independent of the choice of a set $\{g_1,\ldots,g_q\}\subset\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ of elements defining the closed subscheme $\mathcal{Z}\cap\mathcal{U}$ of $\mathcal{U}$. Let $\{h_1,\ldots,h_r\}\subset\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ be another set of such elements. Then, for every $i$, we have

$$g_i=\Sigma_{1\leq j\leq r}s_{ij}h_j$$

for some elements $\{s_{ij}\}\subset\mathcal{O}_{\mathcal{U}}(\mathcal{U})$. Since we have $|s_{ij}(x)|\leq 1$ for every $x\in d(\widehat{\mathcal{U}})$ and every $s_{ij}$, the assertion follows.

(2) We may assume that $\mathcal{X}$ is affine. The subset $T(\mathcal{Z},\epsilon)$ is a rational subset of the affinoid adic space $d(\widehat{\mathcal{X}})$, and hence it is open and quasi-compact. The subset $S(\mathcal{Z},\epsilon)$ is the complement of the union of finitely many rational subsets. It follows that $S(\mathcal{Z},\epsilon)$ is closed and constructible.

(3) We may assume that $\mathcal{X}$ and $\mathcal{Y}$ are affine. Then the assertion follows from the descriptions given in (1). □

The subsets $T(\mathcal{Z},\epsilon)$ and $S(\mathcal{Z},\epsilon)$ in Proposition 3.4.1 are called an *open tubular neighborhood* and a *closed tubular neighborhood* of $d(\widehat{\mathcal{Z}})$ in $d(\widehat{\mathcal{X}})$, respectively. For an element $\epsilon\in|K^\times|$, we also consider the following subsets:

$$Q(\mathcal{Z},\epsilon):=d(\widehat{\mathcal{X}})\backslash S(\mathcal{Z},\epsilon).$$

This is a quasi-compact open subset of $d(\widehat{\mathcal{X}})$.

For a locally closed subset $S$ of an adic space $X$, the pseudo-adic space $(X,S)$ is often denoted by $S$ for simplicity. For example, the pseudo-adic spaces $(d(\widehat{\mathcal{X}}),S(\mathcal{Z},\epsilon))$ and $(d(\widehat{\mathcal{X}}),T(\mathcal{Z},\epsilon))$ are denoted by $S(\mathcal{Z},\epsilon)$ and $T(\mathcal{Z},\epsilon)$, respectively.

**Remark 3.4.2.** For a formal scheme $\mathscr{X}$ of finite type over Spf $\mathcal{O}$ and a closed immersion $\mathscr{Z}\hookrightarrow\mathscr{X}$ of finite presentation (in the sense of [**43**, Chapter I, Definition 2.2.1]), we can also define tubular neighborhoods of $d(\mathscr{Z})$ in $d(\mathscr{X})$ in the same way. However, we will always work with algebraizable formal schemes of finite type over $\mathcal{O}$ in this chapter.

We end this subsection with the following lemma.

**Lemma 3.4.3.** *Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. For a constructible subset $W \subset d(\widehat{\mathcal{X}})$ containing $d(\widehat{\mathcal{Z}})$, there is an element $\epsilon \in |K^\times|$ such that $T(\mathcal{Z}, \epsilon) \subset W$.*

Proof. We may assume that $\mathcal{X}$ is affine. Then the underlying topological space of $d(\widehat{\mathcal{X}})$ is a spectral space. We have

$$d(\widehat{\mathcal{Z}}) = \bigcap_{\epsilon \in |K^\times|} T(\mathcal{Z}, \epsilon).$$

Hence the intersection

$$\bigcap_{\epsilon \in |K^\times|} T(\mathcal{Z}, \epsilon) \cap (d(\widehat{\mathcal{X}}) \backslash W)$$

is empty. In the constructible topology, the subsets $T(\mathcal{Z}, \epsilon)$ and $d(\widehat{\mathcal{X}}) \backslash W$ are closed, and $d(\widehat{\mathcal{X}})$ is quasi-compact. It follows that there is an element $\epsilon \in |K^\times|$ such that the intersection $T(\mathcal{Z}, \epsilon) \cap d(\widehat{\mathcal{X}}) \backslash W$ is empty, that is $T(\mathcal{Z}, \epsilon) \subset W$. □

**3.4.3. Main results on tubular neighborhoods.** In this subsection, let $K$ be an algebraically closed complete non-archimedean field with ring of integers $\mathcal{O}$.

To state the main results on tubular neighborhoods, we need étale cohomology and étale cohomology with proper support of pseudo-adic spaces. See [**52**, Section 2.3] for definition of the étale site of a pseudo-adic space. As shown in [**52**, Proposition 2.3.7], for an adic space $X$ and an open subset $U \subset X$, the étale topos of the adic space $U$ is naturally equivalent to the étale topos of the pseudo-adic space $(X, U)$. For a commutative ring $\Lambda$, let $D^+(X, \Lambda)$ denote the derived category of bounded below complexes of étale sheaves of $\Lambda$-modules on a pseudo-adic space $X$.

Let $f : X \to Y$ be a morphism of analytic pseudo-adic spaces. We assume that $f$ is separated, locally of finite type, and *taut*. (See [**52**, Definition 5.1.2] for the definitions of a taut pseudo-adic space and a taut morphism of pseudo-adic spaces. For example, if $f$ is separated and quasi-compact, then $f$ is taut.) For such a morphism $f$, the direct image functor with proper support

$$Rf_! : D^+(X, \Lambda) \to D^+(Y, \Lambda)$$

is defined in [**52**, Definition 5.4.4], where $\Lambda$ is a torsion commutative ring. Moreover, if $Y = \mathrm{Spa}(K, \mathcal{O})$, we obtain for a complex $\mathcal{K} \in D^+(X, \Lambda)$ the cohomology group with proper support

$$H^i_c(X, \mathcal{K}).$$

**Example 3.4.4.** Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation.

(1) The adic spaces $d(\widehat{\mathcal{Z}})$ and $d(\widehat{\mathcal{X}})$ are separated and of finite type over $\mathrm{Spa}(K, \mathcal{O})$. The morphism $d(\widehat{\mathcal{X}}) \backslash d(\widehat{\mathcal{Z}}) \to \mathrm{Spa}(K, \mathcal{O})$ is separated, locally of finite type, and taut; see [**52**, Lemma 5.1.4].
(2) The pseudo-adic spaces $S(\mathcal{Z}, \epsilon)$, $T(\mathcal{Z}, \epsilon)$, and $Q(\mathcal{Z}, \epsilon)$ are separated and of finite type (and hence taut) over $\mathrm{Spa}(K, \mathcal{O})$.
(3) For a subset $S$ of an analytic adic space $X$, the interior of $S$ in $X$ is denoted by $S^\circ$. The morphism $S(\mathcal{Z}, \epsilon)^\circ \to \mathrm{Spa}(K, \mathcal{O})$ is separated, locally of finite type, and taut [**53**, Lemma 1.3 iii)].

Let us recall the following results due to Huber in our setting.

**Theorem 3.4.5 (Huber [53, Theorem 2.5], [54, Theorem 3.6]).** *We assume that $K$ is of characteristic 0. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. Let $n$ be a positive integer invertible in $\mathcal{O}$ and let $\mathcal{F}$ be a constructible étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $d(\widehat{\mathcal{X}})$ in the sense of* [52, Definition 2.7.2].

(1) *There exists an element $\epsilon_0 \in |K^\times|$ such that, for every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$, the following natural maps are isomorphisms for every $i$:*
   (a) $H^i_c(S(\mathcal{Z},\epsilon), \mathcal{F}|_{S(\mathcal{Z},\epsilon)}) \xrightarrow{\cong} H^i_c(d(\widehat{\mathcal{Z}}), \mathcal{F}|_{d(\widehat{\mathcal{Z}})})$.
   (b) $H^i_c(T(\mathcal{Z},\epsilon), \mathcal{F}) \xrightarrow{\cong} H^i_c(T(\mathcal{Z},\epsilon_0), \mathcal{F})$.
   (c) $H^i_c(Q(\mathcal{Z},\epsilon), \mathcal{F}) \xrightarrow{\cong} H^i_c(d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}), \mathcal{F})$.
(2) *We assume further that $\mathcal{F}$ is locally constant. Then there exists an element $\epsilon_0 \in |K^\times|$ such that, for every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$, the restriction maps*

$$H^i(T(\mathcal{Z},\epsilon), \mathcal{F}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon), \mathcal{F}|_{S(\mathcal{Z},\epsilon)}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon)^\circ, \mathcal{F}) \xrightarrow{\cong} H^i(d(\widehat{\mathcal{Z}}), \mathcal{F}|_{d(\widehat{\mathcal{Z}})})$$

*are isomorphisms for every $i$.*

Proof. See [53, Theorem 2.5] for the proof of (1) and a more general result. (See [52, Remark 5.5.11] for the constructions of the natural maps.) See [54, Theorem 3.6] for the proof (2) and a more general result. □

**Remark 3.4.6.** For an algebraically closed complete non-archimedean field $K$ of positive characteristic, an analogous statement to Theorem 3.4.5 (1) is proved in [56, Corollary 5.8].

**Remark 3.4.7.** If the residue field of $\mathcal{O}$ is of positive characteristic $p > 0$, the assumption that $n$ is invertible in $\mathcal{O}$ in Theorem 3.4.5 is essential. For example, the étale cohomology group $H^1(\mathbb{B}(1), \mathbb{Z}/p\mathbb{Z})$ is an infinite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space; see the computations in [5, Remark 6.4.2]. However we have $H^1(\{0\}, \mathbb{Z}/p\mathbb{Z}) = 0$ for the origin $0 \in \mathbb{B}(1)$.

The main objective of this chapter is to prove uniform variants of Theorem 3.4.5 for constant sheaves. The main result on étale cohomology groups with proper support of tubular neighborhoods is as follows.

**Theorem 3.4.8.** *Let $K$ be an algebraically closed complete non-archimedean field with ring of integers $\mathcal{O}$. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. Then there exists an element $\epsilon_0 \in |K^\times|$ such that, for every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$ and for every positive integer $n$ invertible in $\mathcal{O}$, the following natural maps are isomorphisms for every $i$:*

(1) $H^i_c(S(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i_c(d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z})$.
(2) $H^i_c(T(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i_c(T(\mathcal{Z},\epsilon_0), \mathbb{Z}/n\mathbb{Z})$.
(3) $H^i_c(Q(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i_c(d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z})$.

The main result on étale cohomology groups of tubular neighborhoods is as follows.

**Theorem 3.4.9.** *Let $K$ be an algebraically closed complete non-archimedean field with ring of integers $\mathcal{O}$. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. Then there exists an element $\epsilon_0 \in |K^\times|$ such that, for* *every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$ and for every positive integer $n$ invertible in $\mathcal{O}$, the restriction maps*

$$H^i(T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon)^\circ,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z})$$

*are isomorphisms for every $i$.*

**Remark 3.4.10.** In Theorem 3.4.8 and Theorem 3.4.9, the assumption that the closed immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ is of finite presentation is not important in practice. Indeed, if we are only interested in the adic spaces $d(\widehat{\mathcal{Z}})$ and $d(\widehat{\mathcal{X}})$, then by replacing $\mathcal{Z}$ with the closed subscheme $\mathcal{Z}' \hookrightarrow \mathcal{Z}$ defined by the sections killed by a power of a non-zero element in the maximal ideal of $\mathcal{O}$, we can reduce to the case where $\mathcal{Z}$ is flat over $\mathcal{O}$ without changing $d(\widehat{\mathcal{Z}})$. Then $\mathcal{Z}$ is of finite presentation over $\mathcal{O}$ by [**108**, Première partie, Corollaire 3.4.7], and hence $\mathcal{Z} \hookrightarrow \mathcal{X}$ is also of finite presentation.

The proofs of Theorem 3.4.8 and Theorem 3.4.9 will be given in Section 3.7. In the rest of this section, we will restate Theorem 3.4.9 for proper schemes over $K$.

Let $L \subset K$ be a subfield of $K$ which is a complete non-archimedean field with the induced topology. Let $\mathcal{O}_L$ be the ring of integers of $L$. For a scheme $X$ of finite type over $L$, the adic space associated with $X$ is denoted by

$$X^{\mathrm{ad}} := X \times_{\operatorname{Spec} L} \operatorname{Spa}(L,\mathcal{O}_L);$$

see [**51**, Proposition 3.8]. For an adic space $Y$ locally of finite type over $\operatorname{Spa}(L,\mathcal{O}_L)$, we denote by

$$Y_K := Y \times_{\operatorname{Spa}(L,\mathcal{O}_L)} \operatorname{Spa}(K,\mathcal{O})$$

the base change of $Y$ to $\operatorname{Spa}(K,\mathcal{O})$, which exists by [**52**, Proposition 1.2.2].

**Corollary 3.4.11.** *Let $X$ be a proper scheme over $L$ and $Z \hookrightarrow X$ a closed immersion. We have a closed immersion $Z^{\mathrm{ad}} \hookrightarrow X^{\mathrm{ad}}$ of adic spaces over $\operatorname{Spa}(L,\mathcal{O}_L)$. Then, there is a quasi-compact open subset $V$ of $X^{\mathrm{ad}}$ containing $Z^{\mathrm{ad}}$ such that, for every positive integer $n$ invertible in $\mathcal{O}$, the restriction map*

$$H^i(V_K,\mathbb{Z}/n\mathbb{Z}) \to H^i((Z^{\mathrm{ad}})_K,\mathbb{Z}/n\mathbb{Z})$$

*is an isomorphism for every $i$.*

Proof. There exist a proper scheme $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_L$ and a closed immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ such that the base change of it to $\operatorname{Spec} L$ is isomorphic to the closed immersion $Z \hookrightarrow X$ by Nagata's compactification theorem; see [**43**, Chapter II, Theorem F.1.1] for example. As in Remark 3.4.10, we may assume that $\mathcal{Z} \hookrightarrow \mathcal{X}$ is of finite presentation. Let

$$\overline{\mathcal{X}} := \mathcal{X} \times_{\operatorname{Spec}\mathcal{O}_L} \operatorname{Spec}\mathcal{O} \quad \text{and} \quad \overline{\mathcal{Z}} := \mathcal{Z} \times_{\operatorname{Spec}\mathcal{O}_L} \operatorname{Spec}\mathcal{O}$$

denote the fiber products. We have $d(\widehat{\overline{\mathcal{Z}}}) \cong d(\widehat{\mathcal{Z}})_K$ and $d(\widehat{\overline{\mathcal{X}}}) \cong d(\widehat{\mathcal{X}})_K$. For an element $\epsilon \in |L^\times|$, we have $T(\mathcal{Z},\epsilon)_K = T(\overline{\mathcal{Z}},\epsilon)$ in $d(\widehat{\overline{\mathcal{X}}})$. By [**52**, Proposition 1.9.6], we have $d(\widehat{\mathcal{Z}}) = Z^{\mathrm{ad}}$ and $d(\widehat{\mathcal{X}}) = X^{\mathrm{ad}}$. Therefore, the assertion follows from Theorem 3.4.9. □

## 3.5. Étale cohomology with proper support of adic spaces and nearby cycles

In this section, we study the relation between the compatibility of the sliced nearby cycles functors with base change and the bijectivity of specialization maps on stalks of higher direct image sheaves with proper support for adic spaces by using a comparison theorem of Huber [**52**, Theorem 5.7.8].

**3.5.1. Analytic adic spaces associated with formal schemes.** In this subsection, we recall the functor $d(-)$ from a certain category of formal schemes to the category of analytic adic spaces defined in [**52**, Section 1.9].

Following [**52**], for a commutative ring $A$ and an element $s \in A$, let

$$A(s/s)$$

denote the localization $A[1/s]$ equipped with the structure of a Tate ring such that the image $A_0$ of the map $A \to A[1/s]$ is a ring of definition and $sA_0$ is an ideal of definition.

We record the following well known results.

**Lemma 3.5.1.** *Let $A$ be a commutative ring endowed with the $\varpi$-adic topology for an element $\varpi \in A$ satisfying the following two properties:*

*(i) $A$ is $\varpi$-adically complete, i.e. the following natural map is an isomorphism:*

$$A \to \widehat{A} := \varprojlim_n A/\varpi^n A.$$

*(ii) Let $A\langle X_1, \ldots, X_n\rangle$ be the $\varpi$-adic completion of $A[X_1, \ldots, X_n]$, called the restricted formal power series ring. Then $A\langle X_1, \ldots, X_n\rangle[1/\varpi]$ is Noetherian for every $n \geq 0$.*

*Then the following assertions hold:*

*(1) For every ideal $I \subset A\langle X_1, \ldots, X_n\rangle$, the quotient $A\langle X_1, \ldots, X_n\rangle/I$ is $\varpi$-adically complete.*

*(2) Let $B$ be an $A$-algebra such that the $\varpi$-adic completion $\widehat{B}$ of $B$ is isomorphic to $A\langle X_1, \ldots, X_n\rangle$. Let $I \subset B$ be an ideal. Then, the $\varpi$-adic completion $\widehat{B/I}$ of $B/I$ is isomorphic to $\widehat{B}/I\widehat{B}$.*

*(3) The Tate ring $A(\varpi/\varpi)$ is complete and we have for every $n \geq 0$*

$$A\langle X_1, \ldots, X_n\rangle[1/\varpi] \cong A(\varpi/\varpi)\langle X_1, \ldots, X_n\rangle.$$

*Here $A(\varpi/\varpi)\langle Y_1, \ldots, Y_m\rangle$ is the ring defined in* [**52**, Section 1.1] *for the Tate ring $A(\varpi/\varpi)$. In particular, the Tate ring $A(\varpi/\varpi)$ is strongly Noetherian.*

Proof. See [**43**, Chapter 0, Proposition 8.4.4] for (1). The rest of the proposition is an immediate consequence of (1). We will sketch the proof for the reader's convenience.

(2) By (1), the ring $\widehat{B}/I\widehat{B}$ is $\varpi$-adically complete. Hence we have

$$\widehat{B/I} = \varprojlim_n (B/I)/\varpi^n \cong \varprojlim_n (\widehat{B}/I\widehat{B})/\varpi^n \cong \widehat{B}/I\widehat{B}.$$

(3) Let $A_0$ be the image of the map $A \to A[1/\varpi]$. By (1), the ring $A_0$ is $\varpi$-adically complete, and hence $A(\varpi/\varpi)$ is complete. It is clear from the definitions that

$$A_0\langle Y_1, \ldots, Y_m\rangle[1/\varpi] \cong A(\varpi/\varpi)\langle Y_1, \ldots, Y_m\rangle.$$

Let $N$ be the kernel of the surjection $B := A[X_1, \ldots, X_n] \to A_0[X_1, \ldots, X_n]$. By using (2), we have the following exact sequence:

$$N \otimes_B \widehat{B} \to \widehat{B} \to A_0\langle Y_1, \ldots, Y_m\rangle \to 0.$$

Since $N[1/\varpi] = 0$, we have $(N \otimes_B \widehat{B})[1/\varpi] = 0$, and hence

$$\widehat{B}[1/\varpi] \cong A_0\langle Y_1, \ldots, Y_m\rangle[1/\varpi].$$

This completes the proof of (3). $\square$

Let $\mathcal{C}$ be the category whose objects are formal schemes which are locally isomorphic to $\operatorname{Spf} A$ for an adic ring $A$ with an ideal of definition $\varpi A$ such that the pair $(A, \varpi)$ satisfies the conditions in Lemma 3.5.1. The morphisms in $\mathcal{C}$ are adic morphisms. A formal scheme in $\mathcal{C}$ satisfies the condition (S) in [**52**, Section 1.9] by Lemma 3.5.1 (3). In [**52**, Proposition 1.9.1], Huber defined a functor

$$d(-)$$

from $\mathcal{C}$ to the category of analytic adic spaces. For a formal scheme $\mathscr{X}$ in $\mathcal{C}$, the adic space $d(\mathscr{X})$ is equipped with a morphism of ringed spaces

$$\lambda\colon d(\mathscr{X}) \to \mathscr{X}.$$

This map is called a specialization map. If $A$ and $\varpi \in A$ satisfy the conditions in Lemma 3.5.1, then we have

$$d(\operatorname{Spf} A) = \operatorname{Spa}(A(\varpi/\varpi), A^+)$$

where $A^+$ is the integral closure of $A$ in $A(\varpi/\varpi) = A[1/\varpi]$. The map $\lambda\colon d(\operatorname{Spf} A) \to \operatorname{Spf} A$ sends $x \in d(\operatorname{Spf} A)$ to the prime ideal $\{a \in A \,|\, |a(x)| < 1\} \subset A$. If $f\colon \mathscr{X} \to \mathscr{Y}$ is an adic morphism of formal schemes in $\mathcal{C}$, then the induced morphism $d(f)\colon d(\mathscr{X}) \to d(\mathscr{Y})$ fits into the following commutative diagram:

$$\begin{array}{ccc} d(\mathscr{X}) & \xrightarrow{\lambda} & \mathscr{X} \\ \big\downarrow{\scriptstyle d(f)} & & \big\downarrow{\scriptstyle f} \\ d(\mathscr{Y}) & \xrightarrow{\lambda} & \mathscr{Y}. \end{array}$$

For the sake of completeness, we include a proof of the following result on the compatibility of the functor $d(-)$ with fiber products.

**Proposition 3.5.2.** *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism locally of finite type of formal schemes in $\mathcal{C}$. Let $\mathscr{Z} \to \mathscr{Y}$ be an adic morphism of formal schemes in $\mathcal{C}$. Then the morphism*

$$d(\mathscr{X} \times_{\mathscr{Y}} \mathscr{Z}) \to d(\mathscr{X}) \times_{d(\mathscr{Y})} d(\mathscr{Z})$$

*induced by the universal property of the fiber product is an isomorphism.*

Proof. First, we note that the fiber product $d(\mathscr{X}) \times_{d(\mathscr{Y})} d(\mathscr{Z})$ exists by [**52**, Proposition 1.2.2] since $d(f)\colon d(\mathscr{X}) \to d(\mathscr{Y})$ is locally of finite type.

We may assume that $\mathscr{X} = \operatorname{Spf} A$, $\mathscr{Y} = \operatorname{Spf} B$, and $\mathscr{Z} = \operatorname{Spf} C$ are affine, where $B$ and $C$ satisfy the conditions in Lemma 3.5.1 for some element $\varpi \in B$ and for its image in $C$, respectively. We may assume further that $A$ is of the form $B\langle X_1, \ldots, X_n\rangle/I$. We write $D := A \otimes_B C$. The source of the morphism in question is isomorphic to

$$\operatorname{Spa}((\widehat{D})(\varpi/\varpi), E^+)$$

where $\widehat{D}$ is the $\varpi$-adic completion of $D$ and $E^+$ is the integral closure of $\widehat{D}$ in $(\widehat{D})[1/\varpi]$. On the other hand, the target of the morphism in question is isomorphic to

$$\operatorname{Spa}(D(\varpi/\varpi), F^+)$$

where $F^+$ is the integral closure of $D$ in $D[1/\varpi]$. Let $D_0$ be the image of the map $D \to D[1/\varpi]$. Clearly, the completion of $D(\varpi/\varpi)$ is isomorphic to $(\widehat{D}_0)(\varpi/\varpi)$. By a similar argument as in the proof of Lemma 3.5.1 (3), we have $(\widehat{D})(\varpi/\varpi) \cong (\widehat{D}_0)(\varpi/\varpi)$. This completes the proof of the proposition since the adic spaces associated with an affinoid ring and its completion are naturally isomorphic (see [**51**, Lemma 1.5]). $\square$

A valuation ring $R$ is called a *microbial valuation ring* if the field of fractions $L$ of $R$ admits a topologically nilpotent unit $\varpi$ with respect to the valuation topology; see [**52**, Definition 1.1.4]. We equip $R$ with the valuation topology unless explicitly mentioned otherwise. In this case, the element $\varpi$ is contained in $R$, the ideal $\varpi R$ is an ideal of definition of $R$, and we have $L = R[1/\varpi]$. The completion $\widehat{R}$ of $R$ is also a microbial valuation ring.

Let $R$ be a complete microbial valuation ring. It is well known that

$$R\langle X_1, \ldots, X_n\rangle[1/\varpi] \cong L\langle X_1, \ldots, X_n\rangle$$

is Noetherian for every $n \geq 0$; see [**11**, 5.2.6, Theorem 1]. A formal scheme $\mathscr{X}$ locally of finite type over $\operatorname{Spf} R$ is in the category $\mathcal{C}$.

**3.5.2. Étale cohomology with proper support of adic spaces and nearby cycles.** We shall recall a comparison theorem of Huber. To formulate his result, we need some preparations.

Let $R$ be a microbial valuation ring with field of fractions $L$. We assume that $R$ is a strictly Henselian local ring. Let $\varpi$ be a topologically nilpotent unit in $L$. Let $\overline{L}$ be a separable closure of $L$ and let $\overline{R}$ be the valuation ring of $\overline{L}$ which extends $R$.

We will use the following notation. For a scheme $\mathcal{X}$ over $R$, we write

$$\overline{\mathcal{X}} := \mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \overline{R} \quad \text{and} \quad \mathcal{X}' := \mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R/\varpi R.$$

Let $\eta \in \operatorname{Spec} R$ and $\overline{\eta} \in \operatorname{Spec} \overline{R}$ be the generic points. We define

$$\mathcal{X}_\eta := \mathcal{X} \times_{\operatorname{Spec} R} \eta \quad \text{and} \quad \mathcal{X}_{\overline{\eta}} := \mathcal{X} \times_{\operatorname{Spec} R} \overline{\eta}.$$

The $\varpi$-adic formal completion of a scheme (or a ring) $\mathcal{X}$ over $R$ is denoted by $\widehat{\mathcal{X}}$. Let $s \in \operatorname{Spec} R$ be the closed point and $\mathcal{X}_s$ the special fiber of $\mathcal{X}$. We will use the same notation for morphisms of schemes over $R$ when there is no possibility of confusion.

We write

$$S := \operatorname{Spa}(L, R) = d(\operatorname{Spf} \widehat{R}) \quad \text{and} \quad \overline{S} := \operatorname{Spa}(\overline{L}, \overline{R}) = d(\operatorname{Spf} \widehat{\overline{R}}).$$

Let $t \in S$ and $\overline{t} \in \overline{S}$ be the closed points corresponding to the valuation rings $R$ and $\overline{R}$, respectively. The pseudo-adic space $(\overline{S}, \{\overline{t}\})$ is also denoted by $\overline{t}$. The natural morphism $\xi\colon \overline{t} \to S$ is a geometric point with support $t \in S$ in the sense of [**52**, Definition 2.5.1].

Let $f\colon \mathcal{X} \to \operatorname{Spec} R$ be a separated morphism of finite type of schemes. The induced morphism

$$d(f)\colon d(\widehat{\mathcal{X}}) \to S$$

is separated and of finite type; the separatedness can be checked for example by using Proposition 3.5.2. (We often write $d(f)$ instead of $d(\widehat{f})$.)

There is a natural morphism $d(\widehat{\mathcal{X}}) \to \mathcal{X}_\eta$ of locally ringed spaces; see [**52**, (1.9.4)]. An étale morphism $Y \to \mathcal{X}_\eta$ defines an adic space $d(\widehat{\mathcal{X}}) \times_{\mathcal{X}_\eta} Y$, which is étale over $d(\widehat{\mathcal{X}})$; see [**51**, Proposition 3.8] and [**52**, Corollary 1.7.3 i)]. In this way, we get a morphism of étale sites

$$a\colon d(\widehat{\mathcal{X}})_{\text{ét}} \to (\mathcal{X}_\eta)_{\text{ét}}.$$

Let $\Lambda$ be a torsion commutative ring. Let $\mathcal{F}$ be an étale sheaf of $\Lambda$-modules on $\mathcal{X}$. Let $\mathcal{F}^a$ denote the pull-back of $\mathcal{F}$ by the composition

$$d(\widehat{\mathcal{X}})_{\text{ét}} \xrightarrow{a} (\mathcal{X}_\eta)_{\text{ét}} \to \mathcal{X}_{\text{ét}}.$$

Recall that we have the direct image functor with proper support

$$Rd(f)_! \colon D^+(d(\widehat{\mathcal{X}}), \Lambda) \to D^+(S, \Lambda)$$

for $d(f)$ by [**52**, Definition 5.4.4]. We define $R^n d(f)_! \mathcal{F}^a := H^n(Rd(f)_! \mathcal{F}^a)$. We will describe the stalk

$$(R^n d(f)_! \mathcal{F}^a)_{\bar{t}} := \Gamma(\bar{t}, \xi^* R^n d(f)_! \mathcal{F}^a)$$

at the geometric point $\xi \colon \bar{t} \to S$ in terms of the sliced nearby cycles functor relative to $f$. Recall that we defined the sliced nearby cycles functor

$$R\Psi_{f,\overline{\eta}} := i^* Rj_* j^* \colon D^+(\mathcal{X}, \Lambda) \to D^+(\mathcal{X}_s, \Lambda)$$

in Section 3.2. Here we fix the notation by the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{X}_{\overline{\eta}} & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & \mathcal{X}_s \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\eta} & \longrightarrow & \operatorname{Spec} R & \longleftarrow & s. \end{array}$$

Now we can state the following result due to Huber:

**Theorem 3.5.3 (Huber** [**52**, Theorem 5.7.8]**).** *There is an isomorphism*

$$(R^n d(f)_! \mathcal{F}^a)_{\bar{t}} \cong H^n_c(\mathcal{X}_s, R\Psi_{f,\overline{\eta}}(\mathcal{F}))$$

*for every $n$. This isomorphism is compatible with the natural actions of $G_L$ on both sides.*

PROOF. We recall the construction of the isomorphism since it is a key ingredient in this chapter and the construction will make the compatibility of it with specialization maps clear; see the proof of Proposition 3.5.5.

First, we recall the following fact. Let $\overline{\mathcal{F}}$ be the pull-back of $\mathcal{F}$ to $\overline{\mathcal{X}}$. By [**52**, (1) in the proof of Proposition 4.2.4], we see that the base change map

$$R\Psi_{f,\overline{\eta}}(\mathcal{F}) \to R\Psi_{\overline{f},\overline{\eta}}(\overline{\mathcal{F}}) \tag{3.5.1}$$

is an isomorphism, where $\overline{f} \colon \overline{\mathcal{X}} \to \operatorname{Spec} \overline{R}$ is the base change of $f$.

There is a factorization $f = g \circ u$ where $u \colon \mathcal{X} \hookrightarrow \mathcal{P}$ is an open immersion and $g \colon \mathcal{P} \to \operatorname{Spec} R$ is a proper morphism by Nagata's compactification theorem. By using the valuation criterion [**52**, Corollary 1.3.9], we see that the morphism $d(g) \colon d(\widehat{\mathcal{P}}) \to S$ is proper; see also the proof of [**89**, Lemma 3.5].

Let $q \colon \operatorname{Spec} \overline{R} \to \operatorname{Spec} R$ denote the morphism induced by the inclusion $R \subset \overline{R}$. The base change of it will be denoted by the same letter $q$ when there is no possibility of confusion. We have the following Cartesian diagram:

$$\begin{array}{ccc} d(\widehat{\overline{\mathcal{X}}}) & \xrightarrow{d(\overline{u})} & d(\widehat{\overline{\mathcal{P}}}) \\ \downarrow{\scriptstyle d(q)} & & \downarrow{\scriptstyle d(q)} \\ d(\widehat{\mathcal{X}}) & \xrightarrow{d(u)} & d(\widehat{\mathcal{P}}). \end{array}$$

By [**52**, Proposition 2.5.13 i) and Proposition 2.6.1], we have

$$(R^n d(f)_! \mathcal{F}^a)_{\bar{t}} \cong (R^n d(g)_* d(u)_! \mathcal{F}^a)_{\bar{t}} \cong H^n(d(\widehat{\overline{\mathcal{P}}}), d(q)^* d(u)_! \mathcal{F}^a). \tag{3.5.2}$$

(See [**52**, Section 2.7] for the functor $d(u)_!$.) By [**52**, Proposition 5.2.2 iv)], we have $d(q)^*d(u)_!\mathcal{F}^a \cong d(\overline{u})_!d(q)^*\mathcal{F}^a$. The sheaf $d(q)^*\mathcal{F}^a$ is isomorphic to the pull-back of $\overline{\mathcal{F}}$ by the composition

$$d(\widehat{\overline{\mathcal{X}}})_{\text{ét}} \xrightarrow{a} (\mathcal{X}_{\overline{\eta}})_{\text{ét}} \to (\overline{\mathcal{X}})_{\text{ét}}.$$

Therefore, in view of (3.5.1), we reduce to the case where $R = \overline{R}$. Therefore, in view of (3.5.1), we reduce to the case where $R = \overline{R}$. The construction given below shows that the desired isomorphism is $G_L$-equivariant.

Let $\lambda\colon d(\widehat{\mathcal{X}})_{\text{ét}} \to (\mathcal{X}')_{\text{ét}}$ denote the morphism of sites defined by sending an étale morphism $h\colon Y \to \mathcal{X}'$ to $d(\widetilde{Y}) \to d(\widehat{\mathcal{X}})$ where $\widetilde{Y} \to \widehat{\mathcal{X}}$ is an étale morphism of formal schemes lifting $h$; see [**52**, Lemma 3.5.1]. Similarly, we have a morphism $\lambda\colon d(\widehat{\mathcal{P}})_{\text{ét}} \to (\mathcal{P}')_{\text{ét}}$ of sites. By applying [**52**, Corollary 3.5.11 ii)] to the following diagram

$$\begin{array}{ccc} d(\widehat{\mathcal{X}})_{\text{ét}} & \xrightarrow{d(u)} & d(\widehat{\mathcal{P}})_{\text{ét}} \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ (\mathcal{X}')_{\text{ét}} & \xrightarrow{u'} & (\mathcal{P}')_{\text{ét}}, \end{array}$$

we have $R\lambda_*d(u)_!\mathcal{F}^a \cong u'_!R\lambda_*\mathcal{F}^a$. Moreover, by applying [**52**, Theorem 3.5.13] to the following diagram

$$\begin{array}{ccc} d(\widehat{\mathcal{X}})_{\text{ét}} & \xrightarrow{a} & (\mathcal{X}_\eta)_{\text{ét}} \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle j} \\ (\mathcal{X}')_{\text{ét}} & \xrightarrow{i'} & \mathcal{X}_{\text{ét}}, \end{array}$$

we have an isomorphism $R\lambda_*\mathcal{F}^a \cong i'^*Rj_*j^*\mathcal{F}$. So we have

$$\begin{aligned} H^n(d(\widehat{\mathcal{P}}), d(u)_!\mathcal{F}^a) &\cong H^n(\mathcal{P}', R\lambda_*d(u)_!\mathcal{F}^a) \\ &\cong H^n(\mathcal{P}', u'_!R\lambda_*\mathcal{F}^a) \\ &\cong H^n(\mathcal{P}', u'_!i'^*Rj_*j^*\mathcal{F}). \end{aligned}$$

Together with (3.5.2), we obtain the following isomorphism

$$(R^nd(f)_!\mathcal{F}^a)_{\overline{t}} \cong H^n(\mathcal{P}', u'_!i'^*Rj_*j^*\mathcal{F}). \tag{3.5.3}$$

The proper base change theorem for schemes implies that

$$H^n(\mathcal{P}', u'_!i'^*Rj_*j^*\mathcal{F}) \cong H^n_c(\mathcal{X}_s, R\Psi_{f,\eta}(\mathcal{F})).$$

This isomorphism completes the construction of the desired isomorphism. □

**3.5.3. Specialization maps on the stalks of $Rd(f)_!$.** In this subsection, we work over a complete non-archimedean field $K$ with ring of integers $\mathcal{O}$ for simplicity. We fix a topologically nilpotent unit $\varpi$ in $K$.

For an adic space $X$ over $\mathrm{Spa}(K, \mathcal{O})$, we will use the following notation. For a point $x \in X$, let $k(x)$ be the residue field of the local ring $\mathcal{O}_{X,x}$ and $k(x)^+$ the valuation ring corresponding to the valuation $v_x$. We note that $k(x)^+$ is a microbial valuation ring and the image of $\varpi$, also denoted by $\varpi$, is a topologically nilpotent unit in $k(x)$. For a geometric point $\xi$ of $X$, let $\mathrm{Supp}(\xi) \in X$ denote the support of it.

We recall strict localizations of analytic adic spaces. Let $\xi\colon s \to X$ be a geometric point. The strict localization

$$X(\xi)$$

of $X$ at $\xi$ is defined in [**52**, Section 2.5.11]. It is an adic space over $X$ with an $X$-morphism $s \to X(\xi)$. We write $x := \operatorname{Supp}(\xi)$. By [**52**, Proposition 2.5.13], the strict localization $X(\xi)$ is isomorphic to

$$\operatorname{Spa}(\overline{k}(x), \overline{k}(x)^+)$$

over $X$, where $\overline{k}(x)$ is a separable closure of $k(x)$ and $\overline{k}(x)^+$ is a valuation ring extending $k(x)^+$. A specialization morphism $\xi_1 \to \xi_2$ of geometric points of $X$ is by definition a morphism $X(\xi_1) \to X(\xi_2)$ over $X$, and such a morphism exists if and only if we have

$$\operatorname{Supp}(\xi_2) \in \overline{\{\operatorname{Supp}(\xi_1)\}}.$$

Let $\mathcal{F}$ be an abelian étale sheaf on $X$. A specialization morphism $\xi_1 \to \xi_2$ of geometric points of $X$ induces a mapping

$$\mathcal{F}_{\xi_2} \to \mathcal{F}_{\xi_1}$$

on the stalks in the usual way; see [**52**, (2.5.16)].

**Definition 3.5.4.** Let $X$ be an adic space over $\operatorname{Spa}(K, \mathcal{O})$ (or more generally an analytic adic space). Let $\mathcal{F}$ be an abelian étale sheaf on $X$. For a subset $W \subset X$, we say that $\mathcal{F}$ is *overconvergent* on $W$ if, for every specialization morphism $\xi_1 \to \xi_2$ of geometric points of $X$ whose supports are contained in $W$, the induced map $\mathcal{F}_{\xi_2} \to \mathcal{F}_{\xi_1}$ is bijective.

Let $\mathcal{X}$ be a scheme of finite type over $\operatorname{Spec} \mathcal{O}$. We write $\mathcal{X}_K := \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} K$. For an étale sheaf $\mathcal{F}$ on $\mathcal{X}$, let $\mathcal{F}^a$ denote the pull-back of $\mathcal{F}$ by the composition

$$d(\widehat{\mathcal{X}})_{\text{ét}} \xrightarrow{a} (\mathcal{X}_K)_{\text{ét}} \to \mathcal{X}_{\text{ét}}.$$

Let $\Lambda$ be a torsion commutative ring.

**Proposition 3.5.5.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be separated schemes of finite type over $\mathcal{O}$. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism over $\mathcal{O}$. Let $\mathcal{F}$ be an étale sheaf of $\Lambda$-modules on $\mathcal{X}$. We assume that the sliced nearby cycles complexes for $f$ and $\mathcal{F}$ are compatible with any base change; see Definition 3.2.2 (1). Let $s \in \widehat{\mathcal{Y}}$ be a point. We consider the inverse image $\lambda^{-1}(s)$ under the map*

$$\lambda\colon d(\widehat{\mathcal{Y}}) \to \widehat{\mathcal{Y}}.$$

*Then, the sheaf $R^n d(f)_! \mathcal{F}^a$ is overconvergent on $\lambda^{-1}(s)$ for every $n$.*

Proof. Let $\xi_1 \to \xi_2$ be a specialization morphism of geometric points of $d(\widehat{\mathcal{Y}})$ whose supports are contained in $\lambda^{-1}(s)$. We write $y_m := \operatorname{Supp}(\xi_m)$ $(m = 1, 2)$. Let $\overline{k}(y_m)$ be a separable closure of $k(y_m)$ and let $\overline{k}(y_m)^+$ be a valuation ring extending $k(y_m)^+$. We identify $d(\widehat{\mathcal{Y}})(\xi_m)$ with $\operatorname{Spa}(\overline{k}(y_m), \overline{k}(y_m)^+)$. Let $R_m$ be the completion of $\overline{k}(y_m)^+$ and we put $U_m := \operatorname{Spec} R_m$. The morphism $\operatorname{Spa}(\overline{k}(y_m), \overline{k}(y_m)^+) \to d(\widehat{\mathcal{Y}})$ induces a natural morphism

$$q_m\colon U_m \to \mathcal{Y}$$

over $\operatorname{Spec} \mathcal{O}$ and the specialization morphism $d(\widehat{\mathcal{Y}})(\xi_1) \to d(\widehat{\mathcal{Y}})(\xi_2)$ induces a natural $\mathcal{Y}$-morphism

$$r\colon U_1 \to U_2.$$

By the assumption, we have $q_m(s_m) = s$ for the closed point $s_m \in U_m$, where the image of $s \in \widehat{\mathcal{Y}}$ in $\mathcal{Y}$ is denoted by the same letter. Let $\overline{s} \to \mathcal{Y}$ be an algebraic geometric point lying above $s$ and let $U = \operatorname{Spec} R$ be the strict localization of $\mathcal{Y}$ at $\overline{s}$. There are local $\mathcal{Y}$-morphisms $\widetilde{q}_m \colon U_m \to U$ $(m = 1, 2)$ such that the following diagram commutes:

$$\begin{array}{ccc} U_1 & \xrightarrow{r} & U_2 \\ & {\scriptstyle \widetilde{q}_1}\searrow \quad \swarrow{\scriptstyle \widetilde{q}_2} & \\ & U. & \end{array}$$

We remark that $r$ is not a local morphism if $y_1 \neq y_2$. Let $\eta_m$ be the generic point of $U_m$. Then we have $r(\eta_1) = \eta_2$. We write $\eta := \widetilde{q}_1(\eta_1) = \widetilde{q}_2(\eta_2)$. Let $\overline{\eta} \to U$ denote the algebraic geometric point which is the image of $\eta_2$. We fix the notation by the following commutative diagrams:

$$\begin{array}{ccccccc} \mathcal{X} \times_{\mathcal{Y}} \eta_m & \xrightarrow{j_m} & \mathcal{X} \times_{\mathcal{Y}} U_m & \xleftarrow{i'_m} & \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R_m/\varpi R_m & \longleftarrow & \mathcal{X} \times_{\mathcal{Y}} s_m \\ \downarrow{\scriptstyle \widetilde{q}_m} & & \downarrow{\scriptstyle \widetilde{q}_m} & & \downarrow{\scriptstyle \widetilde{q}_m} & & \downarrow{\scriptstyle \widetilde{q}_m} \\ \mathcal{X} \times_{\mathcal{Y}} U_{(\overline{\eta})} & \xrightarrow{j} & \mathcal{X} \times_{\mathcal{Y}} U & \xleftarrow{i'} & \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R/\varpi R & \longleftarrow & \mathcal{X} \times_{\mathcal{Y}} \overline{s}, \end{array}$$

$$\begin{array}{ccccc} \mathcal{X} \times_{\mathcal{Y}} \eta_1 & \xrightarrow{j_1} & \mathcal{X} \times_{\mathcal{Y}} U_1 & \xleftarrow{i'_1} & \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R_1/\varpi R_1 \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ \mathcal{X} \times_{\mathcal{Y}} \eta_2 & \xrightarrow{j_2} & \mathcal{X} \times_{\mathcal{Y}} U_2 & \xleftarrow{i'_2} & \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R_2/\varpi R_2. \end{array}$$

There is a factorization $f = g \circ u$ where $u \colon \mathcal{X} \hookrightarrow \mathcal{P}$ is an open immersion and $g \colon \mathcal{P} \to \mathcal{Y}$ is a proper morphism by Nagata's compactification theorem. Let

$$u'_m \colon \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R_m/\varpi R_m \hookrightarrow \mathcal{P}'_m := \mathcal{P} \times_{\mathcal{Y}} \operatorname{Spec} R_m/\varpi R_m$$

and

$$u' \colon \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R/\varpi R \hookrightarrow \mathcal{P}' := \mathcal{P} \times_{\mathcal{Y}} \operatorname{Spec} R/\varpi R$$

be the morphisms induced by $u$. Let $\mathcal{F}_m$ be the pull-back of $\mathcal{F}$ to $\mathcal{X} \times_{\mathcal{Y}} U_m$ and $\mathcal{F}_U$ the pull-back of $\mathcal{F}$ to $\mathcal{X} \times_{\mathcal{Y}} U$. For $m = 1, 2$, by the isomorphism (3.5.3) in the proof of Theorem 3.5.3, we have

$$(R^n d(f)_! \mathcal{F}^a)_{\xi_m} \cong H^n(\mathcal{P}'_m, (u'_m)_! i'^{*}_m R j_{m*} j_m^* \mathcal{F}_m).$$

Via these isomorphisms, the morphism $(R^n d(f)_! \mathcal{F}^a)_{\xi_2} \to (R^n d(f)_! \mathcal{F}^a)_{\xi_1}$ can be identified with the composition

$$\begin{aligned} \phi \colon H^n(\mathcal{P}'_2, (u'_2)_! i'^{*}_2 R j_{2*} j_2^* \mathcal{F}_2) &\to H^n(\mathcal{P}'_1, (u'_1)_! r^* i'^{*}_2 R j_{2*} j_2^* \mathcal{F}_2) \\ &\to H^n(\mathcal{P}'_1, (u'_1)_! i'^{*}_1 R j_{1*} j_1^* \mathcal{F}_1). \end{aligned}$$

We have the following commutative diagram:

$$\begin{array}{ccc} H^n(\mathcal{P}'_2, (u'_2)_! i'^{*}_2 R j_{2*} j_2^* \mathcal{F}_2) & \xrightarrow{\qquad \phi \qquad} & H^n(\mathcal{P}'_1, (u'_1)_! i'^{*}_1 R j_{1*} j_1^* \mathcal{F}_1) \\ & {\scriptstyle \phi_2}\nwarrow \quad \nearrow{\scriptstyle \phi_1} & \\ & H^n(\mathcal{P}', u'_! i'^{*} R j_* j^* \mathcal{F}_U). & \end{array}$$

By the proper base change theorem, the map $\phi_m$ is identified with the map

$$H^n_c(\mathcal{X} \times_{\mathcal{Y}} \overline{s}, R\Psi_{f_U, \overline{\eta}}(\mathcal{F}_U)) \to H^n_c(\mathcal{X} \times_{\mathcal{Y}} s_m, R\Psi_{f_{U_m}, \eta_m}(\mathcal{F}_m)).$$

For both $m = 1$ and $m = 2$, this map is an isomorphism by our assumption that the sliced nearby cycles complexes for $f$ and $\mathcal{F}$ are compatible with any base change. Thus $\phi$ is also an isomorphism. The proof of the proposition is complete. $\square$

## 3.6. Local constancy of higher direct images with proper support

In this section, we study local constancy of higher direct images with proper support for generically smooth morphisms of adic spaces whose target is one-dimensional. We will formulate and prove the results not only for constant sheaves, but also for non-constant sheaves satisfying certain conditions related to the sliced nearby cycles functors.

Throughout this section, we fix an algebraically closed complete non-archimedean field $K$ with ring of integers $\mathcal{O}$.

**3.6.1. Tame sheaves on annuli.** In this subsection, we recall two theorems on finite étale coverings on annuli and the punctured disc, which are essentially proved in [**82, 105, 83**]. We do not impose any conditions on the characteristic of $K$. Since we can not directly apply some results there and some results are only stated in the case where the base field is of characteristic 0, we will give proofs in Appendix 3.A.

To state the two theorems, we need some preparations. Recall that we defined $\mathbb{B}(1) = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}\langle T\rangle)$. Let

$$\mathbb{B}(1)^* := \mathbb{B}(1)\backslash\{0\}$$

be the *punctured disc*, where $0 \in \mathbb{B}(1)$ is the $K$-rational point corresponding to $0 \in K$. We fix a valuation $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ of rank 1 such that the topology of $K$ is induced by it. For elements $a, b \in |K^\times|$ with $a \leq b \leq 1$, we define

$$\begin{aligned}\mathbb{B}(a, b) &:= \{x \in \mathbb{B}(1) \,|\, a \leq |T(x)| \leq b\} \\ &:= \{x \in \mathbb{B}(1) \,|\, |\varpi_a(x)| \leq |T(x)| \leq |\varpi_b(x)|\},\end{aligned}$$

which is called an *open annulus*. Here $\varpi_a, \varpi_b \in K^\times$ are elements such that $a = |\varpi_a|$ and $b = |\varpi_b|$. It is a rational subset of $\mathbb{B}(1)$, and hence it is an affinoid open subspace of $\mathbb{B}(1)$.

Let $m$ be a positive integer invertible in $K$. The finite étale morphism $\varphi_m\colon \mathbb{B}(1)^* \to \mathbb{B}(1)^*$ defined by $T \mapsto T^m$ is called a *Kummer covering* of degree $m$. For elements $a, b \in |K^\times|$ with $a \leq b \leq 1$, the restriction

$$\varphi_m\colon \mathbb{B}(a^{1/m}, b^{1/m}) \to \mathbb{B}(a, b)$$

of $\varphi_m$ is also called a Kummer covering of degree $m$. (We often call a morphism of affinoid adic spaces of finite type over $\mathrm{Spa}(K, \mathcal{O})$ a Kummer covering if it is isomorphic to $\varphi_m\colon \mathbb{B}(a^{1/m}, b^{1/m}) \to \mathbb{B}(a, b)$ for $a, b \in |K^\times|$ and some $m$ with $a \leq b \leq 1$.)

In this chapter, we use the following notion of tameness for étale sheaves on one-dimensional smooth adic spaces over $\mathrm{Spa}(K, \mathcal{O})$.

**Definition 3.6.1.** Let $X$ be a one-dimensional smooth adic space over $\mathrm{Spa}(K, \mathcal{O})$. Let $x \in X$ be a point which has a proper generalization in $X$, i.e. there exists a point $x' \in X$ with $x \in \overline{\{x'\}}$ and $x \neq x'$. Let

$$k(x)^{\wedge h+}$$

be the Henselization of the completion of the valuation ring $k(x)^+$ of $x$. Let $L(x)$ be a separable closure of the field of fractions $k(x)^{\wedge h}$ of $k(x)^{\wedge h+}$. It induces a geometric point $\overline{x} \to X$ with support $x$. For an étale sheaf $\mathcal{F}$ on $X$, we say that $\mathcal{F}$ is *tame* at $x \in X$ if the action of

$$\mathrm{Gal}(L(x)/k(x)^{\wedge h})$$

on the stalk $\mathcal{F}_{\overline{x}}$ at the geometric point $\overline{x}$ factors through a finite group $G$ such that $\sharp G$ is invertible in $\mathcal{O}$, where $\sharp G$ denotes the cardinality of $G$.

Now we can formulate the results.

**Theorem 3.6.2 ([82, Theorem 2.2], [83, Theorem 4.11], [105, Theorem 2.4.3]).** *Let $f\colon X \to \mathbb{B}(1)^*$ be a finite étale morphism of adic spaces. There exists an element $\epsilon \in |K^\times|$ with $\epsilon \le 1$ such that, for all $a, b \in |K^\times|$ with $a < b \le \epsilon$, we have*

$$f^{-1}(\mathbb{B}(a,b)) \cong \coprod_{i=1}^{n} \mathbb{B}(c_i, d_i)$$

*for some elements $c_i, d_i \in |K^\times|$ with $c_i < d_i \le 1$ $(1 \le i \le n)$. If $K$ is of characteristic 0, then we can take such an element $\epsilon \in |K^\times|$ so that the restriction*

$$\mathbb{B}(c_i, d_i) \to \mathbb{B}(a,b)$$

*of $f$ to every component $\mathbb{B}(c_i, d_i)$ appearing in the above decomposition is a Kummer covering.*

**Theorem 3.6.3.** *Let $a, b \in |K^\times|$ be elements with $a < b \le 1$. Let $\mathcal{F}$ be a locally constant étale sheaf with finite stalks on $\mathbb{B}(a,b)$. We assume that the sheaf $\mathcal{F}$ is tame at every $x \in \mathbb{B}(a,b)$ having a proper generalization in $\mathbb{B}(a,b)$, in the sense of Definition 3.6.1. Let $t \in |K^\times|$ be an element with $a/b < t^2 < 1$. Then the restriction $\mathcal{F}|_{\mathbb{B}(a/t,tb)}$ of $\mathcal{F}$ to $\mathbb{B}(a/t,tb)$ is trivialized by a Kummer covering $\varphi_m$ of degree $m$, i.e. the pull-back*

$$\varphi_m^*(\mathcal{F}|_{\mathbb{B}(a/t,tb)})$$

*is a constant sheaf. Moreover, we can assume that the degree $m$ is invertible in $\mathcal{O}$.*

We will give proofs of Theorem 3.6.2 and Theorem 3.6.3 in Appendix 3.A.

**Remark 3.6.4.** If $K$ is of characteristic 0, then Theorem 3.6.2 is known as the $p$-adic Riemann existence theorem of Lütkebohmert [82].

**3.6.2. Local constancy of higher direct images with proper support.** As in Section 3.5, we use the following notation. Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}$. We write $\mathcal{X}_K := \mathcal{X} \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec} K$. For an étale sheaf $\mathcal{F}$ on $\mathcal{X}$, we denote by $\mathcal{F}^a$ the pull-back of $\mathcal{F}$ by the composition $d(\widehat{\mathcal{X}})_{\text{ét}} \xrightarrow{a} (\mathcal{X}_K)_{\text{ét}} \to \mathcal{X}_{\text{ét}}$. (See Section 3.5.2 for the morphism $a\colon d(\widehat{\mathcal{X}})_{\text{ét}} \to (\mathcal{X}_K)_{\text{ét}}$.)

Let us introduce the following slightly technical definition.

**Definition 3.6.5.** We consider the following diagram:

$$\begin{array}{ccc} & & \mathcal{Z} \\ & & \downarrow{\scriptstyle \pi} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array}$$

where

- $\mathcal{Y} = \operatorname{Spec} A$ is an integral affine scheme of finite type over $\mathcal{O}$ such that $\mathcal{Y}_K$ is one-dimensional and smooth over $K$,
- $f\colon \mathcal{X} \to \mathcal{Y}$ is a separated morphism of finite type, and
- $\pi\colon \mathcal{Z} \to \mathcal{Y}$ is a proper surjective morphism such that $\mathcal{Z}$ is an integral scheme whose generic fiber $\mathcal{Z}_K$ is smooth over $K$, and the base change $\pi_K\colon \mathcal{Z}_K \to \mathcal{Y}_K$ is a finite morphism.

Let $n$ be a positive integer invertible in $\mathcal{O}$ and $\mathcal{F}$ a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$. We say that $\mathcal{F}$ is *adapted* to the pair $(f, \pi)$ if the following conditions are satisfied:

(1) The étale sheaf $\mathcal{F}^a$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $d(\widehat{\mathcal{X}})$ is constructible in the sense of [**52**, Definition 2.7.2].
(2) The sliced nearby cycles complexes for $f_{\mathcal{Z}}$ and $\mathcal{F}_{\mathcal{Z}}$ are compatible with any base change.
(3) The sliced nearby cycles complexes for $f_{\mathcal{Z}}$ and $\mathcal{F}_{\mathcal{Z}}$ are unipotent.

See Definition 3.2.2 for the terminology used in the conditions (2) and (3). Here we retain the notation of Section 3.2. For example $f_{\mathcal{Z}}$ denotes the base change $f_{\mathcal{Z}}\colon \mathcal{X}\times_{\mathcal{Y}}\mathcal{Z} \to \mathcal{Z}$ of $f$ and $\mathcal{F}_{\mathcal{Z}}$ denotes the pull-back of the sheaf $\mathcal{F}$ to $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.

For the proofs of Theorem 3.4.8 and Theorem 3.4.9, we need the following proposition, which is a consequence of Theorem 3.2.7.

**Proposition 3.6.6.** *Let $\mathcal{Y} = \operatorname{Spec} A$ be an integral affine scheme of finite type over $\mathcal{O}$ such that $\mathcal{Y}_K$ is one-dimensional and smooth over $K$. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a separated morphism of finite presentation. Then, there exists a proper surjective morphism $\pi\colon \mathcal{Z} \to \mathcal{Y}$ as in Definition 3.6.5 such that, for every positive integer $n$ invertible in $\mathcal{O}$, the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on $\mathcal{X}$ is adapted to $(f, \pi)$.*

Proof. Let $p \geq 0$ be the characteristic of the residue field of $\mathcal{O}$. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal $(p)$. We may find a finitely generated $\mathbb{Z}_{(p)}$-subalgebra $A_0$ of $A$ and a separated morphism $f_0\colon \mathcal{X}_0 \to \operatorname{Spec} A_0$ of finite type such that the base change $\mathcal{X}_0 \times_{\operatorname{Spec} A_0} \mathcal{Y}$ is isomorphic to $\mathcal{X}$ over $\mathcal{Y}$. By applying Corollary 3.3.17 to $f_0$, we find an alteration $\pi_0\colon \mathcal{Z}_0 \to \operatorname{Spec} A_0$ satisfying the properties stated there. The base change

$$\pi'\colon \mathcal{Z}' := \mathcal{Z}_0 \times_{\operatorname{Spec} A_0} \mathcal{Y} \to \mathcal{Y}$$

to $\mathcal{Y}$ is generically finite, proper, and surjective. By restricting $\pi'$ to an irreducible component $\mathcal{Z}''$ of $\mathcal{Z}'$ dominating $\mathcal{Y}$, we obtain a morphism $\pi''\colon \mathcal{Z}'' \to \mathcal{Y}$. The scheme $\mathcal{Z}''_K$ is one-dimensional since $\pi''$ is generically finite. It follows that $\pi''_K\colon \mathcal{Z}''_K \to \mathcal{Y}_K$ is finite. Let $h\colon Z \to \mathcal{Z}''_K$ be the normalization of $\mathcal{Z}''_K$. There exists a proper surjective morphism $\mathcal{Z} \to \mathcal{Z}''$ such that $\mathcal{Z}$ is integral and the base change $\mathcal{Z}_K \to \mathcal{Z}''_K$ is isomorphic to $h$. Then we define $\pi$ as the composition

$$\pi\colon \mathcal{Z} \to \mathcal{Z}'' \to \mathcal{Y}.$$

By the construction, the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ is adapted to $(f, \pi)$ for every positive integer $n$ invertible in $\mathcal{O}$. □

By using the results in Section 3.5, we prove the following proposition:

**Proposition 3.6.7.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $\pi\colon \mathcal{Z} \to \mathcal{Y}$ be morphisms as in Definition 3.6.5. We have the following diagram:*

$$\begin{array}{ccc} & & d(\widehat{\mathcal{Z}}) \\ & & \big\downarrow{\scriptstyle d(\pi)} \\ d(\widehat{\mathcal{X}}) & \xrightarrow{d(f)} & d(\widehat{\mathcal{Y}}). \end{array}$$

*Then the following assertions hold:*

(1) *Let $\mathcal{F}$ be an étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to $(f, \pi)$ with $n \in \mathcal{O}^{\times}$. For every $i$, the sheaf*

$$d(\pi)^* R^i d(f)_! \mathcal{F}^a$$

*on $d(\widehat{\mathcal{Z}})$ is tame at every $z \in d(\widehat{\mathcal{Z}})$ having a proper generalization in $d(\widehat{\mathcal{Z}})$, in the sense of Definition 3.6.1.*

(2) *Let $y \in d(\widehat{\mathcal{Y}})$ be a $K$-rational point. There exists an open subset $V \subset d(\widehat{\mathcal{Y}})$ containing $y$ such that, for every étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to $(f, \pi)$ with $n \in \mathcal{O}^\times$, the sheaf $d(\pi)^* R^i d(f)_! \mathcal{F}^a$ is overconvergent on $d(\pi)^{-1}(V)$ for every $i$.*

Proof. For an étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ with $n \in \mathcal{O}^\times$, the pull-back of $\mathcal{F}^a$ by $d(\widehat{\mathcal{X}_{\mathcal{Z}}}) \to d(\widehat{\mathcal{X}})$ is isomorphic to $(\mathcal{F}_{\mathcal{Z}})^a$, and hence, by using the base change theorem [**52**, Theorem 5.4.6] for $Rd(f)_!$, we have

$$d(\pi)^* R^i d(f)_! \mathcal{F}^a \cong R^i d(f_{\mathcal{Z}})_! (\mathcal{F}_{\mathcal{Z}})^a.$$

(1) This is an immediate consequence of Theorem 3.5.3. Indeed, let $z \in d(\widehat{\mathcal{Z}})$ be an element having a proper generalization in $d(\widehat{\mathcal{Z}})$. Let

$$R := k(z)^{\wedge h+}$$

be the Henselization of the completion of the valuation ring $k(z)^+$ of $z$. By [**55**, Corollary 5.4], the residue field of $R$ is algebraically closed. We write $U := \operatorname{Spec} R$. Let $L$ be the field of fractions of $R$. The composite

$$\operatorname{Spa}(L, R) \to \operatorname{Spa}(k(z), k(z)^+) \to d(\widehat{\mathcal{Z}})$$

is induced by a natural morphism $q\colon U \to \mathcal{Z}$ of schemes over $\mathcal{O}$. Let $\overline{L}$ be a separable closure of $L$, which induces a geometric point $\overline{t} \to \operatorname{Spa}(L, R)$ and a geometric point $\overline{z} \to d(\widehat{\mathcal{Z}})$ in the usual way.

Let $\mathcal{F}$ be an étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to $(f, \pi)$ with $n \in \mathcal{O}^\times$. By applying Theorem 3.5.3 to $f_U\colon \mathcal{X}_U \to U$, we have $G_L$-equivariant isomorphisms

$$(R^i d(f_{\mathcal{Z}})_! (\mathcal{F}_{\mathcal{Z}})^a)_{\overline{z}} \cong (R^i d(f_U)_! (\mathcal{F}_U)^a)_{\overline{t}} \cong H^i_c((\mathcal{X}_U)_s, R\Psi_{f_U, \overline{\eta}}(\mathcal{F}_U)),$$

where $s \in U$ is the closed point and $\overline{\eta} = \operatorname{Spec} \overline{L} \to U$ is the algebraic geometric point. By [**52**, Corollary 5.4.8 and Proposition 6.2.1 i)], the left hand side is a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module. Moreover, the action of $G_L$ on it factors through a finite group. Since the complex $R\Psi_{f_U, \overline{\eta}}(\mathcal{F}_U)$ is $G_L$-unipotent and the integer $n$ is invertible in $\mathcal{O}$, it follows that the action of $G_L$ on the right hand side factors through a finite group $G$ such that $\sharp G$ is invertible in $\mathcal{O}$. This proves (1).

(2) Since $\pi\colon \mathcal{Z} \to \mathcal{Y}$ is proper, by [**52**, Proposition 1.9.6], we have $d(\widehat{\mathcal{Z}}) \cong d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}} \mathcal{Z}$, where $d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}} \mathcal{Z}$ is the adic space over $d(\widehat{\mathcal{Y}})$ associated with $d(\widehat{\mathcal{Y}}) \to \mathcal{Y}_K \to \mathcal{Y}$ and $\pi\colon \mathcal{Z} \to \mathcal{Y}$; see [**51**, Proposition 3.8]. Since $\pi_K\colon \mathcal{Z}_K \to \mathcal{Y}_K$ is a finite morphism, it follows that

$$d(\widehat{\mathcal{Z}}) \cong d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}} \mathcal{Z} \cong d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}_K} \mathcal{Z}_K$$

is finite over $d(\widehat{\mathcal{Y}})$. The inverse image

$$d(\pi)^{-1}(y) = \{z_1, \ldots, z_m\}$$

of $y \in d(\widehat{\mathcal{Y}})$ consists of finitely many $K$-rational points. Let

$$\lambda\colon d(\widehat{\mathcal{Z}}) \to \widehat{\mathcal{Z}}$$

be the specialization map associated with the formal scheme $\widehat{\mathcal{Z}}$. Since the inverse image $\lambda^{-1}(\lambda(z_j))$ of $\lambda(z_j)$ is a closed constructible subset of $d(\widehat{\mathcal{Z}})$ and $z_j$ is a $K$-rational point, there exists an open neighborhood $V_j \subset d(\widehat{\mathcal{Z}})$ of $z_j$ with

$$V_j \subset \lambda^{-1}(\lambda(z_j))$$

for every $j$; see Lemma 3.4.3. Since $d(\pi)$ is a finite morphism, there is an open neighborhood $V \subset d(\widehat{\mathcal{Y}})$ of $y$ with $d(\pi)^{-1}(V) \subset \cup_j V_j$. By using Proposition 3.5.5, we see that, for every étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to $(f, \pi)$ with $n \in \mathcal{O}^\times$, the sheaf $R^i d(f_{\mathcal{Z}})_!(\mathcal{F}_{\mathcal{Z}})^a$ is overconvergent on $d(\pi)^{-1}(V)$. □

We need the following finiteness result due to Huber.

**Theorem 3.6.8 (Huber [52, Theorem 6.2.2]).** *Let $Y$ be an adic space over $\mathrm{Spa}(K, \mathcal{O})$. Let $f\colon X \to Y$ be a morphism of adic spaces which is smooth, separated, and quasi-compact. Let $\mathcal{F}$ be a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $X$ with $n \in \mathcal{O}^\times$. Then, the sheaf $R^i f_! \mathcal{F}$ on $Y$ is a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for every $i$.*

Proof. See [**52**, Theorem 6.2.2]. □

**Remark 3.6.9.** The assertion of Theorem 3.6.8 can fail for non-smooth morphisms. See the introduction of [**53**] for details. See also [**56**, Proposition 7.1] for a more general result for smooth, separated, and quasi-compact morphisms of analytic pseudo-adic spaces.

As in the previous sections, we write $\mathbb{B}(1) = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}\langle T\rangle)$. For an element $\epsilon \in |K^\times|$ with $\epsilon \leq 1$, we define

$$\mathbb{B}(\epsilon) := \{x \in \mathbb{B}(1) \mid |T(x)| \leq \epsilon\}$$

and $\mathbb{B}(\epsilon)^* := \mathbb{B}(\epsilon)\backslash\{0\}$. Let $X$ be a one-dimensional adic space of finite type over $\mathrm{Spa}(K, \mathcal{O})$. We define an *open disc* $V \subset X$ as an open subset $V$ of $X$ equipped with an isomorphism

$$\phi\colon \mathbb{B}(1) \cong V$$

over $\mathrm{Spa}(K, \mathcal{O})$. For an open disc $V \subset X$, we write

$$V(\epsilon) := \phi(\mathbb{B}(\epsilon)) \quad \text{and} \quad V(\epsilon)^* := \phi(\mathbb{B}(\epsilon)^*).$$

Similarly, we write

$$V(a, b) := \phi(\mathbb{B}(a, b))$$

for an open annulus $\mathbb{B}(a, b) \subset \mathbb{B}(1)$.

The main result of this section is the following theorem.

**Theorem 3.6.10.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $\pi\colon \mathcal{Z} \to \mathcal{Y}$ be morphisms as in Definition 3.6.5. We assume that there is an open disc $V \subset d(\widehat{\mathcal{Y}})$ such that*

$$d(f)\colon d(\widehat{\mathcal{X}}) \to d(\widehat{\mathcal{Y}})$$

*is smooth over $V(1)^*$. Then there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq 1$ such that, for every étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to $(f, \pi)$ with $n \in \mathcal{O}^\times$, the following two assertions hold:*

*(1) The restriction*

$$(R^i d(f)_! \mathcal{F}^a)|_{V(\epsilon_0)^*}$$

*of $R^i d(f)_! \mathcal{F}^a$ to $V(\epsilon_0)^*$ is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for every $i$.*

*(2) For elements $a, b \in |K^\times|$ with $a < b \leq \epsilon_0$, there exists a composition*

$$h\colon \mathbb{B}(c^{1/m}, d^{1/m}) \xrightarrow{\varphi_m} \mathbb{B}(c, d) \xrightarrow{g} V(a, b)$$

*of a Kummer covering $\varphi_m$ of degree $m$, where $m$ is invertible in $\mathcal{O}$, with a finite Galois étale morphism $g$, such that the pull-back*

$$h^*((R^i d(f)_! \mathcal{F}^a)|_{V(a,b)})$$

*is a constant sheaf associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module for every $i$. If $K$ is of characteristic 0, then we can take $g$ as a Kummer covering. (The morphism $h$ possibly depends on $\mathcal{F}$.)*

Proof. Clearly, the first assertion (1) follows from the second assertion (2). We shall prove (2).

*Step 1.* We may assume that, for the dominant morphism $\pi\colon \mathcal{Z} \to \mathcal{Y}$, the separable closure $k(\mathcal{Y})^{\text{sep}}$ of the function field $k(\mathcal{Y})$ of $\mathcal{Y}$ in the function field of $\mathcal{Z}$ is Galois over $k(\mathcal{Y})$.

Indeed, there is a finite surjective morphism $Z' \to \mathcal{Z}_K$ from an integral scheme $Z'$ which is smooth over $K$ such that the separable closure of the function field $k(\mathcal{Y})$ of $\mathcal{Y}$ in the function field of $Z'$ is Galois over $k(\mathcal{Y})$. There exists a proper surjective morphism $\mathcal{Z}' \to \mathcal{Z}$ such that $\mathcal{Z}'$ is integral and the base change $\mathcal{Z}'_K \to \mathcal{Z}_K$ is isomorphic to $Z' \to \mathcal{Z}_K$. We define $\pi'$ as the composition $\pi'\colon \mathcal{Z}' \to \mathcal{Z} \to \mathcal{Y}$. If a sheaf $\mathcal{F}$ is adapted to $(f, \pi)$, then it is also adapted to $(f, \pi')$. Thus it suffices to prove Theorem 3.6.10 for $(f, \pi')$.

*Step 2.* We will choose an appropriate $\epsilon_0 \in |K^\times|$.

Let $W \to \mathcal{Y}_K$ be the normalization of $\mathcal{Y}_K$ in $k(\mathcal{Y})^{\text{sep}}$. Then the induced morphism $\mathcal{Z}_K \to W$ is finite, radicial, and surjective and there is a dense open subset $U \subset \mathcal{Y}_K$ over which $W \to \mathcal{Y}_K$ is a finite Galois étale morphism. Let

$$W' := d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}_K} W$$

be the adic space over $d(\widehat{\mathcal{Y}})$ associated with $d(\widehat{\mathcal{Y}}) \to \mathcal{Y}_K$ and $W \to \mathcal{Y}_K$. Let $g\colon W' \to d(\widehat{\mathcal{Y}})$ denote the structure morphism. The morphism $d(\pi)$ can be written as the composition of finite morphisms

$$d(\widehat{\mathcal{Z}}) \xrightarrow{\alpha} W' \xrightarrow{g} d(\widehat{\mathcal{Y}}).$$

Let $\epsilon_1 \in |K^\times|$ be an element with $\epsilon_1 \leq 1$ such that $V(\epsilon_1)^* \subset d(\widehat{\mathcal{Y}})$ is mapped into $U$ under the map $d(\widehat{\mathcal{Y}}) \to \mathcal{Y}_K$. Then the restriction

$$g^{-1}(V(\epsilon_1)^*) \to V(\epsilon_1)^*$$

is finite and étale. By Theorem 3.6.2, there exists an element $\epsilon_2 \in |K^\times|$ with $\epsilon_2 \leq \epsilon_1$ such that, for all $a, b \in |K^\times|$ with $a < b \leq \epsilon_2$, we have

$$g^{-1}(V(a, b)) \cong \coprod_{j=1}^{N} \mathbb{B}(c_j, d_j)$$

for some elements $c_j, d_j \in |K^\times|$ with $c_j < d_j \leq 1$ $(1 \leq j \leq N)$. If $K$ is of characteristic 0, then we can take such $\epsilon_2 \in |K^\times|$ so that the restriction

$$\mathbb{B}(c_j, d_j) \to V(a, b)$$

of $g$ to every component $\mathbb{B}(c_j, d_j)$ appearing in the above decomposition is a Kummer covering. By Proposition 3.6.7 (2), there exists an element $\epsilon_3 \in |K^\times|$ with $\epsilon_3 \leq \epsilon_2$ such that, for every étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to $(f, \pi)$ with $n \in \mathcal{O}^\times$, the sheaf $d(\pi)^* R^i d(f)_! \mathcal{F}^a$ is overconvergent on $d(\pi)^{-1}(V(\epsilon_3))$ for every $i$. Let $t \in |K^\times|$ be an element with $t < 1$. Then we put $\epsilon_0 := t\epsilon_3$.

*Step 3.* We shall show that $\epsilon_0$ satisfies the condition.

Indeed, let $\mathcal{F}$ be an étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to $(f, \pi)$ with $n \in \mathcal{O}^\times$. Let $a, b \in |K^\times|$ be elements with $a < b \leq \epsilon_0$. We have $g^{-1}(V(ta, b/t)) \cong \coprod_{i=1}^N \mathbb{B}(c_j, d_j)$ for some elements $c_j, d_j \in |K^\times|$ with $c_j < d_j \leq 1$ $(1 \leq j \leq N)$. We take a component $\mathbb{B}(c_1, d_1)$ of the decomposition. The restriction

$$g \colon \mathbb{B}(c_1, d_1) \to V(ta, b/t)$$

of $g$ is denoted by the same letter. By the construction, it is a finite Galois étale morphism.

We remark that, since the morphism $\mathcal{Z}_K \to W$ is finite, radicial, and surjective, it follows that $\alpha \colon d(\widehat{\mathcal{Z}}) \to W'$ is a homeomorphism and, for every $z \in d(\widehat{\mathcal{Z}})$, the extension $k(\alpha(z))^\wedge \to k(z)^\wedge$ of the completions of the residue fields is a finite purely inseparable extension, and hence the extension $k(\alpha(z))^{\wedge h} \to k(z)^{\wedge h}$ of the Henselizations of these fields is also a finite purely inseparable extension.

Since $d(f) \colon d(\widehat{\mathcal{X}}) \to d(\widehat{\mathcal{Y}})$ is smooth over $V(1)^*$, the sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules

$$\mathcal{G} := g^*((R^i d(f)_! \mathcal{F}^a)|_{V(ta, b/t)})$$

on $\mathbb{B}(c_1, d_1)$ is constructible by Theorem 3.6.8. By the construction, it is overconvergent on $\mathbb{B}(c_1, d_1)$. Therefore, by [**52**, Lemma 2.7.11], the sheaf $\mathcal{G}$ is locally constant. Moreover, by Proposition 3.6.7 (1) and the remark above, the sheaf $\mathcal{G}$ is tame at every $x \in \mathbb{B}(c_1, d_1)$ having a proper generalization in $\mathbb{B}(c_1, d_1)$. We have $g^{-1}(V(a, b)) = \mathbb{B}(c, d)$ for some elements $c, d \in |K^\times|$ with $c_1 < c < d < d_1$. Hence, by Theorem 3.6.3, we conclude that the restriction of $\mathcal{G}$ to $g^{-1}(V(a, b)) = \mathbb{B}(c, d)$ is trivialized by a Kummer covering $\varphi_m \colon \mathbb{B}(c^{1/m}, d^{1/m}) \to \mathbb{B}(c, d)$ with $m \in \mathcal{O}^\times$.

The proof of Theorem 3.6.10 is now complete. □

For an element $\epsilon \in |K^\times|$, we define

$$\mathbb{D}(\epsilon) := \{x \in \mathbb{B}(1) \mid |T(x)| < \epsilon\}.$$

This is a closed constructible subset of $\mathbb{B}(1)$. For later use, we record the following results.

**Lemma 3.6.11.** *Let $n$ be a positive integer invertible in $\mathcal{O}$.*

*(1) Let $a, b \in |K^\times|$ be elements with $a < b \leq 1$. Let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathbb{B}(a, b)$. Assume that there exists a finite étale morphism $h \colon \mathbb{B}(c, d) \to \mathbb{B}(a, b)$ such that $h$ is a composition of finite Galois étale morphisms and the pull-back $h^*\mathcal{F}$ is a constant sheaf. Then the following assertions hold:*

*(a) We have*

$$H^i_c(\mathbb{B}(b)\backslash\mathbb{B}(a), \mathcal{F}|_{\mathbb{B}(b)\backslash\mathbb{B}(a)}) = 0 \quad \text{and} \quad H^i_c(\mathbb{D}(b) \cap \mathbb{B}(a, b), \mathcal{F}|_{\mathbb{D}(b)\cap\mathbb{B}(a,b)}) = 0$$

*for every $i$.*

*(b) The restriction map*

$$H^i(\mathbb{B}(a, b), \mathcal{F}) \to H^i(\mathbb{D}(b)^\circ \cap \mathbb{B}(a, b), \mathcal{F})$$

*is an isomorphism for every $i$, where $\mathbb{D}(b)^\circ$ is the interior of $\mathbb{D}(b)$ in $\mathbb{B}(1)$.*

*(2) Let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathbb{B}(1)^*$. Assume that for all $a, b \in |K^\times|$ with $a < b \leq 1$, there exists a finite étale morphism*

*$h\colon \mathbb{B}(c,d)\to\mathbb{B}(a,b)$ such that $h$ is a composition of finite Galois étale morphisms and the pull-back $h^*(\mathcal{F}|_{\mathbb{B}(a,b)})$ is a constant sheaf. Then we have*

$$H^i_c(\mathbb{D}(1)\backslash\{0\},\mathcal{F}|_{\mathbb{D}(1)\backslash\{0\}})=0$$

*for every $i$.*

Proof. (1) After possibly changing the coordinate function of $\mathbb{B}(c,d)$, we may assume that $h^{-1}(\mathbb{B}(a,a))=\mathbb{B}(c,c)$ and $h^{-1}(\mathbb{B}(b,b))=\mathbb{B}(d,d)$. Then, we have

$$h^{-1}(\mathbb{B}(b)\backslash\mathbb{B}(a))=\mathbb{B}(d)\backslash\mathbb{B}(c)\quad\text{and}\quad h^{-1}(\mathbb{D}(b)\cap\mathbb{B}(a,b))=\mathbb{D}(d)\cap\mathbb{B}(c,d).$$

We shall show the first equality of (a). By the spectral sequence in [**53**, 4.2 ii)] and the fact that $h$ is a composition of finite Galois étale morphisms, it is enough to show that, for a constant sheaf $M$ on $\mathbb{B}(d)\backslash\mathbb{B}(c)$ associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module, we have for every $i$

$$H^i_c(\mathbb{B}(d)\backslash\mathbb{B}(c),M)=0. \tag{3.6.1}$$

This is proved in [**53**, (II) in the proof of Theorem 2.5].

We shall show the second equality of (a). Similarly as above, it suffices to show that, for a constant sheaf $M$ on $\mathbb{D}(d)\cap\mathbb{B}(c,d)$ associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module, we have $H^i_c(\mathbb{D}(d)\cap\mathbb{B}(c,d),M)=0$ for every $i$. But this follows from (3.6.1) since $\mathbb{D}(d)\cap\mathbb{B}(c,d)$ is isomorphic to $\mathbb{B}(d)\backslash\mathbb{B}(c)$ as a pseudo-adic space over $\mathrm{Spa}(K,\mathcal{O})$.

We shall prove (b). By the Čech-to-cohomology spectral sequences and by the fact that $h$ is a composition of finite Galois étale morphisms, it is enough to show that, for a constant sheaf $M$ on $\mathbb{B}(c,d)$ associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module, the restriction map

$$H^i(\mathbb{B}(c,d),M)\to H^i(\mathbb{D}(d)^\circ\cap\mathbb{B}(c,d),M)$$

is an isomorphism for every $i$. Let $t\in|K^\times|$ be an element with $t<1$. Then we have

$$\mathbb{D}(d)^\circ\cap\mathbb{B}(c,d)=\bigcup_{m\in\mathbb{Z}_{>0}}\mathbb{B}(c,t^{1/m}d).$$

(See also [**53**, Lemma 1.3].) Moreover $H^i(\mathbb{B}(c,t^{1/m}d),M)$ is a finite group for every $i$; see [**52**, Proposition 6.1.1]. Therefore, by [**52**, Lemma 3.9.2 i)], we obtain that

$$H^i(\mathbb{D}(d)^\circ\cap\mathbb{B}(c,d),M)\cong\varprojlim_m H^i(\mathbb{B}(c,t^{1/m}d),M)$$

for every $i$. Thus it suffices to prove that, for every $m\in\mathbb{Z}_{>0}$, the restriction map

$$H^i(\mathbb{B}(c,d),M)\to H^i(\mathbb{B}(c,t^{1/m}d),M)$$

is an isomorphism for every $i$. By [**52**, Example 6.1.2], both sides vanish when $i\geq 2$. For $i\leq 1$, the assertion can be proved by using the Kummer sequence. (See the last paragraph of the proof of Theorem 3.6.3 in Appendix 3.A.)

(2) Since

$$\mathbb{D}(1)\backslash\{0\}=\bigcup_{\epsilon\in|K^\times|,\,\epsilon<1}\mathbb{D}(1)\cap\mathbb{B}(\epsilon,1),$$

we have

$$\varinjlim_\epsilon H^i_c(\mathbb{D}(1)\cap\mathbb{B}(\epsilon,1),\mathcal{F}|_{\mathbb{D}(1)\cap\mathbb{B}(\epsilon,1)})\cong H^i_c(\mathbb{D}(1)\backslash\{0\},\mathcal{F}|_{\mathbb{D}(1)\backslash\{0\}})$$

by [**52**, Proposition 5.4.5 ii)]. Therefore, the assertion follows from (1). □

## 3.7. Proofs of Theorem 3.4.8 and Theorem 3.4.9

In this section, we shall prove Theorem 3.4.8 and Theorem 3.4.9. Let $K$ be an algebraically closed complete non-archimedean field with ring of integers $\mathcal{O}$.

The main part of the proofs of Theorem 3.4.8 and Theorem 3.4.9 is contained in the following lemma.

**Lemma 3.7.1.** *Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion defined by one global section $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. Let $f\colon \mathcal{X} \to \operatorname{Spec}\mathcal{O}[T]$ be the morphism defined by $T \mapsto f$, which is also denoted by $f$. We assume that there is an element $\epsilon_1 \in |K^\times|$ with $\epsilon_1 \leq 1$ such that*

$$d(f)\colon d(\widehat{\mathcal{X}}) \to d((\operatorname{Spec}\mathcal{O}[T])^\wedge) = \mathbb{B}(1).$$

*is smooth over $\mathbb{B}(\epsilon_1)^* = \mathbb{B}(\epsilon_1)\backslash\{0\}$. Then, there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq \epsilon_1$, such that the following assertions hold for every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$, every positive integer $n$ invertible in $\mathcal{O}$, and every integer $i$.*

*(1) $H^i_c(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) = 0$ and $H^i_c(T(\mathcal{Z},\epsilon_0)\backslash T(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) = 0$.*
*(2) $H^i(T(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon)^\circ, \mathbb{Z}/n\mathbb{Z})$.*
*(3) $H^i(S(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z})$.*
*(4) $H^i(S(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon)^\circ, \mathbb{Z}/n\mathbb{Z})$.*

We first deduce Theorem 3.4.8 and Theorem 3.4.9 from Lemma 3.7.1. We will prove Lemma 3.7.1 in Section 3.7.3.

**3.7.1. Cohomological descent for analytic adic spaces.** We will recall some results on cohomological descent for analytic pseudo-adic spaces. Our basic references are [**SGA 4 II**, Exposé Vbis] and [**33**, Section 5]. See also [**89**, Section 3].

Let $f\colon Y \to X$ be a morphism of finite type of analytic pseudo-adic spaces. Let

$$\beta\colon Y_\bullet := \operatorname{cosq}_0(Y/X) \to X$$

be the augmented simplicial object in the category of analytic pseudo-adic spaces of finite type over $X$ (this category has finite projective limits by [**52**, Proposition 1.10.6]) defined as in [**33**, (5.1.4)]. So $Y_m$ is the $(m+1)$-times fiber product $Y \times_X \cdots \times_X Y$ for $m \geq 0$. As in [**33**, (5.1.6)–(5.1.8)], one can associate to the étale topoi $(Y_m)^\sim_{\text{ét}}$ $(m \geq 0)$ a topos $(Y_\bullet)^\sim$. Moreover, as in [**33**, (5.1.11)], we have a morphism of topoi

$$(\beta_*, \beta^*)\colon (Y_\bullet)^\sim \to X^\sim_{\text{ét}}$$

from $(Y_\bullet)^\sim$ to the étale topos $X^\sim_{\text{ét}}$ of $X$. We say that $f\colon Y \to X$ is a *morphism of cohomological descent* for torsion abelian étale sheaves if for every torsion abelian étale sheaf $\mathcal{F}$ on $X$, the natural morphism

$$\mathcal{F} \to R\beta_*\beta^*\mathcal{F}$$

in the derived category $D^+(X^\sim_{\text{ét}})$ is an isomorphism. See [**SGA 4 II**, Exposé Vbis, Section 2] for details.

As a consequence of the proper base change theorem for analytic pseudo-adic spaces [**52**, Theorem 4.4.1], we have the following proposition. We formulate it in the generality we need.

**Proposition 3.7.2.** *Let $f\colon Y \to X$ be a morphism of analytic adic spaces which is proper, of finite type, and surjective. Then for every morphism $Z \to X$ of analytic pseudo-adic spaces, the base change $f\colon Y \times_X Z \to Z$ is of cohomological descent for torsion abelian étale sheaves.*

PROOF. First, we note that the fiber product $Y \times_X Z \to Z$ exists by [**52**, Proposition 1.10.6]. By the proper base change theorem for analytic pseudo-adic spaces [**52**, Theorem 4.4.1 (b)], it suffices to prove that, for every geometric point $S \to X$, the base change $Y \times_X S \to S$ is of cohomological descent for torsion abelian étale sheaves. It is enough to show that there exists a section $S \to Y \times_X S$; see [**SGA 4 II**, Exposé Vbis, Proposition 3.3.1] for example. The existence of a section can be easily proved in our case: By the properness of $f$ and [**52**, Corollary 1.3.9], we may assume that $S$ is of rank 1. Then it is well known. □

For future reference, we deduce the following corollaries from Proposition 3.7.2.

**Corollary 3.7.3.** *Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$. Let $\beta_0\colon \mathcal{Y} \to \mathcal{X}$ be a proper surjective morphism. We put $\beta\colon \mathcal{Y}_\bullet = \operatorname{cosq}_0(\mathcal{Y}/\mathcal{X}) \to \mathcal{X}$. Let $Z \subset d(\widehat{\mathcal{X}})$ be a taut locally closed subset. Then we have the following spectral sequence:*

$$E_1^{i,j} = H_c^j(Z_i, \mathbb{Z}/n\mathbb{Z}) \Rightarrow H_c^{i+j}(Z, \mathbb{Z}/n\mathbb{Z}),$$

*where $Z_i$ is the inverse image of $Z$ under the morphism $d(\beta_i)\colon d(\widehat{\mathcal{Y}}_i) \to d(\widehat{\mathcal{X}})$.*

PROOF. By Nagata's compactification theorem, there exists a proper scheme $\mathcal{P}$ over $\operatorname{Spec}\mathcal{O}$ with a dense open immersion $u\colon \mathcal{X} \hookrightarrow \mathcal{P}$ over $\operatorname{Spec}\mathcal{O}$. Moreover, there is the following Cartesian diagram of schemes:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{u'} & \mathcal{Q} \\ \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \beta_0'} \\ \mathcal{X} & \xrightarrow{u} & \mathcal{P} \end{array}$$

where $u'$ is an open immersion and $\beta_0'$ is a proper surjective morphism. The morphism $d(\beta_0')\colon d(\widehat{\mathcal{Q}}) \to d(\widehat{\mathcal{P}})$ is proper, of finite type, and surjective; see [**89**, Lemma 3.5] (although the base field is assumed to be a discrete valuation field in [**89**], the same proof works). We put $\beta'\colon \mathcal{Q}_\bullet = \operatorname{cosq}_0(\mathcal{Q}/\mathcal{P}) \to \mathcal{P}$. We have a taut locally closed embedding $j\colon Z \to d(\widehat{\mathcal{P}})$. Let $j_m\colon Z_m \to d(\widehat{\mathcal{Q}}_m)$ be the pull-back of $j$ by $d(\beta_m')$. Then we have $H_c^i(Z, \mathbb{Z}/n\mathbb{Z}) = H^i(d(\widehat{\mathcal{P}}), j_!\mathbb{Z}/n\mathbb{Z})$ and

$$H_c^i(Z_m, \mathbb{Z}/n\mathbb{Z}) = H^i(d(\widehat{\mathcal{Q}}_m), (j_m)_!\mathbb{Z}/n\mathbb{Z}) = H^i(d(\widehat{\mathcal{Q}}_m), d(\beta_m')^* j_!\mathbb{Z}/n\mathbb{Z}).$$

Therefore the assertion follows from Proposition 3.7.2 and [**SGA 4 II**, Exposé Vbis, Proposition 2.5.5]. □

**Corollary 3.7.4.** *Let $\beta_0\colon Y \to X$ be a morphism of analytic adic spaces which is proper, of finite type, and surjective. We put $\beta\colon Y_\bullet = \operatorname{cosq}_0(Y/X) \to X$. Let $i\colon Z \hookrightarrow W$ be an inclusion of locally closed subsets of $X$. For $m \geq 0$, let $i_m\colon Z_m \hookrightarrow W_m$ be the pull-back of $i$ by $\beta_m\colon Y_m \to X$. Let*

$$R\Gamma(W, \mathbb{Z}/n\mathbb{Z}) \to R\Gamma(Z, \mathbb{Z}/n\mathbb{Z}) \to \mathcal{K} \to$$

*and*

$$R\Gamma(W_m, \mathbb{Z}/n\mathbb{Z}) \to R\Gamma(Z_m, \mathbb{Z}/n\mathbb{Z}) \to \mathcal{K}_m \to$$

*be distinguished triangles. Let* $\rho \geq -1$ *be an integer. If* $\tau_{\leq(\rho-m)}\mathcal{K}_m = 0$ *for every* $0 \leq m \leq \rho+1$*, then we have* $\tau_{\leq\rho}\mathcal{K} = 0$.

PROOF. The assertion can be proved by a similar method used in [**100**, Lemma 4.1]. We shall give a brief sketch here.

Let $\beta'\colon W_\bullet = \mathrm{cosq}_0(W_0/W) \to W$ (resp. $\beta''\colon Z_\bullet = \mathrm{cosq}_0(Z_0/Z) \to Z$) be the base change of $\beta$ to $W$ (resp. to $Z$). The morphism $i$ induces a morphism $i_\bullet\colon (Z_\bullet)^\sim \to (W_\bullet)^\sim$ of topoi. For the sheaf $\mathbb{Z}/n\mathbb{Z}$ on $W$, we have $\mathbb{Z}/n\mathbb{Z} \cong R\beta'_*\beta'^*\mathbb{Z}/n\mathbb{Z}$ by Proposition 3.7.2, and we obtain isomorphisms

$$Ri_*i^*R\beta'_*\beta'^*\mathbb{Z}/n\mathbb{Z} \cong Ri_*R\beta''_*(i_\bullet)^*\beta'^*\mathbb{Z}/n\mathbb{Z} \cong R\beta'_*R(i_\bullet)_*(i_\bullet)^*\beta'^*\mathbb{Z}/n\mathbb{Z}$$

by the proper base change theorem [**52**, Theorem 4.4.1 (b)] and by a spectral sequence as in [**33**, (5.2.7.1)] (see also [**120**, Tag 0D7A]). Thus, by applying $R\beta'_*$ to the following distinguished triangle

$$\beta'^*\mathbb{Z}/n\mathbb{Z} \to R(i_\bullet)_*(i_\bullet)^*\beta'^*\mathbb{Z}/n\mathbb{Z} \to \Delta \to,$$

we have the following distinguished triangle

$$\mathbb{Z}/n\mathbb{Z} \to Ri_*i^*\mathbb{Z}/n\mathbb{Z} \to R\beta'_*\Delta \to .$$

This implies that $R\Gamma((W_\bullet)^\sim, \Delta) \cong R\Gamma(W, R\beta'_*\Delta) \cong \mathcal{K}$. By [**SGA 4 II**, Exposé Vbis, Corollaire 1.3.12], we have $R\Gamma(W_m, \Delta|_{W_m}) \cong \mathcal{K}_m$ for every $m \geq 0$, where $\Delta|_{W_m}$ denotes the restriction of $\Delta$ to $(W_m)^\sim_{\text{ét}}$. Now the assertion follows from the following spectral sequence (cf. [**33**, (5.2.3.2)] and [**120**, Tag 09WJ]):

$$E_1^{p,q} = H^q(W_p, \Delta|_{W_p}) \Rightarrow H^{p+q}((W_\bullet)^\sim, \Delta).$$

□

**3.7.2. Reduction to the key case.** In this subsection, we deduce Theorem 3.4.8 and Theorem 3.4.9 from Lemma 3.7.1. A theorem of de Jong [**28**, Theorem 4.1] will again play a key role.

**Lemma 3.7.5.** *To prove Theorem 3.4.8, it suffices to prove the following statement* $\boldsymbol{P}_c(i)$ *for every integer* $i$.

$\boldsymbol{P}_c(i)$: *Let* $\mathcal{X}$ *be a separated scheme of finite type over* $\mathcal{O}$ *and* $\mathcal{Z} \hookrightarrow \mathcal{X}$ *a closed immersion defined by one global section* $f \in \mathcal{O}_\mathcal{X}(\mathcal{X})$*. Then there exists an element* $\epsilon_0 \in |K^\times|$*, such that for every* $\epsilon \in |K^\times|$ *with* $\epsilon \leq \epsilon_0$ *and every positive integer* $n$ *invertible in* $\mathcal{O}$*, we have*

$$H^j_c(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) = 0 \quad \textit{and} \quad H^j_c(T(\mathcal{Z},\epsilon_0)\backslash T(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) = 0$$

*for every* $j \leq i$.

PROOF. *Step 1.* To prove Theorem 3.4.8, it suffices to prove the following statement $\mathbf{P}'_c(i)$ for every $i$.

$\mathbf{P}'_c(i)$: Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. Then there exists an element $\epsilon_0 \in |K^\times|$, such that for every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$ and every positive integer $n$ invertible in $\mathcal{O}$, we have

$$H^j_c(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{and} \quad H^j_c(T(\mathcal{Z},\epsilon_0)\backslash T(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) = 0$$

for every $j \leq i$.

Indeed, let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. By applying [**52**, Remark 5.5.11 iv)] to the following diagram

$$S(\mathcal{Z}, \epsilon)\backslash d(\widehat{\mathcal{Z}}) \hookrightarrow S(\mathcal{Z}, \epsilon) \hookleftarrow d(\widehat{\mathcal{Z}}),$$

we have the following long exact sequence:

$$\begin{aligned}\cdots \to H^i_c(S(\mathcal{Z}, \epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) \to & H^i_c(S(\mathcal{Z}, \epsilon), \mathbb{Z}/n\mathbb{Z}) \\ \to H^i_c(d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) \to & H^{i+1}_c(S(\mathcal{Z}, \epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) \to \cdots .\end{aligned}$$

We note that $S(\mathcal{Z}, \epsilon)\backslash d(\widehat{\mathcal{Z}}) = (d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}))\backslash Q(\mathcal{Z}, \epsilon)$. Hence we have a similar spectral sequence for the diagram

$$Q(\mathcal{Z}, \epsilon) \hookrightarrow d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}) \hookleftarrow S(\mathcal{Z}, \epsilon)\backslash d(\widehat{\mathcal{Z}}).$$

Moreover, for elements $\epsilon, \epsilon' \in |K^\times|$ with $\epsilon \leq \epsilon'$, we have a similar spectral sequence for the diagram

$$T(\mathcal{Z}, \epsilon) \hookrightarrow T(\mathcal{Z}, \epsilon') \hookleftarrow T(\mathcal{Z}, \epsilon')\backslash T(\mathcal{Z}, \epsilon).$$

By [**52**, Proposition 5.5.8], there exists an integer $N$, which is independent of $n$ and $\epsilon, \epsilon' \in |K^\times|$, such that we have

$$H^i_c(S(\mathcal{Z}, \epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{and} \quad H^i_c(T(\mathcal{Z}, \epsilon')\backslash T(\mathcal{Z}, \epsilon), \mathbb{Z}/n\mathbb{Z}) = 0$$

for every $i \geq N$. Our claim follows from these results.

*Step 2.* We suppose that $\mathbf{P}_c(i)$ holds for every $i$. We will prove $\mathbf{P}'_c(i)$ by induction on $i$. The assertion holds trivially for $i = -1$. We assume that $\mathbf{P}'_c(i_0 - 1)$ holds. First, we claim that, to prove $\mathbf{P}'_c(i_0)$, we may assume that $\mathcal{X}$ is integral.

We may assume that $\mathcal{X}$ is flat over $\mathcal{O}$. Then every irreducible component of $\mathcal{X}$ dominates $\operatorname{Spec}\mathcal{O}$, and hence $\mathcal{X}$ has finitely many irreducible components. Let $\mathcal{X}'$ be the disjoint union of the irreducible components of $\mathcal{X}$. Then $\mathcal{X}' \to \mathcal{X}$ is proper and surjective. By $\mathbf{P}'_c(i_0 - 1)$ and Corollary 3.7.3, it suffices to prove $\mathbf{P}'_c(i_0)$ for $\mathcal{X}'$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{X}'$. By considering each component of $\mathcal{X}'$ separately, our claim follows.

*Step 3.* We assume that $\mathcal{X}$ is integral. We may assume further that $\mathcal{Z}$ is not equal to $\mathcal{X}$. Let $\mathcal{Y} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{Z}$, which is proper and surjective. By $\mathbf{P}'_c(i_0 - 1)$ and Corollary 3.7.3, it suffices to prove $\mathbf{P}'_c(i_0)$ for $\mathcal{Y}$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$. Consequently, to prove $\mathbf{P}'_c(i_0)$, we may assume further that $\mathcal{Z} \hookrightarrow \mathcal{X}$ is locally defined by one function.

Finally, let $\mathcal{X} = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$ be a finite affine covering such that $\mathcal{Z} \cap \mathcal{U}_\alpha \hookrightarrow \mathcal{U}_\alpha$ is defined by one global section in $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_\alpha)$ for every $\alpha \in I$. We have the following spectral sequence by [**52**, Remark 5.5.12 iii)]:

$$E_1^{i,j} = \bigoplus_{J \subset I, \sharp J = -i+1} H^j_c(S(\mathcal{Z}_J, \epsilon)\backslash d(\widehat{\mathcal{Z}}_J), \mathbb{Z}/n\mathbb{Z}) \Rightarrow H^{i+j}_c(S(\mathcal{Z}, \epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}).$$

Here we write $\mathcal{U}_J := \cap_{\alpha \in J} \mathcal{U}_\alpha$ and $\mathcal{Z}_J := \mathcal{Z} \times_{\mathcal{X}} \mathcal{U}_J \hookrightarrow \mathcal{U}_J$. We have a similar spectral sequence for $T(\mathcal{Z}, \epsilon')\backslash T(\mathcal{Z}, \epsilon)$. Since $\mathbf{P}_c(i)$ holds for every $i$ by hypothesis, it follows that $\mathbf{P}'_c(i_0)$ holds for $\mathcal{X}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$. □

**Lemma 3.7.6.** *To prove Theorem 3.4.9, it suffices to prove the following statement* $\boldsymbol{P}(i)$ *for every integer* $i$.

$\boldsymbol{P}(i)$: *Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion defined by one global section $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. We consider the following distinguished triangles:*

$$R\Gamma(T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \to R\Gamma(S(\mathcal{Z},\epsilon)^{\circ},\mathbb{Z}/n\mathbb{Z}) \to \mathcal{K}_1(\epsilon,n) \to,$$

$$R\Gamma(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \to R\Gamma(d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) \to \mathcal{K}_2(\epsilon,n) \to,$$

$$R\Gamma(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \to R\Gamma(S(\mathcal{Z},\epsilon)^{\circ},\mathbb{Z}/n\mathbb{Z}) \to \mathcal{K}_3(\epsilon,n) \to .$$

*Then there exists an element $\epsilon_0 \in |K^{\times}|$, such that for every $\epsilon \in |K^{\times}|$ with $\epsilon \leq \epsilon_0$ and every positive integer $n$ invertible in $\mathcal{O}$, we have $\tau_{\leq i}\mathcal{K}_m(\epsilon,n) = 0$ for every $m \in \{1,2,3\}$.*

Proof. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. By [**52**, Corollary 2.8.3], there exists an integer $N$, which is independent of $\epsilon$ and $n$, such that cohomology groups $H^i(T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z})$, $H^i(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z})$, and $H^i(d(\widehat{\mathcal{Z}}),\epsilon),\mathbb{Z}/n\mathbb{Z})$ vanish for every $i \geq N$. Let $t \in |K^{\times}|$ be an element with $t < 1$. Then we have

$$S(\mathcal{Z},\epsilon)^{\circ} = \bigcup_{m \in \mathbb{Z}_{>0}} T(\mathcal{Z}, t^{1/m}\epsilon);$$

see [**53**, Lemma 1.3]. Therefore, by [**52**, Lemma 3.9.2 i)], the same holds for the cohomology group $H^i(S(\mathcal{Z},\epsilon)^{\circ},\mathbb{Z}/n\mathbb{Z})$.

We can prove the assertion by the same argument as in the proof of Lemma 3.7.5 by using the results remarked above instead of [**52**, Proposition 5.5.8] and by using Corollary 3.7.4 instead of Corollary 3.7.3. □

We will now prove the desired statement:

**Lemma 3.7.7.** *To prove Theorem 3.4.8 and Theorem 3.4.9, it suffices to prove Lemma 3.7.1.*

Proof. We suppose that Lemma 3.7.1 holds. By Lemma 3.7.5 and Lemma 3.7.6, it suffices to prove that $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold for every $i$. Let us show the assertions by induction on $i$. The assertions $\mathbf{P}_c(-1)$ and $\mathbf{P}(-2)$ hold trivially. Assume that $\mathbf{P}_c(i-1)$ and $\mathbf{P}(i-1)$ hold. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion defined by one global section $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. We shall show that $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold for $\mathcal{X}$ and $\mathcal{Z}$. As in the proof of Lemma 3.7.5, we may assume that $\mathcal{X}$ is integral.

First, we prove the assertions in the case where $K$ is of characteristic 0. By [**28**, Theorem 4.1], there is an integral alteration $Y \to \mathcal{X}_K$ such that $Y$ is smooth over $K$. By Nagata's compactification theorem, there is a proper surjective morphism $\mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}_K \cong Y$ over $\mathcal{X}_K$ and $\mathcal{Y}$ is integral. By the induction hypothesis, Corollary 3.7.3, and Corollary 3.7.4, it suffices to prove $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ for $\mathcal{Y}$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$. (As we have already seen in the proof of Corollary 3.7.3, the morphism $d(\widehat{\mathcal{Y}}) \to d(\widehat{\mathcal{X}})$ is proper and surjective.) Therefore, we may assume that $\mathcal{X}_K$ is smooth over $K$. Let

$$f \colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}[T]$$

be the morphism defined by $T \mapsto f$. Since $K$ is of characteristic 0, there is a dense open subset $W \subset \operatorname{Spec} K[T]$ such that $f_K$ is smooth over $W$. It follows from [**52**, Proposition 1.9.6] that there exists an open subset $V \subset \mathbb{B}(1)$ whose complement consists of finitely many $K$-rational points of $\mathbb{B}(1)$ such that $d(f) \colon d(\widehat{\mathcal{X}}) \to \mathbb{B}(1)$ is smooth over $V$. Thus $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold for $\mathcal{X}$ and $\mathcal{Z}$ since we suppose that Lemma 3.7.1 holds.

Let us now suppose that $K$ is of characteristic $p > 0$. As above, let $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}[T]$ be the morphism defined by $T \mapsto f$. If $f$ is not dominant, then $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold trivially for $\mathcal{X}$ and $\mathcal{Z}$. Thus we may assume that $f$ is dominant. By applying [**28**, Theorem 4.1] to the underlying reduced subscheme of $\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}[T]} \operatorname{Spec} K(T^{1/p^\infty})$, where $K(T^{1/p^\infty})$ is the perfection of $K(T)$, we find an alteration

$$g_K\colon Y \to \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}[T]} \operatorname{Spec} K(T^{1/p^N})$$

such that $Y$ is integral and smooth over $K(T^{1/p^N})$ for some integer $N \geq 0$. By Nagata's compactification theorem, there is a proper surjective morphism

$$g\colon \mathcal{Y} \to \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}[T]} \operatorname{Spec} \mathcal{O}[T^{1/p^N}]$$

whose base change to $\operatorname{Spec} K(T^{1/p^N})$ is isomorphic to $g_K$. As above, it suffices to prove $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ for $\mathcal{Y}$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$.

Let $f'$ be the image of $T^{1/p^N} \in \mathcal{O}[T^{1/p^N}]$ in $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$ and let $\mathcal{Z}' \hookrightarrow \mathcal{Y}$ be the closed subscheme defined by $f'$. Then we have $(f')^{p^N} = f$, where the image of $f$ in $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$ is also denoted by $f$. Hence the closed immersion $d(\widehat{\mathcal{Z}'}) \hookrightarrow d((\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y})^\wedge)$ is a homeomorphism and we have

$$S(\mathcal{Z}', \epsilon) = S(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}, \epsilon^{p^N}) \quad \text{and} \quad T(\mathcal{Z}', \epsilon) = T(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}, \epsilon^{p^N})$$

for every $\epsilon \in |K^\times|$. Thus, it suffices to prove that $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold for $\mathcal{Y}$ and $\mathcal{Z}'$. (See [**52**, Proposition 2.3.7].) By the construction, the generic fiber of the morphism $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}[T]$ defined by $T \mapsto f'$ is smooth over $K(T)$. Therefore, as in the case of characteristic 0, Lemma 3.7.1 implies $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ for $\mathcal{Y}$ and $\mathcal{Z}'$. $\square$

**3.7.3. Proof of the key case.** In this subsection, we prove Lemma 3.7.1 and finish the proofs of Theorem 3.4.8 and Theorem 3.4.9.

**Proof of Lemma 3.7.1.** We may assume that $\mathcal{X}$ is flat over $\operatorname{Spec} \mathcal{O}$. Then $\mathcal{X}$ is of finite presentation over $\operatorname{Spec} \mathcal{O}$ by [**108**, Première partie, Corollaire 3.4.7]. By Proposition 3.6.6 and Theorem 3.6.10, there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq \epsilon_1$ such that, for all $a, b \in |K^\times|$ with $a < b \leq \epsilon_0$ and every positive integer $n$ invertible in $\mathcal{O}^\times$, there exists a finite étale morphism

$$h\colon \mathbb{B}(c, d) \to \mathbb{B}(a, b)$$

such that $h$ is a composition of finite Galois étale morphisms and the pull-back

$$h^*((R^i d(f)_! \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(a,b)})$$

is a constant sheaf associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module for every $i$. We shall show that $\epsilon_0$ satisfies the desired properties. We fix a positive integer $n$ invertible in $\mathcal{O}$ and an element $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$.

(1) We have

$$S(\mathcal{Z}, \epsilon) \backslash d(\widehat{\mathcal{Z}}) = d(f)^{-1}(\mathbb{D}(\epsilon) \backslash \{0\}) \quad \text{and} \quad T(\mathcal{Z}, \epsilon_0) \backslash T(\mathcal{Z}, \epsilon) = d(f)^{-1}(\mathbb{B}(\epsilon_0) \backslash \mathbb{B}(\epsilon)).$$

Keeping the base change theorem [**52**, Theorem 5.4.6] for $Rd(f)_!$ in mind, we have the following spectral sequences by [**52**, Remark 5.5.12 i)]:

$$E_2^{i,j} = H_c^i(\mathbb{D}(\epsilon) \backslash \{0\}, (R^j d(f)_! \mathbb{Z}/n\mathbb{Z})|_{\mathbb{D}(\epsilon) \backslash \{0\}}) \Rightarrow H_c^{i+j}(S(\mathcal{Z}, \epsilon) \backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}),$$

$$E_2^{i,j} = H_c^i(\mathbb{B}(\epsilon_0) \backslash \mathbb{B}(\epsilon), (R^j d(f)_! \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon_0) \backslash \mathbb{B}(\epsilon)}) \Rightarrow H_c^{i+j}(T(\mathcal{Z}, \epsilon_0) \backslash T(\mathcal{Z}, \epsilon), \mathbb{Z}/n\mathbb{Z}).$$

Hence the assertion (1) follows from Lemma 3.6.11.

(2) We have

$$T(\mathcal{Z},\epsilon) = d(f)^{-1}(\mathbb{B}(\epsilon)) \quad \text{and} \quad S(\mathcal{Z},\epsilon)^\circ = d(f)^{-1}(\mathbb{D}(\epsilon)^\circ).$$

Be the Leray spectral spectral sequences, it suffices to prove that the restriction map

$$H^i(\mathbb{B}(\epsilon), R^j d(f)_*\mathbb{Z}/n\mathbb{Z}) \to H^i(\mathbb{D}(\epsilon)^\circ, R^j d(f)_*\mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all $i,j$. Let $\epsilon' \in |K^\times|$ be an element with $\epsilon' < \epsilon$. Then $\{\mathbb{B}(\epsilon',\epsilon), \mathbb{D}(\epsilon)^\circ\}$ is an open covering of $\mathbb{B}(\epsilon)$. By the Čech-to-cohomology spectral sequences, it is enough to prove that the restriction map

$$H^i(\mathbb{B}(\epsilon',\epsilon), R^j d(f)_*\mathbb{Z}/n\mathbb{Z}) \to H^i(\mathbb{D}(\epsilon)^\circ \cap \mathbb{B}(\epsilon',\epsilon), R^j d(f)_*\mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all $i,j$.

The inverse image $d(f)^{-1}(\mathbb{B}(\epsilon',\epsilon))$ has finitely many connected components. It is enough to show that, for every connected component $W \subset d(f)^{-1}(\mathbb{B}(\epsilon',\epsilon))$ and the restriction $g\colon W \to \mathbb{B}(\epsilon',\epsilon)$ of $d(f)$, the restriction map

$$H^i(\mathbb{B}(\epsilon',\epsilon), R^j g_*\mathbb{Z}/n\mathbb{Z}) \to H^i(\mathbb{D}(\epsilon)^\circ \cap \mathbb{B}(\epsilon',\epsilon), R^j g_*\mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all $i,j$. The morphism $g$ is of pure dimension $N$ for some integer $N \geq 0$. (See [**52**, Section 1.8] for the definition of the dimension of a morphism of adic spaces.) Since $g$ is smooth and $R^i g_!\mathbb{Z}/n\mathbb{Z}$ is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules, Poincaré duality [**52**, Corollary 7.5.5] implies that

$$R^j g_*(\mathbb{Z}/n\mathbb{Z}(N)) \cong (R^{2N-j} g_!\mathbb{Z}/n\mathbb{Z})^\vee$$

for every $j$, where $(N)$ denotes the Tate twist and $()^\vee$ denotes the $\mathbb{Z}/n\mathbb{Z}$-dual. (Here we use the fact that $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$-module.) The right hand side satisfies the assumption of Lemma 3.6.11 (1), and hence the assertion follows from the lemma.

(3) We have $S(\mathcal{Z},\epsilon) = d(f)^{-1}(\mathbb{D}(\epsilon))$. Let

$$d(f)'\colon S(\mathcal{Z},\epsilon) \to \mathbb{D}(\epsilon)$$

be the base change of $d(f)$ to $\mathbb{D}(\epsilon)$. We write

$$\mathcal{F}_i := R^i d(f)'_*\mathbb{Z}/n\mathbb{Z}.$$

We claim that restriction map $H^i(\mathbb{D}(\epsilon), \mathcal{F}_j) \to H^i(\{0\}, \mathcal{F}_j|_{\{0\}})$ is an isomorphism for all $i,j$. Since we have by [**52**, Example 2.6.2]

$$H^i(\{0\}, \mathcal{F}_j|_{\{0\}}) = \begin{cases} H^j(d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) & (i=0) \\ 0 & (i \neq 0), \end{cases}$$

the claim and the Leray spectral spectral sequence for $d(f)'$ imply the assertion (3).

We prove the claim. Since $\mathbb{D}(\epsilon)$ and $\{0\}$ are proper over $\mathrm{Spa}(K,\mathcal{O})$, we have

$$H^i(\mathbb{D}(\epsilon), \mathcal{F}_j) = H^i_c(\mathbb{D}(\epsilon), \mathcal{F}_j) \quad \text{and} \quad H^i(\{0\}, \mathcal{F}_j|_{\{0\}}) = H^i_c(\{0\}, \mathcal{F}_j|_{\{0\}}),$$

and hence it suffices to prove that $H^i_c(\mathbb{D}(\epsilon)\backslash\{0\}, \mathcal{F}_j|_{\mathbb{D}(\epsilon)\backslash\{0\}}) = 0$ for all $i,j$. Moreover, by [**52**, Proposition 5.4.5 ii)], it suffices to prove that, for any $\epsilon' \in |K^\times|$ with $\epsilon' < \epsilon$, we have $H^i_c(\mathbb{D}(\epsilon) \cap \mathbb{B}(\epsilon',\epsilon), \mathcal{F}_j|_{\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon)}) = 0$ for all $i,j$.

Let $\epsilon' \in |K^\times|$ be an element with $\epsilon' < \epsilon$. Let $W$ and $g$ be as in the proof of (2). Let

$$g'\colon g^{-1}(\mathbb{D}(\epsilon) \cap \mathbb{B}(\epsilon',\epsilon)) \to \mathbb{D}(\epsilon) \cap \mathbb{B}(\epsilon',\epsilon)$$

be the base change of $g$. It suffices to prove that $H^i_c(\mathbb{D}(\epsilon) \cap \mathbb{B}(\epsilon', \epsilon), R^j g'_* \mathbb{Z}/n\mathbb{Z}) = 0$ for all $i, j$. By the base change theorem [**52**, Theorem 5.4.6] for $Rg_!$, we have

$$(R^j g_! \mathbb{Z}/n\mathbb{Z})|_{\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon)} \cong R^j g'_! \mathbb{Z}/n\mathbb{Z}.$$

In particular the right hand side is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. As in the proof of (2), Poincaré duality [**52**, Corollary 7.5.5] for $g'$ then implies that

$$R^j g'_*(\mathbb{Z}/n\mathbb{Z}(N)) \cong (R^{2N-j} g'_! \mathbb{Z}/n\mathbb{Z})^\vee,$$

and the assertion follows from Lemma 3.6.11 (1).

(4) Similarly as above, it suffices to prove that the restriction map

$$H^i(\mathbb{D}(\epsilon), \mathcal{F}_j) \to H^i(\mathbb{D}(\epsilon)^\circ, \mathcal{F}_j)$$

is an isomorphism for all $i, j$. Let $\epsilon' \in |K^\times|$ be an element with $\epsilon' < \epsilon$. As in the proof of (2), it suffices to prove that the restriction map

$$H^i(\mathbb{D}(\epsilon) \cap \mathbb{B}(\epsilon', \epsilon), \mathcal{F}_j) \to H^i(\mathbb{D}(\epsilon)^\circ \cap \mathbb{B}(\epsilon', \epsilon), \mathcal{F}_j)$$

is an isomorphism for all $i, j$. By the proof of (3), the sheaf $\mathcal{F}_j|_{\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon)}$ is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. Hence the assertion follows from the proof of [**54**, Lemma 2.5]. (In [**54**], the characteristic of the base field is always assumed to be 0. However [**54**, Lemma 2.5] holds in positive characteristic without changing the proof.)

The proof of Lemma 3.7.1 is complete. □

The proofs of Theorem 3.4.8 and Theorem 3.4.9 are now complete.

## 3.A. Finite étale coverings of annuli

In this appendix, we give proofs of Theorem 3.6.2 and Theorem 3.6.3. We retain the notation of Section 3.6.1. In particular, we fix an algebraically closed complete non-archimedean field $K$ with ring of integers $\mathcal{O}$. We will follow the methods given in Ramero's paper [**105**].

Following [**105**], we will use the following notation in this appendix. Recall that for a morphism of finite type

$$\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(K, \mathcal{O})$$

from an affinoid adic space associated with a complete affinoid ring $(A, A^+)$, the ring $A^+$ coincides with the ring $A^\circ$ of power-bounded elements of $A$; see [**51**, Lemma 4.4] and [**52**, Section 1.2]. We often omit $A^+$ and abbreviate $\mathrm{Spa}(A, A^+)$ to $\mathrm{Spa}(A)$. If $A$ is reduced, then $A^\circ$ is topologically of finite type over $\mathcal{O}$, i.e. $A^\circ \cong \mathcal{O}\langle T_1, \ldots, T_n\rangle / I$ for some ideal $I \subset \mathcal{O}\langle T_1, \ldots, T_n\rangle$ by [**11**, 6.4.1, Corollary 4]. Let $\mathfrak{m} \subset \mathcal{O}$ be the maximal ideal and let $\kappa := \mathcal{O}/\mathfrak{m}$ be the residue field. The quotient

$$A^\sim := A^\circ / \mathfrak{m} A^\circ$$

is a finitely generated algebra over $\kappa$. We note that the ideal $\mathfrak{m}A^\circ$ coincides with the set of topologically nilpotent elements in $A$. In particular, the ring $A^\sim$ is reduced.

**3.A.1. Open annuli in the unit disc.** We recall some basic properties of open annuli in the unit disc $\mathbb{B}(1) = \mathrm{Spa}(K\langle T\rangle)$.

Let $a, b \in |K^\times|$ be elements with $a \leq b \leq 1$. Recall that we defined

$$\begin{aligned}\mathbb{B}(a,b) &:= \{x \in \mathbb{B}(1) \,|\, a \leq |T(x)| \leq b\}\\ &:= \{x \in \mathbb{B}(1) \,|\, |\varpi_a(x)| \leq |T(x)| \leq |\varpi_b(x)|\},\end{aligned}$$

where $\varpi_a, \varpi_b \in K^\times$ are elements such that $a = |\varpi_a|$ and $b = |\varpi_b|$. We have

$$\mathbb{B}(a,b) \cong \mathrm{Spa}(K\langle T, T_a, T_b\rangle/(T_aT - \varpi_a, T - \varpi_b T_b))$$

as an adic space over $\mathbb{B}(1)$. The adic space $\mathbb{B}(a,b)$ is isomorphic to $\mathbb{B}(a/b,1)$ as an adic space over $\mathrm{Spa}(K)$. We write $A(a,b) := \mathcal{O}_{\mathbb{B}(1)}(\mathbb{B}(a,b))$.

We will focus on the following points of the unit disc $\mathbb{B}(1)$. Let $r \in |K^\times|$ be an element with $r \leq 1$.

- Let
$$\eta(r)^\flat \colon K\langle T\rangle \to |K^\times|, \quad \sum_{i\geq 0} a_iT^i \mapsto \max_{i\geq 0}\{|a_i|r^i\}$$
be the Gauss norm of radius $r$ centered at 0. The corresponding point $\eta(r)^\flat \in \mathbb{B}(1)$ is denoted by the same letter.
- Let $\langle\delta\rangle$ be an infinite cyclic group with generator $\delta$. We equip $|K^\times| \times \langle\delta\rangle$ with a total order such that
$$(s,\delta^m) < (t,\delta^n) \iff s < t, \quad \text{or} \quad s = t \quad \text{and} \quad m > n.$$
So we have $(1,\delta) < (1,1)$ and $(s,1) < (1,\delta)$ for every $s \in |K^\times|$ with $s < 1$. We identify $\delta$ with $(1,\delta)$ and $r$ with $(r,1)$. The valuation
$$\eta(r) \colon K\langle T\rangle \to |K^\times| \times \langle\delta\rangle, \quad \sum_{i\geq 0} a_iT^i \mapsto \max_{i\geq 0}\{|a_i|r^i\delta^i\}$$
gives a point $\eta(r) \in \mathbb{B}(1)$. The point $\eta(r)$ is a specialization of $\eta(r)^\flat$, i.e. we have
$$\eta(r) \in \overline{\{\eta(r)^\flat\}}.$$
- Similarly, if $r < 1$, the valuation
$$\eta(r)' \colon K\langle T\rangle \to |K^\times| \times \langle\delta\rangle, \quad \sum_{i\geq 0} a_iT^i \mapsto \max_{i\geq 0}\{|a_i|r^i\delta^{-i}\}$$
gives a point $\eta(r)' \in \mathbb{B}(1)$. The point $\eta(r)'$ is a specialization of $\eta(r)^\flat$.

We can use the points $\eta(r)$ and $\eta(r)'$ to describe the closure of an annulus $\mathbb{B}(a,b)$.

**Example 3.A.1.** Let $a, b \in |K^\times|$ be elements with $a \leq b \leq 1$.

(1) For an element $r \in |K^\times|$ with $a \leq r \leq b$, we have $\eta(r)^\flat \in \mathbb{B}(a,b)$. If $a < r \leq b$, we have $\eta(r) \in \mathbb{B}(a,b)$. Similarly, if $a \leq r < b$, we have $\eta(r)' \in \mathbb{B}(a,b)$.

(2) We assume that $a \leq b < 1$. Let $\mathbb{B}(a,b)^c$ be the closure of $\mathbb{B}(a,b)$ in $\mathbb{B}(1)$. Then we have $\mathbb{B}(a,b)^c \backslash \mathbb{B}(a,b) = \{\eta(a), \eta(b)'\}$. In particular, the complement $\mathbb{B}(a,b)^c\backslash\mathbb{B}(a,b)$ consists of two points.

(3) We have some kind of converse to (2). We define $\mathbb{D}(1) := \{x \in \mathbb{B}(1) \,|\, |T(x)| < 1\}$, which is a closed subset of $\mathbb{B}(1)$. Let $X \subset \mathbb{B}(1)$ be a connected affinoid open subset contained in $\mathbb{D}(1)$. Let $X^c$ be the closure of $X$ in $\mathbb{B}(1)$. If the complement $X^c\backslash X$ consists of two points, then there exists an isomorphism
$$X \cong \mathbb{B}(a,1)$$

of adic spaces over $\mathrm{Spa}(K)$ for some element $a \in |K^\times|$ with $a \leq 1$. This can be proved by using [**11**, 9.7.2, Theorem 2].

We recall the following example from [**105**], which is useful to study finite étale coverings of $\mathbb{B}(a,b)$.

**Example 3.A.2 ([105, Example 2.1.12]).** We assume that $a < b$. Let

$$\Psi\colon \mathbb{B}(a,b) = \mathrm{Spa}(A(a,b)) \to \mathbb{B}(1) = \mathrm{Spa}(K\langle S\rangle)$$

be the morphism over $\mathrm{Spa}(K)$ defined by the following homomorphism

$$\psi\colon \mathcal{O}\langle S\rangle \to \mathcal{O}\langle T_a, T_b\rangle/(T_aT_b - \varpi_a/\varpi_b) \cong A(a,b)^\circ, \quad S \mapsto T_a + T_b.$$

The homomorphism $\psi$ makes $A(a,b)^\circ$ into a free $\mathcal{O}\langle S\rangle$-module of rank 2.

**Remark 3.A.3.** In the rest of this section, we shall study finite étale coverings of $\mathbb{B}(a,b)$. We recall the following fact from [**52**, Example 1.6.6 ii)], which we will use freely: Let $X$ be an affinoid adic space of finite type over $\mathrm{Spa}(K,\mathcal{O})$. Let $Y \to X$ be a finite étale morphism of adic spaces. Then $Y$ is affinoid and the induced morphism $\mathrm{Spec}\,\mathcal{O}_Y(Y) \to \mathrm{Spec}\,\mathcal{O}_X(X)$ of schemes is finite and étale. This construction gives an equivalence of categories between the category of adic spaces which are finite and étale over $X$ and the category of schemes which are finite and étale over $\mathrm{Spec}\,\mathcal{O}_X(X)$.

In the rest of this subsection, we will give two lemmas about the connected components of a finite étale covering of $\mathbb{B}(a,b)$.

**Lemma 3.A.4.** *We assume that $a < b$. Let $f\colon X \to \mathbb{B}(a,b)$ be a finite étale morphism of adic spaces. For every $t \in |K^\times|$ with $a/b < t^2 < 1$, there exists an element $s_0 \in |K^\times|$ with $t < s_0 \leq 1$ such that every connected component of $f^{-1}(\mathbb{B}(a/s_0, s_0b))$ remains connected after restricting to $\mathbb{B}(a/s, sb)$ for every $s \in |K^\times|$ with $t < s \leq s_0$.*

Proof. The number of the connected components of $f^{-1}(\mathbb{B}(a/s, sb))$ increases with decreasing $s$ and it is bounded above by the degree of $f$ (i.e. the rank of $\mathcal{O}_X(X)$ as an $A(a,b)$-module) for every $s \in |K^\times|$ with $a/b < s^2 < 1$. The assertion follows from these properties. □

**Lemma 3.A.5.** *We assume that $a < b$. Let $f\colon X \to \mathbb{B}(a,b)$ be a finite étale morphism of adic spaces. We write $B := \mathcal{O}_X(X)$ and consider the composition*

$$\mathcal{O}\langle S\rangle \xrightarrow{\psi} A(a,b)^\circ \to B^\circ,$$

*where $\psi$ is the homomorphism defined in Example 3.A.2 and the second homomorphism is the one induced by $f$. Let $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\} \subset \mathrm{Spec}\,B^\circ$ be the set of the prime ideals of $B^\circ$ lying above the maximal ideal $\mathfrak{m}\mathcal{O}\langle S\rangle + S\mathcal{O}\langle S\rangle \subset \mathcal{O}\langle S\rangle$. Then, for every $t \in |K^\times|$ with $a/b < t^2 < 1$, the adic space $f^{-1}(\mathbb{B}(a/t, tb))$ has at least $n$ connected components.*

Proof. This lemma is proved in the proof of [**105**, Theorem 2.4.3]. We recall the arguments for the reader's convenience.

We define $g$ as the composition

$$g\colon X \xrightarrow{f} \mathbb{B}(a,b) \xrightarrow{\Psi} \mathbb{B}(1) = \mathrm{Spa}(K\langle S\rangle).$$

Let $t \in |K^\times|$ be an element with $a/b < t^2 < 1$. We define $\mathbb{B}(t) := \{x \in \mathbb{B}(1) \mid |S(x)| \leq t\}$. Since $\Psi^{-1}(\mathbb{B}(t)) = \mathbb{B}(a/t, tb)$, it is enough to show that $g^{-1}(\mathbb{B}(t))$ has at least $n$ connected components. The ring $\mathcal{O}[[S]]$ is a Henselian local ring. Since $B^\circ$ is a free $\mathcal{O}\langle S\rangle$-module of finite rank by [**105**, Proposition 2.3.5], we have a decomposition

$$B^\circ \otimes_{\mathcal{O}\langle S\rangle} \mathcal{O}[[S]] \cong R_1 \times \cdots \times R_n,$$

where $R_1, \ldots, R_n$ are local rings, which are free $\mathcal{O}[[S]]$-modules of finite rank. Since the natural homomorphism $\mathcal{O}\langle S\rangle = \mathcal{O}_{\mathbb{B}(1)}(\mathbb{B}(1))^\circ \to \mathcal{O}_{\mathbb{B}(t)}(\mathbb{B}(t))^\circ$ factors through $\mathcal{O}\langle S\rangle \to \mathcal{O}[[S]]$, we have

$$g^{-1}(\mathbb{B}(t)) \cong \mathrm{Spa}(B_1) \coprod \cdots \coprod \mathrm{Spa}(B_n),$$

where $B_i := R_i \otimes_{\mathcal{O}[[S]]} \mathcal{O}_{\mathbb{B}(t)}(\mathbb{B}(t))$. This proves our claim since $\mathrm{Spa}(B_i)$ is non-empty for every $i$. □

**3.A.2. Discriminant functions and finite étale coverings of open annuli.** Let $a, b \in |K^\times|$ be elements with $a < b \leq 1$. Let $f\colon X \to \mathbb{B}(a, b)$ be a finite étale morphism of adic spaces. Let us briefly recall the definition of the *discriminant function*

$$\delta_f\colon [-\log b, -\log a] \to \mathbb{R}_{\geq 0}$$

associated with $f$ following [**105**].

Let $\mathcal{O}_X^+$ be the subsheaf of $\mathcal{O}_X$ defined by

$$\mathcal{O}_X^+(U) = \{g \in \mathcal{O}_X(U) \,|\, |g(x)| \leq 1 \text{ for every } x \in U\}$$

for every open subset $U \subset X$. For an element $r \in |K^\times|$ with $a \leq r \leq b$, let

$$\mathscr{A}(r)^\flat := (f_*\mathcal{O}_X^+)_{\eta(r)^\flat}$$

be the stalk of $f_*\mathcal{O}_X^+$ at the point $\eta(r)^\flat$. The maximal ideal of the stalk $\mathcal{O}_{\mathbb{B}(a,b),\eta(r)^\flat}$ at $\eta(r)^\flat$ is zero, in other words, we have $\mathcal{O}_{\mathbb{B}(a,b),\eta(r)^\flat} = k(\eta(r)^\flat)$. Hence there is a natural morphism $k(\eta(r)^\flat)^+ \to \mathscr{A}(r)^\flat$. By applying [**105**, Proposition 2.3.5] to the restriction $f^{-1}(\mathbb{B}(r, r)) \to \mathbb{B}(r, r)$ of $f$, we see that $\mathscr{A}(r)^\flat$ is a free $k(\eta(r)^\flat)^+$-module of finite rank. Then we can define the valuation

$$v_{\eta(r)^\flat}(\mathfrak{d}_f^\flat(r)) \in \mathbb{R}_{>0}$$

of the discriminant $\mathfrak{d}_f^\flat(r) \in k(\eta(r)^\flat)^+$ of $\mathscr{A}(r)^\flat$ over $k(\eta(r)^\flat)^+$ (in the sense of [**105**, Section 2.1]) and we define

$$\delta_f\colon [-\log b, -\log a] \cap -\log |K^\times| \to \mathbb{R}_{\geq 0}, \quad -\log r \mapsto -\log(v_{\eta(r)^\flat}(\mathfrak{d}_f^\flat(r))) \in \mathbb{R}_{\geq 0}.$$

See [**105**, 2.3.12] for details.

**Theorem 3.A.6 (Ramero [105, Theorem 2.3.25]).** *The function $\delta_f$ extends uniquely to a continuous, piecewise linear, and convex function*

$$\delta_f\colon [-\log b, -\log a] \to \mathbb{R}_{\geq 0}.$$

*Moreover, the slopes of $\delta_f$ are integers.*

PROOF. See [**105**, Theorem 2.3.25]. □

Many basic properties of the discriminant function $\delta_f$ were studied in detail in [**105**]. Here, we are interested in the case where $\delta_f$ is linear.

**Proposition 3.A.7 (Ramero [105]).** *Let $f\colon X = \mathrm{Spa}(B, B^\circ) \to \mathbb{B}(a, b)$ be a finite étale morphism of adic spaces with a complete affinoid ring $(B, B^\circ)$. Define $g$ as the composition*

$$g\colon X \xrightarrow{f} \mathbb{B}(a, b) \xrightarrow{\Psi} \mathbb{B}(1) = \mathrm{Spa}(K\langle S\rangle),$$

*where $\Psi$ is the morphism defined in Example 3.A.2. The map $g$ induces a homomorphism $\mathcal{O}\langle S\rangle \to B^\circ$. Let us suppose that the following two conditions hold:*

- *There is only one prime ideal $\mathfrak{q} \subset B^\circ$ lying above the maximal ideal $\mathfrak{m}\mathcal{O}\langle S\rangle + S\mathcal{O}\langle S\rangle \subset \mathcal{O}\langle S\rangle$.*
- *The discriminant function $\delta_f$ is linear.*

*Then the inverse image $g^{-1}(\eta(1))$ consists of two points, or equivalently, the inverse images $f^{-1}(\eta(a)')$ and $f^{-1}(\eta(b))$ both consist of one point. Moreover, the closed point*

$$x \in \mathrm{Spec}\, B^\sim = \mathrm{Spec}\, B^\circ/\mathfrak{m}B^\circ$$

*corresponding to the prime ideal $\mathfrak{q}$ is an ordinary double point.*

Proof. The first assertion is a consequence of [**105**, (2.4.4) in the proof of Theorem 2.4.3]. (The assumption that the characteristic of the base field is zero in *loc. cit.* is not needed here. Moreover, the morphism $f$ need not be Galois.) The second assertion is claimed in [**105**, Remark 2.4.8] (at least when $K$ is of characteristic 0) without proof. Indeed, the hard parts of the proof were already done in [**105**]. We shall explain how to use the results in *loc. cit.* to deduce the second assertion.

Before giving the proof of the second assertion, let us prepare some notation. We write $A := K\langle S\rangle$. For the points $\eta(1), \eta(1)^\flat \in \mathbb{B}(1)$, we write

$$(k(1), k(1)^+) := (k(\eta(1)), k(\eta(1))^+) \quad \text{and} \quad (k(1^\flat), k(1^\flat)^+) := (k(\eta(1)^\flat), k(\eta(1)^\flat)^+).$$

Note that $\mathcal{O}_{\mathbb{B}(1),\eta(1)} = k(1)$ and $\mathcal{O}_{\mathbb{B}(1),\eta(1)^\flat} = k(1^\flat)$. We have a natural inclusion

$$k(1)^+ \to k(1^\flat)^+.$$

The residue field $k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+$ of $k(1^\flat)^+$ is naturally isomorphic to the field of fractions $\kappa(S)$ of $\kappa[S]$. The image of $k(1)^+$ in $\kappa(S)$ is the localization $\kappa[S]_{(S)}$ of $\kappa[S]$ at the maximal ideal $(S) \subset \kappa[S]$. More precisely, we have the following commutative diagram:

$$\begin{array}{ccccc} A^\sim = A^\circ/\mathfrak{m}A^\circ & \longrightarrow & k(1)^+/\mathfrak{m}k(1)^+ & \longrightarrow & k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+ \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \kappa[S] & \longrightarrow & \kappa[S]_{(S)} & \longrightarrow & \kappa(S). \end{array}$$

Let

$$\mathscr{B}(1)^+ := (g_*\mathcal{O}_X^+)_{\eta(1)}$$

be the stalk of $g_*\mathcal{O}_X^+$ at the point $\eta(1) \in \mathbb{B}(1)$. We have a map

$$i\colon B^\circ \otimes_{A^\circ} k(1)^+ \to \mathscr{B}(1)^+.$$

The target and the source of $i$ are both free $k(1)^+$-modules of finite rank by [**105**, Proposition 2.3.5 and Lemma 2.2.17], and clearly $i$ becomes an isomorphism after tensoring with $k(1)$. In particular, the map $i$ is injective. By [**105**, Proposition 2.3.5] again, it follows that $i$ becomes an isomorphism after tensoring with $k(1^\flat)^+$. Consequently, we have inclusions

$$B^\circ \otimes_{A^\circ} k(1)^+ \hookrightarrow \mathscr{B}(1)^+ \hookrightarrow B^\circ \otimes_{A^\circ} k(1^\flat)^+,$$

$$B^\circ \otimes_{A^\circ} (k(1)^+/\mathfrak{m}k(1)^+) \hookrightarrow \mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+ \hookrightarrow B^\circ \otimes_{A^\circ} (k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+).$$

We shall prove the second assertion. First, we prove that the ring $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ is integrally closed in $B^\circ \otimes_{A^\circ} (k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+)$. Let

$$(k(1)^\wedge, k(1)^{\wedge+}) \quad \text{and} \quad (k(1^\flat)^\wedge, k(1^\flat)^{\wedge+})$$

be the completions with respect to the valuation topologies. By [**52**, Lemma 1.1.10 iii)], we have $k(1)^\wedge \xrightarrow{\cong} k(1^\flat)^\wedge$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{B}(1)^+ \otimes_{k(1)^+} k(1)^{\wedge+} & \longrightarrow & B^\circ \otimes_{A^\circ} k(1^\flat)^{\wedge+} \\ \downarrow & & \downarrow \\ \mathscr{B}(1)^+ \otimes_{k(1)^+} k(1)^\wedge & \xrightarrow{\cong} & \mathscr{B}(1)^+ \otimes_{k(1)^+} k(1^\flat)^\wedge, \end{array}$$

where the vertical maps and the top horizontal map are injective. By using [**105**, Proposition 1.3.2 (iii)], we see that $\mathscr{B}(1)^+ \otimes_{k(1)^+} k(1)^{\wedge+}$ is integrally closed in $\mathscr{B}(1)^+ \otimes_{k(1)^+} k(1)^\wedge$, and hence integrally closed in $B^\circ \otimes_{A^\circ} k(1^\flat)^{\wedge+}$. This implies that $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ is integrally closed in $B^\circ \otimes_{A^\circ} (k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+)$.

By the assumptions, the ring

$$R := B^\circ \otimes_{A^\circ} (k(1)^+/\mathfrak{m}k(1)^+) \cong B^\sim \otimes_{\kappa[S]} \kappa[S]_{(S)}$$

is the local ring of $\operatorname{Spec} B^\sim$ at the closed point $x \in \operatorname{Spec} B^\sim$. Moreover, the ring

$$B^\circ \otimes_{A^\circ} (k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+) \cong B^\sim \otimes_{\kappa[S]} \kappa(S)$$

is the total ring of fractions of $R$. Hence the ring $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ is the normalization of $R$. By using [**105**, (2.4.4) in the proof of Theorem 2.4.3], one can show that there are exactly two maximal ideals of $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ and the length of $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ as an $R$-module is one. In other words, the closed point $x$ is an ordinary double point.

The proof of Proposition 3.A.7 is complete. □

We deduce the following result from Proposition 3.A.7, which is used in the proof of Theorem 3.6.2 (in the case where $K$ is of positive characteristic).

**Proposition 3.A.8.** *Let $f\colon X \to \mathbb{B}(a,b)$ be a finite étale morphism of adic spaces. We assume that the discriminant function $\delta_f$ is linear. Let $t \in |K^\times|$ be an element with $a/b < t^2 < 1$. Then we have*

$$f^{-1}(\mathbb{B}(a/t, tb)) \cong \coprod_{i=1}^n \mathbb{B}(c_i, 1)$$

*for some elements $c_i \in |K^\times|$ with $c_i < 1$ ($1 \leq i \leq n$).*

Proof. By Lemma 3.A.4, without loss of generality, we may assume that every connected component of $X$ remains connected after restricting to $\mathbb{B}(a/s, sb)$ for every $s \in |K^\times|$ with $t < s \leq 1$. Let $X_1, \ldots, X_m$ be the connected components of $X$ and let $f_i\colon X_i \to \mathbb{B}(a,b)$ be the restriction of $f$. By Theorem 3.A.6, each discriminant function $\delta_{f_i}$ associated with $f_i$ is a continuous, piecewise linear, and convex function. Since $\delta_f = \sum_{i=1}^m \delta_{f_i}$, it follows that $\delta_{f_i}$ is linear for every $i$. Thus we may further assume that $X$ is connected.

Let $(B, B^\circ)$ be a complete affinoid ring such that $X = \operatorname{Spa}(B, B^\circ)$. Define $g$ as the composition

$$g\colon X \xrightarrow{f} \mathbb{B}(a,b) \xrightarrow{\Psi} \mathbb{B}(1) = \operatorname{Spa}(K\langle S\rangle).$$

By Lemma 3.A.5, there is only one prime ideal $\mathfrak{q} \subset B^\circ$ lying above the maximal ideal $\mathfrak{m}\mathcal{O}\langle S\rangle + S\mathcal{O}\langle S\rangle \subset \mathcal{O}\langle S\rangle$. Let $x \in \operatorname{Spec} B^\sim$ be the closed point corresponding to the prime ideal $\mathfrak{q}$, which is an ordinary double point by Proposition 3.A.7. Let

$$\lambda\colon X = d(\mathrm{Spf}(B^\circ)) \to \mathrm{Spf}(B^\circ)$$

be the specialization map associated with the formal scheme $\mathrm{Spf}(B^\circ)$; see Section 3.5.2. By the proof of [**12**, Proposition 2.3], the interior $\lambda^{-1}(x)^\circ$ of the inverse image $\lambda^{-1}(x)$ in $X$ is isomorphic to the interior $\mathbb{D}(d,1)^\circ$ of $\mathbb{D}(d,1)$ in $\mathbb{B}(1)$ for some element $d \in |K^\times|$ with $d < 1$ as an adic space over $\mathrm{Spa}(K)$, where we define

$$\mathbb{D}(d,1) := \{x \in \mathbb{B}(1) = \mathrm{Spa}(K\langle T\rangle) \mid d < |T(x)| < 1\} \subset \mathbb{B}(1).$$

We fix such an isomorphism. For every $s \in |K^\times|$ with $t \le s < 1$, we have

$$f^{-1}(\mathbb{B}(a/s, sb)) = g^{-1}(\mathbb{B}(s)) \subset g^{-1}(\mathbb{D}(1)^\circ) = \lambda^{-1}(x)^\circ \cong \mathbb{D}(d,1)^\circ \subset \mathbb{B}(1).$$

Thus we may consider $f^{-1}(\mathbb{B}(a/s, sb))$ as a connected affinoid open subset of $\mathbb{B}(1)$.

We write $X_t := f^{-1}(\mathbb{B}(a/t, tb))$. Let $X_t^c$ be the closure of $X_t$ in $\mathbb{B}(1)$, which is contained in $g^{-1}(\mathbb{D}(1)^\circ)$. In view of Example 3.A.1 (3), to prove the assertion, it suffices to prove that the complement $X_t^c \backslash X_t$ consists of exactly two points. The map $f$ induces a map

$$f'\colon X_t^c \backslash X_t \to \mathbb{B}(a/t, tb)^c \backslash \mathbb{B}(a/t, tb) = \{\eta(a/t), \eta(tb)'\}.$$

We prove that $f'$ is bijective. Since $f$ is surjective and specializing by [**52**, Lemma 1.4.5 ii)], it follows that the map $f'$ is surjective. To show that the map $f'$ is injective, it suffices to prove the following claim:

**Claim 3.A.9.** *The inverse images $f^{-1}(\eta(a/t))$ and $f^{-1}(\eta(tb)')$ both consist of one point.*

Proof. Recall that we assume that $X$ remains connected after restricting to $\mathbb{B}(a/s, sb)$ for every $s \in |K^\times|$ with $t < s \le 1$. Thus, by Lemma 3.A.5 and Proposition 3.A.7, the inverse images $f^{-1}(\eta(a/s)')$ and $f^{-1}(\eta(sb))$ both consist of one point for every $s \in |K^\times|$ with $t < s \le 1$. This fact implies that

$$f^{-1}(\mathbb{B}(s_1 b, s_2 b))$$

is connected for all $s_1, s_2 \in |K^\times|$ with $t \le s_1 < s_2 \le 1$. (Indeed, if it is not connected, then there exist at least two points mapped to $\eta(s_2 b)$.) By applying Lemma 3.A.5 and Proposition 3.A.7 to

$$f^{-1}(\mathbb{B}(tb, b)) \to \mathbb{B}(tb, b),$$

we see that $f^{-1}(\eta(tb)')$ consists of one point. The same arguments show that $f^{-1}(\eta(a/t))$ consists of one point. □

The proof of Proposition 3.A.8 is complete. □

**Remark 3.A.10.** Here we prove Proposition 3.A.7 and Proposition 3.A.8 in the context of adic spaces as in [**105**]. It is probably possible to prove these results by using the methods of [**82, 83**].

We now give a proof of Theorem 3.6.2.

**Proof of Theorem 3.6.2.** Let $f\colon X \to \mathbb{B}(1)^*$ be a finite étale morphism. Clearly, the discriminant functions on open annuli constructed in Theorem 3.A.6 can be glued to a continuous, piecewise linear, and convex function

$$\delta_f\colon [0,\infty) \to \mathbb{R}_{\ge 0}.$$

Moreover, the slopes of $\delta_f$ are integers. By [**105**, Lemma 2.1.10], the function $\delta_f$ is bounded above by some positive real number (depending only on the degree of $f$). It follows that there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq 1$ such that the restriction of $\delta_f$ to $[-\log \epsilon_0, \infty)$ is constant. Let $t \in |K^\times|$ be an element with $t < 1$. We put $\epsilon := t\epsilon_0$. Then, for elements $a, b \in |K^\times|$ with $a < b \leq \epsilon$, we have

$$f^{-1}(\mathbb{B}(a,b)) \cong \coprod_{i=1}^{n} \mathbb{B}(c_i, d_i)$$

for some elements $c_i, d_i \in |K^\times|$ with $c_i < d_i \leq 1$ $(1 \leq i \leq n)$ by Proposition 3.A.8. If $K$ is of characteristic 0, after replacing $\epsilon$ by a smaller one, we can easily show that the restriction $\mathbb{B}(c_i, d_i) \to \mathbb{B}(a, b)$ of $f$ is a Kummer covering by using [**105**, Claim 2.4.5]. (See also the proofs of [**82**, Theorem 2.2] and [**105**, Theorem 2.4.3].) □

**3.A.3. Galois coverings and discriminant functions.** Let $a, b \in |K^\times|$ be elements with $a < b \leq 1$. Let

$$f\colon X = \mathrm{Spa}(B, B^\circ) \to \mathbb{B}(a, b)$$

be a finite étale morphism of adic spaces with a complete affinoid ring $(B, B^\circ)$. Let

$$G := \mathrm{Aut}(X/\mathbb{B}(a,b))^\circ \cong \mathrm{Aut}(B/A(a,b))$$

be the opposite of the group of $\mathbb{B}(a, b)$-automorphisms on $X$, or equivalently, the group of $A(a, b)$-automorphisms of $B$. We assume that $f$ is Galois, i.e. $A(a, b)$ coincides with the ring $B^G$. (This is equivalent to saying that the finite étale morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A(a, b)$ of schemes is Galois; see Remark 3.A.3.) In this case, we call $G$ the Galois group of $f$.

We assume that $X$ is connected. Let $r \in |K^\times|$ be an element with $a < r \leq b$ and let $x \in f^{-1}(\eta(r))$ be an element. Let

$$\mathrm{Stab}_x := \{g \in G \mid g(x) = x\}$$

be the stabilizer of $x$. Let $k(x)^{\wedge h+}$ (resp. $k(r)^{\wedge h+}$) be the Henselization of the completion of the valuation ring $k(x)^+$ (resp. $k(\eta(r))^+$). Let $k(x)^{\wedge h}$ and $k(r)^{\wedge h}$ be the fields of fractions of $k(x)^{\wedge h+}$ and $k(r)^{\wedge h+}$, respectively. Then by [**55**, 5.5] the extension of fields

$$k(r)^{\wedge h} \to k(x)^{\wedge h}$$

is finite and Galois, and we have a natural isomorphism

$$\mathrm{Stab}_x \xrightarrow{\cong} \mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h}).$$

In [**55**], Huber defined higher ramification subgroups and the Swan character of the Galois group $\mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h})$. In [**105**], Ramero investigated the relation between the discriminant functions and the Swan characters. We are interested in the case where all higher ramification subgroups and the Swan character of $\mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h})$ are trivial. All we need is the following lemma:

**Lemma 3.A.11** ([**105**, Lemma 3.3.10])**.** *Let $f\colon X \to \mathbb{B}(a, b)$ be a finite Galois étale morphism such that $X$ is connected. We assume that $\sharp\, \mathrm{Stab}_x$ is invertible in $\mathcal{O}$ for every $r \in |K^\times|$ with $a < r \leq b$ and every $x \in f^{-1}(\eta(r))$. Then the discriminant function*

$$\delta_f\colon [-\log b, -\log a] \to \mathbb{R}_{\geq 0}$$

*associated with $f$ is constant.*

Proof. This follows from the second equality of [**105**, Lemma 3.3.10]. Indeed, under the assumption, we have $\mathrm{Sw}_x^\natural = 0$ for the Swan character $\mathrm{Sw}_x^\natural$ attached to $x \in f^{-1}(\eta(r))$ defined in [**105**, Section 3.3]. □

Finally, we prove Theorem 3.6.3.

**Proof of Theorem 3.6.3.** In fact, we will show that if a locally constant étale sheaf $\mathcal{F}$ with finite stalks on $\mathbb{B}(a,b)$ is tame at $\eta(r) \in \mathbb{B}(a,b)$ for every $r \in |K^\times|$ with $a < r \leq b$, then, for every $t \in |K^\times|$ with $a/b < t^2 < 1$, there exists an integer $m$ invertible in $\mathcal{O}$ such that the restriction $\mathcal{F}|_{\mathbb{B}(a/t,tb)}$ is trivialized by a Kummer covering $\varphi_m$.

There is a finite Galois étale morphism $f\colon X \to \mathbb{B}(a,b)$ such that $X$ is connected and $f^*\mathcal{F}$ is a constant sheaf. Let $G$ be the Galois group of $f$. By replacing $X$ by a quotient of it by a subgroup of $G$ (this makes sense by Remark 3.A.3), we may assume that the induced homomorphism

$$\rho\colon G \to \mathrm{Aut}(\Gamma(X, f^*\mathcal{F}))$$

is injective. Let $t \in |K^\times|$ be an element with $a/b < t^2 < 1$. By Lemma 3.A.4, we may assume that $X$ remains connected after restricting to $\mathbb{B}(a/s, sb)$ for every $s \in |K^\times|$ with $t < s \leq 1$.

We claim that $\sharp\,\mathrm{Stab}_x$ is invertible in $\mathcal{O}$ for every $r \in |K^\times|$ with $a < r \leq b$ and every $x \in f^{-1}(\eta(r))$. Let $L(r)$ be a separable closure of $k(x)^{\wedge h}$. It induces a geometric point $\overline{x} \to X$ with support $x$. Let $\overline{r} \to \mathbb{B}(a,b)$ denote the composition $\overline{x} \to X \to \mathbb{B}(a,b)$. Since $f^*\mathcal{F}$ is a constant sheaf, we have the following identifications

$$\Gamma(X, f^*\mathcal{F}) \cong (f^*\mathcal{F})_{\overline{x}} \cong \mathcal{F}_{\overline{r}}.$$

Recall that we have $\mathrm{Stab}_x \cong \mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h})$. Via these identifications, the action of $\mathrm{Stab}_x \subset G$ on $\Gamma(X, f^*\mathcal{F})$ is compatible with the action of $\mathrm{Gal}(L(r)/k(r)^{\wedge h})$ on $\mathcal{F}_{\overline{r}}$. Since $\mathcal{F}$ is tame at $\eta(r)$ and $\rho$ is injective, it follows that $\sharp\,\mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h})$ is invertible in $\mathcal{O}$. This proves our claim.

By Lemma 3.A.11, it follows that the discriminant function $\delta_f$ is constant. By Lemma 3.A.5 and Proposition 3.A.7, there is exactly one point $x$ in $f^{-1}(\eta(b))$. Therefore the Galois group $G$ is isomorphic to $\mathrm{Stab}_x \cong \mathrm{Gal}(k(x)^{\wedge h}/k(b)^{\wedge h})$.

Theorem 3.6.3 now follows from [**82**, Theorem 2.11]. Alternatively, we can argue as follows. By [**55**, Proposition 2.5, Corollary 2.7, and Corollary 5.4], we see that $G$ is a cyclic group. Let us write $G \cong \mathbb{Z}/m\mathbb{Z}$. We consider $f\colon X \to \mathbb{B}(a,b)$ as a $\mathbb{Z}/m\mathbb{Z}$-torsor. As in the proof of [**105**, Theorem 2.4.3], since the Picard group of $\mathbb{B}(a,b)$ is trivial, the Kummer sequence gives an isomorphism

$$A(a,b)^\times/(A(a,b)^\times)^m \cong H^1(\mathbb{B}(a,b), \mathbb{Z}/m\mathbb{Z}).$$

Since $m$ is invertible in $\mathcal{O}$, the left hand side is a cyclic group of order $m$ generated by the coordinate function $T \in A(a,b)^\times$. It follows that every $\mathbb{Z}/m\mathbb{Z}$-torsor over $\mathbb{B}(a,b)$ is the disjoint union of Kummer coverings. This completes the proof of Theorem 3.6.3. □

CHAPTER 4

# A torsion analogue of the weight-monodromy conjecture

## 4.1. Introduction

In this chapter, we formulate and study a torsion analogue of the weight-monodromy conjecture for a proper smooth scheme over a non-archimedean local field. We prove it for proper smooth schemes over equal characteristic non-archimedean local fields, abelian varieties, surfaces, varieties uniformized by Drinfeld upper half spaces, and set-theoretic complete intersections in toric varieties. In the equal characteristic case, our methods rely on an ultraproduct variant of Weil II established by Cadoret. As applications, we discuss some finiteness properties of the Brauer group and the codimension two Chow group of a proper smooth scheme over a non-archimedean local field. This chapter is based on the preprint [**64**].

**4.1.1. Main results.** Let $K$ be a non-archimedean local field with finite residue field $k$. Let $p > 0$ be the characteristic of $k$, and $q$ the number of elements in $k$. Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Let $\ell \neq p$ be a prime number. Let $\overline{K}$ be an algebraic closure of $K$ and $K^{\mathrm{sep}}$ the separable closure of $K$ in $\overline{K}$. The absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ naturally acts on the $\ell$-adic cohomology $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, where we put $X_{\overline{K}} := X \otimes_K \overline{K}$.

By Grothendieck's quasi-unipotence theorem, the action of an open subgroup of the inertia group $I_K$ of $K$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ defines the monodromy filtration

$$\{M_{i,\mathbb{Q}_\ell}\}_i$$

on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. It is an increasing filtration stable by the action of $G_K$. (See Section 4.3.1 for details.) The *weight-monodromy conjecture* due to Deligne states that the $i$-th graded piece

$$\mathrm{Gr}^M_{i,\mathbb{Q}_\ell} := M_{i,\mathbb{Q}_\ell}/M_{i-1,\mathbb{Q}_\ell}$$

of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is of weight $w + i$, i.e. every eigenvalue of a lift of the geometric Frobenius element $\mathrm{Frob}_k \in \mathrm{Gal}(\overline{k}/k)$ is an algebraic integer such that the complex absolute values of its conjugates are $q^{(w+i)/2}$. When $X$ has good reduction over the ring of integers $\mathcal{O}_K$ of $K$, it is nothing more than the Weil conjecture [**32, 34**]. In general, the weight-monodromy conjecture is still open. In this chapter, we shall propose a torsion analogue of the weight-monodromy conjecture and prove it in some cases.

By the work of Rapoport-Zink [**106**] and de Jong's alteration [**28**], we can take an open subgroup $J \subset I_K$ so that the action of $J$ on the étale cohomology group with $\mathbb{F}_\ell$-coefficients $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ is unipotent for every $\ell \neq p$. By the same construction as in the $\ell$-adic case, we can define the monodromy filtration

$$\{M_{i,\mathbb{F}_\ell}\}_i$$

on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$, which is stable by the action of $G_K$; see Section 4.3.2 for details. We propose the following conjecture.

**Conjecture 4.1.1 (A torsion analogue of the weight-monodromy conjecture, Conjecture 4.3.4).** *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. For every $i$, there exists a non-zero monic polynomial $P_i(T) \in \mathbb{Z}[T]$ satisfying the following conditions:*

- *The roots of $P_i(T)$ have complex absolute values $q^{(w+i)/2}$.*
- *We have $P_i(\mathrm{Frob}) = 0$ on the $i$-th graded piece*

$$\mathrm{Gr}^M_{i,\mathbb{F}_\ell} := M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}$$

  *for all but finitely many $\ell \neq p$ and for every lift $\mathrm{Frob} \in G_K$ of the geometric Frobenius element.*

**Remark 4.1.2.** It is a theorem of Gabber that the étale cohomology group with $\mathbb{Z}_\ell$-coefficients $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$ is torsion-free for all but finitely many $\ell \neq p$ [**44**]. (See also [**121**, Theorem 1.4], [**101**, Théorème 6.2.2].) When $X$ has good reduction over $\mathcal{O}_K$, Conjecture 4.1.1 follows from the Weil conjecture and Gabber's theorem.

The main theorem of this chapter is as follows:

**Theorem 4.1.3 (Theorem 4.3.6).** *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Assume that one of the following conditions holds:*

*(A) $K$ is of equal characteristic, i.e. the characteristic of $K$ is $p$.*
*(B) $X$ is an abelian variety.*
*(C) $w \leq 2$ or $w \geq 2\dim X - 2$.*
*(D) $X$ is uniformized by a Drinfeld upper half space.*
*(E) $X$ is geometrically connected and is a set-theoretic complete intersection in a projective smooth toric variety.*

*Then the assertion of Conjecture 4.1.1 for $(X, w)$ is true.*

The weight-monodromy conjecture for $\mathbb{Q}_\ell$-coefficients is known to be true for $(X, w)$ if one of the above conditions (A)–(E) holds. However, it seems that the weight-monodromy conjecture for $\mathbb{Q}_\ell$-coefficients does not automatically imply Conjecture 4.1.1. The problem is that, in general, we do not know the torsion-freeness of the cokernel of the monodromy operator acting on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$. (See Section 4.3.3 for details.)

**Remark 4.1.4.** There are other cases in which the weight-monodromy conjecture is known to be true; see Remark 4.3.3. In this chapter, we will restrict ourselves to the cases (A)–(E) for the sake of simplicity.

We shall give two applications of our results. The first one is an application to the finiteness of the Brauer group of a proper smooth scheme over $K$ for which the $\ell$-adic Chern class map for divisors is surjective; see Corollary 4.10.2. As the second application, we will show the finiteness of the $G_K$-fixed part of the prime-to-$p$ torsion part of the Chow group $\mathrm{CH}^2(X_{\overline{K}})$ of codimension two cycles on $X_{\overline{K}}$ if $(X, w = 3)$ satisfies one of the conditions (A)–(E); see Corollary 4.10.7.

**4.1.2. Outline of this chapter.** The outline of this chapter is as follows. In Section 4.2, we define a notion of weight for a family of $G_K$-representations over $\mathbb{F}_\ell$ and prepare some elementary lemmas used in this chapter. In Section 4.3, we define the monodromy filtration with coefficients in $\mathbb{F}_\ell$ for all but finitely many $\ell \neq p$ and propose a torsion analogue of the weight-monodromy conjecture. We also discuss a relation between the weight-monodromy conjecture and Conjecture 4.1.1. In Section 4.4, we discuss some torsion-freeness properties of the weight spectral sequence and their relation to Conjecture 4.1.1. In Section 4.5–4.9, we prove Theorem 4.1.3. In Section 4.10, as applications of Theorem 4.1.3, we discuss some finiteness properties of the Brauer group and the codimension two Chow group of a proper smooth scheme over a non-archimedean local field.

## 4.2. Preliminaries

**4.2.1. Weights.** Let $p$ be a prime number. In this subsection, we fix a finitely generated field $k$ over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number. We call a finitely generated $\mathbb{Z}_\ell$-module endowed with a continuous action of $G_k$ a *$G_k$-module over $\mathbb{Z}_\ell$* for simplicity. Let $q$ be a power of $p$. For a non-zero monic polynomial $P(T) \in \mathbb{Z}[T]$, we say that $P(T)$ is a *Weil $q$-polynomial* if the complex absolute value of every root of $P(T)$ is $q^{1/2}$.

For a finite dimensional representation of $G_k$ over $\mathbb{Q}_\ell$, there is a notion of weight; see [**69**, Section 2.2] for example. In this chapter, we will use the following notion of weight for a family of $G_k$-modules over $\mathbb{Z}_\ell$. Let $\mathfrak{L}$ be an infinite set of prime numbers $\ell \neq p$. Let $w$ be an integer.

- Let $U$ be an integral scheme of finite type over $\mathbb{F}_p$ with function field $k$. We say that a family $\{\mathcal{F}_\ell\}_{\ell \in \mathfrak{L}}$ of locally constant constructible $\mathbb{Z}_\ell$-sheaves on $U$ is of *weight $w$* if, for every closed point $x \in U$, there is a Weil $(q_x)^w$-polynomial $P_x(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \in \mathfrak{L}$, we have $P_x(\mathrm{Frob}_x) = 0$ on $\mathcal{F}_{\ell,\overline{x}}$. Here $q_x$ is the cardinality of the residue field $\kappa(x)$ of $x$, $\overline{x}$ is a geometric point of $U$ above $x$, and $\mathrm{Frob}_x \in G_{\kappa(x)} = \mathrm{Gal}(\overline{\kappa(x)}/\kappa(x))$, $a \mapsto a^{1/q_x}$ is the geometric Frobenius element at $x$.
- We say that a family $\{H_\ell\}_{\ell \in \mathfrak{L}}$ of $G_k$-modules over $\mathbb{Z}_\ell$ is of *weight $w$* if there is an integral scheme $U$ of finite type over $\mathbb{F}_p$ with function field $k$ such that the family $\{H_\ell\}_{\ell \in \mathfrak{L}}$ comes from a family $\{\mathcal{F}_\ell\}_{\ell \in \mathfrak{L}}$ of locally constant constructible $\mathbb{Z}_\ell$-sheaves on $U$ of weight $w$.

When there is no possibility of confusion, we will omit $\mathfrak{L}$ from the notation and write $\{H_\ell\}_\ell$ in place of $\{H_\ell\}_{\ell \in \mathfrak{L}}$.

**Lemma 4.2.1.** *Let $\{H_{1,\ell}\}_{\ell \in \mathfrak{L}}$ and $\{H_{2,\ell}\}_{\ell \in \mathfrak{L}}$ be families of $G_k$-modules over $\mathbb{Z}_\ell$ of weight $w_1$ and $w_2$, respectively. We assume $w_1 \neq w_2$. Then, for all but finitely many $\ell \in \mathfrak{L}$, every map $H_{1,\ell} \to H_{2,\ell}$ of $G_k$-modules over $\mathbb{Z}_\ell$ is zero.*

Proof. We may assume that $\{H_{1,\ell}\}_{\ell \in \mathfrak{L}}$ and $\{H_{2,\ell}\}_{\ell \in \mathfrak{L}}$ come from families $\{\mathcal{F}_{1,\ell}\}_{\ell \in \mathfrak{L}}$ and $\{\mathcal{F}_{2,\ell}\}_{\ell \in \mathfrak{L}}$ of locally constant constructible $\mathbb{Z}_\ell$-sheaves on $U$ of weight $w_1$ and $w_2$, respectively. Here $U$ is an integral scheme of finite type over $\mathbb{F}_p$ with function field $k$. Take a closed point $x \in U$. Let $P_{1,x}(T) \in \mathbb{Z}[T]$ be a Weil $(q_x)^{w_1}$-polynomial such that, for all but finitely many $\ell \in \mathfrak{L}$, we have $P_{1,x}(\mathrm{Frob}_x) = 0$ on $(\mathcal{F}_{1,\ell})_{\overline{x}}$. Let $P_{2,x}(T) \in \mathbb{Z}[T]$ be a Weil $(q_x)^{w_2}$-polynomial which satisfies the same condition for $\{\mathcal{F}_{2,\ell}\}_{\ell \in \mathfrak{L}}$. The polynomials $P_{1,x}(T)$ and $P_{2,x}(T)$ are relatively prime. For $\ell \in \mathfrak{L}$ such that $P_{1,x}(\mathrm{Frob}_x) = 0$ on $(\mathcal{F}_{1,\ell})_{\overline{x}}$ and $P_{2,x}(\mathrm{Frob}_x) = 0$ on $(\mathcal{F}_{2,\ell})_{\overline{x}}$, we have $P_{1,x}(\mathrm{Frob}_x) = P_{2,x}(\mathrm{Frob}_x) = 0$ on the stalk of the image of any map $\mathcal{F}_{1,\ell} \to \mathcal{F}_{2,\ell}$ at $\overline{x}$. Therefore, the assertion follows from Lemma 4.2.2 below. □

**Lemma 4.2.2.** *Let $P_1(T), P_2(T) \in \mathbb{Q}[T]$ be two relatively prime polynomials. For all but finitely many prime numbers $\ell$, every $\mathbb{Z}_\ell[T]$-module $H_\ell$ such that $P_1(T) = P_2(T) = 0$ on $H_\ell$ is zero.*

Proof. There exist polynomials $Q_1(T), Q_2(T) \in \mathbb{Q}[T]$ satisfying

$$P_1(T)Q_1(T) + P_2(T)Q_2(T) = 1$$

in $\mathbb{Q}[T]$ since $P_1(T)$ and $P_2(T)$ are relatively prime. Thus, for all but finitely many prime numbers $\ell$, we have $P_1(T), P_2(T) \in \mathbb{Z}_\ell[T]$, and they generate the unit ideal of $\mathbb{Z}_\ell[T]$. The assertion follows from this fact. □

We need the following theorem to define monodromy filtrations with coefficients in $\mathbb{F}_\ell$ for all but finitely many $\ell$ and to prove main results in this chapter.

**Theorem 4.2.3 (Gabber [44], Suh [121, Theorem 1.4]).** *Let $X$ be a proper smooth scheme over a separably closed field of characteristic $p \geq 0$. For all but finitely many $\ell \neq p$, the $\mathbb{Z}_\ell$-module $H^w_{\text{ét}}(X, \mathbb{Z}_\ell)$ is torsion-free for every $w$. In particular, for all but finitely many $\ell \neq p$, the natural map $H^w_{\text{ét}}(X, \mathbb{Z}_\ell) \to H^w_{\text{ét}}(X, \mathbb{F}_\ell)$ gives an isomorphism*

$$H^w_{\text{ét}}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \simeq H^w_{\text{ét}}(X, \mathbb{F}_\ell)$$

*for every $w$.*

Proof. If $X$ is projective, this is a theorem of Gabber [**44**, Theorem]. (An alternative proof using ultraproduct Weil cohomology theory was obtained by Orgogozo; see [**101**, Théorème 6.2.2].) By using de Jong's alteration [**28**, Theorem 4.1], the general case can be deduced from the projective case; see the proof of [**121**, Theorem 1.4] for details. □

**Corollary 4.2.4.** *Let $X$ be a proper smooth scheme over $k$. Then $\{H^w_{\text{ét}}(X_{\overline{k}}, \Lambda_\ell)\}_{\ell \neq p}$ is a family of $G_k$-modules of weight $w$, where $\Lambda_\ell$ is either $\mathbb{Z}_\ell$ or $\mathbb{F}_\ell$.*

Proof. This follows from the Weil conjecture [**34**, Corollaire (3.3.9)] and Theorem 4.2.3. □

Let $K$ be a non-archimedean local field with finite residue field $\mathbb{F}_q$. Assume that the characteristic of $\mathbb{F}_q$ is $p$. Similarly, we will use the following notion of weight for representations of $G_K$. Let $I_K \subset G_K$ be the inertia group of $K$. Let $\{H_\ell\}_{\ell \in \mathfrak{L}}$ be a family of $G_K$-modules over $\mathbb{Z}_\ell$. We assume that there is an open subgroup $J \subset I_K$ such that the action of $J$ on $H_\ell$ is trivial for all but finitely many $\ell \in \mathfrak{L}$.

**Definition 4.2.5.** We say that the family $\{H_\ell\}_{\ell \in \mathfrak{L}}$ is of *weight $w$* if there is a Weil $q^w$-polynomial $P(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \in \mathfrak{L}$, we have $P(\mathrm{Frob}) = 0$ on $H_\ell$ for every lift $\mathrm{Frob} \in G_K$ of the geometric Frobenius element $\mathrm{Frob}_q \in G_{\mathbb{F}_q}$, $a \mapsto a^{1/q}$.

**Remark 4.2.6.** Since the action of the open subgroup $J \subset I_K$ on $H_\ell$ is trivial for all but finitely many $\ell \in \mathfrak{L}$, it follows that the family $\{H_\ell\}_{\ell \in \mathfrak{L}}$ is of weight $w$ if and only if, for *one* lift $\mathrm{Frob} \in G_K$ of the geometric Frobenius element, there is a Weil $q^w$-polynomial $P(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \in \mathfrak{L}$, we have $P(\mathrm{Frob}) = 0$ on $H_\ell$.

**Lemma 4.2.7.** *Let $L$ be a finite extension of $K$. Then $\{H_\ell\}_{\ell \in \mathfrak{L}}$ is of weight $w$ as a family of $G_K$-modules over $\mathbb{Z}_\ell$ if and only if $\{H_\ell\}_{\ell \in \mathfrak{L}}$ is of weight $w$ as a family of $G_L$-modules over $\mathbb{Z}_\ell$.*

Proof. Let $f$ be the residue degree of the extension $L/K$. Let $\mathrm{Frob} \in G_K$ and $\mathrm{Frob}' \in G_L$ be lifts of the geometric Frobenius elements. There is a positive integer $n$ such that, for all but finitely many $\ell \in \mathfrak{L}$, the action of $\mathrm{Frob}^{fn}$ on $H_\ell$ coincides with that of $(\mathrm{Frob}')^n$ on $H_\ell$.

We assume that $\{H_\ell\}_{\ell \in \mathfrak{L}}$ is of weight $w$ as a family of $G_L$-modules. Let $P(T) \in \mathbb{Z}[T]$ be a Weil $q^{fw}$-polynomial satisfying the condition in Definition 4.2.5. We write $P(T)$ in the form $P(T) = \prod_i (T - \alpha_i)$ with $\alpha_i \in \overline{\mathbb{Q}}$. We put $Q(T) := \prod_i (T^{fn} - \alpha_i^n) \in \mathbb{Z}[T]$, which is a Weil $q^w$-polynomial. Then we have $Q(\mathrm{Frob}) = 0$ on $H_\ell$ for all but finitely many $\ell \in \mathfrak{L}$. Therefore $\{H_\ell\}_{\ell \in \mathfrak{L}}$ is of weight $w$ as a family of $G_K$-modules.

The converse can be proved in a similar way. □

**4.2.2. Some elementary lemmas on nilpotent operators.** We collect some elementary lemmas on nilpotent operators, which will be used in the sequel.

**Lemma 4.2.8.** *Let $\ell$ be a prime number. Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be a complex of free $\mathbb{Z}_\ell$-modules of finite rank. The reduction modulo $\ell$ of $f$ and $g$ will be denoted by $\overline{f}$ and $\overline{g}$, respectively. Hence we have the following complex of $\mathbb{F}_\ell$-vector spaces:*

$$M_1 \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\overline{f}} M_2 \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\overline{g}} M_3 \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell.$$

*Then we have*

$$\mathrm{rank}_{\mathbb{Z}_\ell}(\mathrm{Ker}\, g/\mathrm{Im}\, f) \leq \dim_{\mathbb{F}_\ell}(\mathrm{Ker}\, \overline{g}/\mathrm{Im}\, \overline{f}).$$

*The equality holds if and only if the $\mathbb{Z}_\ell$-modules* $\mathrm{Coker}\, f$ *and* $\mathrm{Coker}\, g$ *are torsion-free. If this is the case, then we have* $(\mathrm{Ker}\, g/\mathrm{Im}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \cong \mathrm{Ker}\, \overline{g}/\mathrm{Im}\, \overline{f}$.

Proof. By the theory of elementary divisors, we have

$$\mathrm{rank}_{\mathbb{Z}_\ell}(\mathrm{Ker}\, g/\mathrm{Im}\, f) \leq \dim_{\mathbb{F}_\ell} (\mathrm{Ker}\, g/\mathrm{Im}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell,$$

and the equality holds if and only if $\mathrm{Ker}\, g/\mathrm{Im}\, f$ is torsion-free. Since $M_3$ is torsion-free, we see that $\mathrm{Ker}\, g/\mathrm{Im}\, f$ is torsion-free if and only if $\mathrm{Coker}\, f$ is torsion-free. Moreover, we have inclusions $\mathrm{Im}\, \overline{f} \subset (\mathrm{Ker}\, g) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \subset \mathrm{Ker}\, \overline{g}$. Hence we have

$$\dim_{\mathbb{F}_\ell} (\mathrm{Ker}\, g/\mathrm{Im}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \leq \dim_{\mathbb{F}_\ell} (\mathrm{Ker}\, \overline{g}/\mathrm{Im}\, \overline{f}),$$

and the equality holds if and only if $(\mathrm{Ker}\, g) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = \mathrm{Ker}\, \overline{g}$. It is easy to see that $(\mathrm{Ker}\, g) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = \mathrm{Ker}\, \overline{g}$ if and only if $\mathrm{Coker}\, g$ is torsion-free. This fact completes the proof of the lemma. □

**Lemma 4.2.9.** *Let $V$ be a vector space of dimension $n$ over a field $\mathbb{F}$ of positive characteristic $\ell$. We assume that $\ell \geq n$. For a unipotent operator $U$ on $V$, we define*

$$\log(U) := \sum_{1 \leq i \leq n-1} \frac{(-1)^{i+1}}{i} (U - 1)^i.$$

*For a nilpotent operator $N$ on $V$, we define*

$$\exp(N) := \sum_{0 \leq i \leq n-1} \frac{1}{i!} N^i.$$

*Then the following assertions hold.*

(1) $\log(-)$ *defines a bijection from the set of unipotent operators on $V$ to the set of nilpotent operators on $V$ with inverse map* $\exp(-)$.

(2) *For two unipotent operators $U, U'$ (resp. two nilpotent operators $N, N'$) such that they commute, we have* $\log(UU') = \log(U) + \log(U')$ *(resp.* $\exp(N + N') = \exp(N)\exp(N')$*).*

Proof. Although this lemma is well known, we recall the proof for the reader's convenience.

(1) Let $\mathbb{Z}_{(\ell)}$ be the localization of $\mathbb{Z}$ at the prime ideal $(\ell)$. It suffices to prove that the homomorphism $\mathbb{Z}_{(\ell)}[S]/(S-1)^n \to \mathbb{Z}_{(\ell)}[T]/(T)^n$, $S \mapsto \exp(T)$ and the homomorphism

$\mathbb{Z}_{(\ell)}[T]/(T)^n \to \mathbb{Z}_{(\ell)}[S]/(S-1)^n$, $T \mapsto \log(S)$ are inverse to each other, where $\exp(-)$ and $\log(-)$ are defined by the same formulas as above. Since both rings are torsion-free over $\mathbb{Z}_{(\ell)}$, it suffices to prove the claim after tensoring with $\mathbb{Q}$. Then it follows from the fact that the map $\mathbb{Q}[[S-1]] \to \mathbb{Q}[[T]]$, $S-1 \mapsto \exp(T)-1$ and the map $\mathbb{Q}[[T]] \to \mathbb{Q}[[S-1]]$, $T \mapsto \log(S)$ are inverse to each other, where $\exp(-)$ and $\log(-)$ are defined in the usual way.

(2) By (1), we only need to prove that, for two nilpotent operators $N, N'$ such that they commute, we have $\exp(N+N') = \exp(N)\exp(N')$. We have $N^i(N')^j = 0$ on $V$ for $i, j \geq 0$ with $i+j \geq n$. Thus, it suffices to prove $\exp(T+T') = \exp(T)\exp(T')$ in $\mathbb{Z}_{(\ell)}[T,T']/(T,T')^n$, where $\exp(-)$ is defined by the same formula as above. As in (1), this can be deduced from an analogous statement for $\mathbb{Q}[[T,T']]$. □

Let $R$ be a principal ideal domain, and $F$ its field of fractions. Let $H$ be a free $R$-module of finite rank. Let $N\colon H \to H$ be a nilpotent homomorphism. By [**34**, Proposition (1.6.1)], the nilpotent homomorphism $N_F := N \otimes_R F$ on $H_F := H \otimes_R F$ determines a unique increasing, separated, exhaustive filtration $\{M_{i,F}\}_i$ on $H_F$ characterized by the following properties:

- $N_F(M_{i,F}) \subset M_{i-2,F}$ for every $i$.
- For every integer $i \geq 0$, the $i$-th iterate $N_F^i$ induces an isomorphism $\mathrm{Gr}^M_{i,F} \cong \mathrm{Gr}^M_{-i,F}$. Here we put $\mathrm{Gr}^M_{i,F} := M_{i,F}/M_{i-1,F}$.

We call $\{M_{i,F}\}_i$ the filtration on $H_F$ associated with $N_F$. Let $\{M_i\}_i$ be a filtration on the $R$-module $H$ defined by

$$M_i := H \cap M_{i,F}$$

for every $i$. The $R$-module $\mathrm{Gr}^M_i := M_i/M_{i-1}$ is torsion-free for every $i$.

**Lemma 4.2.10.** *Let the notation be as above. The cokernel of the $i$-th iterate $N^i\colon H \to H$ of $N$ is torsion-free for every $i \geq 0$ if and only if $N^i$ induces an isomorphism $\mathrm{Gr}^M_i \cong \mathrm{Gr}^M_{-i}$ for every $i \geq 0$.*

Proof. Assume that the cokernel of $N^i\colon H \to H$ is torsion-free for every $i \geq 0$. Let $d \geq 0$ be the smallest integer such that $N^{d+1} = 0$. The cokernel of the $i$-th iterate of the homomorphism

$$\mathrm{Ker}\, N^d / \mathrm{Im}\, N^d \to \mathrm{Ker}\, N^d / \mathrm{Im}\, N^d$$

induced by $N$ is torsion-free for every $i \geq 0$. Thus, by the same argument as in the proof of [**34**, Proposition (1.6.1)], we can construct inductively an increasing, separated, exhaustive filtration $\{M'_i\}_i$ on $H$ satisfying the following properties:

- $\mathrm{Gr}^{M'}_i := M'_i/M'_{i-1}$ is torsion-free for every $i$.
- $N(M'_i) \subset M'_{i-2}$ for every $i$.
- For every integer $i \geq 0$, the $i$-th iterate $N^i$ induces an isomorphism $\mathrm{Gr}^{M'}_i \cong \mathrm{Gr}^{M'}_{-i}$.

By uniqueness, the filtration $\{M_{i,F}\}_i$ coincides with $\{M'_i \otimes_R F\}_i$. Since both $\mathrm{Gr}^M_i$ and $\mathrm{Gr}^{M'}_i$ are torsion-free for every $i$, the filtration $\{M_i\}_i$ coincides with $\{M'_i\}_i$ and we have an isomorphism $N^i\colon \mathrm{Gr}^M_i \cong \mathrm{Gr}^M_{-i}$ for every $i \geq 0$.

Conversely, we assume that $N^i$ induces an isomorphism $\mathrm{Gr}^M_i \cong \mathrm{Gr}^M_{-i}$ for every $i \geq 0$. We fix an integer $i \geq 0$. For every $j \leq i$, the $i$-th iterate $N^i\colon \mathrm{Gr}^M_j \to \mathrm{Gr}^M_{j-2i}$ is surjective. It follows that $N^i\colon M_j \to M_{j-2i}$ is surjective for every $j \leq i$ since it is surjective for sufficiently small $j$. For every $j \geq i$, the $i$-th iterate $N^i\colon \mathrm{Gr}^M_j \to \mathrm{Gr}^M_{j-2i}$ is a split injection.

It follows that the cokernel of $N^i\colon M_j \to M_{j-2i}$ is torsion-free for every $j \geq i$ since we have shown that it is zero for $j = i$. Hence the cokernel of $N^i\colon H \to H$ is torsion-free. □

## 4.3. A torsion analogue of the weight-monodromy conjecture

In this section, let $K$ be a non-archimedean local field with ring of integers $\mathcal{O}_K$. Let $\mathbb{F}_q$ be the residue field of $\mathcal{O}_K$. Let $p > 0$ be the characteristic of $\mathbb{F}_q$. Let $I_K \subset G_K$ be the inertia group of $K$. For a prime number $\ell \neq p$, the group of $\ell^n$-th roots of unity in $\overline{K}$ is denoted by $\mu_{\ell^n}$. Let

$$t_\ell\colon I_K \to \mathbb{Z}_\ell(1) := \varprojlim_n \mu_{\ell^n}$$

be the map defined by $g \mapsto \{g(\varpi^{1/\ell^n})/\varpi^{1/\ell^n}\}_n$ for a uniformizer $\varpi \in \mathcal{O}_K$. This map is independent of the choice of $\varpi$ and gives the maximal pro-$\ell$ quotient of $I_K$. Let $X$ be a proper smooth scheme over $K$ and $w$ an integer.

**4.3.1. The weight-monodromy conjecture.** We shall recall the definition of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ for every $\ell \neq p$. The absolute Galois group $G_K$ naturally acts on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ via the natural isomorphism $\mathrm{Aut}(\overline{K}/K) \cong G_K$.

By Grothendieck's quasi-unipotence theorem, there is an open subgroup $J$ of $I_K$ such that the action of $J$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unipotent and factors through $t_\ell$. Take an element $\sigma \in J$ such that $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a non-zero element. We define

$$N_\sigma := \log(\sigma) := \sum_{1 \leq i} \frac{(-1)^{i+1}}{i}(\sigma - 1)^i\colon H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell).$$

Let $\{M_{i,\mathbb{Q}_\ell}\}_i$ be the filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ associated with $N_\sigma$; see [**34**, Proposition (1.6.1)]. The filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$ is independent of $J$ and $\sigma \in J$. It is called the *monodromy filtration*. We have $\chi_{\mathrm{cyc}}(g)N_\sigma g = gN_\sigma$ for every $g \in G_K$, where $\chi_{\mathrm{cyc}}\colon G_K \to \mathbb{Z}_\ell^\times$ is the $\ell$-adic cyclotomic character. It follows from the uniqueness of the monodromy filtration that $\{M_{i,\mathbb{Q}_\ell}\}_i$ is stable by the action of $G_K$. We note that the filtration associated with $\sigma - 1$ coincides with $\{M_{i,\mathbb{Q}_\ell}\}_i$. We put

$$\mathrm{Gr}^M_{i,\mathbb{Q}_\ell} := M_{i,\mathbb{Q}_\ell}/M_{i-1,\mathbb{Q}_\ell}.$$

We recall the weight-monodromy conjecture due to Deligne.

**Conjecture 4.3.1 (Deligne [30]).** *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Let $\ell \neq p$ be a prime number. Then the $i$-th graded piece $\mathrm{Gr}^M_{i,\mathbb{Q}_\ell}$ of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is of weight $w+i$, i.e. every eigenvalue of every lift* $\mathrm{Frob} \in G_K$ *of the geometric Frobenius element is an algebraic integer such that the complex absolute values of its conjugates are $q^{(w+i)/2}$.*

When $X$ has good reduction over $\mathcal{O}_K$, it is nothing more than the Weil conjecture. Conjecture 4.3.1 is known to be true in the following cases.

**Theorem 4.3.2.** *Conjecture 4.3.1 for $(X, w)$ is true in the following cases:*

*(A) $K$ is of equal characteristic (*[**34, 129, 69**]*).*
*(B) $X$ is an abelian variety (*[**SGA 7 I**, Exposé IX]*).*
*(C) $w \leq 2$ or $w \geq 2\dim X - 2$ (*[**106, 28, 113**]*).*
*(D) $X$ is uniformized by a Drinfeld upper half space (*[**68, 27**]*).*
*(E) $X$ is geometrically connected and is a set-theoretic complete intersection in a projective smooth toric variety (*[**114**]*).*

Proof. See the references given above. □

We will prove a torsion analogue of Conjecture 4.3.1 in each of the above cases.

**Remark 4.3.3.** There are other cases in which Conjecture 4.3.1 is known to be true. For example, in [**67**], it is proved for a certain projective threefold with strictly semistable reduction, and in [**90**], it is proved for a variety which is uniformized by a product of Drinfeld upper half spaces. We will not discuss a torsion analogue of Conjecture 4.3.1 for these varieties in this chapter for the sake of simplicity.

**4.3.2. A torsion analogue of the weight-monodromy conjecture.** Let $\{H_\ell\}_{\ell \neq p}$ be a family of finite dimensional $G_K$-representations over $\mathbb{F}_\ell$. We define the monodromy filtrations when the family $\{H_\ell\}_{\ell \neq p}$ satisfies the following two conditions:

- There is an open subgroup $J$ of $I_K$ such that, for every $\ell \neq p$, the action of $J$ on $H_\ell$ is unipotent (i.e. $\sigma$ is a unipotent operator on $H_\ell$ for every $\sigma \in J$).
- $n := \sup_{\ell \neq p} \dim_{\mathbb{F}_\ell} H_\ell < \infty$.

The action of $J$ factors through $t_\ell$ for every $\ell \neq p$. Take an element $\sigma \in J$ such that, for all but finitely many $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. For a prime number $\ell \neq p$ with $\ell \geq n$, we define

$$N_\sigma := \log(\sigma) := \sum_{1 \leq i \leq n-1} \frac{(-1)^{i+1}}{i} (\sigma - 1)^i \colon H_\ell \to H_\ell.$$

See also Lemma 4.2.9. Let $\{M_{i,\mathbb{F}_\ell}\}_i$ be the filtration on $H_\ell$ associated with $N_\sigma$. The filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ is independent of $J$ and $\sigma \in J$ up to excluding finitely many $\ell \neq p$. Moreover, for all but finite many $\ell \neq p$, we have $\overline{\chi_{\mathrm{cyc}}(g)} N_\sigma g = g N_\sigma$ for every $g \in G_K$, where $\overline{\chi_{\mathrm{cyc}}(g)}$ is the reduction modulo $\ell$ of $\chi_{\mathrm{cyc}}(g)$, and $\{M_{i,\mathbb{F}_\ell}\}_i$ is stable by the action of $G_K$. We note that the filtration induced by $\sigma - 1$ coincides with $\{M_{i,\mathbb{F}_\ell}\}_i$ up to excluding finitely many $\ell \neq p$. We call $\{M_{i,\mathbb{F}_\ell}\}_i$ the *monodromy filtration with coefficients in* $\mathbb{F}_\ell$ on $H_\ell$. For all but finitely many $\ell \neq p$, the action of $J$ is trivial on $M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}$ for every $i$, and we can ask whether the family $\{M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}\}_\ell$ of $G_K$-representations over $\mathbb{F}_\ell$ is of weight $w$ for some integer $w$ in the sense of Definition 4.2.5.

Now let us come back to our original setting. By the work of Rapoport-Zink [**106**] and de Jong's alteration [**28**, Theorem 6.5], there is an open subgroup $J$ of $I_K$ such that, for every $\ell \neq p$, the action of $J$ on $H^w_{\text{ét}}(X_{\overline{K}}, \Lambda_\ell)$ is unipotent and factors through $t_\ell$, where $\Lambda_\ell$ is $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, or $\mathbb{F}_\ell$. (See also [**6**, Proposition 6.3.2].) By Theorem 4.2.3, we have

$$\sup_{\ell \neq p} \dim_{\mathbb{F}_\ell} H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell) < \infty.$$

(Alternatively, this fact can be proved by using the argument in [**101**, Section 6.2.4].) Therefore, the family $\{H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)\}_{\ell \neq p}$ satisfies the above two conditions, and we have the monodromy filtration

$$\{M_{i,\mathbb{F}_\ell}\}_i$$

with coefficients in $\mathbb{F}_\ell$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$. We put

$$\mathrm{Gr}^M_{i,\mathbb{F}_\ell} := M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}.$$

Here we omit $X$ and $w$ from the notation. This will not cause any confusion in the context.

A torsion analogue of Conjecture 4.3.1 can be formulated as follows:

**Conjecture 4.3.4.** *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. The family $\{\mathrm{Gr}^M_{i,\mathbb{F}_\ell}\}_\ell$ of finite dimensional $G_K$-representations over $\mathbb{F}_\ell$ defined above is of weight $w+i$ for every $i$ in the sense of Definition 4.2.5.*

**Remark 4.3.5.**

(1) For a finite extension $L$ of $K$, Conjecture 4.3.4 for $(X, w)$ is equivalent to Conjecture 4.3.4 for $(X_L, w)$ by Lemma 4.2.7.
(2) When $X$ has good reduction over $\mathcal{O}_K$, Conjecture 4.3.4 is a consequence of the Weil conjecture and Theorem 4.2.3; see Corollary 4.2.4.

The main theorem of this chapter is as follows.

**Theorem 4.3.6.** *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. We assume that $(X, w)$ satisfies one of the conditions (A)–(E) in Theorem 4.3.2. Then the assertion of Conjecture 4.3.4 for $(X, w)$ is true.*

We will prove Theorem 4.3.6 in Section 4.5–4.9.

**4.3.3. Torsion-freeness of monodromy operators.** In this subsection, we discuss a relation between Conjecture 4.3.1 and Conjecture 4.3.4.

Let $J$ be an open subgroup of $I_K$ such that the action of $J$ on $H^w_{\text{ét}}(X_{\overline{K}}, \Lambda_\ell)$ is unipotent for every $\ell \neq p$, where $\Lambda_\ell$ is $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, or $\mathbb{F}_\ell$. Take an element $\sigma \in J$ such that, for all but finitely many $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator.

**Lemma 4.3.7.** *By pulling back the monodromy filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, we define a filtration $\{M_{i,\mathbb{Z}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$. Then the following two statements for $(X, w)$ are equivalent:*

*(1) For all but finitely many $\ell \neq p$, the reduction modulo $\ell$ of $\{M_{i,\mathbb{Z}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ with coefficients in $\mathbb{F}_\ell$ via the isomorphism $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \cong H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ in Theorem 4.2.3.*
*(2) The cokernel of*

$$(\sigma - 1)^i \colon H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) \to H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$$

*is torsion-free for all but finitely many $\ell \neq p$ and every $i \geq 0$.*

PROOF. Use Theorem 4.2.3, Lemma 4.2.10 and Nakayama's lemma. □

**Definition 4.3.8.** If the two equivalent statements in Lemma 4.3.7 hold for $(X, w)$, then we say that $(X, w)$ *satisfies the property* (t-f).

**Proposition 4.3.9.**

*(1) If $(X, w)$ satisfies the property (t-f), then for all but finitely many $\ell \neq p$, we have*

$$H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \cong H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)^{I_K}$$

*and the $\mathbb{Z}_\ell$-module $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)_{I_K}$ is torsion-free.*
*(2) If Conjecture 4.3.4 for $(X, w)$ is true, then $(X, w)$ satisfies the property (t-f).*
*(3) Assume that Conjecture 4.3.1 for $(X, w)$ is true and $(X, w)$ satisfies the property (t-f). Then Conjecture 4.3.4 for $(X, w)$ is true.*

PROOF. In the proof, we will use the weight filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, which we will recall in Remark 4.4.10.

(1) We may assume that $J = I_K$. Then the assertion follows from the torsion-freeness of the cokernel of $\sigma - 1 \colon H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) \to H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$ and Lemma 4.2.8.

(2) Let $\{W_{i,\mathbb{Q}_\ell}\}_i$ be the weight filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. By pulling buck $\{W_{i,\mathbb{Q}_\ell}\}_i$ to $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$, we have a filtration $\{W_{i,\mathbb{Z}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$. We have $(\sigma-1)(W_{i,\mathbb{Z}_\ell}) \subset W_{i-2,\mathbb{Z}_\ell}$ and the $i$-th graded piece $\mathrm{Gr}^W_{i,\mathbb{Z}_\ell} := W_{i,\mathbb{Z}_\ell}/W_{i-1,\mathbb{Z}_\ell}$ is torsion-free for every $i$. By Theorem 4.2.3, for all but finitely many $\ell \neq p$, we can define a filtration $\{W_{i,\mathbb{F}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ by taking the reduction modulo $\ell$ of the filtration $\{W_{i,\mathbb{Z}_\ell}\}_i$. We define $\mathrm{Gr}^W_{i,\mathbb{F}_\ell} := W_{i,\mathbb{F}_\ell}/W_{i-1,\mathbb{F}_\ell}$. Then the family $\{\mathrm{Gr}^W_{i,\mathbb{F}_\ell}\}_\ell$ is of weight $w+i$; see Proposition 4.4.9 (2) in Section 4.4.

Now we assume that Conjecture 4.3.4 for $(X, w)$ is true. Then $\{W_{i,\mathbb{F}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ with coefficients in $\mathbb{F}_\ell$ for all but finitely many $\ell \neq p$ by Lemma 4.2.2. Thus, the $i$-th iterate $(\sigma-1)^i$ of $\sigma - 1$ induces an isomorphism

$$(\sigma-1)^i\colon \mathrm{Gr}^W_{i,\mathbb{F}_\ell} \cong \mathrm{Gr}^W_{-i,\mathbb{F}_\ell}$$

for every $i \geq 0$ and all but finitely many $\ell \neq p$. By Nakayama's lemma, we have

$$(\sigma-1)^i\colon \mathrm{Gr}^W_{i,\mathbb{Z}_\ell} \cong \mathrm{Gr}^W_{-i,\mathbb{Z}_\ell}$$

for every $i \geq 0$ and all but finitely many $\ell \neq p$. It follows that the weight filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$ for all but finitely many $\ell \neq p$, and the condition (1) in Lemma 4.3.7 is satisfied.

(3) Assume that Conjecture 4.3.1 for $(X, w)$ is true. Then the weight filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$ for every $\ell \neq p$. Assume further that $(X, w)$ satisfies the property (t-f). Then it follows that the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ coincides with $\{W_{i,\mathbb{F}_\ell}\}_i$ for all but finitely many $\ell \neq p$. Thus, Conjecture 4.3.4 for $(X, w)$ is true. □

For later use, we state the following result as a corollary.

**Corollary 4.3.10.** *Assume that $(X, w)$ satisfies one of conditions (A)–(E) in Theorem 4.3.2. Then, for all but finitely many $\ell \neq p$, we have $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \cong H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)^{I_K}$ and the $\mathbb{Z}_\ell$-module $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)_{I_K}$ is torsion-free.*

PROOF. Use Theorem 4.3.2, Theorem 4.3.6, and Proposition 4.3.9. □

**Remark 4.3.11.** Corollary 4.3.10 is a local analogue of a theorem of Cadoret-Hui-Tamagawa [**20**, Theorem 4.5]. (See also Remark 2.9.3.)

## 4.4. Torsion-freeness of the weight spectral sequence

Let $K$ be a Henselian discrete valuation field with ring of integers $\mathcal{O}_K$. The residue field of $\mathcal{O}_K$ is denoted by $k$. Let $p \geq 0$ be the characteristic of $k$. For a prime number $\ell \neq p$, let $t_\ell\colon I_K \to \mathbb{Z}_\ell(1)$ be a map defined in the same way as in Section 4.3. Let $\varpi \in \mathcal{O}_K$ be a uniformizer.

Let $\mathcal{X}$ be a proper scheme over $\mathcal{O}_K$. We assume that $\mathcal{X}$ is *strictly semi-stable* over $\mathcal{O}_K$ purely of relative dimension $d$, i.e. it is, Zariski locally on $\mathcal{X}$, étale over

$$\operatorname{Spec} \mathcal{O}_K[T_0, \ldots, T_d]/(T_0 \cdots T_r - \varpi)$$

for an integer $r$ with $0 \leq r \leq d$.

Let $X$ and $Y$ be the generic fiber and the special fiber of $\mathcal{X}$, respectively. Let $D_1, \ldots, D_m$ be the irreducible components of $Y$. We equip each $D_i$ with the reduced induced subscheme structure. Following [**113**], we introduce some notation. Let $v$ be a non-negative integer. For a non-empty subset $I \subset \{1, \ldots, m\}$ of cardinality $v+1$, we define $D_I := \cap_{i \in I} D_i$

(scheme-theoretic intersection). If $D_I$ is non-empty, then it is purely of codimension $v$ in $Y$. Moreover, we put

$$Y^{(v)} := \coprod_{I\subset\{1,\dots,m\},\, \sharp I=v+1} D_I,$$

where $\sharp I$ denotes the cardinality of $I$.

**Theorem 4.4.1 (Rapoport-Zink [106, Satz 2.10], Saito [113, Corollary 2.8]).** *Let the notation be as above. Let $\ell \neq p$ be a prime number. Let $\Lambda_\ell$ be $\mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Z}_\ell$, or $\mathbb{Q}_\ell$.*

*(1) We have a spectral sequence*

$$E_{1,\Lambda_\ell}^{v,w} = \bigoplus_{i\geq \max(0,-v)} H^{w-2i}_{\text{ét}}(Y^{(v+2i)}_{\overline{k}}, \Lambda_\ell(-i)) \Rightarrow H^{v+w}_{\text{ét}}(X_{\overline{K}}, \Lambda_\ell),$$

*which is compatible with the action of $G_K$. Here $(-i)$ denotes the Tate twist.*

*(2) Let $\sigma \in I_K$ be an element such that $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. There exists the following homomorphism of spectral sequences:*

$$\begin{array}{ccc} E_{1,\Lambda_\ell}^{v,w} = \bigoplus_{i\geq \max(0,-v)} H^{w-2i}_{\text{ét}}(Y^{(v+2i)}_{\overline{k}}, \Lambda_\ell(-i)) & \Longrightarrow & H^{v+w}_{\text{ét}}(X_{\overline{K}}, \Lambda_\ell) \\ \downarrow{\scriptstyle 1\otimes t_\ell(\sigma)} & & \downarrow{\scriptstyle \sigma-1} \\ E_{1,\Lambda_\ell}^{v+2,w-2} = \bigoplus_{i-1\geq \max(0,-v-2)} H^{w-2i}_{\text{ét}}(Y^{(v+2i)}_{\overline{k}}, \Lambda_\ell(-i+1)) & \Longrightarrow & H^{v+w}_{\text{ét}}(X_{\overline{K}}, \Lambda_\ell). \end{array}$$

PROOF. For (1), see [**106**, Satz 2.10] and [**113**, Corollary 2.8 (1)]. We remark that the spectral sequence constructed in [**106**] coincides with that constructed in [**113**] up to signs; see [**113**, p.613]. In this chapter, we use the spectral sequence constructed in [**113**]. The assertion (2) follows from [**113**, Corollary 2.8 (2)]. □

The spectral sequence in Theorem 4.4.1 is called the *weight spectral sequence with $\Lambda_\ell$-coefficients.*

We will discuss the degeneracy of the weight spectral sequence. For the weight spectral sequence with $\mathbb{Q}_\ell$-coefficients, we have the following theorem:

**Theorem 4.4.2.** *The weight spectral sequence with $\mathbb{Q}_\ell$-coefficients degenerates at $E_2$ for every $\ell \neq p$.*

PROOF. See [**95**, Theorem 0.1] or [**69**, Theorem 1.1 (1)]. □

For the weight spectral sequence with $\Lambda_\ell$-coefficients, where $\Lambda_\ell$ is either $\mathbb{F}_\ell$ or $\mathbb{Z}_\ell$, we can prove the following theorem, which relies on the Weil conjecture and Theorem 4.2.3.

**Theorem 4.4.3.**

*(1) Let $\Lambda_\ell$ be either $\mathbb{F}_\ell$ or $\mathbb{Z}_\ell$. For all but finitely many $\ell \neq p$, the weight spectral sequence with $\Lambda_\ell$-coefficients degenerates at $E_2$.*

*(2) For all but finitely many $\ell \neq p$, the $\mathbb{Z}_\ell$-modules $E_{1,\mathbb{Z}_\ell}^{v,w}$ and $E_{2,\mathbb{Z}_\ell}^{v,w}$ are torsion-free and we have $E_{2,\mathbb{Z}_\ell}^{v,w} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \cong E_{2,\mathbb{F}_\ell}^{v,w}$ for all $v, w$.*

PROOF. The torsion-freeness of $E_{1,\mathbb{Z}_\ell}^{v,w}$ for all but finitely many $\ell \neq p$ follows from Theorem 4.2.3.

If $p = 0$, by the comparison of étale and singular cohomology for varieties over $\mathbb{C}$, it follows that the cokernel of the map $d_1^{v,w} \colon E_{1,\mathbb{Z}_\ell}^{v,w} \to E_{1,\mathbb{Z}_\ell}^{v+1,w}$ is torsion-free for all but finitely many $\ell \neq p$ and all $v, w$. Thus Theorem 4.4.3 is a consequence of Theorem 4.4.2.

We assume that $p > 0$. First, we assume that $k$ is finitely generated over $\mathbb{F}_p$. The family $\{E_{1,\Lambda_\ell}^{v,w}\}_{\ell\neq p}$ of $G_k$-modules over $\Lambda_\ell$ is of weight $w$ by Corollary 4.2.4. Since $E_{2,\Lambda_\ell}^{v,w}$ is a subquotient of $E_{1,\Lambda_\ell}^{v,w}$, the family $\{E_{2,\Lambda_\ell}^{v,w}\}_{\ell\neq p}$ is also of weight $w$. Since the map $d_2^{v,w}\colon E_{2,\Lambda_\ell}^{v,w} \to E_{2,\Lambda_\ell}^{v+2,w-1}$ is $G_k$-equivariant, it is a zero map for all but finitely many $\ell \neq p$ by Lemma 4.2.1. This proves the first assertion.

We shall prove the second assertion. By the degeneracy of the weight spectral sequence with $\mathbb{F}_\ell$-coefficients, we have

$$\sum_{v,w} \dim_{\mathbb{F}_\ell} E_{2,\mathbb{F}_\ell}^{v,w} = \sum_i \dim_{\mathbb{F}_\ell} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$$

for all but finitely many $\ell \neq p$. By Theorem 4.4.2, we have

$$\sum_{v,w} \operatorname{rank}_{\mathbb{Z}_\ell} E_{2,\mathbb{Z}_\ell}^{v,w} = \sum_i \operatorname{rank}_{\mathbb{Z}_\ell} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell).$$

By Theorem 4.2.3 and Lemma 4.2.8, for all but finitely many $\ell \neq p$, we have

$$\sum_i \operatorname{rank}_{\mathbb{Z}_\ell} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) = \sum_i \dim_{\mathbb{F}_\ell} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$$

and

$$\operatorname{rank}_{\mathbb{Z}_\ell} E_{2,\mathbb{Z}_\ell}^{v,w} \leq \dim_{\mathbb{F}_\ell} E_{2,\mathbb{F}_\ell}^{v,w}$$

for all $v, w$. It follows that, for all but finitely many $\ell \neq p$, we have

$$\operatorname{rank}_{\mathbb{Z}_\ell} E_{2,\mathbb{Z}_\ell}^{v,w} = \dim_{\mathbb{F}_\ell} E_{2,\mathbb{F}_\ell}^{v,w}$$

for all $v, w$. Now the second assertion follows from Lemma 4.2.8.

The general case can be deduced from the case where $k$ is finitely generated over $\mathbb{F}_p$ by using Néron's blowing up as in [**69**, Section 4] and by using an argument in the proof of [**69**, Lemma 3.2]. □

In the rest of this section, we discuss a relation between Conjecture 4.3.4 and the weight spectral sequence.

Let $\sigma \in I_K$ be an element such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. Let $i \geq 0$ be an integer. The $i$-th iterate of $(1\otimes t_\ell(\sigma))^i$ induces a homomorphism

$$(1 \otimes t_\ell(\sigma))^i\colon E_{2,\Lambda_\ell}^{-i,w+i} \to E_{2,\Lambda_\ell}^{i,w-i},$$

see Theorem 4.4.1 (2). Then we have the following conjecture.

**Conjecture 4.4.4.** *Let $\mathcal{X}$ be a proper strictly semi-stable scheme over* $\operatorname{Spec} \mathcal{O}_K$ *purely of relative dimension $d$. Let the notation be as above. We put $\Lambda_\ell = \mathbb{Q}_\ell$ (resp. $\Lambda_\ell = \mathbb{F}_\ell, \mathbb{Z}_\ell$). Let $w$ be an integer. Then for every $\ell \neq p$ (resp. all but finitely many $\ell \neq p$), the above morphism $(1 \otimes t_\ell(\sigma))^i\colon E_{2,\Lambda_\ell}^{-i,w+i} \to E_{2,\Lambda_\ell}^{i,w-i}$ is an isomorphism for every $i \geq 0$.*

**Remark 4.4.5.** Assume that $p = 0$. Then Conjecture 4.4.4 for $(\mathcal{X}, \Lambda_\ell = \mathbb{Q}_\ell)$ is true; see [**69**, Theorem 1.1 (2)]. Therefore, by a similar argument as in the proof of Theorem 4.4.3, we see that Conjecture 4.4.4 for $(\mathcal{X}, \Lambda_\ell = \mathbb{F}_\ell)$ and $(\mathcal{X}, \Lambda_\ell = \mathbb{Z}_\ell)$ also holds.

**Lemma 4.4.6.** *Conjecture 4.4.4 for $(\mathcal{X}, w, \Lambda_\ell = \mathbb{F}_\ell)$ is equivalent to Conjecture 4.4.4 for $(\mathcal{X}, w, \Lambda_\ell = \mathbb{Z}_\ell)$.*

PROOF. By Theorem 4.4.3, it follows that, for all but finitely many $\ell \neq p$, the $\mathbb{Z}_\ell$-module $E_{2,\mathbb{Z}_\ell}^{v,w}$ is torsion-free and we have $E_{2,\mathbb{Z}_\ell}^{v,w} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \cong E_{2,\mathbb{F}_\ell}^{v,w}$ for all $v, w$. Therefore the assertion follows from Nakayama's lemma. □

In the rest of this section, we assume that $K$ is a non-archimedean local field with residue field $\mathbb{F}_q$.

**Remark 4.4.7.** It is well known that Conjecture 4.3.1 for $(X, w)$ is equivalent to Conjecture 4.4.4 for $(\mathcal{X}, w, \Lambda_\ell = \mathbb{Q}_\ell)$; see [**69**, Proposition 2.5] for example.

Similarly to Remark 4.4.7, we have the following lemma.

**Lemma 4.4.8.** *Let $\mathcal{X}$ be a proper strictly semi-stable scheme over $\operatorname{Spec} \mathcal{O}_K$ purely of relative dimension $d$ with generic fiber $X$ and let $w$ be an integer. Then Conjecture 4.3.4 for $(X, w)$ is equivalent to Conjecture 4.4.4 for $(\mathcal{X}, w, \Lambda_\ell = \mathbb{F}_\ell)$.*

Proof. By Theorem 4.4.3, the weight spectral sequence with $\mathbb{F}_\ell$-coefficients degenerates at $E_2$ for all but finitely many $\ell \neq p$. Hence the claim follows from Lemma 4.2.2, Corollary 4.2.4, Theorem 4.4.1 (2), and the definition of the monodromy filtration. □

Finally, we record the following well known proposition.

**Proposition 4.4.9.** *Let $Z$ be a proper smooth scheme over $\operatorname{Spec} K$ and $w$ an integer. Let $\mathrm{Frob} \in G_K$ be a lift of the geometric Frobenius element.*

*(1) There is a non-zero monic polynomial $P(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \neq p$, we have $P(\mathrm{Frob}) = 0$ on $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Z}_\ell)$.*

*(2) For every $\ell \neq p$, there exists a unique increasing, separated, exhaustive filtration*

$$\{W_{i,\mathbb{Q}_\ell}\}_i$$

*on $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$ which is stable by the action of $G_K$ and satisfies the following property. For every $i$, there exists a Weil $q^{w+i}$-polynomial $P_i(T) \in \mathbb{Z}[T]$ such that $P_i(\mathrm{Frob}) = 0$ on the $i$-th graded piece $\mathrm{Gr}^W_{i,\mathbb{Q}_\ell} := W_{i,\mathbb{Q}_\ell}/W_{i-1,\mathbb{Q}_\ell}$. Moreover, we can take the polynomial $P_i(T) \in \mathbb{Z}[T]$ independent of $\ell \neq p$.*

*(3) Assume that Conjecture 4.3.1 for $(Z, w)$ is true. Then, for every $i$, there exists a Weil $q^{w+i}$-polynomial $P_i(T) \in \mathbb{Z}[T]$ such that, for every $\ell \neq p$, we have $P_i(\mathrm{Frob}) = 0$ on the $i$-th graded piece $\mathrm{Gr}^M_{i,\mathbb{Q}_\ell}$ of the monodromy filtration on $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$.*

Proof. We may assume that $Z$ is geometrically connected. By de Jong's alteration [**28**, Theorem 6.5], there exist a finite extension $L$ of $K$ and a proper strictly semi-stable scheme $\mathcal{X}$ over $\operatorname{Spec} \mathcal{O}_L$ such that the generic fiber $X$ of $\mathcal{X}$ is an alteration of $Z$. We may assume further that $K = L$ and $X$ is geometrically connected. By a trace argument, we see that $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$ is a direct summand of $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ as a $G_K$-representation for every $\ell \neq p$.

Let $\{F^i_{\mathbb{Q}_\ell}\}_i$ be the decreasing filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ arising from the weight spectral sequence. The filtration $\{F^i_{\mathbb{Q}_\ell}\}_i$ defines a decreasing filtration on $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$, which is also denoted by $\{F^i_{\mathbb{Q}_\ell}\}_i$. Let $\{W_{i,\mathbb{Q}_\ell}\}_i$ be the increasing filtration on $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$ defined by

$$W_{i,\mathbb{Q}_\ell} := F^{-i}_{\mathbb{Q}_\ell}.$$

Since the $i$-th graded piece $\mathrm{Gr}^W_{i,\mathbb{Q}_\ell} := W_{i,\mathbb{Q}_\ell}/W_{i-1,\mathbb{Q}_\ell}$ is a subquotient of $E^{-i,w+i}_{1,\mathbb{Q}_\ell}$, by the Weil conjecture, there exists a Weil $q^{w+i}$-polynomial $P_i(T) \in \mathbb{Z}[T]$ such that, for every $\ell \neq p$, we have $P_i(\mathrm{Frob}) = 0$ on $\mathrm{Gr}^W_{i,\mathbb{Q}_\ell}$. Thus the assertion (2) follows.

The assertion (1) follows from (2) and Theorem 4.2.3. If Conjecture 4.3.1 for $(Z, w)$ is true, the filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$. Therefore the assertion (3) follows from (2). □

**Remark 4.4.10.** We call the filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ in Proposition 4.4.9 the *weight filtration* on $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$. (The numbering used here differs from it of [**34**, Proposition-définition (1.7.5)].)

**Remark 4.4.11.** Let $\mathrm{Frob} \in G_K$ be a lift of the geometric Frobenius element. Let $Z$ be a proper smooth scheme over $K$. It is conjectured that the characteristic polynomial $P_{\mathrm{Frob},\ell}(T)$ of Frob acting on $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$ is in $\mathbb{Z}[T]$ and independent of $\ell \neq p$. If $Z$ is a surface or $K$ is of equal characteristic, this conjecture is true; see [**98**, Corollary 2.5] and [**81**, Theorem 1.4]. (See also [**129**, Theorem 3.3].) If this conjecture and Conjecture 4.3.1 for $(Z, w)$ are true, then we can take $P_i(T)$ in Proposition 4.4.9 (3) as the characteristic polynomial of Frob acting on $\mathrm{Gr}^M_{i,\mathbb{Q}_\ell}$.

## 4.5. Equal characteristic cases

In this section, we will prove Theorem 4.3.6 in the case (A). We will use the language of ultraproducts following [**19**]. We first recall some properties of ultraproducts which we need. For details, see [**21**, Appendix] for example. The notation used here is similar to that of [**19**].

**4.5.1. Ultraproducts.** Let $\mathfrak{L}$ be an infinite set of prime numbers. We define

$$\underline{F} := \prod_{\ell \in \mathfrak{L}} \overline{\mathbb{F}}_\ell,$$

where $\overline{\mathbb{F}}_\ell$ is an algebraic closure of $\mathbb{F}_\ell$. For a subset $S \subset \mathfrak{L}$, let $e_S$ be the characteristic function of $\mathfrak{L}\backslash S$, which we consider as an element of $\underline{F}$. Attaching to an ultrafilter $\mathfrak{u}$ on $\mathfrak{L}$ a prime ideal

$$\mathfrak{m}_\mathfrak{u} := \langle e_S \mid S \in \mathfrak{u} \rangle \subset \underline{F}$$

defines a bijection from the set of ultrafilters on $\mathfrak{L}$ to $\operatorname{Spec} \underline{F}$. Note that every prime ideal of $\underline{F}$ is a maximal ideal. An ultrafilter $\mathfrak{u}$ corresponding to a principal ideal is called principal. For a non-principal ultrafilter $\mathfrak{u}$, we define

$$\overline{\mathbb{Q}}_\mathfrak{u} := \underline{F}/\mathfrak{m}_\mathfrak{u}.$$

It is a field of characteristic 0 and is isomorphic to the field of complex numbers $\mathbb{C}$. The field $\overline{\mathbb{Q}}_\mathfrak{u}$ is called the *ultraproduct* of $\{\overline{\mathbb{F}}_\ell\}_{\ell \in \mathfrak{L}}$ (with respect to the non-principal ultrafilter $\mathfrak{u}$). The ring homomorphism $\underline{F} \to \overline{\mathbb{Q}}_\mathfrak{u}$ is flat; see [**21**, Lemma in Section 4.1.4].

**Remark 4.5.1.** Let $\mathfrak{L}' \subset \mathfrak{L}$ be a subset such that $\mathfrak{L}\backslash\mathfrak{L}'$ is finite. The projection $\underline{F} \to \prod_{\ell \in \mathfrak{L}'} \overline{\mathbb{F}}_\ell$ defines a bijection from the set of non-principal ultrafilters on $\mathfrak{L}'$ to the set of non-principal ultrafilters on $\mathfrak{L}$.

Let $\{M_\ell\}_{\ell \in \mathfrak{L}}$ be a family of $\overline{\mathbb{F}}_\ell$-vector spaces. We define

$$\underline{M} := \prod_{\ell \in \mathfrak{L}} M_\ell.$$

For the $\underline{F}$-module $\underline{M}$, the following assertions are equivalent.

- $\underline{M}$ is a finitely generated $\underline{F}$-module.
- $\underline{M}$ is a finitely presented $\underline{F}$-module.
- $\sup_{\ell \in \mathfrak{L}} \dim_{\overline{\mathbb{F}}_\ell} M_\ell < \infty$.

We put $M_\mathfrak{u} := \underline{M} \otimes_{\underline{F}} \overline{\mathbb{Q}}_\mathfrak{u}$ for a non-principal ultrafilter $\mathfrak{u}$. We will use a similar notation for a family $\{f_\ell\}_{\ell \in \mathfrak{L}}$ of maps of $\overline{\mathbb{F}}_\ell$-vector spaces.

**Lemma 4.5.2.** *Let $\{M_\ell\}_{\ell\in\mathfrak{L}}$ and $\{N_\ell\}_{\ell\in\mathfrak{L}}$ be families of $\overline{\mathbb{F}}_\ell$-vector spaces. Assume that $\underline{M}$ and $\underline{N}$ are finitely generated $\underline{F}$-modules. Let $\{f_\ell\}_{\ell\in\mathfrak{L}}$ be a family of maps $f_\ell\colon M_\ell \to N_\ell$ of $\overline{\mathbb{F}}_\ell$-vector spaces. Then the following assertions are equivalent.*

*(1) $f_\mathfrak{u}\colon M_\mathfrak{u} \to N_\mathfrak{u}$ is an isomorphism for every non-principal ultrafilter $\mathfrak{u}$.*
*(2) $f_\ell\colon M_\ell \to N_\ell$ is an isomorphism for all but finitely many $\ell \in \mathfrak{L}$.*

Proof. For a subset $S \subset \mathfrak{L}$ which is contained in every non-principal ultrafilter, the complement $\mathfrak{L}\backslash S$ is finite. Hence the lemma follows from [**21**, Lemma 4.3.3]. □

Let $p$ be a prime number and let $\mathfrak{L}$ be the set of prime numbers $\ell \neq p$. Let $K$ be a Henselian discrete valuation field. Assume that the characteristic of the residue field $k$ of $K$ is $p$. Let $\mathcal{X}$ be a proper strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension $d$. We retain the notation of Section 4.4.

Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathfrak{L}$. Since the map $\underline{F} \to \overline{\mathbb{Q}}_\mathfrak{u}$ is flat, we have the following weight spectral sequence with $\overline{\mathbb{Q}}_\mathfrak{u}$-coefficients:

$$E_{1,\overline{\mathbb{Q}}_\mathfrak{u}}^{v,w} = \bigoplus_{i\geq \max(0,-v)} H^{w-2i}_{\text{ét}}(Y^{(v+2i)}_{\overline{k}}, \overline{\mathbb{Q}}_\mathfrak{u}(-i)) \Rightarrow H^{v+w}_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_\mathfrak{u}).$$

Here we define

$$H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_\mathfrak{u}) := (\prod_{\ell\neq p} H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{F}}_\ell)) \otimes_{\underline{F}} \overline{\mathbb{Q}}_\mathfrak{u},$$

and similarly for $H^{w-2i}_{\text{ét}}(Y^{(v+2i)}_{\overline{k}}, \overline{\mathbb{Q}}_\mathfrak{u}(-i))$. For an element $\sigma \in I_K$ such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator, we have a monodromy operator

$$(1 \otimes t_\ell(\sigma))^i\colon E_{2,\overline{\mathbb{Q}}_\mathfrak{u}}^{-i,w+i} \to E_{2,\overline{\mathbb{Q}}_\mathfrak{u}}^{i,w-i}$$

for all $w, i \geq 0$.

**Lemma 4.5.3.** *Conjecture 4.4.4 for $(\mathcal{X}, w, \Lambda_\ell = \mathbb{F}_\ell)$ is equivalent to the assertion that the morphism*

$$(1 \otimes t_\ell(\sigma))^i\colon E_{2,\overline{\mathbb{Q}}_\mathfrak{u}}^{-i,w+i} \to E_{2,\overline{\mathbb{Q}}_\mathfrak{u}}^{i,w-i}$$

*is an isomorphism for every non-principal ultrafilter $\mathfrak{u}$ on $\mathfrak{L}$ and every $i \geq 0$.*

Proof. The $\underline{F}$-module $\prod_{\ell\neq p}(E^{v,w}_{2,\mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell)$ is finitely generated for all $v, w$ by Theorem 4.2.3. Hence the assertion follows from Lemma 4.5.2. □

Finally, we define an ultraproduct variant of the notion of weight. Let $k$ be a finitely generated field over $\mathbb{F}_p$ and let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathfrak{L}$. Let $\{H_\ell\}_{\ell\in\mathfrak{L}}$ be a family of finite dimensional $G_k$-representations over $\overline{\mathbb{F}}_\ell$ such that the $\underline{F}$-module $\underline{H}$ is finitely generated. Then $H_\mathfrak{u}$ is a finite dimensional representation of $G_k$ over $\overline{\mathbb{Q}}_\mathfrak{u}$. (We do not impose any continuity conditions here.) Let $w$ be an integer. Let $\iota\colon \overline{\mathbb{Q}}_\mathfrak{u} \cong \mathbb{C}$ be an isomorphism. We say that $H_\mathfrak{u}$ is *$\iota$-pure of weight $w$* if the following conditions are satisfied:

- There is an integral scheme $U$ of finite type over $\mathbb{F}_p$ with function field $k$ such that the family $\{H_\ell\}_{\ell\in\mathfrak{L}}$ comes from a family $\{\mathcal{F}_\ell\}_{\ell\in\mathfrak{L}}$ of locally constant constructible $\overline{\mathbb{F}}_\ell$-sheaves on $U$.
- Moreover, for every closed point $x \in U$ and for every eigenvalue $\alpha$ of $\mathrm{Frob}_x$ acting on $H_\mathfrak{u}$, we have $|\iota(\alpha)| = (q_x)^{w/2}$.

(See also Section 4.2.1.) We say that $H_\mathfrak{u}$ is *pure of weight $w$* if it is $\iota$-pure of weight $w$ for every $\iota\colon \overline{\mathbb{Q}}_\mathfrak{u} \cong \mathbb{C}$.

**4.5.2. Proof of Theorem 4.3.6 in the case (A).** We shall prove Theorem 4.3.6 in the case (A). By de Jong's alteration [**28**, Theorem 6.5], a trace argument, and Lemma 4.4.8, it suffices to prove the following theorem.

**Theorem 4.5.4.** *Let $K$ be a Henselian discrete valuation field of equal characteristic $p > 0$. Then Conjecture 4.4.4 for $\Lambda_\ell = \mathbb{F}_\ell$ is true.*

Proof. We use the same strategy as in [**69**]. The only problem is that we cannot use Weil II [**34**] directly since it works with étale cohomology with $\mathbb{Q}_\ell$-coefficients. However, Cadoret recently established an ultraproduct variant of Weil II in [**19**]. By using her results, the same method as in [**69**] can be carried out. We shall explain it.

Let $\mathfrak{L}$ be the set of prime numbers $\ell \neq p$. Let $\mathcal{X}$ be a proper strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension $d$. We retain the notation of Section 4.5.1. By Lemma 4.5.3, it suffices to prove that the morphism $(1 \otimes t_\ell(\sigma))^i \colon E_{2,\overline{\mathbb{Q}}_{\mathfrak{u}}}^{-i,w+i} \to E_{2,\overline{\mathbb{Q}}_{\mathfrak{u}}}^{i,w-i}$ is an isomorphism for every non-principal ultrafilter $\mathfrak{u}$ on $\mathfrak{L}$ and for all $w, i \geq 0$.

By using Néron's blowing up as in [**69**, Section 4] and by using an argument in the proof of [**69**, Lemma 3.2], we may assume that there exist a connected smooth scheme $\operatorname{Spec} A$ over $\mathbb{F}_p$ and an element $\varpi \in A$ satisfying the following properties:

- $D := \operatorname{Spec} A/(\varpi)$ is an irreducible divisor on $A$ which is smooth over $\mathbb{F}_p$ and $\mathcal{O}_K$ is the Henselization of the local ring of $\operatorname{Spec} A$ at the prime ideal $(\varpi) \subset A$.
- There is a proper scheme $\widetilde{\mathcal{X}}$ over $\operatorname{Spec} A$ which is smooth over $\operatorname{Spec} A[1/\varpi]$ such that $\widetilde{\mathcal{X}} \otimes_A \mathcal{O}_K \cong \mathcal{X}$.

Let $f \colon \widetilde{\mathcal{X}} \to \operatorname{Spec} A$ be the structure morphism. The function field of $D$ is the residue field $k$ of $K$, which is finitely generated over $\mathbb{F}_p$.

Let $w \geq 0$ be an integer. By the same construction as in Section 4.3.2, after removing finitely many $\ell \neq p$ from $\mathfrak{L}$, we can construct the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ with coefficients in $\mathbb{F}_\ell$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for every $\ell \in \mathfrak{L}$. We have $\sup_{\ell \in \mathfrak{L}} \dim_{\mathbb{F}_\ell} \operatorname{Gr}^M_{i,\mathbb{F}_\ell} < \infty$, where $\operatorname{Gr}^M_{i,\mathbb{F}_\ell} := M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}$ is the $i$-th graded piece. Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathfrak{L}$. By an analogue of Lemma 4.4.8, it suffices to prove that the $G_k$-representation over $\overline{\mathbb{Q}}_{\mathfrak{u}}$

$$(\prod_{\ell \in \mathfrak{L}} \operatorname{Gr}^M_{i,\mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell) \otimes_{\underline{F}} \overline{\mathbb{Q}}_{\mathfrak{u}}$$

is pure of weight $w + i$ for every $i$.

By applying a construction given in [**34**, Variante (1.7.8)] to the higher direct image sheaf $R^w f_* \mathbb{F}_\ell$ and by using a similar construction as in Section 4.3.2, after removing finitely many $\ell \neq p$ from $\mathfrak{L}$, we get a locally constant constructible $\overline{\mathbb{F}}_\ell$-sheaf $\mathcal{F}_\ell[D]$ on $D$ with a filtration $\{\mathcal{M}_{i,\ell}\}_i$. For every $i$, the stalk of the quotient

$$\operatorname{Gr}^{\mathcal{M}}_{i,\ell} := \mathcal{M}_{i,\ell}/\mathcal{M}_{i-1,\ell}$$

at the geometric generic point of $U$ is isomorphic to $\operatorname{Gr}^M_{i,\mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell$ as a $G_k$-representation for every $\ell \in \mathfrak{L}$.

Let $x \in D$ be a closed point. We can find a connected smooth one-dimensional scheme $C \subset \operatorname{Spec} A$ over $\mathbb{F}_p$ such that $C \cap D = \{x\}$ and the image of $\varpi \in A$ in $\mathcal{O}_{C,x}$ is a uniformizer. Let $L$ be the field of fractions of the completion $\widehat{\mathcal{O}}_{C,x}$ of $\mathcal{O}_{C,x}$. We write $Z := \widetilde{\mathcal{X}} \otimes_A L$. By the construction, for all but finitely many $\ell \in \mathfrak{L}$, the stalk $(\operatorname{Gr}^{\mathcal{M}}_{i,\ell})_{\overline{x}}$ is isomorphic to the base change of the $i$-th graded piece of the monodromy filtration with coefficients in $\mathbb{F}_\ell$ on $H^w_{\text{ét}}(Z_{\overline{L}}, \mathbb{F}_\ell)$ as a $G_{\kappa(x)}$-representation. Thus we see that $(\prod_{\ell \in \mathfrak{L}} (\operatorname{Gr}^{\mathcal{M}}_{i,\ell})_{\overline{x}}) \otimes_{\underline{F}} \overline{\mathbb{Q}}_{\mathfrak{u}}$ is pure of weight $w+i$ by [**19**, Corollary 5.3.2.4] together with Corollary 4.2.4 and [**19**, Lemma in 11.3]. This fact completes the proof of Theorem 4.5.4. □

## 4.6. The case of set-theoretic complete intersections in toric varieties

**4.6.1. Proof of Theorem 4.3.6 in the case (E).** We shall prove Theorem 4.3.6 in the case (E). The proof is the same as that of [**114**, Theorem 9.6], except that we use Corollary 3.4.11 instead of Huber's theorem (Theorem 3.4.5). We shall give a sketch here. In this section, we will freely use the theory of adic spaces developed by Huber and the terminology in [**114**].

Let $K$ be a non-archimedean local field with ring of integers $\mathcal{O}_K$. Let $\mathbb{F}_q$ be the residue field of $\mathcal{O}_K$. Let $p > 0$ be the characteristic of $\mathbb{F}_q$. Since we have already shown that Theorem 4.3.6 holds in the equal characteristic case (A), we may assume that $K$ is of characteristic 0. Let $\mathbb{C}_p$ be the completion of an algebraic closure $\overline{K}$ of $K$.

Let $X$ be a geometrically connected projective smooth scheme over $K$ which is a set-theoretic complete intersection in a projective smooth toric variety $Y_{\Sigma,K}$ over $K$ associated with a fan $\Sigma$. After replacing $K$ by its finite extension, we may assume that the action of $I_K$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ is unipotent and factors through $t_\ell$ for every $w$ and for every $\ell \neq p$.

Let $\varpi$ be a uniformizer of $K$. We fix a system $\{\varpi^{1/p^n}\}_{n\geq 0} \subset \overline{K}$ of $p^n$-th roots of $\varpi$. Let $L$ be the completion of $\bigcup_{n\geq 0} K(\varpi^{1/p^n})$, which is a perfectoid field. Let $G_L = \mathrm{Aut}(\overline{L}/L)$ be the absolute Galois group of $L$, where $\overline{L}$ is the algebraic closure of $L$ in $\mathbb{C}_p$. Then we have a surjection $G_L \to G_{\mathbb{F}_q}$. Thus there exists a lift $\mathrm{Frob} \in G_L$ of the geometric Frobenius element $\mathrm{Frob}_q$. Let $I_L$ be the kernel of the map $G_L \to G_{\mathbb{F}_q}$. We have $I_L \subset I_K$. Since $\bigcup_{n\geq 0} K(\varpi^{1/p^n})$ is a pro-$p$ extension of $K$, there exists an element $\sigma \in I_L$ such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. In other words, there exists an element $\sigma \in I_L$ such that it defines the monodromy filtration with coefficients in $\mathbb{F}_\ell$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$. Therefore, it suffices to prove a natural analogue of Theorem 4.3.6 for the family $\{H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)\}_{\ell\neq p}$ of $G_L$-representations. Moreover, in order to prove this, we can replace $L$ by its finite extension if necessary.

Let $L^\flat$ be the tilt of $L$. We have an identification $G_L = G_{L^\flat}$. The choice of the system $\{\varpi^{1/p^n}\}_{n\geq 0} \subset \overline{K}$ gives an identification between $L^\flat$ and the completion of the perfection of the field of formal Laurent series $\mathbb{F}_q((t))$ over $\mathbb{F}_q$.

Let $Y_{\Sigma,L}$ be the toric variety over $L$ associated with the fan $\Sigma$ and let $Y^{\text{ad}}_{\Sigma,L}$ be the adic space associated with $Y_{\Sigma,L}$. We define $Y^{\text{ad}}_{\Sigma,L^\flat}$ similarly. By [**114**, Theorem 8.5 (iii)], we have a projection

$$\pi\colon Y^{\text{ad}}_{\Sigma,L^\flat} \to Y^{\text{ad}}_{\Sigma,L}$$

of topological spaces. By Corollary 3.4.11, there exists an open subset $V$ of $Y^{\text{ad}}_{\Sigma,L}$ containing $X^{\text{ad}}_L$ such that, for every prime number $\ell \neq p$, the pull-back map

$$H^w_{\text{ét}}(V_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(X^{\text{ad}}_{\mathbb{C}_p}, \mathbb{F}_\ell)$$

is an isomorphism for every $w$. By [**114**, Corollary 8.8], there exists a closed subscheme $Z$ of $Y_{\Sigma,L^\flat}$, which is defined over a global field (i.e. the function field of a smooth connected curve over a finite field), such that $Z^{\text{ad}}$ is contained in $\pi^{-1}(V)$ and $\dim Z = \dim X$. We may assume that $Z$ is irreducible. By [**28**, Theorem 4.1], there exists an alteration $Z' \to Z$, which is defined over a global field, such that $Z'$ is projective and smooth over $L^\flat$. (We note that $L^\flat$ is a perfect field.) We may assume further that $Z'$ and $Z$ are geometrically irreducible after replacing $L^\flat$ by its finite extension.

We have the following composition for every $\ell \neq p$ and every $w$:

$$H^w_{\text{ét}}(X^{\text{ad}}_{\mathbb{C}_p}, \mathbb{F}_\ell) \xrightarrow{\cong} H^w_{\text{ét}}(V_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(\pi^{-1}(V)_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(Z^{\text{ad}}_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell) \to H^w_{\text{ét}}((Z'_{\mathbb{C}_p^\flat})^{\text{ad}}, \mathbb{F}_\ell),$$

where the first map is the inverse map of the pull-back map, the second map is induced by [**114**, Theorem 8.5 (v)], and the last two maps are the pull-back maps. By using a comparison theorem of Huber [**52**, Theorem 3.8.1], we obtain a map

$$H^w_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(Z'_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell)$$

for every $\ell \neq p$ and every $w$. This map is compatible with the actions of $G := G_L = G_{L^\flat}$ on both sides and compatible with the cup products.

For $w = 2\dim X$, by the same argument as in the proof of [**114**, Lemma 9.9], we conclude that the above map is an isomorphism for all but finitely many $\ell \neq p$ from the fact that the image of the $(\dim X)$-th power of the Chern class of an ample line bundle on $Y_{\Sigma,\mathbb{C}_p^\flat}$ under the map

$$H^{2\dim X}_{\text{ét}}(Y_{\Sigma,\mathbb{C}_p^\flat}, \mathbb{F}_\ell) \to H^{2\dim X}_{\text{ét}}(Z'_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell)$$

is not zero for all but finitely many $\ell \neq p$. By Poincaré duality, it follows that $H^w_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{F}_\ell)$ is a direct summand of $H^w_{\text{ét}}(Z'_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell)$ as a $G$-representation for every $w$ and for all but finitely many $\ell \neq p$. Since $Z'$ is defined over a global field, a natural analogue of Theorem 4.3.6 holds for the family $\{H^w_{\text{ét}}(Z'_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell)\}_{\ell \neq p}$ of $G$-representations by the case (A). This fact completes the proof of Theorem 4.3.6 in the case (E).

## 4.7. The case of abelian varieties

**4.7.1. Proof of Theorem 4.3.6 in the case (B).** We use the same notation as in Section 4.3. Let $A$ be an abelian variety over $K$. Let $\mathcal{A}$ be the Néron model of $A$. After replacing $K$ by its finite extension, we may assume that $A$ has semi-abelian reduction, i.e. the identity component $\mathcal{A}^0_s$ of the special fiber $\mathcal{A}_s$ of $\mathcal{A}$ is a semi-abelian variety over $\mathbb{F}_q$. In this case, the action of $I_K$ on the $\ell$-adic Tate module $T_\ell A_{\overline{K}}$ of $A$ is unipotent and factors through $t_\ell \colon I_K \to \mathbb{Z}_\ell(1)$ for every $\ell \neq p$. Let $\sigma \in I_K$ be an element such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator.

Since the quotient $\mathcal{A}_s/\mathcal{A}^0_s$ is a finite étale group scheme over $\mathbb{F}_q$, for all but finitely many $\ell \neq p$, we have

$$A[\ell^n](\overline{K})^{I_K} = \mathcal{A}^0_s[\ell^n](\overline{\mathbb{F}}_q)$$

for every $n \geq 1$ by the Néron mapping property and [**13**, Section 7.3, Proposition 3]. It follows that

$$(T_\ell A_{\overline{K}})^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = (T_\ell A_{\overline{K}} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell)^{I_K}$$

for all but finitely many $\ell \neq p$. For such $\ell \neq p$, the cokernel of $\sigma - 1$ acting on $T_\ell A_{\overline{K}}$ is torsion-free by Lemma 4.2.8. Note that we have $(\sigma - 1)^2 = 0$ on $T_\ell A_{\overline{K}}$. Therefore we see that Conjecture 4.3.4 for $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ is true by Theorem 4.3.2 and Proposition 4.3.9.

Let $w$ be an integer. We can define the monodromy filtration on $\otimes^w H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$; see Section 4.3.2. The assertion of Conjecture 4.3.4 also holds for $\otimes^w H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ by the formula in [**34**, Proposition (1.6.9)(i)]. (Although the base field is of characteristic 0 in *loc. cit.*, the same formula holds with $\mathbb{F}_\ell$-coefficients for all but finitely many $\ell \neq p$.) Since $H^w_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell) \cong \wedge^w H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ is a direct summand of $\otimes^w H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$, it follows that Conjecture 4.3.4 holds for $H^w_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$.

The proof of Theorem 4.3.6 in the case (B) is complete.

## 4.8. The cases of surfaces

**4.8.1. Proof of Theorem 4.3.6 in the case (C).** We shall prove Theorem 4.3.6 in the case (C). We retain the notation of Section 4.3.

By Poincaré duality, it is enough to prove the case $w \leq 2$. By the theory of Picard varieties, the case $w = 1$ follows from the case (B). We may assume $w = 2$. Since we have already proved Theorem 4.3.6 in the case (A), we may assume that the characteristic of $K$ is 0. By de Jong's alteration, we may assume that $X$ is connected and projective over $K$; see [**28**, Theorem 4.1].

Since the hard Lefschetz theorem with $\mathbb{Q}$-coefficients holds for singular cohomology of projective smooth varieties over $\mathbb{C}$, the hard Lefschetz theorem with $\mathbb{Z}_\ell$-coefficients holds for étale cohomology of projective smooth varieties over $K$ for all but finitely many $\ell$. (See also Remark 4.8.1.) Therefore we may assume $\dim X = 2$.

We may assume further that there exists a proper strictly semi-stable scheme $\mathcal{X}$ over $\mathcal{O}_K$ purely of relative dimension 2 with generic fiber $X$ by de Jong's alteration; see [**28**, Theorem 6.5]. By Lemma 4.4.6 and Lemma 4.4.8, it suffices to prove Conjecture 4.4.4 for $(\mathcal{X}, w = 2, \Lambda_\ell = \mathbb{Z}_\ell)$. We use the same notation as in Section 4.4.

We fix an element $\sigma \in I_K$ such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. Using the generator $t_\ell(\sigma)$, we identify $\mathbb{Z}_\ell(i)$ with $\mathbb{Z}_\ell$. We shall prove that the map $(1 \otimes t_\ell(\sigma))^2 \colon E_{2,\mathbb{Z}_\ell}^{-2,4} \to E_{2,\mathbb{Z}_\ell}^{2,0}$ is an isomorphism for all but finitely many $\ell \neq p$. This map is identified with the map

$$\mathrm{Ker}(d_1^{-2,4} \colon H^0(Y^{(2)}, \mathbb{Z}_\ell) \to H^2(Y^{(1)}, \mathbb{Z}_\ell)) \to \mathrm{Coker}(d_1^{1,0} \colon H^0(Y^{(1)}, \mathbb{Z}_\ell) \to H^0(Y^{(2)}, \mathbb{Z}_\ell))$$

induced by the identity map on $H^0(Y^{(2)}, \mathbb{Z}_\ell)$. Here we put $H^i(Y^{(j)}, \mathbb{Z}_\ell) := H^i_{\text{ét}}(Y^{(j)}_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell)$ for simplicity. The map $d_1^{-2,4}$ is a linear combination of Gysin maps and the map $d_1^{1,0}$ is a linear combination of restriction maps. Since $\dim Y^{(1)} = 1$ and $\dim Y^{(2)} = 0$, each cohomology group is the base change of a finitely generated $\mathbb{Z}$-module and the above morphism is defined over $\mathbb{Z}$. These $\mathbb{Z}$-structures are independent of $\ell \neq p$. Hence $E_{2,\mathbb{Z}_\ell}^{-2,4}$, $E_{2,\mathbb{Z}_\ell}^{2,0}$, and the cokernel of the map $E_{2,\mathbb{Z}_\ell}^{-2,4} \to E_{2,\mathbb{Z}_\ell}^{2,0}$ are torsion-free for all but finitely many $\ell \neq p$. Therefore the assertion follows from the fact that the map $E_{2,\mathbb{Q}_\ell}^{-2,4} \to E_{2,\mathbb{Q}_\ell}^{2,0}$ is an isomorphism for every $\ell \neq p$; see Theorem 4.3.2 and Remark 4.4.7.

To prove that the map $1 \otimes t_\ell(\sigma) \colon E_{2,\mathbb{Z}_\ell}^{-1,3} \to E_{2,\mathbb{Z}_\ell}^{1,1}$ is an isomorphism for all but finitely many $\ell \neq p$, it suffices to prove that the restriction of the canonical pairing on $H^1(Y^{(1)}, \mathbb{Z}_\ell)$ to the image of the boundary map

$$d_1^{0,1} \colon E_{1,\mathbb{Z}_\ell}^{0,1} = H^1(Y^{(0)}, \mathbb{Z}_\ell) \to E_{1,\mathbb{Z}_\ell}^{1,1} = H^1(Y^{(1)}, \mathbb{Z}_\ell)$$

is perfect for all but finitely many $\ell \neq p$. For every $i$, let $\mathrm{Pic}^0_{D_i}$ be the Picard variety of $D_i$, i.e. the underlying reduced subscheme of the identity component of the Picard scheme associated with $D_i$. Similarly, let $\mathrm{Pic}^0_{D_i \cap D_j}$ be the Picard variety of $D_i \cap D_j$ for every $i < j$. Since $D_i$ and $D_i \cap D_j$ are proper smooth schemes, the group schemes $\mathrm{Pic}^0_{D_i}$ and $\mathrm{Pic}^0_{D_i \cap D_j}$ are abelian varieties. The Kummer sequence gives isomorphisms $H^1(D_i, \mathbb{Z}_\ell) \cong T_\ell(\mathrm{Pic}^0_{D_i})_{\overline{\mathbb{F}}_q}$ and $H^1(D_i \cap D_j, \mathbb{Z}_\ell) \cong T_\ell(\mathrm{Pic}^0_{D_i \cap D_j})_{\overline{\mathbb{F}}_q}$. (Recall that we have fixed the isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$.) Under these isomorphisms, the map $d_1^{0,1}$ can be identified with the homomorphism of Tate modules induced by a linear combination of pull-back maps

$$\rho \colon \times_i \mathrm{Pic}^0_{D_i} \to \times_{i<j} \mathrm{Pic}^0_{D_i \cap D_j}.$$

We write $A := \times_{i<j} \mathrm{Pic}^0_{D_i \cap D_j}$. Let $B \subset A$ be the image of $\rho$. By the Poincaré complete reducibility theorem, the image of $d_1^{0,1}$ coincides with $T_\ell B_{\overline{\mathbb{F}}_q}$ for all but finitely many $\ell \neq p$. The canonical pairing on $H^1(Y^{(1)}, \mathbb{Z}_\ell)$ is equal to the pairing on $T_\ell A_{\overline{\mathbb{F}}_q}$ induced by a principal polarization on $A_{\overline{\mathbb{F}}_q}$. The restriction of the pairing on $T_\ell A_{\overline{\mathbb{F}}_q}$ to $T_\ell B_{\overline{\mathbb{F}}_q}$ is induced by a polarization on $B_{\overline{\mathbb{F}}_q}$, which is perfect for all but finitely many $\ell \neq p$. This proves our assertion.

The proof of Theorem 4.3.6 in the case (C) is complete.

**Remark 4.8.1.** In [**44**, Complément 6], Gabber announced the hard Lefschetz theorem with $\mathbb{Z}_\ell$-coefficients (for all but finitely many $\ell$) for étale cohomology of projective smooth varieties in positive characteristic.

## 4.9. The cases of varieties uniformized by Drinfeld upper half spaces

**4.9.1. The $\ell$-independence of the weight-monodromy conjecture in certain cases.** In this subsection, we make some preparations for the proof of Theorem 4.3.6 in the case (D). Let $\mathbb{F}_q$ be a finite field of characteristic $p$. Let $Y$ be a projective smooth scheme over $\mathbb{F}_q$. Let $\ell \neq p$ be a prime number. The cycle map for codimension $w$ cycles is denoted by

$$\mathrm{cl}^w_\ell \colon Z^w(Y) \to H^{2w}_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(w)),$$

where $Z^w(Y)$ is the group of algebraic cycles of codimension $w$ on $Y$. We denote by $N^w(Y) := Z^w(Y)/\sim_{\text{num}}$ the group of algebraic cycles of codimension $w$ on $Y$ modulo numerical equivalence. It is known that $N^w(Y)$ is a finitely generated $\mathbb{Z}$-module [**SGA 6**, Exposé XIII, Proposition 5.2].

**Assumption 4.9.1 (Assumption $(*)$).** We say that $Y$ *satisfies the assumption* $(*)$ if, for every $\ell \neq p$, we have $H^w_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = 0$ for every odd integer $w$ and the $\mathbb{Q}_\ell$-vector space $H^{2w}_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(w))$ is spanned by the image of $\mathrm{cl}^w_\ell$ for every $w \geq 0$.

**Lemma 4.9.2.** *Let $Y$ be a projective smooth scheme over $\mathbb{F}_q$. Assume that $Y$ satisfies the assumption $(*)$.*

*(1) The cycle map $\mathrm{cl}^w_\ell$ induces an isomorphism*

$$N^w(Y) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \cong H^{2w}_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(w))$$

*for every $\ell \neq p$ and $w \geq 0$.*

*(2) For all but finitely many $\ell \neq p$, we have $H^w_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell) = 0$ for every odd integer $w$ and the cycle map $\mathrm{cl}^w_\ell$ induces an isomorphism*

$$N^w(Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong H^{2w}_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(w))$$

*for every $w \geq 0$.*

Proof. The assertions can be proved by using the same argument as in [**68**, Lemma 2.1] together with Theorem 4.2.3. □

Let $K$ be a non-archimedean local field with residue field $\mathbb{F}_q$. Let $\mathcal{X}$ be a projective strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension $d$. We use the same notation as in Section 4.4. So $D_1, \ldots, D_m$ are the irreducible components of the special fiber $Y$ of $\mathcal{X}$ and for every non-empty subset $I \subset \{1, \ldots, m\}$, we define $D_I := \cap_{i \in I} D_i$. We will consider the weight spectral sequences arising from $\mathcal{X}$. We fix an element $\sigma \in I_K$ such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator.

**Proposition 4.9.3.** *Let the notation be as above. Assume that for every non-empty subset $I \subset \{1, \dots, m\}$, the intersection $D_I$ satisfies the assumption $(*)$. We assume further that, for some prime number $\ell' \neq p$, the map*

$$(1 \otimes t_{\ell'}(\sigma))^i \colon E_{2,\mathbb{Q}_{\ell'}}^{-i,w+i} \to E_{2,\mathbb{Q}_{\ell'}}^{i,w-i}$$

*is an isomorphism for all $w, i \geq 0$. Then Conjecture 4.4.4 for $\mathcal{X}$ is true.*

PROOF. Using the generator $t_\ell(\sigma)$, we identify $\mathbb{Z}_\ell(i)$ with $\mathbb{Z}_\ell$. Let $\Lambda_\ell$ be $\mathbb{Q}_\ell$ (resp. $\mathbb{Z}_\ell$). The map $d_1^{v,w} \colon E_{1,\Lambda_\ell}^{v,w} \to E_{1,\Lambda_\ell}^{v+1,w}$ is a linear combination of Gysin maps and restriction maps, whose coefficients are in $\mathbb{Z}$ and independent of $\ell \neq p$; see [**113**, Proposition 2.10]. By Lemma 4.9.2, for every $\ell \neq p$ (resp. all but finitely many $\ell \neq p$), this map is the base change of a homomorphism of finitely generated $\mathbb{Z}$-modules which is independent of $\ell \neq p$. Moreover, the same holds for the map $(1 \otimes t_\ell(\sigma))^i \colon E_{2,\Lambda_\ell}^{-i,w+i} \to E_{2,\Lambda_\ell}^{i,w-i}$. Conjecture 4.4.4 for $\mathcal{X}$ follows from this fact. □

**4.9.2. Proof of Theorem 4.3.6 in the case ((D)).** We shall explain the precise statement. Let $K$ be a non-archimedean local field of characteristic 0 with residue field $\mathbb{F}_q$. Let $\Omega_K^d$ be the Drinfeld upper half space over $K$ of dimension $d$. It is a rigid analytic variety over $K$. Let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a discrete cocompact torsion-free subgroup. It is known that the quotient $\Gamma \backslash \Omega_K^d$ is the rigid analytic variety associated with a projective smooth scheme $X$ over $K$. In this case, we say that $X$ is uniformized by a Drinfeld upper half space. We shall prove Conjecture 4.3.4 for $X$.

Let $\widehat{\Omega}_K^d$ be the formal model of $\Omega_K^d$ considered in [**94**], which is a flat formal scheme locally of finite type over $\mathrm{Spf}\, \mathcal{O}_K$. We can take the quotient $\Gamma \backslash \widehat{\Omega}_K^d$. There is a flat projective scheme $\mathcal{X}$ over $\mathrm{Spec}\, \mathcal{O}_K$ whose $\varpi$-adic formal completion is isomorphic to $\Gamma \backslash \widehat{\Omega}_K^d$. Here $\varpi$ is a uniformizer of $K$. The generic fiber of $\mathcal{X}$ is isomorphic to $X$. Let $D_1, D_2, \dots, D_m$ be the irreducible components of the special fiber of $\mathcal{X}$. As in [**68**, Proof of Theorem 1.1], after replacing $\Gamma$ by its finite index subgroup, we may assume that $\mathcal{X}$ is a projective strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension $d$ and, for every non-empty subset $I \subset \{1, 2, \dots, m\}$, the intersection $D_I := \cap_{i \in I} D_i$ satisfies the assumption $(*)$. Since the weight-monodromy conjecture for $X$ is true, we see that Conjecture 4.3.4 for $X$ is true by Lemma 4.4.8 and Proposition 4.9.3.

## 4.10. Applications to Brauer groups and Chow groups of codimension two

In this section, let $K$ be a non-archimedean local field with residue field $\mathbb{F}_q$. Let $p > 0$ be the characteristic of $\mathbb{F}_q$. Let $\mathrm{char}(F)$ denote the characteristic of a field $F$.

**4.10.1. Brauer groups.** In this subsection, we retain the notation of Section 2.9. Let $X$ be a proper smooth scheme over the non-archimedean local field $K$. Let $\ell \neq \mathrm{char}(K)$ be a prime number. Let

$$\mathrm{ch}_{\mathbb{Q}_\ell} \colon \mathrm{Pic}(X)_{\mathbb{Q}_\ell} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))$$

be the $\ell$-adic Chern class map and let

$$\mathrm{ch}_{\mathbb{F}_\ell} \colon \mathrm{Pic}(X) \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell(1))$$

be the $\ell$-torsion Chern class map. They induce homomorphisms

$$\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell} \colon \mathrm{Pic}(X)_{\mathbb{Q}_\ell} \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K} \quad \text{and} \quad \widetilde{\mathrm{ch}}_{\mathbb{F}_\ell} \colon \mathrm{Pic}(X) \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell(1))^{G_K}.$$

We will also call $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ (resp. $\widetilde{\mathrm{ch}}_{\mathbb{F}_\ell}$) the $\ell$-adic (resp. $\ell$-torsion) Chern class map. We shall study the relation between the Chern class maps and the $G_K$-fixed part of the cohomological Brauer group $\mathrm{Br}(X_{\overline{K}})$ of $X_{\overline{K}}$. (Here $G_K$ acts on $\mathrm{Br}(X_{\overline{K}})$ via $\mathrm{Aut}(\overline{K}/K) \cong G_K$.)

**Theorem 4.10.1.** *Let $X$ be a proper smooth scheme over $K$. Assume that the $\ell$-adic Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ is surjective for all but finitely many $\ell \neq p$. Then the following assertions hold:*

*(1) The $\ell$-torsion Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{F}_\ell}$ is surjective for all but finitely many $\ell \neq p$.*
*(2) The $G_K$-fixed part $\mathrm{Br}(X_{\overline{K}})[\ell]^{G_K}$ is zero for all but finitely many $\ell \neq p$.*

PROOF. (1) By Lemma 2.9.2 in Chapter 2, there is a decomposition

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1)) \cong \mathrm{NS}(X_{\overline{K}})_{\mathbb{Z}_\ell} \oplus M_\ell$$

as a $G_K$-module for all but finitely many $\ell \neq p$. By the assumption, we have $M_\ell[1/\ell]^{G_K} = 0$ for all but finitely many $\ell \neq p$. It follows that, for all but finitely many $\ell \neq p$, every eigenvalue of a lift $\mathrm{Frob} \in G_K$ of the geometric Frobenius element acting on $M_\ell[1/\ell]^{I_K}$ is different from 1.

By Proposition 4.4.9 (1), there exists a non-zero monic polynomial $P(T) \in \mathbb{Z}[1/p][T]$ such that, for all but finitely many $\ell \neq p$, we have $P(\mathrm{Frob}) = 0$ on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1))$. We write $P(T)$ in the form $(T-1)^m Q(T)$ for some non-negative integer $m$ and $Q(T) \in \mathbb{Z}[1/p][T]$ with $Q(1) \neq 0$. Then $Q(\mathrm{Frob}) = 0$ on $M_\ell[1/\ell]^{I_K}$, and hence $Q(\mathrm{Frob}) = 0$ on $M_\ell^{I_K}$ for all but finitely many $\ell \neq p$. By Corollary 4.3.10, we have $Q(\mathrm{Frob}) = 0$ on $(M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell)^{I_K} = 0$ for all but finitely many $\ell \neq p$. Since $Q(T)$ and $T-1$ are relatively prime in $\mathbb{Q}[T]$, we have $(M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell)^{G_K} = 0$ for all but finitely many $\ell \neq p$ by Lemma 4.2.2. Now, the assertion follows from the fact that the natural map $\mathrm{Pic}(X) \to (\mathrm{NS}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{F}_\ell)^{G_K}$ is surjective for all but finitely many $\ell \neq p$.

(2) As in the proof of Proposition 2.9.4, the assertion follows from (1). □

**Corollary 4.10.2.** *Assume that* $\mathrm{char}(K) = 0$ *(resp.* $\mathrm{char}(K) = p$*). Let $X$ be a proper smooth scheme over $K$. Assume that the $\ell$-adic Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ is surjective for every $\ell \neq \mathrm{char}(K)$. Then* $\mathrm{Br}(X_{\overline{K}})^{G_K}$ *(resp.* $\mathrm{Br}(X_{\overline{K}})[p']^{G_K}$*) is finite.*

PROOF. This follows from Theorem 4.10.1 and the fact that the union $\cup_n \mathrm{Br}(X_{\overline{K}})[\ell^n]^{G_K}$ is finite for every $\ell \neq \mathrm{char}(K)$, which can be proved by the same argument as in the proof of Proposition 2.9.4. □

Here we give an example of a projective smooth scheme over $K$ for which $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ is surjective for every $\ell \neq \mathrm{char}(K)$.

**Corollary 4.10.3.** *Let $X$ be a projective smooth scheme over $K$ which is uniformized by the Drinfeld upper half space $\Omega^d_K$ of dimension $d \geq 1$, i.e. the rigid analytic variety associated with $X$ is isomorphic to $\Gamma\backslash\Omega^d_K$ for some discrete cocompact torsion-free subgroup $\Gamma \subset \mathrm{PGL}_{d+1}(K)$; see also Section 4.9.2.*

*(1) The $\ell$-adic Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ is surjective for every $\ell \neq \mathrm{char}(K)$.*
*(2) The $G_K$-fixed part* $\mathrm{Br}(X_{\overline{K}})^{G_K}$ *(resp.* $\mathrm{Br}(X_{\overline{K}})[p']^{G_K}$*) is finite if* $\mathrm{char}(K) = 0$ *(resp.* $\mathrm{char}(K) = p$*).*

PROOF. (1) If $d \neq 2$, then $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))$ is one-dimensional for every $\ell \neq \mathrm{char}(K)$. If $d = 2$, then the $G_K$-fixed part $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K}$ is one-dimensional for every $\ell \neq \mathrm{char}(K)$ by [**68**, Lemma 7.1]. Therefore, for any $d \geq 1$, the $\ell$-adic Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ is surjective for every $\ell \neq \mathrm{char}(K)$.

(2) The assertion follows from (1) and Corollary 4.10.2. □

**Remark 4.10.4.** Let $X$ be a proper smooth scheme over $K$. Assume that $\mathrm{char}(K) = p$ or $\dim X = 2$. If the $\ell'$-adic Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{Q}_{\ell'}}$ is surjective for some $\ell' \neq p$, then the same holds for every prime number $\ell \neq \mathrm{char}(K)$. For $\ell \neq p$, this fact can be proved by using Lemma 2.9.2 and the $\ell$-independence conjecture stated in Remark 4.4.11 (it is a theorem under the assumptions). If $\mathrm{char}(K) = 0$, $\dim X = 2$, and $\ell = p$, we use a $p$-adic analogue of the $\ell$-independence conjecture for the Weil-Deligne representation associated with $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$; see [**98**, Theorem 3.1].

**4.10.2. Chow groups of codimension two cycles.** In this subsection, following the strategy of Colliot-Thélène and Raskind [**25**], we show some finiteness properties of the Chow group of codimension two cycles on a proper smooth scheme over $K$.

First we briefly recall a $p$-adic analogue of the weight-monodromy conjecture. Assume that $\mathrm{char}(K) = 0$. Let $W_K$ be the Weil group of $K$. Let $X$ be a proper smooth scheme over $K$. Let

$$\mathrm{WD}(H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_p))$$

be the Weil-Deligne representation of $W_K$ over $\overline{\mathbb{Q}}_p$ associated with $H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_p)$; see [**128**, p.469]. We say that the $p$-adic analogue of the weight-monodromy conjecture holds for $(X, w)$ if $\mathrm{WD}(H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_p))$ is pure of weight $w$ in the sense of [**128**, p.471].

Assume that there exists a proper strictly semi-stable scheme $\mathcal{X}$ over $\mathcal{O}_K$ purely of relative dimension $d$ whose generic fiber is isomorphic to $X$. Let $Y$ be the special fiber of $\mathcal{X}$. Then, by the semi-stable comparison isomorphism [**133**, Theorem 0.2], the $p$-adic analogue of the weight-monodromy conjecture holds for $(X, w)$ if and only if the assertion of [**91**, Conjecture 3.27] holds for the logarithmic crystalline cohomology group $H^w_{\log \mathrm{cris}}(Y/W(\mathbb{F}_q))[1/p]$, where we endow $Y$ with the canonical log structure arising from the strictly semi-stable scheme $\mathcal{X}$.

The following results are analogues of [**25**, Theorem 1.5 and Theorem 1.5.1].

**Proposition 4.10.5.** *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Let $i$ be an integer with $w < 2i$.*

1. *Assume that Conjecture 4.3.1 holds for $(X, w)$. Then $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^{G_K}$ is finite for every $\ell \neq p$. Assume further that Conjecture 4.3.4 for $(X, w)$ is true. Then we have $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^{G_K} = 0$ for all but finitely many $\ell \neq p$.*
2. *If $\mathrm{char}(K) = 0$ and the $p$-adic analogue of the weight-monodromy conjecture is true for $(X, w)$, then $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(i))^{G_K}$ is finite.*

Proof. For every $\ell \neq \mathrm{char}(K)$, we have the following exact sequence of $G_K$-modules:

$$H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i)) \to H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(i)) \xrightarrow{f_\ell} H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to H^{w+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))_{\mathrm{tor}} \to 0.$$

Here $H^{w+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))_{\mathrm{tor}}$ is the torsion part of $H^{w+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))$. Let $H_\ell$ denote the free part of $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))$. We will use the continuous cohomology group $H^j(G_K, H_\ell)$ defined in [**126**, Section 2]. It is a finitely generated $\mathbb{Z}_\ell$-module for every $\ell \neq \mathrm{char}(K)$.

(1) We assume that Conjecture 4.3.1 holds for $(X, w)$. Since $w < 2i$, it follows that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(i))^{G_K} = 0$ for every $\ell \neq p$. To show that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^{G_K}$ is finite for every $\ell \neq p$, it suffices to show that $(\mathrm{Im}\, f_\ell)^{G_K}$ is finite for every $\ell \neq p$. For every $\ell \neq p$, since $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(i))^{G_K} = 0$, we see that $(\mathrm{Im}\, f_\ell)^{G_K}$ is isomorphic to the torsion part of $H^1(G_K, H_\ell)$ by [**126**, Proposition (2.3)]. Hence $(\mathrm{Im}\, f_\ell)^{G_K}$ is finite.

Assume further that Conjecture 4.3.4 for $(X, w)$ is true. Since $H^{w+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))_{\text{tor}} = 0$ for all but finitely many $\ell \neq p$ by Theorem 4.2.3, we have $\operatorname{Im} f_\ell = H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ for all but finitely many $\ell \neq p$. Thus, to show that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^{G_K} = 0$ for all but finitely many $\ell \neq p$, it suffices to prove that the $\mathbb{Z}_\ell$-module $H^1(G_K, H_\ell)$ is torsion-free for all but finitely many $\ell \neq p$. We have the following exact sequence:

$$0 \to H^1(G_k, H_\ell^{I_K}) \to H^1(G_K, H_\ell) \to H^1(I_K, H_\ell).$$

Let $\mathrm{Frob} \in G_K$ be a lift of the geometric Frobenius element. We have

$$H^1(G_k, H_\ell^{I_K}) = \operatorname{Coker}(\mathrm{Frob} - 1\colon H_\ell^{I_K} \to H_\ell^{I_K})$$

and

$$H^1(I_K, H_\ell) = (H_\ell)_{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(-1).$$

We have $H_\ell^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \subset M_{0,\mathbb{Q}_\ell} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(i)$, where $M_{0,\mathbb{Q}_\ell}$ is the 0-th part of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. By Proposition 4.4.9 (3), there exists a non-zero monic polynomial $P(T) \in \mathbb{Z}[1/p][T]$ such that every root of $P(T)$ has complex absolute values $q^{(w+j)/2}$ with $j \leq -2i$ and, for every $\ell \neq p$, we have $P(\mathrm{Frob}) = 0$ on $H_\ell^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Thus we also have $P(\mathrm{Frob}) = 0$ on $H_\ell^{I_K}$ for every $\ell \neq p$. Since $w < 2i$, the polynomials $P(T)$ and $T - 1$ are relatively prime in $\mathbb{Q}[T]$. Thus, we have $H^1(G_k, H_\ell^{I_K}) = 0$ for all but finitely many $\ell \neq p$ by Lemma 4.2.2. Now, it remains to prove that the $\mathbb{Z}_\ell$-module $H^1(I_K, H_\ell)$ is torsion-free for all but finitely many $\ell \neq p$. This follows from Proposition 4.3.9.

(2) If $\operatorname{char}(K) = 0$ and the $p$-adic analogue of the weight-monodromy conjecture holds for $(X, w)$, then we have $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(i))^{G_K} = 0$ if $w < 2i$. Then the same argument as above shows that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(i))^{G_K}$ is finite.

The proof of Proposition 4.10.5 is complete. □

Let $X$ be a proper smooth scheme over $K$. The Chow group of codimension two cycles on $X_{\overline{K}}$ is denoted by $\mathrm{CH}^2(X_{\overline{K}})$. By combining Proposition 4.10.5 and [**25**, Proposition 3.1], we have the following results on the torsion part of $\mathrm{CH}^2(X_{\overline{K}})$, which are local analogues of [**25**, Theorem 3.3 and Theorem 3.4].

**Corollary 4.10.6.** *Let $X$ be a proper smooth scheme over $K$.*

*(1) Assume that Conjecture 4.3.1 and Conjecture 4.3.4 hold for $(X, w = 3)$. The prime-to-p torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ is finite.*

*(2) Assume that $\operatorname{char}(K) = 0$ and the p-adic analogue of the weight-monodromy conjecture holds for $(X, w = 3)$. Then $\cup_n \mathrm{CH}^2(X_{\overline{K}})[p^n]^{G_K}$ is finite.*

Proof. By [**25**, Proposition 3.1], there is a $G_K$-equivariant injection

$$\cup_n \mathrm{CH}^2(X_{\overline{K}})[\ell^n] \hookrightarrow H^3_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

for every $\ell \neq \operatorname{char}(K)$. Thus the assertions follow from Proposition 4.10.5. □

**Corollary 4.10.7.**

*(1) If $(X, w = 3)$ satisfies one of the conditions (A)–(E) in Theorem 4.3.2, then the prime-to-p torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ is finite.*

*(2) Assume that $\operatorname{char}(K) = 0$ and $(X, w = 3)$ satisfies one of the conditions (B)–(D) in Theorem 4.3.2. Then the torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ is finite.*

Proof. (1) Use Theorem 4.3.2, Theorem 4.3.6, and Corollary 4.10.6 (1).

(2) Under the assumptions, the $p$-adic analogue of the weight-monodromy conjecture holds for $(X, w = 3)$. Indeed, if $X$ is an abelian variety over $K$, then this is well known;

since we have $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \wedge^w H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, it suffices to prove the $p$-adic analogue of the weight-monodromy conjecture for $(Z, w = 1)$ for every proper smooth scheme $Z$ over $K$, and this follows from the hard Lefschetz theorem and [**91**, Théorème 5.3]. If $X$ is a proper smooth surface over $K$, by Poincaré duality, this follows from what we have just seen. If $X$ is uniformized by a Drinfeld upper half space, this follows from [**68**, Theorem 6.3]. (See also [**91**, 3.33].)

Therefore, the assertion follows from (1) and Corollary 4.10.6 (2). □

**Remark 4.10.8.**

(1) If $X$ has good reduction over $\mathcal{O}_K$, the finiteness of the prime-to-$p$ torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ was known; see the proof of [**25**, Theorem 3.4].

(2) Assume that $\mathrm{char}(K) = 0$. Moreover, we assume that $\dim X = 2$ or $H^3_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) = 0$ for some (and hence every) $\ell$. The finiteness of the torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ was known; see [**26**, Section 4] and the proof of [**112**, Theorem 4.1]. When $\dim X = 2$, it is a consequence of Roitman's theorem; see [**25**, Remark 3.5] for details.

CHAPTER 5

# Deformations of rational curves

## 5.1. Introduction

In this chapter, we study deformations of rational curves and their singularities in positive characteristic. We use this to prove that if a proper smooth surface in positive characteristic $p$ is dominated by a family of rational curves such that one member has all $\delta$-invariants (resp. Jacobian numbers) strictly less than $(p-1)/2$ (resp. $p$), then the surface has negative Kodaira dimension. This chapter is based on the joint work with Tetsushi Ito and Christian Liedtke [**66**].

**5.1.1. Main results.** A *rational curve* is a proper integral curve over an algebraically closed field whose normalization is isomorphic to the projective line $\mathbb{P}^1$. Rational curves are central to higher dimensional algebraic geometry, as already indicated by the title of Kollár's fundamental book [**75**]. Let us shortly recall the situation in dimension two.

(1) In characteristic 0, it is well known that rational curves on surfaces of non-negative Kodaira dimension are *topologically rigid*, i.e., do not deform in positive-dimensional families.
(2) In characteristic $p > 0$, then the situation is different: Zariski gave examples of unirational surfaces of non-negative Kodaira dimension [**139**]. Therefore, rational curves on surfaces of non-negative Kodaira dimension may *not* be topologically rigid. However, in this case, the general member of such a positive-dimensional family of rational curves is *not* smooth.

This poses the interesting question what can be said about topological (non-)rigidity of rational curves on varieties of non-negative Kodaira dimension in positive characteristics.

We recall some classical invariants of singularities. Let $C$ be an integral curve over an algebraically closed field $k$ of characteristic $p > 0$. For each closed point $x \in C$, the $\delta$-invariant and the Jacobian number of $C$ at $x$ are defined as follows:

- The *$\delta$-invariant* is defined by
$$\delta(C, x) := \dim_k(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)_x,$$
where $\pi\colon \widetilde{C} \to C$ is the normalization morphism.
- The *Jacobian number* is defined by
$$\mathrm{jac}(C, x) := \dim_k \left(\mathcal{O}_C/\operatorname{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k})\right)_x,$$
where $\Omega^1_{C/k}$ is the sheaf of Kähler differentials on $C$ and $\operatorname{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k}) \subset \mathcal{O}_C$ is the first Fitting ideal of $\Omega^1_{C/k}$. There are several definitions of Jacobian numbers in the literature. (See Section 5.3 for details.)

Next, we recall the notion of families of rational curves and uniruledness of varieties following [**75**]. Let $X$ be a proper smooth variety over $k$.

- A *family of rational curves on* $X$ means a closed subvariety $\mathscr{C} \subset U \times X$ with projections $\pi\colon \mathscr{C} \to U$ and $\varphi\colon \mathscr{C} \to X$ such that $U$ is an integral variety over $k$, $\pi$ is proper flat, and every geometric fiber of $\pi$ is an integral rational curve. We say that a rational curve $C \subset X$ is *topologically non-rigid* if there exists a family of rational curves $(\pi, \varphi)$ on $X$ with $\dim \varphi(\mathscr{C}) \geq 2$ such that $\varphi(\mathscr{C}_u) = C$ for some closed point $u \in U$. Otherwise, we say that $C$ is *topologically rigid*.
- We say that $X$ is *uniruled* if there exist an integral variety $Y$ with $\dim Y = \dim X - 1$ and a dominant rational map $\psi\colon \mathbb{P}^1 \times Y \dashrightarrow X$. If there exists a such a rational map $\psi$ inducing a separable extension of function fields, we say that $X$ is *separably uniruled*.

Here is the statement of our main theorem in this chapter.

**Theorem 5.1.1.** *Let $X$ be a proper smooth surface over an algebraically closed field $k$ of characteristic $p > 0$. Assume that $X$ contains a topologically non-rigid rational curve $C \subset X$ satisfying one of the following conditions:*

*(1) The $\delta$-invariants of $C$ are strictly less than $(p-1)/2$ at every closed point.*
*(2) The Jacobian numbers of $C$ are strictly less than $p$ at every closed point.*

*Then, $X$ is separably uniruled and thus, has negative Kodaira dimension.*

**Remark 5.1.2.** Let $X$ be a proper smooth surface over an algebraically closed field $k$. Then the following assertions are equivalent:

- $X$ is birationally equivalent to a ruled surface.
- $X$ is separably uniruled.
- $X$ has negative Kodaira dimension.

This follows from the classification of surfaces; see [**3**, Theorem 13.2] for example.

Theorem 5.1.1 implies the following corollary.

**Corollary 5.1.3.** *Let $X$ be a proper smooth surface of non-negative Kodaira dimension over $k$. Let $C \subset X$ be a rational curve.*

*(1) If every singularity of $C$ is a node, then $C$ is topologically rigid.*
*(2) If $p \geq 5$ and every singularity of $C$ is either a node or an ordinary cusp, then $C$ is topologically rigid.*
*(3) If $C^2 + K_X \cdot C < p - 3$, then $C$ is topologically rigid (see Corollary 5.7.1).*

**Remark 5.1.4.** We believe that both invariants are useful: $\delta$-invariants are more often used in the literature and they can be bounded using intersection theory; see Corollary 5.7.1. On the other hand, a node $x \in C$ satisfies $\delta(C, x) = \mathrm{jac}(C, x) = 1$, i.e., to conclude the topological rigidity in Corollary 5.1.3 (1) in small characteristics, we have to use the criterion in terms of Jacobian numbers, since the criterion in terms $\delta$-invariants would only give topological rigidity for $p \geq 5$.

**Remark 5.1.5.** Our results are optimal in some sense; see Proposition 5.7.7.

**Remark 5.1.6.** In the course of the proofs of main results, we give a sufficient criterion in terms of Jacobian numbers (resp. $\delta$-invariants) for the smoothness of the normalization of a curve over an imperfect field of characteristic $p > 0$. Such results might be of independent interest. See Theorem 5.3.10 and Theorem 5.4.7 for details.

**5.1.2. Outline of this chapter.** This chapter is organized as follows. In Section 5.2, we discuss rational curves and topological rigidity. In Section 5.3, we collect some basic properties of Jacobian numbers of curves over arbitrary fields. In Section 5.4, we recall the definition and basic properties of $\delta$-invariants of curves over arbitrary fields. In Section 5.5, we prove a lemma that is used in the proof of Theorem 5.1.1. In Section 5.6, we prove Theorem 5.1.1. Finally, in Section 5.7, we give some examples of topologically rigid and topologically non-rigid rational curves on surfaces.

## 5.2. Families of rational curves

In this section, we fix some definitions and recall some basic properties of topologically non-rigid rational curves. The standard reference for rational curves on varieties is Kollár's book [**75**].

**Definition 5.2.1.** Let $X$ be a proper variety over an algebraically closed field $k$. Let $U$ be an integral variety over $k$.

(1) A *rational curve* on $X$ is an integral closed subvariety $C \subset X$ of dimension 1, whose normalization is isomorphic to $\mathbb{P}^1$ over $k$.
(2) A *flat family of rational curves* parameterized by $U$ is a proper flat morphism $\pi\colon \mathscr{C} \to U$ such that, for every geometric point $s \to U$, the geometric fiber $\mathscr{C}_s := s \times_U \mathscr{C}$ is an integral rational curve over the residue field $\kappa(s)$.
(3) A *family of rational curves on* $X$ parametrized by $U$ is a closed subscheme $\mathscr{C} \subset U \times X$ such that the projection $\pi\colon \mathscr{C} \to U$ is a flat family of rational curves. Let $\varphi\colon \mathscr{C} \to X$ be the projection to $X$.
   A rational curve $C \subset X$ is said to be *topologically non-rigid* if there exists a family of rational curves $(\pi, \varphi)$ on $X$ with $\dim \varphi(\mathscr{C}) \geq 2$ and $\varphi(\mathscr{C}_u) = C$ for some closed point $u \in U$. Otherwise, we say that $C$ is *topologically rigid*. (In particular, a topologically rigid curve in our sense is allowed to deform infinitesimally on $X$, but not in a positive dimensional family.)
(4) A *map from a family of rational curves* to $X$ parametrized by $U$ is a pair of morphisms $\pi\colon \mathscr{C} \to U$ and $\varphi\colon \mathscr{C} \to X$ over $k$ such that
   (a) $\pi$ is a flat family of rational curves, and
   (b) $\dim \varphi(\mathscr{C}_s) = 1$ for every geometric point $s \to U$.

**Lemma 5.2.2.** *Let $X$ be a proper integral variety over an algebraically closed field $k$, and let $U$ be an integral variety over $k$. Let $\pi\colon \mathscr{C} \to U$ and $\varphi\colon \mathscr{C} \to X$ be morphisms over $k$. Assume that $\mathscr{C}$ is reduced and that $\pi$ is proper and flat with one-dimensional fibers. Let $W := (\pi \times \varphi)(\mathscr{C})$ be the image of $\pi \times \varphi\colon \mathscr{C} \to U \times X$ endowed with the reduced induced subscheme structure. Let $\mathrm{pr}_1\colon W \to U$ be the projection onto the first factor.*

*(1) Assume that the fiber $\mathscr{C}_u$ is reduced for some closed point $u \in U$. Then, there exists a dense open subset $U' \subset U$ such that the fiber $\mathrm{pr}_1^{-1}(s) := s \times_U W$ is reduced for every geometric point $s \to U'$.*
*(2) Assume moreover that $X$ is a proper smooth surface, $U$ is a smooth curve, $\mathscr{C}$ is irreducible, and the fiber $\mathscr{C}_{u_0}$ is generically reduced and $\varphi_{u_0}\colon \mathscr{C}_{u_0} \to X$ is a generic immersion for some closed point $u_0 \in U$. Then, there exists an open neighborhood $u_0 \in U' \subset U$ such that the fiber $\mathrm{pr}_1^{-1}(s)$ is reduced for every geometric point $s \to U'$.*

Proof. (1) Let $\overline{\eta} \to U$ be the geometric generic point. The fiber $\mathrm{pr}_1^{-1}(\overline{\eta})$ is the schematic image of the morphism $\mathscr{C}_{\overline{\eta}} \to X \otimes_k \kappa(\overline{\eta})$. Since $\mathscr{C}_{\overline{\eta}}$ is reduced by [**EGA IV 3**, Théorème 12.2.4 (v)], it follows that $\mathrm{pr}_1^{-1}(\overline{\eta})$ is reduced. After possibly shrinking $U$, the fiber $\mathrm{pr}_1^{-1}(s)$ is reduced for every geometric point $s \to U$ by [**EGA IV 3**, Théorème 12.2.4 (v)].

(2) It is enough to show that the fiber $\mathrm{pr}_1^{-1}(u_0)$ is reduced; see [**EGA IV 3**, Théorème 12.2.4 (v)]. Since $U \times X$ is smooth over $k$, its reduced closed subscheme $W \subset U \times X$ is a Cartier divisor. Moreover $W$ is flat over $U$ since $U$ is a smooth curve over $k$. Since $\mathrm{pr}_1^{-1}(u_0)$ has no embedded points by [**80**, Chapter 8, Proposition 2.15], we only need to prove that it is generically reduced; see [**80**, Chapter 7, Exercise 1.2]. We consider $\mathscr{C}_{u_0} = \pi^{-1}(u_0) \subset \mathscr{C}$ and $\mathrm{pr}_1^{-1}(u_0) \subset W$ as Cartier divisors on $\mathscr{C}$ and $W$, respectively. Since $\mathscr{C}_{u_0} = (\pi \times \varphi)^* \mathrm{pr}_1^{-1}(u_0)$, we have the following equality of 1-cycles on $W$:

$$(\pi \times \varphi)_*[\mathscr{C}_{u_0}] = (\pi \times \varphi)_* \big((\pi \times \varphi)^*[\mathrm{pr}_1^{-1}(u_0)]\big) = d \cdot [\mathrm{pr}_1^{-1}(u_0)],$$

where $d := [k(\mathscr{C}) : k(W)]$ is the extension degree of function fields; see [**80**, Theorem 7.2.18]. By our assumptions on $\varphi_{u_0}$, we have $d = 1$ and thus, the Cartier divisor $\mathrm{pr}_1^{-1}(u_0) \subset W$ has multiplicity one. Consequently, the fiber $\mathrm{pr}_1^{-1}(u_0)$ is generically reduced. □

The following result is well known, at least in characteristic 0. We give a brief sketch of the proof for the reader's convenience. (See also [**75**, Proposition IV.1.3].)

**Proposition 5.2.3.** *For a proper integral variety $X$ with $\dim X \geq 2$ over an algebraically closed field $k$, the following conditions are equivalent:*

*(1) $X$ is uniruled.*

*(2) $X$ is dominated by a family of rational curves on $X$, i.e., there exist an integral variety $U$ with $\dim U = \dim X - 1$ and a closed subvariety $\mathscr{C} \subset U \times X$ as in Definition 5.2.1 (3) such that $\varphi\colon \mathscr{C} \to X$ is dominant.*

*(3) $X$ is dominated by a map from a family of rational curves, i.e., there exist an integral variety $U$ with $\dim U = \dim X - 1$ and a pair $(\pi, \varphi)$ as in Definition 5.2.1 (4) such that $\varphi\colon \mathscr{C} \to X$ is dominant.*

Proof. (1) ⇒ (3): Assume that $X$ is uniruled. Then there exists a dominant rational map $\psi\colon \mathbb{P}^1 \times Y \dashrightarrow X$ with $\dim Y = \dim X - 1$. Shrinking $Y$ if necessary, we may assume $Y$ is smooth. Then, $\psi$ is defined in codimension 1 and thus, there exists a closed subvariety $Z \subset \mathbb{P}^1 \times Y$ with $\dim Z \leq \dim Y - 1$ such that $\psi$ is defined outside $Z$. Removing $\mathrm{pr}_2(Z)$ from $Y$, we may assume that $\psi$ is defined everywhere. Then, $\psi\colon \mathbb{P}^1 \times Y \to X$ gives rise to a map from a family of rational curves parametrized by $Y$ and dominating $X$.

(3) ⇒ (2): Take a pair $(\pi, \varphi)$ as in Definition 5.2.1 (4). After shrinking $U$, we may assume that the fiber $\mathrm{pr}_1^{-1}(s)$ of the image $W := (\pi \times \varphi)(\mathscr{C})$ is an integral rational curve for every geometric point $s \to U$ by Lemma 5.2.2 (1).

(2) ⇒ (1): Choose a closed subvariety $\mathscr{C} \subset U \times X$ as in Definition 5.2.1 (3). Let $K := k(U)$ be the function field of $U$. After replacing $U$ by a finite covering $U' \to U$ and replacing $\mathscr{C}$ by the normalization of the base change $\mathscr{C} \times_U U'$, we find a dominant morphism $\mathscr{C} \to X$ such that the generic fiber $\mathscr{C}_K$ is a geometrically irreducible and smooth curve over $K$; see [**EGA IV 4**, Proposition 17.15.14]. Moreover, shrinking $U$ further and replacing $U$ by an étale covering, we may assume that $\mathscr{C} \to U$ is a $\mathbb{P}^1$-bundle. Hence, $X$ is uniruled. □

## 5.3. Jacobian numbers of curves over arbitrary fields

In this section, we fix an arbitrary field $k$ of characteristic $p \geq 0$ and we recall the definition and basic properties of Jacobian numbers of curves over $k$. Most of the results in this section are well known if $k = \mathbb{C}$. We also give brief proofs of the results recalled in this section because we need to apply them to curves over function fields of curves, for which we could not find appropriate references.

**5.3.1. Definition of Jacobian numbers.** Let $\overline{k}$ be an algebraic closure of $k$ and let $k^{\mathrm{sep}}$ be the separable closure of $k$ in $\overline{k}$. Let $C$ be a curve over $k$. Let $\Omega^1_{C/k}$ be the sheaf of Kähler differentials on $C$ and let $\mathrm{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k}) \subset \mathcal{O}_C$ be the first Fitting ideal of $\Omega^1_{C/k}$. (For the definition and basic properties of Fitting ideals, we refer to [**45**, Section 16.29].)

**Definition 5.3.1.** For a closed point $x \in C$, the *Jacobian number* of $C$ at $x$ is defined by $\mathrm{jac}(C, x) := \dim_k \left(\mathcal{O}_C / \mathrm{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k})\right)_x$.

**Remark 5.3.2.** There are several definitions of Jacobian numbers in the literature. In this chapter, we adopt Schröer's definition in terms of Fitting ideals; see [**115**, Section 3, p. 64]. The advantage of this definition is that it makes sense for all curves and that it behaves well scheme-theoretically. For plane curves, it coincides with the more traditional definition of Jacobian numbers (as in [**17**], [**46**], or [**130**]) in terms of the dimension of $\mathrm{Ext}^1$ of sheaves, or in terms of the ideal generated by partial derivatives of the defining equation; see Proposition 5.3.7 and Corollary 5.3.13 for details.

**Proposition 5.3.3.** *Let $C$ be a curve over $k$. Then, for a closed point $x \in C$, we have* $\mathrm{jac}(C, x) = 0$ *if and only if $C$ is smooth at $x$.*

Proof. We have $\mathrm{jac}(C, x) = 0$ if and only if $\Omega^1_{C/k}$ is locally free of rank 1 in an open neighborhood of $x$; see [**45**, Remark 16.30]. Hence, by [**80**, Chapter 6, Proposition 2.2], we have $\mathrm{jac}(C, x) = 0$ if and only if $C$ is smooth at $x$. □

The closed subscheme of $C$ defined by $\mathrm{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k})$ is called the *Jacobian subscheme* and by the previous proposition, its support coincides with the non-smooth locus of $C$ over $k$. If $C$ is geometrically reduced, then the smooth locus of $C$ over $k$ is open and dense. It follows that we have $\mathrm{jac}(C, x) = 0$ for all but finitely many closed points $x \in C$ and that we have $\mathrm{jac}(C, x) < \infty$ for every closed point $x \in C$.

**5.3.2. Regular curves over imperfect fields with small Jacobian numbers.**

**Proposition 5.3.4.** *Let $k$ be a field of characteristic $p > 0$ that is not necessarily perfect. Let $C$ be a curve over $k$. Assume that the Jacobian numbers of $C$ are strictly less than $p$ at every closed point of $C$, and $C$ is a regular scheme. Then, $C$ is smooth over $k$.*

Proof. Seeking a contradiction, assume that there exists a closed point $x \in C$ such that $C \to \operatorname{Spec} k$ is not smooth around $x$. Since $C$ was assumed to be regular, it follows that the residue field extension $\kappa(x)/k$ is not separable; see [**115**, Proposition 3.2]. In particular, $[\kappa(x) : k]$ is divisible by $p$. Hence the dimension of the stalk $(\mathcal{O}_C / \mathrm{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k}))_x$ as a $k$-vector space is divisible by $p$. (See also [**115**, Lemma 3.3 and Proposition 3.6].) On the other hand, since $\mathrm{jac}(C, x) < p$, we have $\mathrm{jac}(C, x) = 0$. Since $C$ is not smooth around $x$, this contradicts Proposition 5.3.3. □

**5.3.3. Closed subschemes and finite base change.**

**Proposition 5.3.5.** *Let $C$ and $C'$ be curves over a field $k$ together with a closed immersion $i \colon C' \hookrightarrow C$. For every closed point $x \in C'$, we have* $\mathrm{jac}(C', x) \leq \mathrm{jac}(C, x)$.

Proof. Since $i$ is a closed immersion, we have a natural surjection $i^*\Omega^1_{C/k} \to \Omega^1_{C'/k}$; see [**80**, Chapter 6, Proposition 1.24(d)]. Hence, we have

$$i^* \operatorname{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k}) = \operatorname{Fitt}^1_{\mathcal{O}_{C'}}(i^*\Omega^1_{C/k}) \subset \operatorname{Fitt}^1_{\mathcal{O}_{C'}}(\Omega^1_{C'/k}),$$

from which the assertion follows. □

**Proposition 5.3.6.** *Let $C$ be a curve over a field $k$ and let $k'/k$ be a field extension. Let $p\colon C_{k'} \to C$ be the projection. For every closed point $x \in C$, we have*

$$\operatorname{jac}(C, x) = \sum_{y \in p^{-1}(x)} \operatorname{jac}(C_{k'}, y).$$

Proof. Since $p^*\Omega^1_{C/k} = \Omega^1_{C_{k'}/k'}$, we have

$$p^* \operatorname{Fitt}^1_{\mathcal{O}_C}(\Omega^1_{C/k}) = \operatorname{Fitt}^1_{\mathcal{O}_{C_{k'}}}(\Omega^1_{C_{k'}/k'});$$

see [**45**, Proposition 16.29 (3)]. The assertion easily follows from this equality. □

**5.3.4. Local complete intersection curves.** In this subsection, we study Jacobian numbers of curves that are *local complete intersections*; for the definition of this notion, see [**80**, Chapter 6, Definition 3.17]. For a local complete intersection curve $C$, we can calculate Jacobian numbers using dualizing sheaves: we have a canonical map $c_{C/k}\colon \Omega^1_{C/k} \to \omega_{C/k}$, called the *class map*, from the sheaf of Kähler differentials $\Omega^1_{C/k}$ to the dualizing sheaf $\omega_{C/k}$; see [**80**, Chapter 6, Corollary 4.13].

**Proposition 5.3.7.** *Let $C$ be a local complete intersection curve over $k$. For a closed point $x \in C$, we have $\operatorname{jac}(C, x) = \dim_k \operatorname{Coker}(c_{C/k})_x$. Moreover, if $C$ is geometrically reduced, then we have*

$$\operatorname{jac}(C, x) = \dim_k \operatorname{Coker}(c_{C/k})_x = \dim_k \operatorname{Ext}^1_{\mathcal{O}_{C,x}}(\Omega^1_{C/k,x}, \mathcal{O}_{C,x}).$$

Proof. The first equality follows from an explicit description of the class map $c_{C/k}$ as in [**80**, Chapter 6, Section 6.4.2]. Let us briefly recall it. Since $C$ is a local complete intersection over $k$, there exists an affine open neighborhood $U$ of $x \in C$ such that $U \cong \operatorname{Spec} A$ , where $A = k[T_1, \ldots, T_{n+1}]/I$ for an ideal $I = (F_1, \ldots, F_n)$ of $k[T_1, \ldots, T_{n+1}]$. The dualizing module $\omega_{A/k} := \Gamma(U, \omega_{U/k})$ is given by

$$\det(I/I^2)^\vee \otimes_A \left(\det(\Omega^1_{k[T_1,\ldots,T_{n+1}]/k}) \otimes_{k[T_1,\ldots,T_{n+1}]} A\right).$$

It is a free $A$-module of rank one with basis

$$e := (\overline{F}_1 \wedge \cdots \wedge \overline{F}_n)^\vee \otimes ((dT_1 \wedge \cdots \wedge dT_{n+1}) \otimes 1_A),$$

where $\overline{F}_i$ is the image of $F_i$ in $I/I^2$ and where $1_A \in A$ is the identity; see [**80**, Chapter 6, Lemma 4.12]. With respect to this basis, the class map $c_{U/k}$ is given by

$$c_{U/k}\colon \Omega^1_{A/k} \to \omega_{A/k}, \quad dt_i \mapsto \Delta_i \cdot e.$$

Here, $dt_i \in \Omega^1_{A/k}$ is the image of $T_i$ in $\Omega^1_{A/k}$ under the universal derivation and $\Delta_i \in A$ denotes the determinant of the Jacobian matrix $(\partial F_i/\partial T_j)_{i,j}$ with $i$.th column removed. Therefore, under the isomorphism $A \cong \omega_{A/k}$ that sends $1_A$ to $e$, the ideal of $A$ corresponding to the image of the class map $c_{U/k}$ is equal to the first Fitting ideal $\operatorname{Fitt}^1_A(\Omega^1_{A/k})$; see [**45**, Section 16.9]. This establishes the first equality.

The second equality was essentially proved by Rim in [**109**]. However, there it is somewhat implicit in the proofs of the main theorems of [**109**], as well as under the additional assumption that $k$ is perfect. Let us briefly explain how to deduce the second equality from the results in [**109**]: since the statement is local, we may use the same setup and notation as before. Shrinking $U$ if necessary, we may assume $U\setminus\{x\}$ is smooth over $k$. Let $(\Omega^1_{A/k})_{\mathrm{tors}}$ be the torsion submodule of $\Omega^1_{A/k}$, i.e.,

$$(\Omega^1_{A/k})_{\mathrm{tors}} := \left\{\, m \in \Omega^1_{A/k} \mid \text{there is a regular element } a \in A \text{ such that } am = 0 \,\right\}.$$

Rim proved the equality

$$\operatorname{length}_A(\Omega^1_{A/k})_{\mathrm{tors}} = \operatorname{length}_A(\operatorname{Coker} c_{U/k})$$

using the generalized Koszul complexes of Buchsbaum and Rim; see [**109**, Theorem 1.2 (ii)] and [**109**, Corollary 1.3 (ii)]. (In fact, [**109**, Theorem 1.2 (ii)] is a general result in commutative algebra, which is valid for Cohen-Macaulay algebras. The perfectness of the base field was not used there.) Let $\mathfrak{m}_x \subset A$ be the maximal ideal corresponding to $x \in C$ and let $A_x$ be the localization of $A$. Since $U\setminus\{x\}$ is smooth over $k$, we have $(\Omega^1_{A/k})_{\mathrm{tors}} = (\Omega^1_{A_x/k})_{\mathrm{tors}}$, which is an $A_x$-module of finite length. Since $A_x$ is a one-dimensional Gorenstein local ring, Grothendieck's local duality gives an isomorphism

$$\operatorname{Ext}^1_{A_x}\left(\Omega^1_{A_x/k}, \omega_{A_x/k}\right) \cong \operatorname{Hom}_{A_x}\left(H^0_{\mathfrak{m}_x}(\Omega^1_{A_x/k}), E(A_x/\mathfrak{m}_x)\right),$$

where $E(A_x/\mathfrak{m}_x)$ is an injective hull of the residue field $A_x/\mathfrak{m}_x$; see [**16**, Theorem 3.5.8]. Since $A_x$ is a Cohen-Macaulay ring and $(\Omega^1_{A_x/k})_{\mathrm{tors}}$ is a module of finite length, we have

$$H^0_{\mathfrak{m}_x}(\Omega^1_{A_x/k}) = (\Omega^1_{A_x/k})_{\mathrm{tors}}.$$

The right hand side is identified with the Matlis dual of $(\Omega^1_{A_x/k})_{\mathrm{tors}}$. Since Matlis duality preserves the length of Artinian modules (see [**16**, Proposition 3.2.12]), we have

$$\operatorname{length}_{A_x}\operatorname{Ext}^1_{A_x}(\Omega^1_{A_x/k}, \omega_{A_x/k}) = \operatorname{length}_{A_x}(\Omega^1_{A_x/k})_{\mathrm{tors}}.$$

Since $\omega_{A_x/k}$ is a free $A_x$-module of rank one, the second equality of this proposition follows from above results. ☐

**Remark 5.3.8.** Proposition 5.3.7 is well known if $C$ is a plane curve over the complex numbers; see, for example [**17**, Lemma 1.1.2, Corollary 6.1.6] or [**46**, Chapter II, p. 317, the proof of Lemma 2.32]. The second equality in Proposition 5.3.7 was attributed to Rim in [**39**, Proposition 2.2]. We refer to the proof of [**39**, Proposition 2.2] for a historical account.

**Proposition 5.3.9.** *Let $C$ and $C'$ be two reduced local complete intersection curves over $k$. Let $f\colon C' \to C$ be a finite morphism over $k$ and assume that there exists a dense open subset $U \subset C$ such that the restriction $f^{-1}(U) \to U$ is an isomorphism. Let $g$ be the composition $\Omega^1_{C/k} \to f_*\Omega^1_{C'/k} \to f_*\big(\Omega^1_{C'/k}/\operatorname{Ker}(c_{C'/k})\big)$. If $x \in C$ is a closed point, then*

$$\operatorname{jac}(C,x) = \dim_k \operatorname{Coker}(g)_x + \dim_k\left((f_*\mathcal{O}_{C'})/\mathcal{O}_C\right)_x + \sum_{y \in f^{-1}(x)} \operatorname{jac}(C', y).$$

*In particular, for every closed point $y \in C'$ lying above $x$, we have* $\operatorname{jac}(C', y) \le \operatorname{jac}(C, x)$.

PROOF. We have the following the sequence of homomorphisms of $\mathcal{O}_C$-modules

$$\Omega^1_{C/k} \xrightarrow{\;g\;} f_*\big(\Omega^1_{C'/k}/\operatorname{Ker}(c_{C'/k})\big) \xrightarrow{f_*c_{C'/k}} f_*\omega_{C'/k} \xrightarrow{\;h\;} \omega_{C/k},$$

whose composition is equal to the class map $c_{C/k}$. By assumption, $h$ is generically an isomorphism. Since $C'$ is reduced and a local complete intersection, $\omega_{C'/k}$ is torsion free.

Hence, $h$ is injective. Moreover, $f_* c_{C'/k}$ is injective. We obtain the two short exact sequences

$$0 \longrightarrow \operatorname{Coker}(g) \longrightarrow \operatorname{Coker}(c_{C/k}) \longrightarrow \operatorname{Coker}(h \circ f_* c_{C'/k}) \longrightarrow 0$$

and

$$0 \longrightarrow f_* \operatorname{Coker}(c_{C'/k}) \longrightarrow \operatorname{Coker}(h \circ f_* c_{C'/k}) \longrightarrow \operatorname{Coker}(h) \longrightarrow 0.$$

Taking dimensions of the stalks, we obtain the following equality

$$\operatorname{jac}(C, x) = \dim_k \operatorname{Coker}(g)_x + \dim_k \operatorname{Coker}(h)_x + \dim_k \left(f_* \operatorname{Coker}(c_{C'/k})\right)_x .$$

By Proposition 5.3.7, the last term is equal to the sum $\sum_{y \in f^{-1}(x)} \operatorname{jac}(C', y)$.

It remains show

$$\dim_k \operatorname{Coker}(h)_x = \dim_k \left((f_* \mathcal{O}_{C'})/\mathcal{O}_C\right)_x .$$

We put $A := \mathcal{O}_{C,x}$. Then, $B := \Gamma(C' \otimes_C \operatorname{Spec} A, \mathcal{O}_{C' \otimes_C \operatorname{Spec} A})$ is a finite semi-local $A$-algebra. We have a short exact sequence of $A$-modules $0 \to A \to B \to B/A \to 0$. By assumption, the $A$-module $B/A$ is of finite length. Since $A$ is a local complete intersection, it is Gorenstein, and thus, the dualizing module $\omega_A$ is a free $A$-module of rank 1. Hence $\operatorname{Hom}_A(B/A, \omega_A) = 0$ and $H^0_{\mathfrak{m}_A}(B/A) = B/A$, where $\mathfrak{m}_A$ denotes the maximal ideal of $A$. By Grothendieck's local duality [**16**, Theorem 3.5.8], we have $\operatorname{Ext}^1_A(B, \omega_A) = \operatorname{Hom}_A\left(H^0_{\mathfrak{m}_A}(B), E(A/\mathfrak{m}_A)\right) = 0$. We also have $\operatorname{Hom}_A(A, \omega_A) = \omega_A$. Hence, we obtain the following short exact sequence:

$$0 \longrightarrow \operatorname{Hom}_A(B, \omega_A) \xrightarrow{h} \omega_A \longrightarrow \operatorname{Ext}^1_A(B/A, \omega_A) \longrightarrow 0.$$

Grothendieck's local duality gives an isomorphism

$$\operatorname{Ext}^1_A(B/A, \omega_A) \cong \operatorname{Hom}_A\left(H^0_{\mathfrak{m}_A}(B/A), E(A/\mathfrak{m}_A)\right).$$

The right hand side is the Matlis dual of $B/A$ because $H^0_{\mathfrak{m}_A}(B/A) = B/A$. Since Matlis duality preserves lengths (see [**16**, Proposition 3.2.12]), we have

$$\operatorname{length}_A(\operatorname{Coker}(h)) = \operatorname{length}_A \operatorname{Ext}^1_A(B/A, \omega_A) = \operatorname{length}_A(B/A).$$

Therefore we have $\dim_k \operatorname{Coker}(h)_x = \dim_k \left((f_* \mathcal{O}_{C'})/\mathcal{O}_C\right)_x$. ☐

**5.3.5. A criterion for the normalization of a curve being smooth.** In this subsection, we give a sufficient condition in terms of Jacobian numbers for the smoothness of the normalization of local complete intersection curves.

**Theorem 5.3.10.** *Let $C$ be an integral curve over a (possibly imperfect) field $k$ of characteristic $p > 0$. Assume that $C$ is local complete intersection over $k$, and the Jacobian numbers of $C$ are strictly less than $p$ at every closed point of $C$. Let $\pi\colon \widetilde{C} \to C$ be the normalization morphism. Then $\widetilde{C}$ is smooth over $k$.*

Proof. Since $\widetilde{C}$ and $\operatorname{Spec} k$ are both regular schemes, the morphism $\widetilde{C} \to \operatorname{Spec} k$ is a local complete intersection; see [**80**, Chapter 6, Example 3.18]. By Proposition 5.3.9, we have $\operatorname{jac}(\widetilde{C}, y) \leq \operatorname{jac}(C, x) < p$ for all closed points $x \in C$ and $y \in \widetilde{C}$ with $\pi(y) = x$. Hence, $\widetilde{C}$ is smooth over $k$ by Proposition 5.3.4. ☐

**Remark 5.3.11.** Theorem 5.3.10 is in some sense optimal; see Lemma 5.7.6 for the construction of a non-smooth regular curve over an imperfect field which has a singular point of Jacobian number $p$.

### 5.3.6. Jacobian numbers and completions.

**Proposition 5.3.12.** *Let $C$ and $C'$ be curves over $k$. Let $x \in C$ and $x' \in C'$ be closed points, such that the completed local rings are isomorphic, i.e., there exists an isomorphism of $k$-algebras $\widehat{\mathcal{O}}_{C,x} \cong \widehat{\mathcal{O}}_{C',x'}$. Then, we have* $\mathrm{jac}(C,x) = \mathrm{jac}(C',x')$.

Proof. It is enough to show that the Jacobian number $\mathrm{jac}(C,x)$ can be calculated in terms of the completed local ring $\widehat{\mathcal{O}}_{C,x}$. We set $A := \mathcal{O}_{C,x}$ and $\widehat{A} := \widehat{\mathcal{O}}_{C,x}$ and let $\Omega^1_{C/k,x}$ be the stalk of $\Omega^1_{C/k}$ at $x$. By definition, we have

$$\mathrm{jac}(C,x) = \dim_k \left(A/\operatorname{Fitt}^1_A(\Omega^1_{C/k,x})\right).$$

If $A/\operatorname{Fitt}^1_A(\Omega^1_{C/k,x})$ is a finite dimensional $k$-vector space, then there exists some $N \geq 1$ such that

$$\mathfrak{m}_A^N \subset \operatorname{Fitt}^1_A(\Omega^1_{C/k,x}),$$

where $\mathfrak{m}_A \subset A$ is the maximal ideal. Hence, we find

$$A/\operatorname{Fitt}^1_A(\Omega^1_{C/k,x}) \cong \left(A/\operatorname{Fitt}^1_A(\Omega^1_{C/k,x})\right) \otimes_A \widehat{A} \cong \widehat{A}/\left(\operatorname{Fitt}^1_A(\Omega^1_{C/k,x}) \otimes_A \widehat{A}\right)$$

Moreover, we have

$$\operatorname{Fitt}^1_A(\Omega^1_{C/k,x}) \otimes_A \widehat{A} = \operatorname{Fitt}^1_{\widehat{A}}(\Omega^1_{C/k,x} \otimes_A \widehat{A})$$

since the formation of Fittings ideals commutes with base change. Combining these results, we find

$$\mathrm{jac}(C,x) = \dim_k \left(\widehat{A}/\operatorname{Fitt}^1_{\widehat{A}}(\Omega^1_{C/k,x} \otimes_A \widehat{A})\right).$$

Note that the right hand side of this equation depends only on the completion $\widehat{A} = \widehat{\mathcal{O}}_{C,x}$ since

$$\Omega^1_{C/k,x} \otimes_A \widehat{A} \cong \varprojlim_n \left(\Omega^1_{\widehat{A}/k}/\mathfrak{m}^n_{\widehat{A}}\Omega^1_{\widehat{A}/k}\right),$$

where $\mathfrak{m}_{\widehat{A}} \subset \widehat{A}$ is the maximal ideal; see [**76**, Corollary 12.10]. If $\mathrm{jac}(C,x) = \infty$, then we also have

$$\mathrm{jac}(C,x) = \dim_k \left(\widehat{A}/\operatorname{Fitt}^1_{\widehat{A}}(\Omega^1_{C/k,x} \otimes_A \widehat{A})\right),$$

because the homomorphism

$$A/\operatorname{Fitt}^1_A(\Omega^1_{C/k,x}) \to \left(A/\operatorname{Fitt}^1_A(\Omega^1_{C/k,x})\right) \otimes_A \widehat{A} \cong \widehat{A}/\operatorname{Fitt}^1_{\widehat{A}}(\Omega^1_{C/k,x} \otimes_A \widehat{A})$$

is faithfully flat and hence, injective. □

**Corollary 5.3.13.** *Let $C$ be a curve over $k$ and let $x \in C(k)$ be a $k$-rational point. If the complete local ring $\widehat{\mathcal{O}}_{C,x}$ is isomorphic to $k[[S,T]]/(f)$ for some non-zero formal power series $f \in k[[S,T]]$ with $f(0,0) = 0$, then we have* $\mathrm{jac}(C,x) = \dim_k (k[[S,T]]/(f_S, f_T, f))$. *Here, $f_S$ and $f_T$ are the derivatives of $f$ with respect to $S$ and $T$, respectively.*

Proof. We set $\widehat{A} := \widehat{\mathcal{O}}_{C,x} \cong k[[S,T]]/(f)$. By the proof of Proposition 5.3.12, it suffices to calculate the first Fitting ideal of

$$\varprojlim_n \left(\Omega^1_{\widehat{A}/k}/\mathfrak{m}^n_{\widehat{A}}\Omega^1_{\widehat{A}/k}\right),$$

where $\mathfrak{m}_{\widehat{A}} \subset \widehat{A}$ is the maximal ideal. This $\widehat{A}$-module can be calculated as follows: we have

$$\Omega^1_{\widehat{A}/k}/\mathfrak{m}^N_{\widehat{A}}\Omega^1_{\widehat{A}/k} \cong \left((\widehat{A}/\mathfrak{m}^N_{\widehat{A}})dS \oplus (\widehat{A}/\mathfrak{m}^N_{\widehat{A}})dT\right)/J_N,$$

where $J_N$ is the $(\widehat{A}/\mathfrak{m}_{\widehat{A}}^N)$-module generated by the image of $df := f_S\, dS + f_T\, dT$. Taking the projective limit with respect to $N$, we have

$$\varprojlim_{N}(\Omega^1_{\widehat{A}/k}/\mathfrak{m}_{\widehat{A}}^N\Omega^1_{\widehat{A}/k}) \cong (\widehat{A}\, dS \oplus \widehat{A}\, dT)/\widehat{J},$$

where $\widehat{J}$ is the $\widehat{A}$-module generated by the image of $df$ in $\widehat{A}\, dS \oplus \widehat{A}\, dT$. From this, we see that the first Fitting ideal of the above $\widehat{A}$-module is generated by $f_S$ and $f_T$. □

### 5.3.7. Upper semicontinuity of Jacobian numbers.

**Proposition 5.3.14.** *Let $U$ be a Noetherian integral scheme. Let $\pi\colon \mathscr{C} \to U$ be a flat family of proper and geometrically reduced curves parameterized by $U$. Let $u_0 \in U$ be a closed point, let $N$ be a non-negative integer, and assume that the Jacobian numbers of $\mathscr{C}_{u_0}$ are smaller than or equal to $N$ at every closed point. Then, there exists a non-empty open subset $U' \subset U$ such that for every point $x \in U'$ (not necessarily closed) the Jacobian numbers of the curve $\mathscr{C}_x \otimes_{\kappa(x)} \kappa(x)^{\mathrm{sep}}$ over $\kappa(x)^{\mathrm{sep}}$ are smaller than or equal to $N$ at every closed point. (We do not require the open subset $U'$ contains the closed point $u_0$.)*

Proof. Since the smooth locus of $\pi\colon \mathscr{C} \to U$ is dense in every fiber, the support of $\mathcal{O}_{\mathscr{C}}/\operatorname{Fitt}^1_{\mathcal{O}_{\mathscr{C}}}(\Omega^1_{\mathscr{C}/U})$ has only finitely many closed points in each fiber of $\pi$. Let $i_Z\colon Z \hookrightarrow \mathscr{C}$ be the closed subscheme of $\mathscr{C}$ defined by $\operatorname{Fitt}^1_{\mathcal{O}_{\mathscr{C}}}(\Omega^1_{\mathscr{C}/U})$. The morphism $Z \to U$ is finite because it is both proper and quasi-finite. Let $\eta \in U$ be the generic point, let $\mathscr{C}_\eta := \mathscr{C} \times_U \eta$ be the generic fiber, and let $t_1, \dots, t_n$ be closed points of $\mathscr{C}_\eta$ such that $Z_\eta = \{t_1, \dots, t_n\}$.

We put $\overline{u}_0 := \operatorname{Spec} \kappa(u_0)^{\mathrm{sep}} \to U$, which is a geometric point above $u_0$. Let $\widetilde{U}_{\overline{u}_0}$ be the strict Henselization of $U$ relative to $\overline{u}_0$. Then, every connected component of $Z \times_U \widetilde{U}_{\overline{u}_0}$ has a unique element above the closed point of $\widetilde{U}_{\overline{u}_0}$. Since the strict Henselization is a direct limit of étale neighborhoods of $u_0$, we may assume, after possibly replacing $U$ by an étale neighborhood of $u_0$, that every connected component of $Z \times_U \operatorname{Spec} \mathcal{O}_{U,u_0}$ has a unique element above $u_0$. (Here we use Proposition 5.3.6: for a field extension $k'/\kappa(u_0)$, the Jacobian numbers of $\mathscr{C}_{u_0} \otimes_{\kappa(u_0)} k'$ are also smaller than or equal to $N$. Hence, it is enough to prove the assertion after shrinking $U$ and replacing $U$ by an étale neighborhood of $u_0$.)

Let $W$ be a connected component of $Z \times_U \operatorname{Spec} \mathcal{O}_{U,u_0}$ that intersects non-trivially with the generic fiber $Z_\eta$. We put $A := \mathcal{O}_{U,u_0}$ and $B := \mathcal{O}_W$. Then, $B$ is a finite $A$-algebra. Let $s \in W$ be the unique element above $u_0$. We have $\operatorname{jac}(\mathscr{C}_{u_0}, s) = \dim_{A/\mathfrak{m}_A}(B \otimes_A (A/\mathfrak{m}_A))$, where $\mathfrak{m}_A \subset A$ is the maximal ideal corresponding to $u_0$. Similarly, for every point $t_i \in W$ in the generic fiber, we have

$$\operatorname{jac}(\mathscr{C}_\eta, t_i) = \dim_{\operatorname{Frac}(A)}(B \otimes_A \operatorname{Frac}(A))_{\mathfrak{p}_i},$$

where $\mathfrak{p}_i$ is the prime ideal of $B \otimes_A \operatorname{Frac}(A)$ corresponding to $t_i$. Then, we have

$$\begin{aligned}\sum_{t_i \in W} \operatorname{jac}(\mathscr{C}_\eta, t_i) &= \dim_{\operatorname{Frac}(A)}(B \otimes_A \operatorname{Frac}(A)) \\ &\leq \dim_{A/\mathfrak{m}_A}(B \otimes_A (A/\mathfrak{m}_A)) = \operatorname{jac}(\mathscr{C}_{u_0}, s)\end{aligned}$$

by Nakayama's lemma. Since we assumed $\operatorname{jac}(\mathscr{C}_{u_0}, s) \leq N$, we have $\operatorname{jac}(\mathscr{C}_\eta, t_i) \leq N$ for every $t_i \in W$. This shows the assertion of this proposition for the generic fiber.

Replacing $U$ by an étale neighborhood of $\eta$ if necessary, we may assume that the following three conditions are satisfied:

(1) the residue fields at $t_i$ for $i = 1, \dots, n$ are purely inseparable extensions of $\kappa(\eta)$,

(2) the Zariski closures $\overline{\{t_i\}} \subset Z$ for $i = 1, \dots, n$, do not intersect with each other over $U$, and
(3) the morphism $Z \to U$ is flat.

Let $u \in U$ be a point, which is not necessarily closed. Since the $\overline{\{t_i\}}$ $(1 \le i \le n)$ do not intersect over $u \in U$ and since the morphism $Z \to U$ is flat, for each element $s \in Z$ above $u$, there is a unique integer $i$ with $s \in \overline{\{t_i\}}$. We note that the Zariski closure $\overline{\{t_i\}}$ of $t_i$ in $Z \times_U \operatorname{Spec} \mathcal{O}_{U,u}$ is a connected component of $Z \times_U \operatorname{Spec} \mathcal{O}_{U,u}$ and $s$ is the unique element of $\overline{\{t_i\}}$ above $u$. Since $\Gamma(\overline{\{t_i\}}, \mathcal{O}_{Z \times_U \operatorname{Spec} \mathcal{O}_{U,u}})$ is a free $\mathcal{O}_{U,u}$-module of finite rank, by the same argument as before, we have $\operatorname{jac}(\mathscr{C}_\eta, t_i) = \operatorname{jac}(\mathscr{C}_u, s)$. Hence, we have $\operatorname{jac}(\mathscr{C}_u, s) \le N$. $\square$

## 5.4. $\delta$-invariants of curves over arbitrary fields

In this section, we briefly recall the definition and the basic properties of $\delta$-invariants which we need. For a curve over an algebraically closed field, we define $\delta$-invariants in the usual way. For a curve over an imperfect field, we basically only consider the $\delta$-invariants of the base change of the curve to an algebraically closed field because we want to study non-smooth points rather than non-regular points. Therefore, it is useful to introduce a variant of the $\delta$-invariant, which we call the *geometric* $\delta$-invariant, of a closed point of a curve over an arbitrary field.

Let $C$ be a geometrically reduced curve over a field $k$. We put $\overline{C} := C \otimes_k \overline{k}$. Let $\pi\colon \widetilde{\overline{C}} \to \overline{C}$ be the normalization morphism. Let $p\colon \overline{C} \to C$ be the projection.

**Definition 5.4.1.** For a closed point $x \in \overline{C}$, the *$\delta$-invariant* of $\overline{C}$ at $x$ is defined to be

$$\delta(\overline{C}, x) := \dim_{\overline{k}} (\pi_* \mathcal{O}_{\widetilde{\overline{C}}} / \mathcal{O}_{\overline{C}})_x.$$

For a closed point $x \in C$, the *geometric $\delta$-invariant* of $C$ at $x$ is defined to be

$$\delta(C, x) := \sum_{y \in p^{-1}(x)} \delta(\overline{C}, y).$$

We collect some basic properties, which can be verified immediately.

**Proposition 5.4.2.** *Let $C$ be a geometrically integral curve over a field $k$. Let $\pi\colon \widetilde{C} \to C$ be the normalization morphism. We assume that $\widetilde{C}$ is smooth over $k$. Then we have $\delta(C, x) = \dim_k (\pi_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C)_x$.*

**Proposition 5.4.3.** *Let $C$ be a geometrically reduced curve over a field $k$. For a closed point $x \in C$, we have $\delta(C, x) = 0$ if and only if $C$ is smooth at $x$.*

**Proposition 5.4.4.** *Let $C$ and $C'$ be geometrically reduced curves over a field $k$ together with a closed immersion $i\colon C' \hookrightarrow C$. For every closed point $x \in C'$, we have the inequality $\delta(C', x) \le \delta(C, x)$.*

**Proposition 5.4.5.** *Let $C$ be a geometrically reduced curve over a field $k$ and let $k'/k$ be a field extension. Let $p\colon C_{k'} \to C$ be the projection. For every closed point $x \in C$, we have $\delta(C, x) = \sum_{y \in p^{-1}(x)} \delta(C_{k'}, y)$.*

**Proposition 5.4.6.** *Let $C$ and $C'$ be two geometrically reduced curves over a field $k$. Let $f\colon C' \to C$ be a finite morphism over $k$ and assume that there exists a dense open subset $U \subset C$ such that the restriction $f^{-1}(U) \to U$ is an isomorphism. If $x \in C$ is a closed point, then $\delta(C, x) = \dim_k (f_* \mathcal{O}_{C'} / \mathcal{O}_C)_x + \sum_{y \in f^{-1}(x)} \delta(C', y)$.*

We give a sufficient criterion in terms of $\delta$-invariants for the smoothness of the normalization of a curve over a possibly imperfect field.

**Theorem 5.4.7.** *Let $C$ be a regular and geometrically integral curve over a (possibly imperfect) field $k$ of characteristic $p > 0$. Assume that $\delta(\overline{C}, x) < (p-1)/2$ for every closed point $x \in \overline{C}$. Then $C$ is smooth over $k$.*

Proof. By Proposition 5.4.5, after replacing $k$ by its finite separable extension, we may assume that $\delta(C, x) < (p-1)/2$ for every closed point $x \in C$. We choose a finite extension $k'$ of $k$, such that the normalization $\widetilde{C_{k'}}$ of $C_{k'}$ is smooth over $k'$. We have to show that $\mathcal{O}_{C_{k'},x}$ is regular for every closed point $x \in C_{k'}$. We fix a closed point $x \in C_{k'}$ and set $A := \mathcal{O}_{C_{k'},x}$. Let $B$ be the normalization of $A$, which is a finite semi-local $A$-module. We will use the same notation as in the proof of Proposition 5.3.9. The conductor ideal $I \subset A$ is defined by the image of the map $h\colon \operatorname{Hom}_A(B, A) \to A$, $\phi \mapsto \phi(1)$. It turns out that $I$ is an ideal of $B$. As in the proof of Proposition 5.3.9, we have

$$\operatorname{length}_A(\operatorname{Coker}(h)) = \operatorname{length}_A(B/A).$$

Since $A/I$ is isomorphic to $\operatorname{Coker}(h)$ as an $A$-module, we have

$$\operatorname{length}_A(A/I) = \operatorname{length}_A(B/A).$$

By the following short exact sequence of $A$-modules

$$0 \to A/I \to B/I \to B/A \to 0,$$

we have

$$\operatorname{length}_A(B/I) = 2 \cdot \operatorname{length}_A(B/A).$$

If $A$ is not regular, we have $\operatorname{length}_A(B/I) \geq p-1$ by [**102**, Theorem 1.2]. This implies

$$\dim_{k'}(B/A) = [\kappa(x) : k'] \cdot \operatorname{length}_A(B/A) \geq (p-1)/2.$$

Since $\widetilde{C_{k'}}$ is smooth, we have

$$\delta(C_{k'}, x) = \dim_{k'}(B/A)$$

by Proposition 5.4.2. Thus, we find $\delta(C_{k'}, x) \geq (p-1)/2$. This contradicts the assumption by Proposition 5.4.5. Hence $A$ is regular. $\square$

**Remark 5.4.8.** Theorem 5.4.7 is in some sense optimal; see Lemma 5.7.6 for the construction of a non-smooth regular curve over an imperfect field which has a singular point of geometric $\delta$-invariant $(p-1)/2$.

**Remark 5.4.9.** Theorem 5.4.7 is a classical result of Tate [**124**] if the sum over all $\delta$-invariants of $\overline{C}$ is strictly less than $(p-1)/2$; see also [**115**]. Thus, our result is a slight improvement over Tate's result. In [**116**], Schröer gave a simple proof of Tate's theorem. Our proof, which is in terms of the *local* $\delta$-invariants $\delta(\overline{C}, x)$, relies on the work of Patakfalvi-Waldron [**102**].

The upper semicontinuity of geometric $\delta$-invariants is presumably well known to the experts. The following is all we need.

**Proposition 5.4.10.** *Let $U$ be a Noetherian integral scheme, and let $\eta \in U$ be the generic point. Let $\pi\colon \mathscr{C} \to U$ be a flat family of proper and geometrically reduced curves parameterized by $U$ such that the generic fiber $\mathscr{C}_\eta$ is geometrically irreducible over $\kappa(\eta)$. Let $u_0 \in U$ be a closed point, let $N$ be a non-negative integer, and assume that the geometric $\delta$-invariants of $\mathscr{C}_{u_0}$ are smaller than or equal to $N$ at every closed point. Then, there exists* *a non-empty open subset $U' \subset U$ such that for every point $x \in U'$ (not necessarily closed) the geometric $\delta$-invariants of the curve $\mathscr{C}_x \otimes_{\kappa(x)} \kappa(x)^{\mathrm{sep}}$ over $\kappa(x)^{\mathrm{sep}}$ are smaller than or equal to $N$ at every closed point.*

Proof. First, we show that the geometric $\delta$-invariants of the curve $\mathscr{C}_\eta \otimes_{\kappa(\eta)} \kappa(\eta)^{\mathrm{sep}}$ over $\kappa(\eta)^{\mathrm{sep}}$ are smaller than or equal to $N$ at every closed point. By Proposition 5.4.5, we may assume that $U$ is the spectrum of a complete discrete valuation ring $A$, whose residue field corresponds to the closed point $u_0$ of $U$. Moreover, after replacing $A$ by a finite extension, we may assume that the normalization of $\mathscr{C}_\eta$ (resp. $\mathscr{C}_{u_0}$) is smooth over $\kappa(\eta)$ (resp. $\kappa(u_0)$); see [**EGA IV 4**, Proposition 17.15.14]. Let $\pi\colon \widetilde{\mathscr{C}} \to \mathscr{C}$ be the normalization morphism. By [**80**, Theorem 7.2.18], we have the following equality of 1-cycles on $\mathscr{C}$:

$$\pi_*[(\widetilde{\mathscr{C}})_{u_0}] = [\mathscr{C}_{u_0}].$$

From this equality, we see that $(\widetilde{\mathscr{C}})_{u_0}$ is generically reduced. Since $(\widetilde{\mathscr{C}})_{u_0}$ has no embedded points by [**80**, Proposition 7.2.15 and Corollary 7.2.22], it follows that $(\widetilde{\mathscr{C}})_{u_0}$ is reduced. Let now $\widetilde{\mathscr{C}_{u_0}}$ be the normalization of $\mathscr{C}_{u_0}$. By the above equality again, the normalization morphism factors as

$$\widetilde{\mathscr{C}_{u_0}} \to (\widetilde{\mathscr{C}})_{u_0} \to \mathscr{C}_{u_0}.$$

Thus, we have

$$\dim_{\kappa(u_0)}(\mathscr{F}|_{\mathscr{C}_{u_0}})_x \leq \delta(\mathscr{C}_{u_0}, x) \leq N$$

for every closed point $x \in \mathscr{C}_{u_0}$. By considering a closed subscheme $i_Z\colon Z \hookrightarrow \mathscr{C}$ such that $\mathscr{F}$ comes from a coherent sheaf $\mathscr{F}_Z$ on $Z$ and $Z = \mathrm{Supp}(\mathscr{F})$, the similar arguments as in the proof of Proposition 5.3.14 show that the claim is true.

Next, we show that the just established result implies the existence of an open subset $U' \subset U$ as in the assertion. There is a flat morphism of finite type $f\colon U'' \to U$, such that the normalization of $\mathscr{C} \times_U U''$ is smooth over $U''$. Since $f$ is an open map, we may assume that the normalization $\widetilde{\mathscr{C}}$ of $\mathscr{C}$ is smooth over $U$. Let $\pi\colon \widetilde{\mathscr{C}} \to \mathscr{C}$ be the normalization morphism. Now, by considering $\mathscr{F} := \pi_*\mathcal{O}_{\widetilde{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}}$ and a closed subscheme $i_Z\colon Z \hookrightarrow \mathscr{C}$ as above, similar arguments as in the proof of Proposition 5.3.14 show the existence of an open subset $U' \subset U$ as desired. □

## 5.5. The key lemma

In this section, we prove a lemma, which is used in the proof of Theorem 5.1.1.

**Lemma 5.5.1.** *Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $X$ be a proper smooth variety $X$ over $k$ with $\dim X \geq 2$ that is dominated by a map from a family of rational curves, i.e., there exists a pair $(\pi, \varphi)$ as in Definition 5.2.1 (4) such that $\dim U = \dim X - 1$ and $\varphi\colon \mathscr{C} \to X$ is dominant. Assume moreover that $\mathscr{C}$ and $U$ are normal. Then, after possibly shrinking $U$, there exists a commutative diagram*

$$\begin{array}{ccccc} & & \overset{\varphi}{\frown} & & \\ \mathscr{C} & \longrightarrow & \mathscr{C}' & \longrightarrow & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle s} & & \\ U & \xrightarrow{t} & U' & & \end{array}$$

*satisfying the following conditions:*

*(1) $\mathscr{C}'$ and $U'$ are normal varieties over $k$.*

(2) *$s\colon \mathscr{C}' \to U'$ is a proper flat morphism, and $t\colon U \to U'$ is a finite morphism.*
(3) *$\mathscr{C} \to \mathscr{C}'$ is a finite morphism.*
(4) *$k(\mathscr{C}')$ is the separable closure of $k(X)$ in $k(\mathscr{C})$.*
(5) *$k(U')$ is algebraically closed in $k(\mathscr{C}')$.*
(6) *For every closed point $u' \in U'$, $s^{-1}(u')_{\mathrm{red}}$ is a (possibly singular) rational curve.*

**Lemma 5.5.2.** *Let $X$ and $Y$ be integral schemes of characteristic $p > 0$, and let $f\colon X \to Y$ be a finite and dominant morphism. Assume that $Y$ is normal and that $f$ is purely inseparable, i.e., the finite extension $k(X)/k(Y)$ of function fields induced by $f$ is purely inseparable. Then $f$ is radicial.*

Proof. We may assume that $X$ and $Y$ are affine, say, $X := \operatorname{Spec} A$ and $Y := \operatorname{Spec} B$. Since the extension $k(X)/k(Y)$ is finite and purely inseparable, there exists a positive integer $e \geq 1$, such that $k(X)^{p^e} \subset k(Y)$. Since $A$ is integral over $B$ and $B$ is normal, we have $A^{p^e} \subset B$. Let $\mathfrak{q}$ be a prime ideal of $B$. It follows that $\mathfrak{p} := \sqrt{\mathfrak{q}A}$ is the unique prime ideal of $A$ above $\mathfrak{q}$. Hence, $\operatorname{Spec} A \to \operatorname{Spec} B$ is bijective. Let $\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q})$ be the residue fields of $\mathfrak{p}$ and $\mathfrak{q}$, respectively. Since $A^{p^e} \subset B$, we have $\kappa(\mathfrak{p})^{p^e} \subset \kappa(\mathfrak{q})$, and the extension $\kappa(\mathfrak{p})/\kappa(\mathfrak{q})$ is purely inseparable. This concludes that $f$ is radicial. □

Now, we shall prove Lemma 5.5.1. With the assumptions and notations as in Lemma 5.5.1, we can compactify $U$ and $\mathscr{C}$ compatibly by the following claim.

**Claim 5.5.3.** *There exists a commutative diagram*

$$\begin{array}{ccccc} \mathscr{C} & \hookrightarrow & \overline{\mathscr{C}} & \xrightarrow{\overline{\varphi}} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \\ U & \hookrightarrow & \overline{U} & & \end{array} \qquad (\varphi\colon \mathscr{C} \to X)$$

*satisfying the following conditions:*
(1) *$\overline{\mathscr{C}}$ is a proper normal variety over $k$ and $\mathscr{C} \subset \overline{\mathscr{C}}$ is an open subset.*
(2) *$\overline{U}$ is a proper normal variety over $k$ and $U \subset \overline{U}$ is an open subset.*

Proof. Choose a compactification $\overline{\mathscr{C}} \supset \mathscr{C}$. Replacing $\overline{\mathscr{C}}$ by the Zariski closure of the image of $\mathscr{C} \to \overline{\mathscr{C}} \times X$, we may assume that $\varphi$ extends to a morphism $\overline{\varphi}\colon \overline{\mathscr{C}} \to X$. Take a normal compactification $\overline{U} \supset U$. Replacing $\overline{\mathscr{C}}$ by the normalization of the Zariski closure of the image of $\mathscr{C} \to \overline{U} \times \overline{\mathscr{C}}$, we may assume that $\overline{\mathscr{C}}$ is normal and that $\pi$ extends to a morphism $\overline{\pi}\colon \overline{\mathscr{C}} \to \overline{U}$. □

The next step is to shrink $U$ and to replace $\overline{\mathscr{C}}$ further in order to find a nice factorization $\overline{\mathscr{C}} \to \overline{\mathscr{C}}' \to X$ of $\overline{\varphi}$. The rough idea is to take a proper and normal model of the separable closure of the function field $k(X)$ in $k(\overline{\mathscr{C}})$. But the actual argument given below is more involved because we also want to ensure that the intermediate variety $\overline{\mathscr{C}}'$ admits an open and dense subset $\mathscr{C}' \subset \overline{\mathscr{C}}'$ that is equipped with a fibration $s\colon \mathscr{C}' \to U'$ over a normal variety $U'$ such that for every closed point $u' \in U'$, the reduced closed subscheme $s^{-1}(u')_{\mathrm{red}}$ of the fiber $s^{-1}(u')$ is a (possibly singular) rational curve. The delicate point is that such a fibration might not exist if we start from an arbitrary normal and proper model.

Let $\overline{\mathscr{C}}'$ be the normalization of $X$ in the separable closure of $k(X)$ in $k(\overline{\mathscr{C}})$. We denote by $\overline{\varphi}'\colon \overline{\mathscr{C}} \to \overline{\mathscr{C}}'$ and $\overline{\varphi}''\colon \overline{\mathscr{C}}' \to X$ the induced morphisms. Then, we have

$$\overline{\varphi} = \overline{\varphi}'' \circ \overline{\varphi}'.$$

The morphism $\overline{\varphi}''$ is finite and $k(\overline{\mathscr{C}}')/k(X)$ is separable. On the other hand, $\overline{\varphi}'$ is a generically finite and proper morphism and $k(\overline{\mathscr{C}})/k(\overline{\mathscr{C}}')$ is purely inseparable.

The morphism $\overline{\varphi}'$ might not be flat. We now modify $\overline{\mathscr{C}}$ and $\overline{\mathscr{C}}'$ to obtain a flat morphism as follows: we apply the flattening theorem of Raynaud-Gruson [**108**, Théorème 5.2.2]. (See also [**28**, Section 2.19].) Then, we obtain a proper and birational morphism $\overline{g}'\colon \overline{\mathscr{C}}'_2 \to \overline{\mathscr{C}}'$ such that the strict transform $\overline{\mathscr{C}}_2 \subset \overline{\mathscr{C}} \times_{\overline{\mathscr{C}}'} \overline{\mathscr{C}}'_2$ is *flat* over $\overline{\mathscr{C}}'_2$. We denote by $\overline{\varphi}'_2\colon \overline{\mathscr{C}}_2 \to \overline{\mathscr{C}}'_2$ and $\overline{g}\colon \overline{\mathscr{C}}_2 \to \overline{\mathscr{C}}$ the induced morphisms. Then, the following diagram commutes:

$$\begin{array}{ccccc} \overline{\mathscr{C}}_2 & \xrightarrow{\overline{\varphi}'_2} & \overline{\mathscr{C}}'_2 & & \\ \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle \overline{g}'} & & \\ \overline{\mathscr{C}} & \xrightarrow{\overline{\varphi}'} & \overline{\mathscr{C}}' & \xrightarrow{\overline{\varphi}''} & X. \end{array}$$

Here, $\overline{\varphi}'_2\colon \overline{\mathscr{C}}_2 \to \overline{\mathscr{C}}'_2$ is *finite* because it is proper, flat, and generically finite.

The varieties $\overline{\mathscr{C}}_2$ and $\overline{\mathscr{C}}'_2$ might not be normal. Passing to normalizations, we find proper normal varieties $\overline{\mathscr{C}}_3$ and $\overline{\mathscr{C}}'_3$ and a morphism $\overline{\psi}\colon \overline{\mathscr{C}}_3 \to \overline{\mathscr{C}}'_3$ over $k$ and we obtain the following commutative diagram:

$$\begin{array}{ccccc} \overline{\mathscr{C}}_3 & \xrightarrow{\overline{\psi}} & \overline{\mathscr{C}}'_3 & & \\ \downarrow{\scriptstyle \overline{h}} & & \downarrow{\scriptstyle \overline{h}'} & & \\ \overline{\mathscr{C}}_2 & \xrightarrow{\overline{\varphi}'_2} & \overline{\mathscr{C}}'_2 & & \\ \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle \overline{g}'} & & \\ \overline{\mathscr{C}} & \xrightarrow{\overline{\varphi}'} & \overline{\mathscr{C}}' & \xrightarrow{\overline{\varphi}''} & X. \end{array}$$

Here, $\overline{h}$ and $\overline{h}'$ are proper birational morphisms. Since $\overline{\varphi}'_2 \circ \overline{h}$ is finite and $\overline{h}'$ is separated, the morphism $\overline{\psi}$ is finite. Let us summarize the situation:

(1) $\overline{\mathscr{C}}, \overline{\mathscr{C}}', \overline{\mathscr{C}}_3, \overline{\mathscr{C}}'_3$ are proper normal varieties over $k$.
(2) $\overline{g}, \overline{g}', \overline{h}, \overline{h}'$ are proper birational morphisms.
(3) $\overline{\varphi}'_2$ and $\overline{\psi}$ are finite morphisms.

Since $\overline{g} \circ \overline{h}$ is an isomorphism outside a closed subset of codimension $\geq 2$ in $\overline{\mathscr{C}}$, after removing its image in $\overline{U}$ from $U$, we may assume that the restriction $\mathscr{C}_3 := (\overline{g} \circ \overline{h})^{-1}(\mathscr{C}) \to \mathscr{C}$ of $\overline{g} \circ \overline{h}$ is an isomorphism. By Lemma 5.5.2, the morphism $\overline{\psi}$ is radicial. Hence it is a homeomorphism. The image $V' := \overline{\psi}(\mathscr{C}_3) \subset \overline{\mathscr{C}}'_3$ is open, and the induced morphism $\overline{\psi}\colon \mathscr{C}_3 \to V'$ is finite. We obtain the following diagram:

$$\begin{array}{ccccc} \mathscr{C} \cong \mathscr{C}_3 & \xrightarrow{\overline{\psi}} & V' & \xrightarrow{\overline{\varphi}'' \circ \overline{g}' \circ \overline{h}'} & X \\ \downarrow{\scriptstyle \pi} & & & & \\ U & & & & \end{array}$$

Shrinking $U$ further, we may assume that $U$ is affine. We set $U' := \operatorname{Spec} H^0(V', \mathcal{O}_{V'})$. By a lemma of Tanaka [**123**, Lemma A.1], $U'$ is a normal variety over $k$ equipped with a proper surjective morphism $s\colon V' \to U'$ and a finite surjective morphism $t\colon U \to U'$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathscr{C} \cong \mathscr{C}_3 & \xrightarrow{\overline{\psi}} & V' & \xrightarrow{\overline{\varphi}''\circ\overline{g}'\circ\overline{h}'} & X \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle s} & & \\ U & \xrightarrow{t} & U' & & \end{array}$$

By construction, we have $s_*\mathcal{O}_{V'} \cong \mathcal{O}_{U'}$. It follows that $k(U')$ is algebraically closed in $k(V')$. For every closed point $u' \in U'$, the scheme $s^{-1}(u')_{\mathrm{red}}$ is a (possibly singular) rational curve because it is dominated by a geometric fiber of $\pi$. After possibly shrinking $U'$ further, we may assume $s$ is flat. Putting $\mathscr{C}' := V'$, all the assertions are proved, which establishes Lemma 5.5.1.

## 5.6. Proof of the main theorems

In this section, we will prove Theorem 5.1.1.

**Proof of Theorem 5.1.1.** By the assumption of Theorem 5.1.1, there is a family of rational curves $\mathscr{C} \subset U \times X$ on $X$ with a closed point $u_0 \in U$ such that the projection $\mathscr{C} \to X$ is dominant and such that the rational curve $\mathscr{C}_{u_0} \subset X$ satisfies one of the following conditions:

- The $\delta$-invariants of $\mathscr{C}_{u_0}$ are strictly less than $(p-1)/2$ at every closed point.
- The Jacobian numbers of $\mathscr{C}_{u_0}$ are strictly less than $p$ at every closed point.

By Proposition 5.3.14 and Proposition 5.4.10, after shrinking $U$, we may assume that one of the following conditions is satisfied:

(1) For every closed point $u \in U$, the $\delta$-invariants of $\mathscr{C}_u$ are strictly less than $(p-1)/2$ at every closed point.
(2) For every closed point $u \in U$, the Jacobian numbers of $\mathscr{C}_u$ are strictly less than $p$ at every closed point.

We may assume that $U$ is a smooth curve. Let $\widetilde{\mathscr{C}} \to \mathscr{C}$ be the normalization of $\mathscr{C}$. After possibly shrinking $U$, we may assume that the pair $\pi\colon \widetilde{\mathscr{C}} \to U$ and $\varphi\colon \widetilde{\mathscr{C}} \to X$ is a map from a family of rational curves to $X$ in the sense of Definition 5.2.1 (4). After possibly shrinking $U$ even further, there exists a commutative diagram

$$\begin{array}{ccccc} & & \overbrace{\qquad\qquad}^{\varphi} & & \\ \widetilde{\mathscr{C}} & \longrightarrow & \mathscr{C}' & \longrightarrow & X \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle s} & & \\ U & \xrightarrow{t} & U' & & \end{array}$$

as in Lemma 5.5.1. Since $U'$ is a curve, it follows that $k(\mathscr{C}')$ is a separable extension of $k(U')$; see [**84**, Theorem 2] (see also [**3**, Lemma 7.2]). After shrinking $U'$ again, we may assume that the varieties $\mathscr{C}'$ and $U'$ are smooth over $k$. By [**3**, Theorem 7.1], after shrinking $U'$, the fibers $s^{-1}(u')$ are geometrically reduced for every closed point $u' \in U'$; see also [**EGA IV 3**, Théorème 12.2.4].

Now we assume that the condition (2) is satisfied. Since $X$ is a smooth surface, $\mathscr{C}_u$ is a local complete intersection for every closed point $u \in U$. By Proposition 5.3.9, for every closed point $u' \in U$, the Jacobian numbers of $\mathscr{C}'_{u'}$ are also strictly less than $p$ at every closed point. (The fiber $\mathscr{C}'_{u'}$ is a local complete intersection because it is the fiber of $s\colon \mathscr{C}' \to U'$ and both, $\mathscr{C}'$ and $U'$, are smooth varieties.) Let $K' := k(U')$ be the function field of $U'$

and let $K'^{\mathrm{sep}}$ be a separable closure of $K'$. Let $\mathscr{C}'_{K'}$ be the generic fiber of $s\colon \mathscr{C}' \to U'$. By Proposition 5.3.14, the Jacobian numbers of $\mathscr{C}'_{K'^{\mathrm{sep}}} := (\mathscr{C}'_{K'}) \otimes_{K'} K'^{\mathrm{sep}}$ are strictly less than $p$ at every closed point. Since $\mathscr{C}'$ is a smooth variety, it is regular. Hence, the generic fiber $\mathscr{C}'_{K'}$ is regular. By [**EGA IV 2**, Proposition 6.7.4 (a)], $\mathscr{C}'_{K'^{\mathrm{sep}}}$ is also regular. By Proposition 5.3.4, $\mathscr{C}'_{K'^{\mathrm{sep}}}$ is smooth over $K'^{\mathrm{sep}}$ and therefore, $\mathscr{C}'_{K'}$ is smooth over $K'$. After replacing $U'$ by an étale neighborhood, we may assume that $\mathscr{C}'_{K'}$ has a $K'$-rational point. (In fact, by Tsen's theorem, we need not replace $U'$ by an étale neighborhood.) Then, $\mathscr{C}'_{K'}$ is isomorphic to the projective line $\mathbb{P}^1_{K'}$ over $K'$; see [**3**, Lemma 11.8]. This implies that $\mathscr{C}'$ is birationally equivalent to $\mathbb{P}^1 \times U'$ over $k$. Since $k(\mathscr{C}')/k(X)$ is separable, we conclude that $X$ is separably uniruled, as desired.

When the condition (1) is satisfied, we can argue similarly by using Theorem 5.4.7, Proposition 5.4.6, and Proposition 5.4.10 instead of Proposition 5.3.4, Proposition 5.3.9, and Proposition 5.3.14, respectively. □

## 5.7. Examples

In this section, we give some examples illustrating Theorem 5.1.1. In particular, we show that the result is in some sense optimal. We work over an algebraically closed field $k$ of characteristic $p \geq 0$.

### 5.7.1. An easy corollary.

**Corollary 5.7.1.** *Assume $p > 0$. Let $X$ be a proper smooth surface of non-negative Kodaira dimension over $k$. Let $C \subset X$ be a rational curve with*

$$C^2 + K_X \cdot C < p - 3.$$

*Then, $C$ is topologically rigid.*

PROOF. By the adjunction formula [**80**, Theorem 9.1.37], the arithmetic genus of $C$ satisfies $p_a(C) < (p-1)/2$. This implies that we have $\delta(C,x) < (p-1)/2$ for every closed point $x \in C$. Thus, $C$ is topologically rigid by Theorem 5.1.1. □

**5.7.2. Nodal and cuspidal curves.** Let $C$ be a reduced curve over $k$. If $x \in C$ is a closed point, the $\delta$-invariant depends only on the completion $\widehat{\mathcal{O}}_{C,x}$. Indeed, if $(\widehat{\mathcal{O}}_{C,x})'$ is the integral closure of $\widehat{\mathcal{O}}_{C,x}$ in its total ring of fractions, we have $\delta(C,x) = \dim_k (\widehat{\mathcal{O}}_{C,x})'/\widehat{\mathcal{O}}_{C,x}$. The Jacobian number also depends only on the completion; see Proposition 5.3.12 and Corollary 5.3.13. We leave the easy computations of the following result to the reader.

**Proposition 5.7.2.** *Let $C$ be a reduced curve over $k$ and let $x \in C$ be a closed point.*

*(1) If $x \in C$ is a node, we have $\delta(C,x) = 1$ and $\mathrm{jac}(C,x) = 1$.*

*(2) We say that $x \in C$ is an ordinary cusp if we have $\widehat{\mathcal{O}}_{C,x} \cong k[[S,T]]/(S^2 + T^3)$. The $\delta$-invariant is $\delta(C,x) = 1$. The Jacobian number depends on $p = \mathrm{char}(k)$ as follows:*

| | $p = 0$ | $p = 2$ | $p = 3$ | $p \geq 5$ |
|---|---|---|---|---|
| $\mathrm{jac}(C,x)$ | 2 | 4 | 3 | 2 |

*(3) Let $F_1, F_2, F_3 \in k[S,T]$ be distinct linear forms over $k$. We put $F = F_1 \cdot F_2 \cdot F_3$. If we have $\widehat{\mathcal{O}}_{C,x} \cong k[[S,T]]/(F)$, then we have $\delta(C,x) = 3$ and $\mathrm{jac}(C,x) = 4$.*

Theorem 5.1.1 implies the following corollary.

**Corollary 5.7.3.** *Let $X$ be a proper smooth surface over $k$.*

(1) *If $X$ contains a topologically non-rigid rational curve $C \subset X$ such that every singularity of $C$ is a node, then $X$ has negative Kodaira dimension.*
(2) *If $p \geq 5$ and $X$ contains a topologically non-rigid rational curve $C \subset X$ such that every singularity of $C$ is a node or an ordinary cusp, then $X$ has negative Kodaira dimension.*

**5.7.3. Topologically non-rigid rational curves and supersingular surfaces.** In positive characteristic, there exist surfaces of non-negative Kodaira dimension containing topologically non-rigid rational curves. However, such surfaces have special properties: if $\rho(X)$ denotes the Picard number and if $b_2(X) := \dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ (which is independent of $\ell$ as long as $\ell \neq \text{char}(k)$) denotes the second Betti number, then *Igusa's inequality* states that $\rho(X) \leq b_2(X)$; see [**59**]. If $X$ contains a topologically non-rigid rational curve, then equality holds.

**Proposition 5.7.4.** *Let $X$ be a proper smooth surface over an algebraically closed field $k$ of characteristic $p \geq 0$. Assume that $X$ contains a topologically non-rigid rational curve $C \subset X$. Then, $X$ is uniruled and the equality $\rho(X) = b_2(X)$ holds.*

Proof. The assertion follows from Shioda's results in [**117**] as follows. First, $X$ is uniruled by Proposition 5.2.3. Then there exists a dominant rational map $\mathbb{P}^1 \times C \dashrightarrow X$ for a smooth and proper curve $C$ over $k$. Since $\rho(\mathbb{P}^1 \times C) = b_2(\mathbb{P}^1 \times C) = 2$, a theorem of Shioda [**117**, Section 2, Lemma] implies $\rho(X) = b_2(X)$. (See also [**10**, Proposition 14].) □

**Remark 5.7.5.** A surface $X$ in positive characteristic satisfying $\rho(X) = b_2(X)$ is called *Shioda-supersingular*. By Proposition 5.7.4, every rational curve on a proper smooth surface that is not Shioda-supersingular is topologically rigid; see [**78**, Section 9] for an overview of supersingular surfaces.

**5.7.4. Topologically non-rigid rational curves with large $\delta$-invariants and Jacobian numbers.** In this subsection, we will see that in characteristic $p \geq 3$, there exist surfaces of non-negative Kodaira dimension that contain topologically non-rigid rational curves that have precisely one singular point, which is of $\delta$-invariant (resp. Jacobian number) equal to $(p-1)/2$ (resp. $p$). Thus, Theorem 5.1.1 is in some sense optimal.

We start with an auxiliary result, which shows that also Theorem 5.3.10 and Theorem 5.4.7 are in some sense optimal.

**Lemma 5.7.6.** *Let $k$ be a field of characteristic $p \geq 3$. Let $K := k(t)$ be the field of rational functions over $k$ of the variable $t$. Then, there exists a proper curve $C$ over $K$ satisfying the following three conditions:*
(1) *$C$ has a unique singular point, whose geometric $\delta$-invariant (resp. Jacobian number) is $(p-1)/2$ (resp. $p$),*
(2) *$C$ is a regular scheme, and*
(3) *$C_{\overline{K}}$ is a rational curve over an algebraic closure $\overline{K}$ of $K$.*

*In particular, the bound of Theorem 5.3.10 is optimal.*

Proof. We consider the two affine curves

$$C_1 : Y^2 = X^p + t \quad \text{and} \quad C_2 : Y'^2 = X' + tX'^{p+1}$$

over $K$. From these, we obtain a curve $C$ over $K$ by gluing the two curves $C_1$ and $C_2$ via the isomorphism

$$\{X \neq 0\} \cap C_1 \xrightarrow{\cong} \{X' \neq 0\} \cap C_2$$

that is defined by $X \mapsto 1/X'$ and $Y \mapsto Y'/X'^{(p+1)/2}$; see [**80**, Proposition 7.4.24]. Moreover, gluing the two morphisms

$$C_1 \to \operatorname{Spec} K[X], \quad (X, Y) \mapsto X$$

and

$$C_2 \to \operatorname{Spec} K[X'], \quad (X', Y') \mapsto X',$$

we obtain a finite morphism $C \to \mathbb{P}^1_K$. In particular, the curve $C$ is proper over $K$. The curve $C$ is regular, but it is not smooth over $K$; see also [**48**, Chapter II, Exercise 6.4]. The closed point $x \in C_1$ corresponding to the maximal ideal $(X^p + t, Y) \subset K[X, Y]$ is the unique singular point of $C$. It is easy to see that $C$ satisfies all the conditions of the lemma. □

We now construct a surface $Y$ of general type over $k$ that contains a topologically non-rigid rational curve, which has one singular point of $\delta$-invariant (resp. Jacobian number) equal to $(p-1)/2$ (resp. $p$). In fact, these constructions are inspired by Raynaud's counterexamples to the Kodaira vanishing theorem in positive characteristic from [**107**, Section 3.1].

**Proposition 5.7.7.** *Let $k$ be an algebraically closed field of characteristic $p \geq 3$. Then, there exists a proper smooth surface $Y$ over $k$ satisfying the following conditions:*

*(1) The Kodaira dimension $\kappa(Y)$ of $Y$ satisfies $\kappa(Y) \geq 1$. If $p \geq 5$, then we may even assume $\kappa(Y) = 2$, i.e., $Y$ is a surface of general type.*
*(2) $Y$ contains a topologically non-rigid rational curve $C \subset Y$, and*
*(3) $C$ has a unique singular point, whose $\delta$-invariant (resp. Jacobian number) is equal to $(p-1)/2$ (resp. $p$).*

Proof. Let $C$ be the proper curve over $K = k(t)$ from Lemma 5.7.6. There exists a proper smooth surface $X$ over $k$ together with a proper flat morphism $X \to \mathbb{P}^1$, whose generic fiber satisfies $X \times_{\mathbb{P}^1} \operatorname{Spec} K \cong C$. We choose a smooth and projective curve $S$ of genus $g(S) \geq 2$ and a generically étale morphism $S \to \mathbb{P}^1$. Let $Y \to X \times_{\mathbb{P}^1} S$ be a resolution of singularities of $X \times_{\mathbb{P}^1} S$. Then, the generic fiber of $Y \to S$ is isomorphic to $C_{k(S)}$. By [**75**, Proposition 2.2] and the proofs of Proposition 5.3.14 and Proposition 5.4.10, there exists an open and dense subset $U \subset S$ such that for every closed point $u \in U$, the fiber $Y_u$ is a rational curve over $k$, and the fiber $Y_u$ has a unique singular point and its $\delta$-invariant (resp. Jacobian number) is $(p-1)/2$ (resp. $p$). Since $g(S) \geq 2$, the Kodaira dimension of $S$ is equal to $\kappa(S) = 1$. If $p = 3$, then the arithmetic genus of $C$ is equal to $p_a(C) = 1$ and if $p \geq 5$, then we even have $p_a(C) \geq 2$. Thus, we find $\kappa(Y) \geq 1$ (resp. $\kappa(Y) = 2$) if $p \geq 3$ (resp. if $p \geq 5$) by a characteristic-$p$ version of Iitaka's $C_{1,1}$-conjecture; see [**24**, Theorem 1.3] for example. □

# Bibliography


[1] André, Y., *On the Shafarevich and Tate conjectures for hyper-Kähler varieties*, Math. Ann. **305** (1996), no. 2, 205–248.

[2] Artin, M., Mazur, B., *Formal groups arising from algebraic varieties*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 1, 87–131.

[3] Bădescu, L., *Algebraic surfaces*, Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author., Universitext., Springer-Verlag, New York, 2001.

[4] Bass, H., *Clifford algebras and spinor norms over a commutative ring*, Amer. J. Math. **96** (1974), 156–206.

[5] Berkovich, V. G., *Étale cohomology for non-Archimedean analytic spaces*, Publ. Math. Inst. Hautes Études Sci. **78** (1993), 5–161.

[6] Berthelot, P., *Altérations de variétés algébriques (d'après A. J. de Jong)*, Séminaire Bourbaki, Astérisque No. 241 (1997), 273–311.

[7] Berthelot, P., Breen, L., Messing, W., *Théorie de Dieudonné cristalline. II*, Lecture Notes in Mathematics, 930. Springer-Verlag, Berlin, 1982.

[8] Berthelot, P., Ogus, A., *F-isocrystals and de Rham cohomology. I*, Invent. Math. **72** (1983), no. 2, 159-199.

[9] Bhatt, B., Morrow, M., Scholze, P., *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 219–397.

[10] Bogomolov, F., Hassett, B., Tschinkel, Y., *Constructing rational curves on K3 surfaces*, Duke Math. J. **157** (2011), no. 3, 535–550.

[11] Bosch, S., Güntzer, U., Remmert, R., *Non-Archimedean analysis*, A systematic approach to rigid analytic geometry. Grundlehren der Mathematischen Wissenschaften, 261. Springer-Verlag, Berlin, 1984.

[12] Bosch, S., Lütkebohmert, W., *Stable reduction and uniformization of abelian varieties. I*, Math. Ann. **270** (1985), no. 3, 349–379.

[13] Bosch, S., Lütkebohmert, W., Raynaud, M., *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21. Springer-Verlag, Berlin, 1990.

[14] Bourbaki, N., *Éléments de mathématique. Algèbre. Chapitre 9*, Reprint of the 1959 original. Springer-Verlag, Berlin, 2007.

[15] Breen, L., *Rapport sur la théorie de Dieudonné*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I, pp. 39–66, Astérisque No. 63 (1979).

[16] Bruns, W., Herzog, J., *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.

[17] Buchweitz, R.-O., Greuel, G.-M., *The Milnor number and deformations of complex curve singularities*, Invent. Math. **58** (1980), no. 3, 241–281.

[18] Buskin, N., *Every rational Hodge isometry between two K3 surfaces is algebraic*, J. Reine Angew. Math. **755** (2019), 127–150.

[19] Cadoret, A., *Ultraproduct Weil II for curves and integral models in compatible families of $\ell$-adic local systems*, preprint, 2019,
`https://webusers.imj-prg.fr/~anna.cadoret/Weil2Ultra.pdf`.

[20] Cadoret, A., Hui, C.-Y., Tamagawa, A., *Geometric monodromy–semisimplicity and maximality*, Ann. of Math. (2) **186** (2017), no. 1, 205–236.

[21] Cadoret, A., Hui, C.-Y., Tamagawa, A., *$\mathbb{Q}_\ell$-versus $\mathbb{F}_\ell$-coefficients in the Grothendieck-Serre/Tate conjectures*, preprint, 2017,
`https://webusers.imj-prg.fr/~anna.cadoret/GST.pdf`.

[22] Chai, C.-L., Conrad, B., Oort, F., *Complex multiplication and lifting problems*, Mathematical Surveys and Monographs, 195. American Mathematical Society, Providence, RI, 2014.

[23] Charles, F., *The Tate conjecture for K3 surfaces over finite fields*, Invent. Math. **194** (2013), no. 1, 119–145.

[24] Chen, Y., Zhang, L., *The subadditivity of the Kodaira dimension for fibrations of relative dimension one in positive characteristics*, Math. Res. Lett. **22** (2015), no. 3, 675–696.

[25] Colliot-Thélène, J.-L., Raskind, W., *$\mathscr{K}_2$-cohomology and the second Chow group*, Math. Ann. **270** (1985), no. 2, 165–199.

[26] Colliot-Thélène, J.-L., Raskind, W., *Groupe de Chow de codimension deux des variétés définies sur un corps de nombres: un théorème de finitude pour la torsion*, Invent. Math. **105** (1991), no. 2, 221–245.

[27] Dat, J.-F., *Espaces symétriques de Drinfeld et correspondance de Langlands locale*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 1, 1–74.

[28] de Jong, A. J., *Smoothness, semi-stability and alterations*, Publ. Math. Inst. Hautes Études Sci. **83** (1996), 51–93.

[29] de Jong, A. J., *Families of curves and alterations*, Ann. Inst. Fourier **47** (1997), no. 2, 599–621.

[30] Deligne, P., *Théorie de Hodge. I*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 425–430.

[31] Deligne, P., *La conjecture de Weil pour les surfaces K3*, Invent. Math. **15** (1972), 206–226.

[32] Deligne, P., *La conjecture de Weil. I*, Publ. Math. Inst. Hautes Études Sci. **43** (1974), 273–307.

[33] Deligne, P., *Théorie de Hodge. III*, Publ. Math. Inst. Hautes Études Sci. **44** (1974), 5–77.

[34] Deligne, P., *La conjecture de Weil. II*, Publ. Math. Inst. Hautes Études Sci. **52** (1980), 137–252.

[35] Deligne, P., *Relèvement des surfaces K3 en caractéristique nulle*, Algebraic surfaces (Orsay, 1976-78), pp. 58–79, Lecture Notes in Mathematics, 868. Springer-Verlag, Berlin-New York, 1981.

[36] Deligne, P., *Hodge cycles on abelian varieties*, Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics, 900. Springer-Verlag, Berlin-New York, 1982.

[37] Deligne, P., Letter to M. Kisin (cc: A. Vasiu), August 26, 2011, `http://people.math.binghamton.edu/adrian/Letter_Deligne.pdf`.

[38] Esnault, H., Oguiso, K., *Non-liftability of automorphism groups of a K3 surface in positive characteristic*, Math. Ann. **363** (2015), no. 3-4, 1187–1206.

[39] Esteves, E., Kleiman, S., *Bounds on leaves of one-dimensional foliations*, Dedicated to the 50th anniversary of IMPA, Bull. Braz. Math. Soc. (N.S.) **34** (2003), no. 1, 145–169.

[40] Faltings, G., *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366.

[41] Faltings, G., *Complements to Mordell*, Rational points (Bonn, 1983/1984), 203–227, Aspects Math., E6, Friedr. Vieweg, Braunschweig, 1984.

[42] Faltings, G., Chai, C.-L., *Degeneration of abelian varieties*, With an appendix by David Mumford, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 22. Springer-Verlag, Berlin, 1990.

[43] Fujiwara, K., Kato, F., *Foundations of rigid geometry. I*, EMS Monographs in Mathematics. European Mathematical Society, Zürich, 2018.

[44] Gabber, O., *Sur la torsion dans la cohomologie l-adique d'une variété*, C. R. Acad. Sci. Paris Sér. I Math. **297** (1983), no. 3, 179–182.

[45] Görtz, U., Wedhorn, T., *Algebraic geometry I. Schemes with examples and exercises*, Advanced Lectures in Mathematics. Vieweg+Teubner, Wiesbaden, 2010.

[46] Greuel, G.-M., Lossen, C., Shustin, E., *Introduction to singularities and deformations*, Springer Monographs in Mathematics. Springer, Berlin, 2007.

[47] Grothendieck, A., *Le groupe de Brauer. II. Théorie cohomologique*, Dix exposés sur la cohomologie des schémas, 67–87, Adv. Stud. Pure Math., **3**, North-Holland, Amsterdam, 1968.

[48] Hartshorne, R., *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[49] Hazewinkel, M., *Formal groups and applications*, Corrected reprint of the 1978 original. AMS Chelsea Publishing, Providence, RI, 2012.

[50] Huber, R., *Continuous valuations*, Math. Z. **212** (1993), no. 3, 455–477.

[51] Huber, R., *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551.

[52] Huber, R., *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[53] Huber, R., *A finiteness result for the compactly supported cohomology of rigid analytic varieties*, J. Algebraic Geom. **7** (1998), no. 2, 313–357.

[54] Huber, R., *A finiteness result for direct image sheaves on the étale site of rigid analytic varieties*, J. Algebraic Geom. **7** (1998), no. 2, 359–403.

[55] Huber, R., *Swan representations associated with rigid analytic curves*, J. Reine Angew. Math. **537** (2001), 165–234.

[56] Huber, R., *A finiteness result for the compactly supported cohomology of rigid analytic varieties. II*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 3, 973–1017.

[57] Huybrechts, D., *Curves and cycles on K3 surfaces*, with an appendix by Claire Voisin, Algebr. Geom. **1** (2014), no. 1, 69–106.

[58] Huybrechts, D., *Motives of isogenous K3 surfaces*, Comment. Math. Helv. **94** (2019), no. 3, 445–458.

[59] Igusa, J., *Betti and Picard numbers of abstract algebraic surfaces*, Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 724–726.

[60] Illusie, L., *Around the Thom-Sebastiani theorem*, with an appendix by Weizhe Zheng, Manuscripta Math. **152** (2017), no. 1-2, 61–125.

[61] Illusie, L., Laszlo, Y., Orgogozo, F., *Travaux de Gabber sur l' uniformisation locale et la cohomologie étale des schémas quasi-excellents. Séminaire à l' École Polytechnique 2006–2008*, in collaboration with Déglise, F., Moreau, A., Pilloni, V., Raynaud, M., Riou, J., Stroh, B., Temkin, M., and Zheng, W., Astérisque No. 363-364 (2014).

[62] Ito, K., *Finiteness of Brauer groups of K3 surfaces in characteristic 2*, Int. J. Number Theory **14** (2018), no. 6, 1813–1825.

[63] Ito, K., *Uniform local constancy of étale cohomology of rigid analytic varieties*, preprint, 2020, `https://arxiv.org/abs/2008.07794`.

[64] Ito, K., *On a torsion analogue of the weight-monodromy conjecture*, preprint, 2020, `https://arxiv.org/abs/2008.11905`.

[65] Ito, K., Ito, T., Koshikawa, T., *CM liftings of K3 surfaces over finite fields and their applications to the Tate conjecture*, preprint, 2018, `https://arxiv.org/abs/1809.09604`.

[66] Ito, K., Ito, T., Liedtke, C., *Deformations of rational curves in positive characteristic*, J. Reine Angew. Math. (2020), `https://doi.org/10.1515/crelle-2020-0003`.

[67] Ito, T., *Weight-monodromy conjecture for certain threefolds in mixed characteristic*, Int. Math. Res. Not. 2004, no. 2, 69–87.

[68] Ito, T., *Weight-monodromy conjecture for p-adically uniformized varieties*, Invent. Math. **159** (2005), no. 3, 607–656.

[69] Ito, T., *Weight-monodromy conjecture over equal characteristic local fields*, Amer. J. Math. **127** (2005), no. 3, 647–658.

[70] Katz, N. M., *Slope filtration of F-crystals*, Journ ées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I, pp. 113–163, Astérisque No. 63 (1971).

[71] Keum, J., *Orders of automorphisms of K3 surfaces*, Adv. Math. **303** (2016), 39–87.

[72] Kim, W., Madapusi Pera, K., *2-adic integral canonical models*, Forum Math. Sigma **4** (2016), e28, 34 pp.

[73] Kisin, M., *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.

[74] Kisin, M., *Mod p points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), no. 3, 819–914.

[75] Kollár, J., *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 32. Springer-Verlag, Berlin, 1996.

[76] Kunz, E., *Kähler differentials*, Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1986.

[77] Li, J., Liedtke, C., *Rational curves on K3 surfaces*, Invent. Math. **188** (2012), no. 3, 713–727.

[78] Liedtke, C., *Algebraic surfaces in positive characteristic*, Birational geometry, rational curves, and arithmetic, 229–292, Simons Symp., Springer, Cham, 2013.

[79] Liedtke, C., *Lectures on supersingular K3 surfaces and the crystalline Torelli theorem*, K3 surfaces and their moduli, 171–235, Progr. Math., 315, Birkhäuser/Springer, 2016.

[80] Liu, Q., *Algebraic geometry and arithmetic curves*, Translated from the French by Reinie Erné, Oxford Graduate Texts in Mathematics, 6., Oxford Science Publications. Oxford University Press, Oxford, 2002.

[81] Lu, Q., Zheng, W., *Compatible systems and ramification*, Compos. Math. **155** (2019), no. 12, 2334–2353.

[82] Lütkebohmert, W., *Riemann's existence problem for a p-adic field*, Invent. Math. **111** (1993), no. 2, 309–330.

[83] Lütkebohmert, W., Schmechta, T., *Étale covers of a punctured p-adic disc*, Quaest. Math. **28** (2005), no. 2, 195–242.

[84] Mac Lane, S., *Modular fields. I. Separating transcendence bases*, Duke Math. J. **5** (1939), no. 2, 372–393.

[85] Madapusi Pera, K., *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. **201** (2015), no. 2, 625–668.

[86] Madapusi Pera, K., *Integral canonical models for spin Shimura varieties*, Compos. Math. **152** (2016), no. 4, 769–824.

[87] Madapusi Pera, K., *Erratum to appendix to '2-adic integral canonical models'*, Forum Math. Sigma **8** (2020), e14, 11 pp.

[88] Maulik, D., *Supersingular K3 surfaces for large primes*, With an appendix by Andrew Snowden. Duke Math. J. **163** (2014), no. 13, 2357–2425.

[89] Mieda, Y., *On the action of the Weil group on the l-adic cohomology of rigid spaces over local fields*, Int. Math. Res. Not. 2006, Art. ID 16429, 11.

[90] Mieda, Y., *Note on weight-monodromy conjecture for p-adically uniformized varieties*, Proc. Amer. Math. Soc. **147** (2019), no. 5, 1911–1920.

[91] Mokrane, A., *La suite spectrale des poids en cohomologie de Hyodo-Kato*, Duke Math. J. **72** (1993), no. 2, 301–337.

[92] Moret-Bailly, L., *Pinceaux de variétés abéliennes*, Astérisque No. 129 (1985).

[93] Mukai, S., *Vector bundles on a K3 surface*, Proceedings of the ICM, Beijing 2002, vol. 2, 495–502.

[94] Mustafin, G. A., *Non-Archimedean uniformization*, Math. USSR-Sb. **34** (1978), no. 2, 187–214.

[95] Nakayama, C., *Degeneration of l-adic weight spectral sequences*, Amer. J. Math. **122** (2000), no. 4, 721–733.

[96] Nygaard, N. O., *The Tate conjecture for ordinary K3 surfaces over finite fields*, Invent. Math. **74** (1983), no. 2, 213–237.

[97] Nygaard, N. O., Ogus, A., *Tate's conjecture for K3 surfaces of finite height*, Ann. of Math. (2) **122** (1985), no. 3, 461–507.

[98] Ochiai, T., *l-independence of the trace of monodromy*, Math. Ann. **315** (1999), no. 2, 321–340.

[99] Ogus, A., *Supersingular K3 crystals*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. II, pp. 3-86, Astérisque, **64**, Soc. Math. France, Paris, 1979.

[100] Orgogozo, F., *Modifications et cycles proches sur une base générale*, Int. Math. Res. Not. 2006, Art. ID 25315, 38.

[101] Orgogozo, F., *Constructibilité et modération uniformes en cohomologie étale*, Compos. Math. **155** (2019), no. 4, 711–757.

[102] Patakfalvi, Z., Waldron, J., *Singularities of general fibers and the LMMP*, preprint, 2017, https://arxiv.org/abs/1708.04268.

[103] Pjateckiĭ-Šapiro, I. I., Šafarevič, I. R., *The arithmetic of surfaces of type K3*, Proceedings of the International Conference on Number Theory (Moscow, 1971). Trudy Mat. Inst. Steklov. **132** (1973), 44–54, 264.

[104] Platonov, V., Rapinchuk, A., *Algebraic groups and number theory*, Translated from the 1991 Russian original by Rachel Rowen. Pure and Applied Mathematics, 139. Academic Press, Inc., Boston, MA, 1994.

[105] Ramero, L., *Local monodromy in non-Archimedean analytic geometry*, Publ. Math. Inst. Hautes Études Sci. **102** (2005), 167–280.

[106] Rapoport, M., Zink, Th., *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), no. 1, 21–101.

[107] Raynaud, M., *Contre-exemple au "vanishing theorem" en caractéristique $p > 0$*, C. P. Ramanujam-a tribute, 273–278, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin-New York, 1978.

[108] Raynaud, M., Gruson, L., *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

[109] Rim, D. S., *Torsion differentials and deformation*, Trans. Amer. Math. Soc. **169** (1972), 257–278.

[110] Rizov, J., *Moduli stacks of polarized K3 surfaces in mixed characteristic*, Serdica Math. J. **32** (2006), no. 2-3, 131–178.

[111] Rizov, J., *Kuga-Satake abelian varieties of K3 surfaces in mixed characteristic*, J. Reine Angew. Math. **648** (2010), 13–67.

[112] Saito, S., *On the cycle map for torsion algebraic cycles of codimension two*, Invent. Math. **106** (1991), no. 3, 443–460.

[113] Saito, T., *Weight spectral sequences and independence of $\ell$*, J. Inst. Math. Jussieu **2** (2003), no. 4, 583–634.

[114] Scholze, P., *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.

[115] Schröer, S., *Singularities appearing on generic fibers of morphisms between smooth schemes*, Michigan Math. J. **56** (2008), no. 1, 55–76.

[116] Schröer, S., *On genus change in algebraic curves over imperfect fields*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1239–1243.

[117] Shioda, T., *An example of unirational surfaces in characteristic $p$*, Math. Ann. **211** (1974), 233–236.

[118] Skorobogatov, A., Zarhin, Yu. G., *A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces*, J. Algebraic Geom. **17** (2008), no. 3, 481–502.

[119] Skorobogatov, A., Zarhin, Yu. G., *A finiteness theorem for the Brauer group of K3 surfaces in odd characteristic*, Int. Math. Res. Not. 2015, no. 21, 11404–11418.

[120] The Stacks Project Authors, *Stacks Project*, 2018, `https://stacks.math.columbia.edu`.

[121] Suh, J., *Symmetry and parity in Frobenius action on cohomology*, Compos. Math. **148** (2012), no. 1, 295–303.

[122] Taelman, L., *K3 surfaces over finite fields with given L-function*, Algebra Number Theory **10** (2016), no. 5, 1133–1146.

[123] Tanaka, H., *Behavior of canonical divisors under purely inseparable base changes*, J. Reine Angew. Math. **744** (2018), 237–264.

[124] Tate, J., *Genus change in inseparable extensions of function fields*, Proc. Amer. Math. Soc. **3**, (1952), 400–406.

[125] Tate, J., *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.

[126] Tate, J., *Relations between $K_2$ and Galois cohomology*, Invent. Math. **36** (1976), 257–274.

[127] Tate, J., *Conjectures on algebraic cycles in $\ell$-adic cohomology*, Motives (Seattle, WA, 1991), 71–83, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[128] Taylor, R., Yoshida, T., *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493.

[129] Terasoma, T., *Monodromy weight filtration is independent of $l$*, preprint, 1998, `https://arxiv.org/abs/math/9802051`.

[130] Tjurina, G. N., *Locally semi-universal flat deformations of isolated singularities of complex spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 1026–1058.

[131] Totaro, B., *Recent progress on the Tate conjecture*, Bull. Amer. Math. Soc. (N.S.) **54** (2017), no. 4, 575–590.

[132] Tretkoff, P., *K3 surfaces with algebraic period ratios have complex multiplication*, Int. J. Number Theory **11** (2015), no. 5, 1709–1724.

[133] Tsuji, T., *$p$-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), no. 2, 233–411.

[134] Yang, Z., *Isogenies between K3 Surfaces over $\overline{\mathbb{F}}_p$*, preprint, 2018, `https://arxiv.org/abs/1810.08546`.

[135] Yu, J-D., Yui, N., *K3 surfaces of finite height over finite fields*, J. Math. Kyoto Univ. **48** (2008), no. 3, 499–519.

[136] Zarhin, Y. G., *Abelian varieties in characteristic $p$*, Mat. Zametki **19** (1976), no. 3, 393–400.

[137] Zarhin, Y. G., *Hodge groups of K3 surfaces*, J. Reine Angew. Math. **341** (1983), 193–220.

[138] Zarhin, Y. G., *The Tate conjecture for powers of ordinary K3 surfaces over finite fields*, J. Algebraic Geom. **5** (1996), no. 1, 151–172.

[139] Zariski, O., *On Castelnuovo's criterion of rationality* $p_a = P_2 = 0$ *of an algebraic surface*, Illinois J. Math. **2** (1958), 303–315.

[EGA II] Grothendieck, A., *Éléments de géométrie algébrique. I. Le langage des schémas*, Publ. Math. Inst. Hautes Études Sci. **4** (1960), 228 pp.

[EGA IV 2] Grothendieck, A., *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II.*, Publ. Math. Inst. Hautes Études Sci. **24** (1965), 231 pp.

[EGA IV 3] Grothendieck, A., *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III.*, Publ. Math. Inst. Hautes Études Sci. **28** (1966), 255 pp.

[EGA IV 4] Grothendieck, A., *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. IV.*, Publ. Math. Inst. Hautes Études Sci. **32** (1967), 361 pp.

[SGA 3] Demazure, M., Grothendieck, A., *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Lecture Notes in Mathematics, 152. Springer-Verlag, Berlin-New York, 1970.

[SGA 4 II] Artin, M., Grothendieck, A., Verdier, J.-L., *Théorie des topos et cohomologie étale des schémas. Tome 2*, in collaboration with Bourbaki, N., Deligne, P., and Saint-Donat, B. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Lecture Notes in Mathematics, 270. Springer-Verlag, Berlin-New York, 1972.

[SGA $4\frac{1}{2}$] Deligne, P., *Cohomologie étale*, Séminaire de géométrie algébrique du Bois-Marie SGA $4\frac{1}{2}$. Lecture Notes in Mathematics, 569. Springer-Verlag, Berlin, 1977.

[SGA 6] *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Mathematics, 225. Springer-Verlag, Berlin-New York, 1971.

[SGA 7 I] *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Mathematics, 288. Springer-Verlag, Berlin-New York, 1972.