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On the homomorphisms between scalar generalized Verma modules

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§ 0. Introduction

In this article, we consider the existence problem of homomorphisms between generalized Verma modules, which are induced from one dimensional representations (such generalized Verma modules are called scalar, cf. [Boe 1985]). In [Matumoto 2003], we classified the homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras.

Here, we announce a classification of homomorphisms between scalar generalized Verma modules for certain non-maximal parabolic subalgebras. The proof will appear elsewhere.

§ 1. Notations and Preliminaries

Let \mathfrak{g} be a complex reductive Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} , and \mathfrak{h} a Cartan subalgebra of \mathfrak{g} . We denote by Δ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix some positive root system Δ^+ and let Π be the set of simple roots. Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let $\langle \cdot, \cdot \rangle$ be a non-degenerate invariant bilinear form on \mathfrak{g} . For $w \in W$, we denote by $\ell(w)$ the length of w as usual. We also denote the inner product on \mathfrak{h}^* which is induced from the above form by the same symbols $\langle \cdot, \cdot \rangle$. For $\alpha \in \Delta$, we denote by s_α the reflection in W with respect to α . We denote by w_0 the longest element of W . For $\alpha \in \Delta$, we define the coroot $\check{\alpha}$ by $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$, as usual. We call $\lambda \in \mathfrak{h}^*$ dominant (resp. anti-dominant), if $\langle \lambda, \check{\alpha} \rangle$ is not a negative (resp. positive) integer, for each $\alpha \in \Delta^+$. We call $\lambda \in \mathfrak{h}^*$ regular, if $\langle \lambda, \alpha \rangle \neq 0$, for each $\alpha \in \Delta$. We denote by P the integral weight lattice, namely $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbf{Z} \text{ for all } \alpha \in \Delta\}$. If $\lambda \in \mathfrak{h}^*$ is contained in P , we call λ an integral weight. We define $\rho \in P$ by $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h} [H, X] = \alpha(H)X\}$, $\mathfrak{u} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$. Then \mathfrak{b} is a Borel subalgebra of \mathfrak{g} . We denote by Q the root lattice, namely \mathbf{Z} -linear span of Δ . We

also denote by Q^+ the linear combination of Π with non-negative integral coefficients. For $\lambda \in \mathfrak{h}^*$, we denote by W_λ the integral Weyl group. Namely,

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in Q\}.$$

We denote by Δ_λ the set of integral roots.

$$\Delta_\lambda = \{\alpha \in \Delta \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}\}.$$

It is well-known that W_λ is the Weyl group for Δ_λ . We put $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. This is a positive system of Δ_λ . We denote by Π_λ the set of simple roots for Δ_λ^+ and denote by Φ_λ the set of reflection corresponding to the elements in Π_λ . So, $(W_\lambda, \Phi_\lambda)$ is a Coxeter system. We denote by Q_λ the integral root lattice, namely $Q_\lambda = \mathbb{Z}\Delta_\lambda^+$ and put $Q_\lambda^+ = \mathbb{N}\Pi_\lambda$.

Next, we fix notations for a parabolic subalgebra (which contains \mathfrak{b}). Hereafter, through this article we fix an arbitrary subset Θ of Π . Let $\check{\Theta}$ be the set of the elements of Δ which are written by linear combinations of elements of Θ over \mathbb{Z} . Put $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall \alpha \in \Theta \alpha(H) = 0\}$, $\mathfrak{l}_\Theta = \mathfrak{h} + \sum_{\alpha \in \check{\Theta}} \mathfrak{g}_\alpha$, $\mathfrak{n}_\Theta = \sum_{\alpha \in \Delta^+ \setminus \check{\Theta}} \mathfrak{g}_\alpha$, $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$. Then \mathfrak{p}_Θ is a parabolic subalgebra of \mathfrak{g} which contains \mathfrak{b} . Conversely, for an arbitrary parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$, there exists some $\Theta \subseteq \Pi$ such that $\mathfrak{p} = \mathfrak{p}_\Theta$. We denote by W_Θ the Weyl group for $(\mathfrak{l}_\Theta, \mathfrak{h})$. W_Θ is identified with a subgroup of W generated by $\{s_\alpha \mid \alpha \in \Theta\}$. We denote by w_Θ the longest element of W_Θ . Using the invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$, we regard \mathfrak{a}_Θ^* as a subspace of \mathfrak{h}^* . It is known that there is a unique nilpotent (adjoint) orbit (say $\mathcal{O}_{\mathfrak{p}_\Theta}$) whose intersection with \mathfrak{n}_Θ is Zariski dense in \mathfrak{n}_Θ . $\mathcal{O}_{\mathfrak{p}_\Theta}$ is called the Richardson orbit with respect to \mathfrak{p}_Θ . We denote by $\bar{\mathcal{O}}_{\mathfrak{p}_\Theta}$ the closure of $\mathcal{O}_{\mathfrak{p}_\Theta}$ in \mathfrak{g} . Put $\rho_\Theta = \frac{1}{2}(\rho - w_\Theta\rho)$ and $\rho^\Theta = \frac{1}{2}(\rho + w_\Theta\rho)$. Then, $\rho^\Theta \in \mathfrak{a}_\Theta^*$.

Define

$$\begin{aligned} P_\Theta^{++} &= \{\lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Theta \langle \lambda, \check{\alpha} \rangle \in \{1, 2, \dots\}\} \\ {}^\circ P_\Theta^{++} &= \{\lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Theta \langle \lambda, \check{\alpha} \rangle = 1\} \end{aligned}$$

We easily have

$${}^\circ P_\Theta^{++} = \{\rho_\Theta + \mu \mid \mu \in \mathfrak{a}_\Theta^*\}.$$

For $\mu \in \mathfrak{h}^*$ such that $\mu + \rho \in P_\Theta^{++}$, we denote by $\sigma_\Theta(\mu)$ the irreducible finite-dimensional \mathfrak{l}_Θ -representation whose highest weight is μ . Let $E_\Theta(\mu)$ be the representation space of $\sigma_\Theta(\mu)$. We define a left action of \mathfrak{n}_Θ on $E_\Theta(\mu)$ by $X \cdot v = 0$ for all $X \in \mathfrak{n}_\Theta$ and $v \in E_\Theta(\mu)$. So, we regard $E_\Theta(\mu)$ as a $U(\mathfrak{p}_\Theta)$ -module.

For $\mu \in P_\Theta^{++}$, we define a generalized Verma module ([Lepowsky 1977]) as follows.

$$M_\Theta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} E_\Theta(\mu - \rho).$$

For all $\lambda \in \mathfrak{h}^*$, we write $M(\lambda) = M_\emptyset(\lambda)$. $M(\lambda)$ is called a Verma module. For $\mu \in P_\Theta^{++}$, $M_\Theta(\mu)$ is a quotient module of $M(\mu)$. Let $L(\mu)$ be the unique highest weight $U(\mathfrak{g})$ -module with the highest weight $\mu - \rho$. Namely, $L(\mu)$ is a unique irreducible quotient of $M(\mu)$. For $\mu \in P_\Theta^{++}$, the canonical projection of $M(\mu)$ to $L(\mu)$ is factored by $M_\Theta(\mu)$.

$\dim E_\Theta(\mu - \rho) = 1$ if and only if $\mu \in {}^\circ P_\Theta^{++}$. If $\mu \in {}^\circ P_\Theta^{++}$, we call $M_\Theta(\mu)$ a scalar generalized Verma module.

§ 2. Reductions of the problem

We retain the notation of §1. In particular, Θ is a subset of Π .

2.1 Basic results of Lepowsky

The following result is one of the fundamental results on the existence problem of homomorphisms between scalar generalized Verma modules.

Theorem 2.1.1. ([Lepowsky 1976])

Let $\mu, \nu \in {}^\circ P_{\Theta}^{++}$.

- (1) $\dim \text{Hom}_{U(\mathfrak{g})}(M_{\Theta}(\mu), M_{\Theta}(\nu)) \leq 1$.
- (2) Any non-zero homomorphism of $M_{\Theta}(\mu)$ to $M_{\Theta}(\nu)$ is injective.

Hence, the existence problem of homomorphisms between scalar generalized Verma modules is reduce to the following problem.

Problem 1 Let $\mu, \nu \in {}^\circ P_{\Theta}^{++}$. When is $M_{\Theta}(\mu) \subseteq M_{\Theta}(\nu)$?

2.2 Reduction to the integral infinitesimal character setting

Since the both $\nu \in W\mu$ and $\nu - \mu \in Q^+$ are necessary condition for the above problem, we can reformulate our problem as follows.

Problem 2 Let $\lambda \in {}^\circ P_{\Theta}^{++}$ be dominant. Let $x, y \in W_{\lambda}$ be such that $x\lambda, y\lambda \in {}^\circ P_{\Theta}^{++}$. When is $M_{\Theta}(x\lambda) \subseteq M_{\Theta}(y\lambda)$?

We fix $\lambda \in {}^\circ P_{\Theta}^{++}$. Then, we can construct a suralgebra \mathfrak{g}' of \mathfrak{h} such that the corresponding Coxeter system is $(W_{\lambda}, \Phi_{\lambda})$. Since $\Theta \subseteq \Pi_{\lambda}$ holds, we can construct corresponding parabolic subalgebra \mathfrak{p}'_{Θ} of \mathfrak{g}' . For $\mu \in P_{\Theta}^{++}$, we denote by $M'_{\Theta}(\mu)$ the corresponding generalized Verma module of \mathfrak{g}' . We consider the category \mathcal{O} in the sense of [Bernstein-Gelfand-Gelfand 1976] corresponding to our particular choice of positive root system. More precisely, we denote by \mathcal{O} (respectively \mathcal{O}') "the category \mathcal{O} " for \mathfrak{g} (respectively \mathfrak{g}'). We denote by \mathcal{O}_{λ} (respectively, \mathcal{O}'_{λ}) the full subcategory of \mathcal{O} (respectively \mathcal{O}') consisting of the objects with a generalized infinitesimal character λ . Soegel's celebrated theorem ([Soegel 1990] Theorem 11) says that there is a Category equivalence between \mathcal{O}_{λ} and \mathcal{O}'_{λ} . Under the equivalence a Verma module $M(x\lambda)$ ($x \in W_{\lambda}$) corresponds to $M'(x\lambda)$. From Lepowsky's generalized BGG resolutions of the generalized Verma modules and their rigidity, we easily see $M_{\Theta}(x\lambda)$ corresponds to $M'_{\Theta}(x\lambda)$ under Soegel's category equivalence. So, we have the following lemma as a corollary of Soergel's theorem.

Lemma 2.2.1. Let $\lambda \in \mathfrak{h}^*$ be dominant. Let $x, y \in W_{\lambda}$ be such that $x\lambda, y\lambda \in {}^\circ P_{\Theta}^{++}$. Then, the following two conditions are equivalent.

- (1) $M_{\Theta}(x\lambda) \subseteq M_{\Theta}(y\lambda)$.
- (2) $M'_{\Theta}(x\lambda) \subseteq M'_{\Theta}(y\lambda)$.

This lemma tells us that we may reduce Problem 2 to the case that λ is integral.

We put

$$W(\Theta) = \{w \in W \mid w\Theta = \Theta\}.$$

§ 3. Excellent parabolic subalgebras

3.1 θ -acceptable positive roots

Hereafter, we fix a subset Θ of Π . For $\alpha \in \Delta$, we put

$$\Delta(\alpha) = \{\beta \in \Delta \mid \exists c \in \mathbb{R} \ \beta|_{\mathfrak{a}_\Theta} = c\alpha|_{\mathfrak{a}_\Theta}\},$$

$$\Delta^+(\alpha) = \Delta(\alpha) \cap \Delta^+,$$

$$U_\alpha = \mathbb{C}S + \mathbb{C}\alpha \subseteq \mathfrak{h}^*.$$

Then $(U_\alpha, \Delta(\alpha), \langle \cdot, \cdot \rangle)$ is a subroot system of $(\mathfrak{h}^*, \Delta, \langle \cdot, \cdot \rangle)$. The set of simple roots for $\Delta^+(\alpha)$ is denoted by $\Pi(\alpha)$. If $\alpha|_{\mathfrak{a}_\Theta} = 0$, then $S = \Pi(\alpha)$. If $\alpha|_{\mathfrak{a}_\Theta} \neq 0$, then $\Pi(\alpha)$ is written as $S \cup \{\tilde{\alpha}\}$. If $\alpha \in \Delta$ satisfies $\alpha|_{\mathfrak{a}_\Theta} \neq 0$ and $\alpha = \tilde{\alpha}$, then we call α Θ -reduced. For $\alpha \in \Delta^+$, we denote by $W_\Theta(\alpha)$ the Weyl group of $(\mathfrak{h}^*, \Delta(\alpha))$. Clearly, $W_\Theta \subseteq W_\Theta(\alpha) \subseteq W$. We denote by w^α the longest element of $W_\Theta(\alpha)$. We call $\alpha \in \Delta$ Θ -acceptable iff $w^\alpha w_\Theta = w_\Theta w^\alpha$. We denote by Δ^Θ (resp. Δ_r^Θ) the set of Θ -acceptable roots (resp. Θ -reduced Θ -acceptable roots). Put $(\Delta^\Theta)^+ = \Delta^+ \cap \Delta^\Theta$ and $(\Delta_r^\Theta)^+ = \Delta^+ \cap \Delta_r^\Theta$. For $\alpha \in \Delta^\Theta$, we define

$$\sigma_\alpha = w^\alpha w_\Theta = w_\Theta w^\alpha.$$

Clearly, $\sigma_\alpha^2 = 1$, $\sigma_\alpha = \sigma_{\tilde{\alpha}}$. If $\alpha|_{\mathfrak{a}_\Theta} = 0$, then $\sigma_\alpha = 1$. If $\alpha \in \Delta$ is orthogonal to all the elements in Θ , then we can easily see α is Θ -reduced and $s_\alpha = \sigma_\alpha$. For $\alpha \in \Delta$, we put

$$V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \alpha \rangle = 0\}.$$

For $\alpha \in \Delta^\Theta$, we put $\hat{\alpha} = \tilde{\alpha}|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^*$. We can easily see:

Lemma 3.1.1. *Let $\alpha \in \Delta_r^\Theta$. Then, we have*

- (1) σ_α preserves \mathfrak{a}_Θ^* .
- (2) $\sigma_\alpha \in W(\Theta)$. In particular, $\sigma_\alpha \rho_\Theta = \rho_\Theta$.
- (3) $\sigma_\alpha \hat{\alpha} = -\hat{\alpha}$.
- (4) $\sigma_\alpha|_{\mathfrak{a}_\Theta^*}$ is the reflection with respect to V_α .

3.2 Excellent parabolic subalgebras

We retain the notations in the previous section.

Let $\Theta \subseteq \Pi$.

A parabolic subalgebra \mathfrak{p}_Θ is called excellent, if all the roots are Θ -acceptable.

Remark If \mathfrak{p}_Θ is a complexified minimal parabolic subalgebra of a real form of \mathfrak{g} such that the m -part of the Langlands decomposition of \mathfrak{p}_Θ is semisimple, then all the roots are Θ -acceptable and σ_α is a reflection with respect to a restricted root $\hat{\alpha}$ for each $\alpha \in \Delta_r^\Theta$.

If \mathfrak{g} is a classical algebra, we can classify excellent parabolic subalgebras of \mathfrak{g} as follows.

(1) Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ (the case of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is similar) and let k be a positive integer dividing n . We consider the following parabolic subalgebras.

$\mathfrak{p}(A_{n-1,k})$: a parabolic subalgebra of \mathfrak{g} whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k, \mathbb{C})}^{n/k}.$$

Then, $\mathfrak{p}(A_{n-1,k})$ is excellent. Conversely any excellent parabolic subalgebra is conjugate to $A_{n,k}$ for some k .

(2) Let \mathfrak{g} be a complex simple Lie algebra of the type X_n . Here, X means one of $B, C,$ and D . Let k and ℓ be positive integers such that k divides $n - \ell$.

We consider the following parabolic subalgebras.

$\mathfrak{p}(X_{n,k,\ell})$: a parabolic subalgebra of \mathfrak{g} whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k, \mathbb{C})}^{(n-\ell)/k} \oplus X_\ell.$$

Here, X_ℓ means that the complex simple Lie algebra of the type X_ℓ . X_0 means the zero Lie algebra.

$\mathfrak{p}(X_{n,k,\ell})$ is excellent unless $X = D, \ell = 0,$ and k is an odd number greater than 1. Any excellent parabolic subalgebra is conjugate to one of such $\mathfrak{p}(X_{n,k,\ell})$ s.

§ 4. Main result

4.1 Elementary homomorphisms

Here, we review some notion in [Matumoto 1993] §3. Hereafter, \mathfrak{g} means a reductive Lie algebra over \mathbb{C} and retain the notations in §1-3. We fix a subset Θ of Π and $\alpha \in \Delta$.

We denote by $\omega_\alpha \in \mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$ the fundamental weight for α with respect to the basis $\Pi(\alpha) = \Theta \cup \{\alpha\}$. Namely ω_α satisfies that $\langle \omega_\alpha, \beta \rangle = 0$ for $\beta \in \Theta, \langle \beta, \check{\alpha} \rangle = 1,$ and $\omega_\alpha|_{\mathfrak{h} \cap \mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Here, $\mathfrak{c}(\mathfrak{g}(\alpha))$ is the center of $\mathfrak{g}(\alpha)$. We see that there is some positive real number a such that $\omega_\alpha = a\alpha|_{\mathfrak{a}_\Theta},$ since $\alpha|_{\mathfrak{h} \cap \mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Hence, we have $V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \omega_\alpha \rangle = 0\}$.

For $\alpha \in (\Delta_r^\Theta)^+,$ we define

$$\mathfrak{g}(\alpha) = \mathfrak{h} + \sum_{\beta \in \Delta(\alpha)} \mathfrak{g}_\beta, \quad \mathfrak{p}_\Theta(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{p}_\Theta.$$

Then, $\mathfrak{g}(\alpha)$ is a reductive Lie subalgebra of \mathfrak{g} whose root system is $\Delta(\alpha)$ and $\mathfrak{p}_\Theta(\alpha)$ is a maximal parabolic subalgebra of $\mathfrak{g}(\alpha)$.

Put $\rho(\alpha) = \frac{1}{2} \sum_{\beta \in \Delta^+(\alpha)} \beta,$ For $\nu \in \mathfrak{a}_\Theta^*,$ we denote by \mathbb{C}_ν the one-dimensional $U(\mathfrak{p}_\Theta(\alpha))$ -module corresponding to $\nu.$ For $\nu \in \mathfrak{a}_\Theta^*$ we define a generalized Verma module for $\mathfrak{g}(\alpha)$ as follows.

$$M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + \nu) = U(\mathfrak{g}(\alpha)) \otimes_{U(\mathfrak{p}_\Theta(\alpha))} \mathbb{C}_{\nu - \rho(\alpha)}.$$

Then, we have:

Theorem 4.1.1. ([Matumoto 2003]) Let ν be an arbitrary element in V_α and let c be either 1 or $\frac{1}{2}$. Assume that $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - nc\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + nc\omega_\alpha)$ for all $n \in \mathbb{N}$. Then, we have $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$ for all $n \in \mathbb{N}$.

We call the above homomorphism of $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha)$ into $M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$ an elementary homomorphism.

The following working hypothesis is proposed in [Matumoto 2003].

Working Hypothesis An arbitrary nontrivial homomorphism between scalar generalized Verma modules is a composition of elementary homomorphisms.

The working hypothesis in the case of the Verma modules is nothing but the result of Bernstein-Gelfand-Gelfand.

I would like to propose :

Conjecture For an excellent parabolic subalgebra, the above working hypothesis is affirmative.

4.2 Main result

Now, we state our main result.

Theorem 4.2.1. Let \mathfrak{g} be a classical Lie algebra and let \mathfrak{p}_Θ be one of the following cases.

- (a) $\mathfrak{p}(A_{n-1,k})$ ($k|n$),
- (b) $\mathfrak{p}(B_{n,2k,m})$ ($k \leq m$),
- (c) $\mathfrak{p}(B_{n,2k+1,m})$ ($k \geq m$),
- (d) $\mathfrak{p}(C_{n,2k,m})$ ($k \leq m$),
- (e) $\mathfrak{p}(C_{n,2k+1,m})$ ($k \geq m$),
- (f) $\mathfrak{p}(D_{n,2k-1,m})$ ($k \leq m$),
- (g) $\mathfrak{p}(D_{n,2k,m})$ ($k \geq m$).

Then, any homomorphism between scalar generalized Verma modules with integral infinitesimal characters is a composition of elementary homomorphisms.

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