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RIMS-1926

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September 2020

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September 30, 2020

Abstract

It is well-known that various profinite groups appearing in anabelian geometry satisfy distinctive group-theoretic properties such as the *slimness* [i.e., the property that every open subgroup is center-free] and the *strong indecomposability* [i.e., the property that every open subgroup has no nontrivial product decomposition]. In the present paper, we consider another group-theoretic property on profinite groups, which we shall refer to as *strong internal indecomposability* — this is a *stronger* property than *both* the slimness and the strong indecomposability — and prove that various profinite groups appearing in anabelian geometry [e.g., the étale fundamental groups of hyperbolic curves over number fields, p-adic local fields, or finite fields; the absolute Galois groups of Henselian discrete valuation fields with positive characteristic residue fields or Hilbertian fields] satisfy this property. Moreover, by applying the *pro-prime-to-p version of the Grothendieck Conjecture for hyperbolic curves over finite fields* of characteristic p [established by Saidi and Tamagawa], together with some considerations on almost surface groups, we also prove that the *Grothendieck-Teichmüller group* satisfies the *strong indecomposability.* This gives an affirmative answer to an open problem posed in a first author's previous work.

2010 Mathematics Subject Classification: Primary 14H30; Secondary 12E30.

Key words and phrases: profinite group; internal indecomposability; absolute Galois group; étale fundamental group; hyperbolic curve; Grothendieck-Teichmüller group; anabelian geometry.

Contents

Introduction

For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. For any field F, we shall write F^{sep} for the separable closure [determined up to isomorphisms] of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let p be a prime number.

Let X be an algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] over a field. In anabelian geometry, we often consider

> *whether or not the algebraic variety X may be "reconstructed" from the étale fundamental group Π_X.*

For instance, if X is a hyperbolic curve over a number field [i.e., a finite extension field of the field of rational numbers $\mathbb{Q}$], then Mochizuki and Tamagawa proved that X may be "reconstructed" from Π_X [cf. [23], Theorem A; [27], Introduction; [36], Theorem 0.4]. However, it seems far-reaching to specify the precise class of algebraic varieties which may be "reconstructed" from their étale fundamental groups [i.e., the class of "anabelian varieties"].

On the other hand, it has been observed that various profinite groups appearing in anabelian geometry [e.g., the étale fundamental groups of hyperbolic curves over number fields] tend to satisfy group-theoretic properties such as the *slimness* and the *strong indecomposability* [cf. [21], [22]]. For our purposes, let us recall the definition of the slimness and the strong indecomposability of profinite groups. Let G be a profinite group. Then we shall say that

- G is *slim* if every open subgroup of G is center-free;
- G is *strongly indecomposable* if every open subgroup of G is indecomposable, i.e., has no nontrivial product decomposition.

However, at the time of writing the present paper, the authors do not know the precise relation between the class of "anabelian varieties" and the class of algebraic varieties that satisfy the above group-theoretic properties. It seems to the authors that a further examination of this relation would be important.

In this context, it is natural to pose the following question:

> Question 1: Do various profinite groups appearing in anabelian geometry satisfy stronger properties than the slimness and the strong indecomposability?

With regard to Question 1, in the present paper, we consider the notion of *strong internal indecomposability*, which is a stronger property than both the slimness and the strong indecomposability [cf. Theorem A below; Remark 1.1.4].

Let $H \subseteq G$ be a normal closed subgroup. Then we shall say that:

- H is *normally decomposable* in G if there exist nontrivial normal closed subgroups $H_1 \subseteq G$ and $H_2 \subseteq G$ such that $H = H_1 \times H_2$.
- H is *normally indecomposable* in G if H is not normally decomposable in G.
- G is *internally indecomposable* if every normal closed subgroup of G is center-free and normally indecomposable in G.
- G is *strongly internally indecomposable* if every open subgroup of G is internally indecomposable.

Note that, if G is strongly internally indecomposable, then it follows immediately from the various definitions involved that G is slim and strongly indecomposable. Moreover, we also note that

> G is internally indecomposable if and only if, for every nontrivial normal closed subgroup $J \subseteq G$, the centralizer of J in G is trivial [cf. Proposition 1.2].

In anabelian geometry, this latter property has been considered and proved for special "$J \subseteq G$" [cf. [12], Lemma 2.13, (ii); [28], Lemma 2.7, (vi)]. Thus, it would be important to establish generalities on this property [cf. §1].

Let n be a positive integer; Σ a nonempty set of prime numbers; k a field; Y a smooth curve over k of type (g, r) [cf. Definition 3.1]. If Y is a hyperbolic curve over k, i.e., $2g - 2 + r > 0$, then we write Y_n for the n-th configuration space associated to Y [cf. Definition 3.8, (i)]. [Note that $Y_1 = Y$.] Then our first main result is the following [cf. Theorems 2.1; 2.3; 2.7; 3.7, (i), (ii); 3.9; 3.11, (i), (ii); Corollary 3.5]:

Theorem A.

(i) *Suppose that k is a Henselian discrete valuation field of residue characteristic p. Then G_k is strongly internally indecomposable. Moreover, if k contains a primitive p-th root of unity in the case where k is of characteristic* 0*, then any almost pro-p-maximal quotient of G_k [cf. Definition 1.4] is strongly internally indecomposable.*

(ii) *Suppose that k is a Hilbertian field [i.e., a field for which Hilbert's irreducibility theorem holds — cf. Remark 2.7.1]. Then G_k is strongly internally indecomposable.*

(iii) *Suppose that*

- *Σ does not contain the characteristic of k;*

- *k is an algebraically closed field;*
- *Y is a hyperbolic curve over k.*

Then Π_Y^{Σ} is strongly internally indecomposable.

(iv) Suppose that

- *$p \in \Sigma$;*
- *k is an algebraically closed field of characteristic p;*
- *$(g, r) \neq (0, 0), (1, 0)$.*

If $r = 0$, i.e., Y is proper, then we write $\sigma(Y)$ for the p-rank of [the Jacobian variety of] Y. Then the following hold:

(a) Suppose that $\Sigma = \{p\}$, and $\sigma(Y) \neq 1$ if $r = 0$. Then Π_Y^p is strongly internally indecomposable.

(b) Suppose that $\Sigma \supsetneq \{p\}$. Then Π_Y^{Σ} is strongly internally indecomposable.

(v) Suppose that k is an algebraically closed field of characteristic 0, and Y is a hyperbolic curve over k. Then Π_{Y_n} and $\Pi_{Y_n}^p$ are strongly internally indecomposable.

(vi) Suppose that

- *k is a number field or a p-adic local field [i.e., a finite extension field of the field of p-adic numbers];*
- *Y is a hyperbolic curve over k.*

Then Π_{Y_n} is strongly internally indecomposable.

(vii) Suppose that

- *k is a finite field of characteristic p;*
- *$(g, r) \neq (0, 0), (1, 0)$ (respectively, $2g - 2 + r > 0$).*

Then Π_Y (respectively, the geometrically pro-prime-to-p quotient of Π_Y [cf. Definition 3.10]) is strongly internally indecomposable.

Next, we consider the Grothendieck-Teichmüller group GT. Let us recall that GT has been considered to be a combinatorial approximation of $G_{\mathbb{Q}}$ [cf. [4]; [8]; [10]; [13], Introduction]. Indeed, the natural faithful outer actions of $G_{\mathbb{Q}}$ and GT on the étale fundamental group of the projective line minus the three points 0, 1, ∞, over $\overline{\mathbb{Q}}$ determine the inclusion

$$G_{\mathbb{Q}} \subseteq \mathrm{GT},$$

and there exists a famous open question concerning this inclusion [cf. [34], §1.4]:

Question 2: Is the natural inclusion $G_{\mathbb{Q}} \subseteq \mathrm{GT}$ bijective?

With regard to Question 2, in the authors' knowledge, there is no [strong] evidence to believe that the inclusion $G_{\mathbb{Q}} \subseteq \mathrm{GT}$ is bijective. Here, we note that André defined a p-adic avatar GT_p of GT and formulated a p-adic analogue of Question 2 by using his theory of tempered fundamental groups [cf. [1], [2]]. In this local setting, the second author constructed a natural splitting $\mathrm{GT}_p \twoheadrightarrow G_{\mathbb{Q}_p}$ of the inclusion $G_{\mathbb{Q}_p} \subseteq \mathrm{GT}_p$ [cf. [37], Corollary B]. It seems to the authors that the existence of such a splitting may be regarded as a strong evidence to believe that the inclusion $G_{\mathbb{Q}_p} \subseteq \mathrm{GT}_p$ is bijective. However, the construction of the splitting $\mathrm{GT}_p \twoheadrightarrow G_{\mathbb{Q}_p}$ heavily depends on a certain rigidity of tempered fundamental groups [cf. [37], Theorem C]. Thus, at the time of writing the present paper, the authors do not regard the existence of the splitting in the local setting as an evidence to believe that the inclusion $G_{\mathbb{Q}} \subseteq \mathrm{GT}$ is bijective.

Since Question 2 is far-reaching, the following question has been considered to be important in the literatures [cf. [34], §1.4]:

Question 3: Let $\mathbb{P}$ be a group-theoretic property that $G_{\mathbb{Q}}$ satisfies. Then does GT satisfy the property $\mathbb{P}$?

Concerning Question 3, for instance, Lochak-Schneps proved a remarkable result that the normalizer of a complex conjugation $\iota \in \mathrm{GT}$ coincides with the group [of order 2] generated by ι [cf. [19], Proposition 4, (ii)]. [Note that the analogous result for $G_{\mathbb{Q}}$ follows from the approximation theorem — cf. [31], Corollary 12.1.4.] On the other hand, the first author posed the following question [cf. [21], Introduction]:

Question 4: Is GT strongly indecomposable?

[Note that the strong indecomposability of $G_{\mathbb{Q}}$ follows from the fact that number fields are Hilbertian — cf. [7], Proposition 13.4.1; [7], Corollary 13.8.4.] We remark that the indecomposability of GT follows from Lochak-Schneps's result [cf. Remark 4.10.1]. However, this argument does not work for open subgroups of GT that do not contain ι. In the present paper, we also give a complete [much more general] affirmative answer to Question 4.

Let K ($\subseteq \overline{\mathbb{Q}}$) be a number field; Z a hyperbolic curve of genus 0 over K. Write $Z_{\overline{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} Z \times_K \overline{\mathbb{Q}}$;

$$\mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}}) \subseteq \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}})$$

for the subgroup of outer automorphisms of $\Pi_{Z_{\overline{\mathbb{Q}}}}$ that induce the identity automorphisms on the set of the conjugacy classes of cuspidal inertia subgroups of $\Pi_{Z_{\overline{\mathbb{Q}}}}$ [i.e., the stabilizer subgroups associated to pro-cusps of the pro-universal covering of the hyperbolic curve $Z_{\overline{\mathbb{Q}}}$]. Then the natural outer action of G_K on $\Pi_{Z_{\overline{\mathbb{Q}}}}$ determines an injection $G_K \hookrightarrow \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}})$ [cf. [14], Theorem C]. We shall regard G_K as a subgroup of $\mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}})$ via this injection. Recall that, if we take Z to be the projective line minus the three points 0, 1, ∞, over K, then GT may be regarded as a closed subgroup of $\mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}})$ [cf. Remark 4.4.1]. Then our second main result is the following [cf. Theorem 4.10]:

Theorem B. *Let* $G \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}})$ *be a closed subgroup such that* G *contains an open subgroup of* G_K*. Then* G *is strongly indecomposable. In particular, the Grothendieck-Teichmüller group* GT *is strongly indecomposable.*

Note that the first author proved that a pro-l analogue of Theorem B holds [cf. Remark 4.10.2; [21], Theorem 6.1]. However, the proof heavily depends on the [easily verified] fact that $\mathbb{Z}_l$ is indecomposable. In contrast, since $\widehat{\mathbb{Z}}$ is decomposable, a similar argument to the argument applied in the proof of [21], Theorem 6.1 does not work in our situation. To overcome this difficulty, we apply [highly nontrivial] Saidi-Tamagawa's result on the pro-prime-to-p version of the Grothendieck Conjecture for hyperbolic curves over finite fields of characteristic p [cf. [33], Theorem 1], together with some considerations on almost surface groups [cf. Definition 3.2].

On the other hand, we recall that $\mathbb{Q}$ is Hilbertian. Then it follows from Theorem A, (ii), that $G_{\mathbb{Q}}$ is strongly internally indecomposable. Thus, from the viewpoint of Question 3, it is natural to pose the following question, which may be regarded as a further generalization of [the second assertion of] Theorem B:

Question 5: Is GT strongly internally indecomposable?

However, at the time of writing the present paper, the authors do not know whether the answer is affirmative or not.

The present paper is organized as follows. In §1, we introduce the notion of internal indecomposability of profinite groups and examine some basic properties of this notion which will be of later use. In §2, by applying the results obtained in §1 of the present paper and [22], we prove that the absolute Galois groups of Henselian discrete valuation fields with positive characteristic residue fields and Hilbertian fields are strongly internally indecomposable [cf. Theorem A, (i), (ii)]. In §3, by applying the results obtained in §1, §2, of the present paper, we prove the strong internal indecomposability of various profinite groups appearing in anabelian geometry [cf. Theorem A, (iii), (iv), (v), (vi), (vii)]. Finally, in §4, we first recall the definition of the Grothendieck-Teichmüller group GT. Then we apply various Grothendieck Conjecture-type results, together with some considerations on almost surface groups, to prove that GT is strongly indecomposable [cf. Theorem B].

Notations and Conventions

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of the underlying additive group of $\mathbb{Z}$. The notation $\mathbb{Z}_{\geq 1}$ will be used to denote the set of positive integers. We shall refer to a finite extension field of $\mathbb{Q}$ as a *number field.* If p is a prime number, then the notation $\mathbb{Q}_p$ will be used to denote the field of p-adic numbers; the notation $\mathbb{Z}_p$ will be used to denote the ring of p-adic integers; the notation $\mathbb{F}_p$ will be used to denote the finite field of cardinality p. We shall refer to a finite extension field of $\mathbb{Q}_p$ as a *p-adic local field*. If A is a commutative ring, then the notation $A^\times$ will be used to denote the group of units of A.

Fields: Let F be a field; F^{sep} a separable closure of F; p a prime number. Then we shall write $\mathrm{char}(F)$ for the characteristic of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$; $F((t))$ for the one parameter formal power series field over F; $F_{p^\infty} \subseteq F^{\mathrm{sep}}$ for the subfield obtained by adjoining p-power roots of unity to F. If $\mathrm{char}(F) \neq p$, then we shall fix a primitive p-th root of unity $\zeta_p \in F^{\mathrm{sep}}$. If F is perfect, then we shall also write $\overline{F} \stackrel{\mathrm{def}}{=} F^{\mathrm{sep}}$.

Schemes: Let S be a scheme. Then we shall write $\mathrm{Aut}(S)$ for the group of automorphisms of S. Let K be a field; $K \subseteq L$ a field extension; X an algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] over K. Then we shall write $X_L \stackrel{\mathrm{def}}{=} X \times_K L$; $\mathrm{Aut}_K(X)$ for the group of automorphisms of X over K; $\mathbb{P}^1_K$ for the projective line over K.

Profinite groups: Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset of prime numbers; G a profinite group. Then we shall write G^Σ for the maximal pro-Σ quotient of G; $\mathrm{Aut}(G)$ for the group of automorphisms of G [in the category of profinite groups], $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G, and $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. If p is a prime number, then we shall also write $G^p \stackrel{\mathrm{def}}{=} G^{\{p\}}$; $G^{(p)'} \stackrel{\mathrm{def}}{=} G^{\mathfrak{Primes}\setminus\{p\}}$.

Suppose that G is topologically finitely generated. Then G admits a basis of *characteristic open subgroups* [cf. [32], Proposition 2.5.1, (b)], which thus induces a *profinite topology* on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$.

Fundamental groups: Let S be a connected locally Noetherian scheme. Then we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. [Note that, for any field F, $\Pi_{\mathrm{Spec}(F)} \cong G_F$.]

1 Basic properties of internal indecomposability

In the present section, we introduce the notion of *internal indecomposability* of profinite groups and examine basic properties.

Let p be a prime number.

Definition 1.1 ([27], Notations and Conventions; [27], Definition 1.1, (ii))**.** Let G be a profinite group; $H \subseteq G$ a closed subgroup of G.

(i) We shall write $Z_G(H)$ for the *centralizer* of H in G, i.e., the closed subgroup $\{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$; $Z(G) \stackrel{\mathrm{def}}{=} Z_G(G)$; $N_G(H)$ for the *normalizer* of H in G, i.e., the closed subgroup $\{g \in G \mid gHg^{-1} = H\}$.

(ii) We shall say that G is *slim* if $Z_G(U) = \{1\}$ for every open subgroup U of G.

(iii) We shall say that G is *elastic* if every nontrivial topologically finitely generated normal closed subgroup of an open subgroup of G is open. If G is elastic, but not topologically finitely generated, then we shall say that G is *very elastic* [cf. [22], Proposition 1.2, (ii)].

(iv) We shall say that G is *decomposable* if there exist nontrivial normal closed subgroups $H_1 \subseteq G$ and $H_2 \subseteq G$ such that $G = H_1 \times H_2$. We shall say that G is *indecomposable* if G is not decomposable. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

(v) We shall say that H is *normally decomposable* in G if there exist nontrivial normal closed subgroups $H_1 \subseteq G$ and $H_2 \subseteq G$ such that $H = H_1 \times H_2$. We shall say that H is *normally indecomposable* in G if H is not normally decomposable in G.

(vi) We shall say that G is *internally indecomposable* if every nontrivial normal closed subgroup of G is center-free and normally indecomposable in G. [Note that the trivial subgroup of G is center-free and normally indecomposable in G.] We shall say that G is *strongly internally indecomposable* if every open subgroup of G is internally indecomposable.

Remark 1.1.1. Let G be a strongly internally indecomposable profinite group. Then it follows immediately from [22], Proposition 1.2, (i), that G is slim.

Remark 1.1.2. Let G be a profinite group. Then it follows immediately from the various definitions involved that:

(i) G is normally decomposable in G if and only if G is decomposable.

(ii) If G is internally indecomposable (respectively, strongly internally indecomposable), then G is indecomposable (respectively, strongly indecomposable).

Remark 1.1.3. Let G be a nonabelian finite simple group. Then it follows immediately from the various definitions involved that G is not strongly internally indecomposable but internally indecomposable.

Remark 1.1.4. Let X be a proper hyperbolic curve over $\mathbb{Q}$; $\mathbb{Q} \subseteq \mathbb{Q}(t)$ a purely transcendental extension of transcendence degree 1. Then $\Pi_{X_{\mathbb{Q}(t)}}$ is slim and strongly indecomposable [cf. [21], Corollary 4.5], but not internally indecomposable. Indeed, it follows immediately from [9], Exposé X, Corollaire 1.7, that the normal closed subgroup $\Pi_{X_{\overline{\mathbb{Q}}(t)}} \subseteq \Pi_{X_{\mathbb{Q}(t)}}$ is isomorphic to the product $\Pi_{X_{\overline{\mathbb{Q}}}} \times G_{\overline{\mathbb{Q}}(t)}$.

Next, we give a useful criterion of the internal indecomposability.

Proposition 1.2. *Let G be a profinite group. Then G is internally indecomposable if and only if $Z_G(H) = \{1\}$ for every nontrivial normal closed subgroup $H \subseteq G$.*

Proof. First, we verify *sufficiency*. Suppose that $Z_G(H) = \{1\}$ for every nontrivial normal closed subgroup $H \subseteq G$. Let $H \subseteq G$ be a nontrivial normal closed subgroup. Then $Z(H) \subseteq Z_G(H) = \{1\}$. On the other hand, let $H_1 \subseteq G$ and $H_2 \subseteq G$ be normal closed subgroups such that $H = H_1 \times H_2$, and $H_1 \neq \{1\}$. Then $H_2 \subseteq Z_G(H_1) = \{1\}$. Thus, we conclude that H is center-free and normally indecomposable in G, hence that G is internally indecomposable.

Next, we verify *necessity*. Suppose that G is internally indecomposable. Let $H \subseteq G$ be a nontrivial normal closed subgroup. Since H is center-free, $H \cap Z_G(H) = Z(H) = \{1\}$. In particular, we obtain a normal closed subgroup $H \times Z_G(H) \subseteq G$. Thus, since G is internally indecomposable, and $H \neq \{1\}$, we conclude that $Z_G(H) = \{1\}$. This completes the proof of Proposition 1.2. $\square$

Next, we recall basic notions concerning profinite groups.

Definition 1.3 ([29], Definition 1.1, (i), (ii))**.** Let $\mathcal{C}$ be a family of finite groups including the trivial group. Then:

(i) We shall refer to a finite group belonging to $\mathcal{C}$ as a $\mathcal{C}$-group.

(ii) We shall refer to $\mathcal{C}$ as a *full-formation* if $\mathcal{C}$ is closed under taking quotients, subgroups, and extensions.

(iii) We shall write $\Sigma_{\mathcal{C}}$ for the set of primes l such that $\mathbb{Z}/l\mathbb{Z}$ is a $\mathcal{C}$-group.

Definition 1.4 ([27], Definition 1.1, (iii))**.** Let G, Q be profinite groups; $q : G \twoheadrightarrow Q$ an epimorphism [in the category of profinite groups]; $\mathcal{C}$ a full-formation; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset of prime numbers. Then we shall say that Q is an *almost pro-$\mathcal{C}$-maximal quotient* of G if there exists a normal open subgroup $N \subseteq G$ such that $\mathrm{Ker}(q)$ coincides with the kernel of the natural surjection $N \twoheadrightarrow N^{\mathcal{C}}$, where $N^{\mathcal{C}}$ denotes the maximal pro-$\mathcal{C}$ quotient of N. Suppose that Q is an almost pro-$\mathcal{C}$-maximal quotient of G, and $\mathcal{C}$ is the family of all finite groups Γ such that every prime divisor of the order of Γ is an element of Σ. Then we shall say that Q is an *almost pro-Σ-maximal quotient* of G. If $\Sigma = \{p\}$, then we shall also say that Q is an *almost pro-p-maximal quotient* of G.

Remark 1.4.1. Let $\mathcal{C}$ be a full-formation. Then it follows immediately from the various definitions involved that the maximal pro-$\mathcal{C}$ quotient of a profinite group is an *almost pro-$\mathcal{C}$-maximal quotient.*

Next, we recall the following result, which is one of the motivations of our study.

Proposition 1.5 ([32], Proposition 8.7.8)**.** *Let $\mathcal{C}$ be a full-formation; F a free pro-$\mathcal{C}$ group of rank ≥ 2. Then F is strongly internally indecomposable [cf. Proposition 1.2; [32], Theorem 3.6.2, (a)].*

Proposition 1.6. *Let G be a slim profinite group. Suppose that there exists an open subgroup $H \subseteq G$ such that H is internally indecomposable (respectively, strongly internally indecomposable). Then G is internally indecomposable (respectively, strongly internally indecomposable).*

Proof. To verify Proposition 1.6, it suffices to prove the non-resp'd case. Fix such an open subgroup $H \subseteq G$. Let $N \subseteq G$ be a nontrivial normal closed subgroup. Write $C \stackrel{\text{def}}{=} Z_G(N)$. Since G is slim, it follows from [22], Lemma 1.3, that $N \cap H \neq \{1\}$. Note that since H is internally indecomposable, it follows from Proposition 1.2 that $C \cap H \subseteq Z_H(N \cap H) = \{1\}$. Again, since G is slim, it follows from [22], Lemma 1.3, that $C = \{1\}$. Thus, we conclude that G is internally indecomposable [cf. Proposition 1.2]. This completes the proof of Proposition 1.6. □

Proposition 1.7. *Let G be a profinite group; $\{G_i\}_{i \in I}$ a directed subset of the set of closed subgroups of G — where $j \geq i \Leftrightarrow G_i \subseteq G_j$ — such that*

$$G = \bigcup_{i \in I} G_i.$$

Suppose that, for each $i \in I$, G_i is internally indecomposable (respectively, strongly internally indecomposable). Then G is internally indecomposable (respectively, strongly internally indecomposable).

Proof. To verify Proposition 1.7, it suffices to prove the non-resp'd case. Let $H \subseteq G$ be a nontrivial normal closed subgroup. Then since $G = \bigcup_{i \in I} G_i$, there exists $i \in I$ such that

$$H \bigcap G_i \neq \{1\}.$$

Fix such $i \in I$. Write $I_i \stackrel{\text{def}}{=} \{j \in I \mid j \geq i\}$; $C \stackrel{\text{def}}{=} Z_G(H)$. Since $\{G_i\}_{i \in I}$ is a directed set, it holds that

$$G = \bigcup_{j \in I_i} G_j.$$

Let $j \in I_i$ be an element. Observe that

- $H \cap G_j \neq \{1\}$,
- $H \cap G_j$ and $C \cap G_j$ are normal closed subgroups of G_j, and

- $H \cap G_j \subseteq Z_{G_j}(C \cap G_j)$.

Then since G_j is internally indecomposable, it holds that $C \cap G_j = \{1\}$. Thus, it follows from the equality

$$C = \bigcup_{j \in I_i} (C \bigcap G_j)$$

that $C = \{1\}$, hence that G is internally indecomposable [cf. Proposition 1.2]. This completes the proof of Proposition 1.7. $\square$

Proposition 1.8. *Let G be a profinite group; $\{G_i\}_{i \in I}$ a directed subset of the set of normal closed subgroups of G — where $j \geq i \Leftrightarrow G_j \subseteq G_i$ — such that the natural homomorphism*

$$G \to \varprojlim_{i \in I} G/G_i$$

is an isomorphism. Suppose that, for each $i \in I$, G/G_i is internally indecomposable (respectively, strongly internally indecomposable). Then G is internally indecomposable (respectively, strongly internally indecomposable).

Proof. To verify Proposition 1.8, it suffices to prove the non-resp'd case. For each $i \in I$, write $\phi_i : G \twoheadrightarrow G/G_i$ for the natural surjection. Let $H \subseteq G$ be a nontrivial normal closed subgroup. Then since $G \overset{\sim}{\to} \varprojlim_{i \in I} G/G_i$, there exists $i \in I$ such that

$$\phi_i(H) \neq \{1\}.$$

Fix such $i \in I$. Write $I_i \overset{\text{def}}{=} \{j \in I \mid j \geq i\}$; $C \overset{\text{def}}{=} Z_G(H)$. Since $\{G_i\}_{i \in I}$ is a directed set, the natural homomorphism

$$G \to \varprojlim_{j \in I_i} G/G_j$$

is an isomorphism. Let $j \in I_i$ be an element. Observe that

- $\phi_j(H) \neq \{1\}$,
- $\phi_j(H)$ and $\phi_j(C)$ are normal closed subgroups of G/G_j, and
- $\phi_j(H) \subseteq Z_{G/G_j}(\phi_j(C))$.

Then since G/G_j is internally indecomposable, it holds that $\phi_j(C) = \{1\}$ [cf. Proposition 1.2]. Thus, it follows from the equality

$$\bigcap_{j \in I_i} G_j = \{1\}$$

that $C = \{1\}$, hence that G is internally indecomposable [cf. Proposition 1.2]. This completes the proof of Proposition 1.8. $\square$

Proposition 1.9. *Let G be an elastic profinite group; $H \subseteq G$ a normal closed subgroup. Suppose that*

- *G is very elastic or slim,*
- *H is internally indecomposable (respectively, strongly internally indecomposable), and*
- *every normal closed subgroup (respectively, every normal closed subgroup of any open subgroup) of G/H is topologically finitely generated.*

Then G is internally indecomposable (respectively, strongly internally indecomposable).

Proof. To verify Proposition 1.9, it suffices to prove the non-resp'd case. Let $N \subseteq G$ be a nontrivial normal closed subgroup. Write

$$C \stackrel{\text{def}}{=} Z_G(N), \quad C_H \stackrel{\text{def}}{=} Z_H(N \cap H).$$

Our goal is to prove that $C = \{1\}$ [cf. Proposition 1.2].

First, we consider the case where G is very elastic. Then since $N \neq \{1\}$, N is not topologically finitely generated. Thus, it follows from our third assumption that $N \cap H \neq \{1\}$. Next, since H is internally indecomposable, it holds that $C_H = \{1\}$. In particular, $C \cap H = \{1\}$. Again, it follows from our third assumption that C is topologically finitely generated. Thus, since G is very elastic, it holds that $C = \{1\}$.

Next, we consider the case where G is slim. If $N \cap H = \{1\}$, then it follows from our third assumption that N is topologically finitely generated. In particular, since G is elastic, and $N \neq \{1\}$, it holds that $N \subseteq G$ is open. Thus, we conclude from our assumption that G is slim that $C = \{1\}$. Here, we note that since G is slim, and $N \neq \{1\}$, it holds that $C \subseteq G$ is a normal closed subgroup of infinite index. Then, if $N \cap H \neq \{1\}$, then it follows from a similar argument to the argument applied in the preceding paragraph that $C = \{1\}$. This completes the proof of Proposition 1.9. ☐

Next, we give a variant of [21], Lemma 1.6.

Lemma 1.10. *Let G be an internally indecomposable profinite group; $H \subseteq G$ a nontrivial normal closed subgroup; $\alpha \in \mathrm{Aut}(G)$. Suppose that, for any $h \in H$, it holds that $\alpha(h) = h$. Then α is the identity automorphism.*

Proof. Lemma 1.10 follows from a similar argument to the argument applied in the proof of [21], Lemma 1.6, together with Proposition 1.2. ☐

Proposition 1.11. *Let*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$$

be an exact sequence of profinite groups. Write $\rho : G_2 \to \mathrm{Out}(G_1)$ for the outer representation associated to this exact sequence. Then the following hold:

(i) *Suppose that*

- G_1 *is internally indecomposable (respectively, strongly internally indecomposable);*
- G_2 *is internally indecomposable (respectively, strongly internally indecomposable);*
- ρ *is injective.*

Then G *is internally indecomposable (respectively, strongly internally indecomposable).*

(ii) *Suppose that*

- G_1 *is internally indecomposable (respectively, strongly internally indecomposable);*
- G_2 *is abelian;*
- ρ *is injective, or* G *is center-free (respectively, slim).*

Then G *is internally indecomposable (respectively, strongly internally indecomposable).*

Proof. It follows immediately from [21], Lemma 1.7, (i), together with Remark 1.1.1, that, to verify Proposition 1.11, it suffices to prove the non-resp'd case. Let $N \subseteq G$ be a nontrivial normal closed subgroup. Write

$$C \stackrel{\text{def}}{=} Z_G(N), \quad C_1 \stackrel{\text{def}}{=} Z_{G_1}(N \cap G_1).$$

Our goal is to prove that $C = \{1\}$ [cf. Proposition 1.2].

First, we verify assertion (i). Let us begin by observing the following assertion:

> Claim 1.11.A: Let $H \subseteq G$ be a nontrivial normal closed subgroup. Suppose that $Z_G(H) \subseteq G_1$. Then $Z_G(H) = \{1\}$.

Indeed, suppose that $Z_G(H) \neq \{1\}$. Then since G_1 is internally indecomposable, and $Z_G(H) \subseteq G_1$ is normal, it holds that $H \subseteq Z_G(G_1)$ [cf. Lemma 1.10]. On the other hand, it follows immediately from our assumption that ρ is injective that $Z_G(G_1) \subseteq Z(G_1)$. Moreover, since G_1 is center-free, it holds that $Z_G(G_1) = \{1\}$, hence that $H = \{1\}$. This is a contradiction. Thus, we conclude that $Z_G(H) = \{1\}$. This completes the proof of Claim 1.11.A.

Suppose that $N \cap G_1 = \{1\}$. Then since $N \neq \{1\}$, and G_2 is internally indecomposable, it holds that $C \subseteq G_1$. Thus, by applying Claim 1.11.A to the nontrivial normal closed subgroup $N \subseteq G$, we conclude that $C = \{1\}$.

Suppose that $N \cap G_1 \neq \{1\}$. Then since G_1 is internally indecomposable, it holds that $C \cap G_1 \subseteq C_1 = \{1\}$. If $C \neq \{1\}$, then since G_2 is internally indecomposable, it holds that

$$\{1\} \neq N \subseteq Z_G(C) \subseteq G_1.$$

However, this contradicts Claim 1.11.A [in the case where $H = C$]. Thus, we conclude that $C = \{1\}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Recall that G_1 is center-free. Then, if ρ is injective, then G is also center-free. Thus, we may assume without loss of generality that G is center-free. Next, we verify the following assertion:

> Claim 1.11.B: Let $H \subseteq G$ be a nontrivial normal closed subgroup. Then $H \cap G_1 \neq \{1\}$.

Indeed, since $H \neq \{1\}$, and $Z(G) = \{1\}$, there exist elements $g \in G$, $h \in H$ such that $1 \neq x \stackrel{\text{def}}{=} g \cdot h \cdot g^{-1} \cdot h^{-1} \in G$. Fix such elements. Then since G_2 is abelian, the image of x via the surjection $G \twoheadrightarrow G_2$ is trivial. Moreover, since $H \subseteq G$ is normal, it holds that $x \in H$. In particular, we have $x \in H \cap G_1 \neq \{1\}$. This completes the proof of Claim 1.11.B.

Then, by applying Claim 1.11.B to the nontrivial normal closed subgroup $N \subseteq G$, we conclude that $N \cap G_1 \neq \{1\}$. Thus, since G_1 is internal indecomposability, it holds that $C \cap G_1 \subseteq C_1 = \{1\}$. Finally, it follows from Claim 1.11.B [in the case where $H = C$] that $C = \{1\}$. This completes the proof of assertion (ii), hence of Proposition 1.11. □

2 Strong internal indecomposability of the absolute Galois groups

In the present section, we prove that the absolute Galois groups of

- Henselian discrete valuation fields with positive characteristic residue fields and
- Hilbertian fields

are strongly internally indecomposable [cf. Definition 1.1, (vi)].

Let p be a prime number.

Theorem 2.1. *Let K be a Henselian discrete valuation field of characteristic p; $N \subseteq G_K$ a normal open subgroup. Then G_K, as well as the almost pro-p-maximal quotient*

$$G_N \stackrel{\text{def}}{=} G_K/\mathrm{Ker}(N \twoheadrightarrow N^p)$$

associated to N, is strongly internally indecomposable.

Proof. Note that $N^p \subseteq G_N$ is an open subgroup. Recall that G_N is slim [cf. [22], Theorem 2.10], and N^p is a free pro-p group of infinite rank [cf. [31], Proposition 6.1.7; [22], Lemma 3.1]. Then it follows immediately from Propositions 1.5, 1.6, that G_N is strongly internally indecomposable. Moreover, by varying N, we conclude that G_K is strongly internally indecomposable [cf. Proposition 1.8]. This completes the proof of Theorem 2.1. □

Next, we recall the following well-known fact [cf. [6], Chapter III, §5; [38]]:

Theorem 2.2. *Let K be a mixed characteristic complete discrete valuation field such that the residue field of K is perfect and of characteristic p. Then the field of norms*

$$N(K_{p^\infty}/K)$$

is isomorphic to $k((t))$, where k denotes the residue field of the [Henselian] valuation field K_{p^∞}. Moreover, $G_{K_{p^\infty}}$ is isomorphic to $G_{k((t))}$.

Theorem 2.3. *Let K be a mixed characteristic Henselian discrete valuation field of residue characteristic p. Then G_K and $G_{K_{p^\infty}}$, as well as any almost pro-p-maximal quotient of $G_{K_{p^\infty}}$, are strongly internally indecomposable. Moreover, if $\zeta_p \in K$, then any almost pro-p-maximal quotient of G_K is strongly internally indecomposable.*

Proof. First, it follows immediately from Proposition 1.6, together with [22], Theorem 2.8, (i), that we may assume without loss of generality that

$$\zeta_p \in K.$$

Moreover, it follows from Propositions 1.6, 1.8, together with [22], Theorem 2.8, (ii), that it suffices to prove that G_K^p and $G_{K_{p^\infty}}^p$ are strongly internally indecomposable. Write k for the residue field of K.

Next, we verify the following assertion:

> Claim 2.3.A: Suppose that k is perfect. Then $G_{K_{p^\infty}}^p$ is strongly internally indecomposable.

Indeed, Claim 2.3.A follows immediately from Theorems 2.1, 2.2, together with [22], Lemma 3.1.

Next, we verify the following assertion:

> Claim 2.3.B: $G_{K_{p^\infty}}^p$ is strongly internally indecomposable.

Indeed, let $\{t_i \ (i \in I)\}$ be a p-basis of k; $\tilde{t}_i \in K$ a lifting of t_i. For each $j \in \mathbb{Z}_{\geq 1}$, let $\tilde{t}_{i,j} \in \overline{K}$ be a p^j-th root of $\tilde{t}_i \in K$ such that $\tilde{t}_{i,j}^p = \tilde{t}_{i,j-1}$, where $\tilde{t}_{i,0} \overset{\text{def}}{=} \tilde{t}_i$. Write

$$L \ (\subseteq \overline{K})$$

for the field obtained by adjoining the elements $\{\tilde{t}_{i,j} \ ((i,j) \in I \times \mathbb{Z}_{\geq 1})\}$ to K. Then L is a mixed characteristic Henselian discrete valuation field such that the residue field of L is perfect and of characteristic p. Therefore, it follows from Claim 2.3.A that $G_{L_{p^\infty}}^p$ $(\subseteq G_{K_{p^\infty}}^p)$ is strongly internally indecomposable. On the other hand, we note that

- $G_{K_{p^\infty}}^p$ is slim [cf. [22], Theorem 2.8, (ii)];

- $\mathrm{Gal}(L_{p^\infty}/K_{p^\infty})$ is abelian.

Thus, it follows immediately from Proposition 1.11, (ii), that $G^p_{K_{p^\infty}}$ is strongly internally indecomposable. This completes the proof of Claim 2.3.B.

Finally, we note that $G^p_K/G^p_{K_{p^\infty}}$ is isomorphic to $\mathbb{Z}_p$ [Recall that $\zeta_p \in K$]. Thus, in light of Claim 2.3.B, it follows immediately from Proposition 1.11, (ii), together with [22], Theorem 2.8, (ii), that G^p_K is strongly internally indecomposable. This completes the proof of Theorem 2.3. □

Remark 2.3.1. It is natural to pose the following questions:

> Question 1: Is the absolute Galois group of any discrete valuation field with a positive characteristic residue field strongly internally indecomposable?
>
> Question 2: More generally, is the absolute Galois group of any subfield of a discrete valuation field with a positive characteristic residue field strongly internally indecomposable?
>
> Question 3: In the notation of Theorem 2.3, can the assumption that $\zeta_p \in K$ be dropped?

However, at the time of writing the present paper, the authors do not know whether these questions are affirmative or not.

Next, we review the definition of higher local fields.

Definition 2.4 ([5], Chapter I, §1.1)**.** Let K be a field; $d \in \mathbb{Z}_{\geq 1}$.

(i) A structure of *local field of dimension* d on K is a sequence of complete discrete valuation fields $K^{(d)} \stackrel{\mathrm{def}}{=} K, K^{(d-1)}, \ldots, K^{(0)}$ such that

- $K^{(0)}$ is a perfect field;
- for each integer $0 \leq i \leq d-1$, $K^{(i)}$ is the residue field of the complete discrete valuation field $K^{(i+1)}$.

(ii) We shall say that K is a *higher local field* if K admits a structure of local field of some positive dimension. In the remainder of the present paper, for each higher local field, we fix a structure of local field of some positive dimension.

Definition 2.5. Let K be a field. Then we shall say that K is *stably* μ_{p^∞}*-finite* if, for every finite extension field M of K, the group of p-power roots of unity $\in M$ is finite.

Corollary 2.6. *Let K be a higher local field. Then the following hold:*

(i) Suppose that the residue characteristic of K is p. Then G_K is strongly internally indecomposable. Moreover, if $\zeta_p \in K$ in the case where $\mathrm{char}(K) = 0$*, then any almost pro-p-maximal quotient of G_K is strongly internally indecomposable.*

(ii) Suppose that $\mathrm{char}(K^{(0)}) \neq 0$*, and $K^{(0)}$ is a stably μ_{l^∞}-finite field for any prime number l. Then G_K is strongly indecomposable. In particular, if $K^{(0)}$ is finite, then G_K is strongly indecomposable.*

Proof. Assertion (i) follows immediately from Theorems 2.1, 2.3.

Next, we verify assertion (ii). It follows immediately from assertion (i) that we may assume without loss of generality that the residue characteristic of K is 0. Since every finite extension of K is a higher local field of residue characteristic 0, it suffices to prove that G_K is indecomposable. Suppose that there exist normal closed subgroups $H_1 \subseteq G_K$ and $H_2 \subseteq G_K$ such that

$$G_K = H_1 \times H_2.$$

Write $i \in \mathbb{Z}_{\geq 1}$ for the positive integer such that $\mathrm{char}(K^{(i+1)}) > 0$. Recall from Cohen's structure theorem that

$$K \cong K^{(i)}((t_1)) \cdots ((t_m)).$$

Then we have an exact sequence of profinite groups

$$1 \longrightarrow \widehat{\mathbb{Z}}(1)^{\oplus m} \longrightarrow G_K \longrightarrow G_{K^{(i)}} \longrightarrow 1,$$

where "(1)" denotes the Tate twist. Here, we note that $G_{K^{(i)}}$ is internally indecomposable [cf. (i)]. In particular, it holds that $H_1 \subseteq \widehat{\mathbb{Z}}(1)^{\oplus m}$ or $H_2 \subseteq \widehat{\mathbb{Z}}(1)^{\oplus m}$. We may assume without loss of generality that $H_1 \subseteq \widehat{\mathbb{Z}}(1)^{\oplus m}$. Then since $G_K = H_1 \times H_2$, and H_1 is abelian, it holds that $H_1 \subseteq Z(G_K)$. Thus, since $Z(G_K) = \{1\}$ [cf. [22], Corollary 2.11, (iii)], we conclude that $H_1 = \{1\}$. This completes the proof of assertion (ii), hence of Corollary 2.6. $\square$

Remark 2.6.1. Let K be a field of characteristic 0. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \widehat{\mathbb{Z}}(1) \longrightarrow G_{K((t))} \longrightarrow G_K \longrightarrow 1.$$

Note that $\widehat{\mathbb{Z}}(1) \subseteq G_{K((t))}$ is a normal closed subgroup, and $\widehat{\mathbb{Z}}(1)$ is not center-free. Thus, we conclude that $G_{K((t))}$ is not internally indecomposable.

Theorem 2.7. *Let K be a Hilbertian field. Then G_K is strongly internally indecomposable.*

Proof. Since every finite separable extension of K is Hilbertian [cf. [7], Corollary 12.2.3], it suffices to prove that G_K is internally indecomposable. Let $N \subseteq G_K$ be a nontrivial normal closed subgroup. Write $C \stackrel{\text{def}}{=} Z_G(N)$. Then it follows immediately from the various definitions involved that

$$C \cap N = Z(N) \subseteq G_K$$

is an abelian normal closed subgroup. Thus, by applying [7], Proposition 16.11.6, we conclude that $C \cap N = \{1\}$, hence that $C \cdot N = C \times N \subseteq G_K$.

Next, we recall that G_K is slim [cf. [21], Theorem 2.1]. Since $N \subseteq G_K$ is nontrivial normal closed subgroup, it follows immediately from [22], Lemma 1.3, that N is infinite. Let $N^\dagger \subsetneq N$ be a proper nontrivial normal open subgroup. Then $C \times N^\dagger \subsetneq C \times N = C \cdot N$ is a proper normal open subgroup. Thus, by applying [7], Theorem 13.9.1, (b), we conclude that $C \times N^\dagger$ is isomorphic to the absolute Galois group of a Hilbertian field. In particular, $C \times N^\dagger$ is indecomposable [cf. [7], Corollary 13.8.4; [21], Theorem 2.1]. Since $N^\dagger \neq \{1\}$, this implies that $C = \{1\}$. Thus, we conclude that G_K is internally indecomposable [cf. Proposition 1.2]. This completes the proof of Theorem 2.7. □

Remark 2.7.1. It is well-known that the following hold:

(i) The field of fractions of an arbitrary integral domain that is finitely generated over $\mathbb{Z}$ is Hilbertian [cf. [7], Proposition 13.4.1].

(ii) Finitely generated transcendental extension field of an arbitrary field is Hilbertian [cf. [7], Proposition 13.4.1].

(iii) The field of fractions of an arbitrary Noetherian integral domain of dimension ≥ 2 is Hilbertian [cf. [7], Theorem 15.4.6; [20], p296, Mori-Nagata's integral closure theorem].

In particular, it follows from Theorem 2.7 that the absolute Galois groups of the above fields are strongly internally indecomposable.

Remark 2.7.2. It is natural to pose the following question:

> Question: In the notation of Theorem 2.7, is any almost pro-p-maximal quotient of G_K is strongly internally indecomposable?

However, at the time of writing the present paper, the authors do not know whether this question is affirmative or not.

3 Strong internal indecomposability of the étale fundamental groups of hyperbolic curves

In the present section, we prove the strong internal indecomposability of various profinite groups appearing in anabelian geometry of hyperbolic curves.

Let p be a prime number. First, we begin by recalling basic notions surrounding hyperbolic curves.

Definition 3.1. Let k be a field; $\overline{k}$ an algebraic closure of k; X a smooth curve [i.e., a one-dimensional, smooth, separated, of finite type, and geometrically connected scheme] over k. Write $\overline{X}_{\overline{k}}$ for the smooth compactification of $X_{\overline{k}}$ over $\overline{k}$. Then we shall say that X is a *smooth curve of type* (g, r) over k if the genus of $X_{\overline{k}}$ is g, and the cardinality of the underlying set of $\overline{X}_{\overline{k}} \setminus X_{\overline{k}}$ is r. If X is a smooth curve of type (g, r) over k, and $2g - 2 + r > 0$, then we shall say that X is a *hyperbolic curve* over k.

Definition 3.2 ([29], Definition 1.2)**.** Let $\mathcal{C}$ be a full-formation; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset of prime numbers; Π a profinite group. Then we shall say that Π is a *pro-$\mathcal{C}$ surface group* (respectively, an *almost pro-$\mathcal{C}$ surface group*) if Π is isomorphic to the maximal pro-$\mathcal{C}$ quotient (respectively, to some almost pro-$\mathcal{C}$-maximal quotient) of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic 0. Suppose that Π is a pro-$\mathcal{C}$ surface group (respectively, an almost pro-$\mathcal{C}$ surface group), and $\mathcal{C}$ is the family of all finite groups Γ such that every prime divisor of the order of Γ is an element of Σ. Then we shall say that Π is a *pro-Σ surface group* (respectively, an *almost pro-Σ surface group*). If $\Sigma = \{p\}$, then we shall also say that Π is a *pro-p surface group* (respectively, an *almost pro-p surface group*).

Next, we prove the strong internal indecomposability of almost pro-$\mathcal{C}$ surface groups, which may be regarded as a partial generalization of Proposition 1.5.

Lemma 3.3. *Let Π be a pro-p surface group; $H \subseteq \Pi$ an abelian normal closed subgroup. Then $H = \{1\}$.*

Proof. Note that every open subgroup of Π is not abelian. Then the subgroup $H \subseteq \Pi$ is of infinite index. Thus, by calculating the second cohomology, we conclude that H is a free pro-p group [cf. [11], Lemma 2.1; [13], Proposition 1.4]. Then since H is abelian, it holds that H is topologically finitely generated [of rank ≤ 1]. Thus, since Π is elastic [cf. [29], Theorem 1.5], and $H \subseteq \Pi$ is of infinite index, we conclude that $H = \{1\}$. This completes the proof of Lemma 3.3. □

Proposition 3.4. *Let $\mathcal{C}$ be a full-formation; Π an almost pro-$\mathcal{C}$ surface group. Then Π is strongly internally indecomposable.*

Proof. Observe that, if $\Sigma_{\mathcal{C}} = \emptyset$, then $\Pi = \{1\}$ is strongly internally indecomposable. Thus, we may assume without loss of generality that $\Sigma_{\mathcal{C}} \neq \emptyset$. Let $l \in \Sigma_{\mathcal{C}}$ be an element. Here, we note that any almost pro-l maximal quotient of Π is an almost pro-l surface group. Then it follows immediately from Proposition 1.8 that we may assume without loss of generality that Π is an almost pro-l surface group. Recall that Π is slim [cf. [29], Proposition 1.4]. Thus, we may assume without loss of generality that Π is a pro-l surface group [cf. Proposition 1.6]. Note that every open subgroup of Π is a pro-l surface group. In particular, it suffices to prove that Π is internally indecomposable.

Let $N \subseteq \Pi$ be a nontrivial normal closed subgroup. Write $C \stackrel{\text{def}}{=} Z_{\Pi}(N)$. Then since $C \cap N = Z(N) \subseteq \Pi$ is an abelian normal closed subgroup, it follows from Lemma 3.3 that $C \cap N = \{1\}$, hence that $C \cdot N = C \times N \subseteq \Pi$.

Suppose that the subgroup $C \cdot N \subseteq \Pi$ is of finite index. Then since Π is strongly indecomposable [cf. [21], Theorem A, (i)], and $N \neq \{1\}$, it holds that $C = \{1\}$.

Suppose that the subgroup $C \cdot N \subseteq \Pi$ is of infinite index. Then, by calculating the second cohomology, we conclude that $C \times N$ is a free pro-l group. Thus, since $N \neq \{1\}$, it follows immediately from Proposition 1.5 that $C = \{1\}$, hence that Π is internally indecomposable [cf. Proposition 1.2]. This completes the proof of Proposition 3.4. □

Corollary 3.5. *Let k be an algebraically closed field; Σ a nonempty set of prime numbers such that* $\mathrm{char}(k) \notin \Sigma$*; X a hyperbolic curve over k. Then Π_X^{Σ} is strongly internally indecomposable.*

Proof. Corollary 3.5 follows immediately from Proposition 3.4, together with [9], EXPOSÉ XIII, Corollaire 2.12. □

On the other hand, we also prove the following generalization of [21], Theorem 3.6.

Lemma 3.6. *Let k be an algebraically closed field of characteristic p; X a smooth curve of type (g, r) over k. Suppose that $(g, r) \neq (0, 0), (1, 0)$, and $g \leq 1$. Then there exists a normal open subgroup $N \subseteq \Pi_X$ such that $N \subseteq \Pi_X$ is of index p, and the domain curve of the covering associated to $N \subseteq \Pi_X$ has genus ≥ 2.*

Proof. Lemma 3.6 follows immediately from the proof of [21], Lemma 3.3. □

Theorem 3.7. *Let Σ be a set of prime numbers such that $p \in \Sigma$; k an algebraically closed field of characteristic p; X a smooth curve of type (g, r) over k. Suppose that $(g, r) \neq (0, 0), (1, 0)$. If $r = 0$, i.e., X is proper, then we write $\sigma(X)$ for the p-rank of [the Jacobian variety of] X. Then the following hold:*

(i) Suppose that $\Sigma = \{p\}$*, and* $\sigma(X) \neq 1$ *if* $r = 0$*. Then* Π_X^p *is strongly internally indecomposable.*

(ii) Suppose that $\Sigma \supsetneq \{p\}$*. Then* Π_X^Σ *is strongly internally indecomposable.*

Proof. First, we verify assertion (i). Recall that, if $r \neq 0$ (respectively, $r = 0$), then Π_X^p is a free pro-p group of infinite rank (respectively, of rank $\sigma(X)$) [cf. [35], Theorem 4.9.4]. Thus, it follows immediately from Proposition 1.5 that Π_X^p is strongly internally indecomposable. This completes the proof of assertion (i).

Next, we verify assertion (ii). By applying Proposition 1.6; Lemma 3.6; the proof of [21], Theorem 3.6, we may assume without loss of generality that $g \geq 2$. Let $l \in \Sigma$ be such that $l \neq p$; Q an almost pro-l-maximal quotient of Π_X^Σ. Then it suffices to verify that Q is strongly internally indecomposable [cf. Proposition 1.8]. Observe that there exists a finite Galois covering $Y \to X$ that determines an exact sequence of profinite groups

$$1 \longrightarrow \Pi_Y^l \longrightarrow Q \longrightarrow \mathrm{Gal}(Y/X) \longrightarrow 1.$$

Note that the outer representation $\mathrm{Gal}(Y/X) \to \mathrm{Out}(\Pi_Y^l)$ associated to the above exact sequence is injective [cf. [21], Lemma 3.4]. In particular, since Π_Y^l is slim [cf. [21], Proposition 3.2], it holds that Q is slim. Thus, since Π_Y^l is strongly internally indecomposable [cf. Corollary 3.5], it follows immediately from Proposition 1.6 that Q is strongly internally indecomposable. This completes the proof of assertion (ii), hence of Theorem 3.7. □

Remark 3.7.1. In the notation of Theorem 3.7, suppose that $r = 0$, and $\sigma(X) = 1$. Then since $\Pi_X^p \cong \mathbb{Z}_p$, it follows immediately from the various definitions involved that Π_X^p is not strongly internally indecomposable.

Next, we recall the definition of a *configuration space group* which plays a central role in combinatorial anabelian geometry [cf. [24], [25], [26], [14], [15], [16], [17], [18]].

Definition 3.8 ([29], Definitions 2.1, 2.3)**.**

(i) Let $n \in \mathbb{Z}_{\geq 1}$ be an element; k a field; X a hyperbolic curve over k. Write

$$X_n \stackrel{\mathrm{def}}{=} X^{\times n} \setminus \Big(\bigcup_{1 \leq i < j \leq n} \Delta_{i,j} \Big),$$

where $X^{\times n}$ denotes the fiber product of n copies of X over k; $\Delta_{i,j}$ denotes the diagonal divisor of $X^{\times n}$ associated to the i-th and j-th components. We shall refer to X_n as the *n-th configuration space* associated to X.

(ii) Let $\mathcal{C}$ be a full-formation; Π a profinite group. Then we shall say that Π is a *pro-$\mathcal{C}$ configuration space group* if Π is isomorphic to the maximal pro-$\mathcal{C}$ quotient of the étale fundamental group of a configuration space associated to a hyperbolic curve over an algebraically closed field of characteristic 0.

Then we observe the following generalization of [21], Theorem C, (i).

Theorem 3.9. *Let $\mathcal{C}$ be a full-formation; Π a pro-$\mathcal{C}$ configuration space group. Suppose that*

- *$\mathcal{C}$ is the family of all finite groups, or*
- *$\Sigma_{\mathcal{C}}$ consists of a single element.*

Then Π is strongly internally indecomposable.

Proof. First, we may assume without loss of generality that $\Pi = \Pi_{X_n}^{\Sigma_{\mathcal{C}}}$, where X_n denotes the n-th configuration space associated to a hyperbolic curve X over an algebraically closed field of characteristic 0. We prove Theorem 3.9 by induction on n.

Recall that $\Pi_X^{\Sigma_{\mathcal{C}}}$ is strongly internally indecomposable [cf. Proposition 3.4]. On the other hand, we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi_Y^{\Sigma_{\mathcal{C}}} \longrightarrow \Pi_{X_n}^{\Sigma_{\mathcal{C}}} \longrightarrow \Pi_{X_{n-1}}^{\Sigma_{\mathcal{C}}} \longrightarrow 1,$$

where the arrow $\Pi_{X_n}^{\Sigma_{\mathcal{C}}} \to \Pi_{X_{n-1}}^{\Sigma_{\mathcal{C}}}$ denotes a surjection induced by the projection $X_n \to X_{n-1}$ obtained by forgetting the final factor; Y denotes an open subscheme of X such that the cardinality of the underlying set of $X \setminus Y$ is $n-1$ [cf. [29], Proposition 2.2, (i)]. Then it is well-known that the outer representation $\Pi_{X_{n-1}}^{\Sigma_{\mathcal{C}}} \to \mathrm{Out}(\Pi_Y^{\Sigma_{\mathcal{C}}})$ determined by the above exact sequence is injective [cf. [3], Theorem 1; [3], Remark following the proof of Theorem 1]. Thus, it follows immediately from Proposition 1.11, (i), together with the induction hypothesis, that $\Pi_{X_n}^{\Sigma_{\mathcal{C}}}$ is strongly internally indecomposable. This completes the proof of Theorem 3.9. □

Definition 3.10. Let k be a field; $\overline{k}$ an algebraic closure of k; Z an algebraic variety over k. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{Z_{\overline{k}}} \longrightarrow \Pi_Z \longrightarrow G_k \longrightarrow 1.$$

We shall write $\rho_Z : G_k \to \mathrm{Out}(\Pi_{Z_{\overline{k}}})$ for the outer representation determined by the above exact sequence. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset of prime numbers. Then we shall write

$$\rho_Z^{\Sigma} : G_k \to \mathrm{Out}(\Pi_{Z_{\overline{k}}}^{\Sigma})$$

for the outer representation induced by ρ_Z;

$$\Pi_Z^{[\Sigma]} \stackrel{\mathrm{def}}{=} \Pi_Z / \mathrm{Ker}(\Pi_{Z_{\overline{k}}} \twoheadrightarrow \Pi_{Z_{\overline{k}}}^{\Sigma}).$$

If $\Sigma = \{p\}$ (respectively, $\Sigma = \mathfrak{Primes} \setminus \{p\}$), then we shall also write $\rho_Z^p \stackrel{\mathrm{def}}{=} \rho_Z^{\Sigma}$; $\Pi_Z^{[p]} \stackrel{\mathrm{def}}{=} \Pi_Z^{[\Sigma]}$ (respectively, $\rho_Z^{(p)'} \stackrel{\mathrm{def}}{=} \rho_Z^{\Sigma}$; $\Pi_Z^{[p]'} \stackrel{\mathrm{def}}{=} \Pi_Z^{[\Sigma]}$).

Next, we prove that the étale fundamental groups of configuration spaces associated to hyperbolic curves over "sufficiently arithmetic" fields such as number fields, p-adic local fields, and finite fields, are also strongly internally indecomposable.

Theorem 3.11. *Let $n \in \mathbb{Z}_{\geq 1}$ be an element; k a field; X a smooth curve of type (g, r) over k. Then the following hold:*

(i) Suppose that

- *k is a number field or a p-adic local field;*
- *X is a hyperbolic curve over k.*

Write X_n for the n-th configuration space associated to X. Then Π_{X_n} is strongly internally indecomposable.

(ii) Suppose that

- *k is a finite field of characteristic p;*
- *$(g, r) \neq (0, 0), (1, 0)$ (respectively, $2g - 2 + r > 0$).*

Then Π_X (respectively, $\Pi_X^{[p]'}$) is strongly internally indecomposable.

Proof. First, we verify assertion (i). Recall that ρ_{X_n} is injective [cf. [14], Theorem C, (ii)]. Thus, since $\Pi_{(X_n)_{\overline{k}}}$ and G_k are strongly internally indecomposable [cf. Theorems 2.3, 2.7, 3.9], it follows immediately from Proposition 1.11, (i), that Π_{X_n} is strongly internally indecomposable. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since $\Pi_{X_{\overline{k}}}$ and $\Pi^{(p)'}_{X_{\overline{k}}}$ are strongly internally indecomposable [cf. Corollary 3.5, Theorem 3.7], and G_k is abelian, it follows from Proposition 1.11, (ii), that it suffices to prove that ρ_X and $\rho_X^{(p)'}$ are injective. Note that since G_k is torsion-free, by applying [21], Lemma 1.7, (i); [21], Lemma 3.3, we may assume without loss of generality that $g \geq 2$.

Write $\overline{X}$ for the smooth compactification of X over k. Then since

$$\mathrm{Hom}(H^2(\Pi^{(p)'}_{\overline{X}_{\overline{k}}}, \widehat{\mathbb{Z}}^{(p)'}), \widehat{\mathbb{Z}}^{(p)'}) \xrightarrow{\sim} \widehat{\mathbb{Z}}^{(p)'}(1)$$

as G_k-modules, where "(1)" denotes the Tate twist, we conclude that $\rho^{(p)'}_{\overline{X}}$ is injective, hence that $\rho_X^{(p)'}$ and ρ_X are also injective. This completes the proof of assertion (ii), hence of Theorem 3.11. □

Remark 3.11.1. In the present remark, we shall use the language of combinatorial anabelian geometry [cf. [24], [25], [26], [14], [15], [16], [17], [18]]. Recall that the notion of an outer representation of NN-type plays a central role. Let

$\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset of prime numbers; $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type such that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$. Write $\Pi_{\mathcal{G}}$ for the fundamental group of $\mathcal{G}$. Note that $\Pi_{\mathcal{G}}$ may be identified with a pro-Σ surface group. Let

$$\rho : I \to \mathrm{Out}(\Pi_{\mathcal{G}})$$

be an outer representation of pro-Σ PSC-type. Then ρ determines an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_I \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} I \longrightarrow I \longrightarrow 1.$$

Suppose that ρ is of NN-type. Then it holds that $I \cong \widehat{\mathbb{Z}}^{\Sigma}$, and ρ is injective [cf. our assumption that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$]. Thus, it follows immediately from Proposition 3.4, together with Proposition 1.11, (ii), that Π_I is strongly internally indecomposable.

4 Strong indecomposability of the Grothendieck-Teichmüller group GT

In the present section, we prove that the Grothendieck-Teichmüller group GT is strongly indecomposable. This gives a complete affirmative solution to the problem posed by the first author of the present paper in [21], Introduction.

First, we begin by recalling the definition of GT.

Definition 4.1. Write $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$; X_2 for the second configuration space associated to X; $p_i : \Pi_{X_2} \to \Pi_X$ for the outer surjection induced by the i-th projection $X_2 \to X$, where $i = 1, 2$. Then we shall denote

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{X_2}) \subseteq \mathrm{Out}(\Pi_{X_2})$$

by the subgroup of outer automorphisms $\sigma \in \mathrm{Out}(\Pi_{X_2})$ such that, for $i = 1, 2$,

- $\sigma(\mathrm{Ker}(p_i)) = \mathrm{Ker}(p_i)$;
- σ induces a permutation on the set of the conjugacy classes of cuspidal inertia subgroups of $\mathrm{Ker}(p_i)$, where we note that $\mathrm{Ker}(p_i)$ may be naturally identified with the étale fundamental group of a hyperbolic curve of type $(0, 4)$ over $\overline{\mathbb{Q}}$. [Recall that the cuspidal inertia subgroups of the étale fundamental group of this hyperbolic curve may be defined as the stabilizer subgroups associated to pro-cusps of the pro-universal covering of the hyperbolic curve.]

Recall that $X_2 \stackrel{\sim}{\to} \mathcal{M}_{0,5}$, where $\mathcal{M}_{0,5}$ denotes the moduli stack over $\overline{\mathbb{Q}}$ of hyperbolic curves of type $(0, 5)$. Then we have a natural action of the symmetric group $\mathfrak{S}_5$ on X_2 by permuting ordered marked points. This action determines an inclusion $\mathfrak{S}_5 \subseteq \mathrm{Out}(\Pi_{X_2})$. Then we shall write

$$\mathrm{GT} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_2}) \cap Z_{\mathrm{Out}(\Pi_{X_2})}(\mathfrak{S}_5) \ (\subseteq \mathrm{Out}(\Pi_{X_2})).$$

We shall refer to GT as the *Grothendieck-Teichmüller group*. Since the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{X_2}) \to \mathrm{Out}(\Pi_X)$ induced by p_1 is injective [cf. [14], Theorem B], GT may be regarded as a closed subgroup of $\mathrm{Out}(\Pi_X)$.

Remark 4.1.1. The Grothendieck-Teichmüller group GT was originally introduced by V.G. Drinfeld [cf. [4]]. Let us note that, a priori, the original definition is different from the above definition. However, it follows from a remarkable theorem proved by Harbater-Schneps [cf. [10]] that these two definitions are equivalent. Moreover, it follows from [13], Theorem C, that

$$\mathrm{Out}(\Pi_{X_2}) = \mathrm{GT} \times \mathfrak{S}_5.$$

Remark 4.1.2. Let us observe that there exists a natural homomorphism $G_{\mathbb{Q}} \to \mathrm{GT}$. Note that it follows from Belyi's theorem that this homomorphism determines an injection

$$G_{\mathbb{Q}} \subseteq \mathrm{GT}.$$

With regard to the above inclusion, let us recall the following famous open question [cf. [34], §1.4]:

Question: Is the inclusion $G_{\mathbb{Q}} \subseteq \mathrm{GT}$ bijective?

From the viewpoint of this question, the comparison of group-theoretic properties of $G_{\mathbb{Q}}$ and GT has been considered to be important.

Lemma 4.2. *Let G be a profinite group; $\{G_i\}_{i\in I}$ a directed subset of the set of characteristic open subgroups of G — where $j \geq i \Leftrightarrow G_j \subseteq G_i$ — such that*

$$\bigcap_{i\in I} G_i = \{1\}.$$

Write $\phi_i : \mathrm{Out}(G) \to \mathrm{Out}(G/G_i)$ for the natural homomorphism. Then

$$\bigcap_{i\in I} \mathrm{Ker}(\phi_i) = \{1\}.$$

Proof. Let $\sigma \in \bigcap_{i\in I} \mathrm{Ker}(\phi_i)$ $(\subseteq \mathrm{Out}(G))$ be an element; $\tilde{\sigma} \in \mathrm{Aut}(G)$ a lifting of $\sigma \in \mathrm{Out}(G)$. For each $i \in I$, write $\tilde{\sigma}_i \in \mathrm{Aut}(G/G_i)$ for the automorphism induced by $\tilde{\sigma}$. Then since $\sigma \in \mathrm{Ker}(\phi_i)$, it holds that $\tilde{\sigma}_i$ is an inner automorphism.

Let $\gamma_i \in G/G_i$ be an element which determines the inner automorphism $\tilde{\sigma}_i$. Write

$$C_i \stackrel{\text{def}}{=} \gamma_i \cdot Z(G/G_i) \subseteq G/G_i.$$

Here, we note that, if $i_1 \geq i_2$ ($i_1, i_2 \in I$), then the natural surjection $G/G_{i_1} \twoheadrightarrow G/G_{i_2}$ induces a map $C_{i_1} \to C_{i_2}$. Observe that since C_i ($i \in I$) is a finite nonempty set, the inverse limit $\varprojlim_{i\in I} C_i$ is nonempty. Let

$$\gamma \in \varprojlim_{i\in I} C_i \quad (\subseteq \varprojlim_{i\in I} G/G_i = G)$$

[cf. [32], Corollary 1.1.6] be an element. Then it follows immediately from the various definitions involved that $\tilde{\sigma}$ is an inner automorphism determined by γ. This completes the proof of Lemma 4.2. $\square$

Lemma 4.3. *Let G be a topologically finitely generated profinite group; $S \subseteq \mathfrak{Primes}$ a finite subset. Then the natural homomorphism*

$$\mathrm{Out}(G) \longrightarrow \prod_{p\in\mathfrak{Primes}\setminus S} \mathrm{Out}(G^{(p)'})$$

is injective.

Proof. Since G is topologically finitely generated, there exists a directed subset $\{G_i\}_{i\in I}$ of the set of characteristic open subgroups of G — where $j \geq i \Leftrightarrow G_j \subseteq G_i$ — such that

$$\bigcap_{i\in I} G_i = \{1\}$$

[cf. [32], Proposition 2.5.1, (b)]. Fix such a family. For each $i \in I$, let $p_i \in \mathfrak{Primes} \setminus S$ be such that p_i does not divide the order of the finite group G/G_i. Then the natural surjection $G \twoheadrightarrow G/G_i$ factors through the natural surjection $G \twoheadrightarrow G^{(p_i)'}$. Thus, Lemma 4.3 follows immediately from Lemma 4.2. $\square$

Definition 4.4. Let k be an algebraically closed field; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset of prime numbers such that $\mathrm{char}(k) \notin \Sigma$; Z a hyperbolic curve over k; Q an almost pro-Σ maximal quotient of Π_Z. Then we shall write

$$\mathrm{Out}^{|\mathrm{C}|}(Q) \subseteq \mathrm{Out}(Q)$$

for the subgroup of outer automorphisms of Q that induce the identity automorphisms on the set of the conjugacy classes of cuspidal inertia subgroups of Q, where the cuspidal inertia subgroups of Q may be defined as the images of the cuspidal inertia subgroups of Π_Z via the natural surjection $\Pi_Z \twoheadrightarrow Q$.

Remark 4.4.1. In the notation of Definitions 4.1, 4.4, we note that since the symmetric group $\mathfrak{S}_3$ is center-free, it follows immediately from the various definitions involved that $\mathrm{GT} \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_X)$.

Next, we observe the following corollaries [cf. Lemmas 4.5, 4.6, 4.7] of highly nontrivial Grothendieck Conjecture-type results [cf. [23], Theorem A; [33], Theorem 1]:

Lemma 4.5. *Let l be a prime number; $n \in \mathbb{Z}_{\geq 1}$; $K \subseteq \overline{\mathbb{Q}}$ a number field; $Z \subseteq \mathbb{P}^1_K \backslash \{0, 1, \infty\}$ an open subscheme obtained by forming the complement of a finite subset of K-rational points of $\mathbb{P}^1_K \backslash \{0, 1, \infty\}$. [In particular, Z is a hyperbolic curve of genus 0 over K.] Write $(\mathbb{P}^1_{\overline{\mathbb{Q}}} \supseteq)\ Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}\ (\subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}})$ for the finite étale Galois covering of $Z_{\overline{\mathbb{Q}}}$ of degree n determined by $t \mapsto t^n$;*

$$Q \stackrel{\mathrm{def}}{=} \Pi_{Z_{\overline{\mathbb{Q}}}}/\mathrm{Ker}(\Pi_{Y_{\overline{\mathbb{Q}}}} \twoheadrightarrow \Pi^l_{Y_{\overline{\mathbb{Q}}}}); \quad \rho : G_K \to \mathrm{Out}(Q)$$

for the homomorphism induced by the outer representation $G_K \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}})$ [where we regard G_K as a subgroup of $\mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}})$ via the natural outer action of G_K on $\Pi_{Z_{\overline{\mathbb{Q}}}}$ — cf. [14], Theorem C]. Then

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(Q)}(\mathrm{Im}(\rho)) = \{1\}.$$

Proof. Let $\sigma \in Z_{\mathrm{Out}^{|\mathrm{C}|}(Q)}(\mathrm{Im}(\rho))$ be an element. Recall that

- σ induces the identity automorphism on the set of the conjugacy classes of cuspidal inertia subgroups [which are pro-cyclic subgroups] of Q;
- the normal open subgroup $\Pi_{Y_{\overline{\mathbb{Q}}}} \subseteq \Pi_{Z_{\overline{\mathbb{Q}}}}$ [determined by the finite étale Galois covering $Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}$] may be characterized as the normal open subgroup topologically generated by the cuspidal inertia subgroups of $\Pi_{Z_{\overline{\mathbb{Q}}}}$ that is not associated to the cusps 0, ∞, and the [unique] closed subgroups of the cuspidal inertia subgroups of $\Pi_{Z_{\overline{\mathbb{Q}}}}$ associated to the cusps 0, ∞, of index n.

Thus, any lifting $\in \mathrm{Aut}(Q)$ of σ induces an automorphism of $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$. Let $\tilde{\sigma} \in \mathrm{Aut}(Q)$ be a lifting of σ such that the automorphism $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^l_{Y_{\overline{\mathbb{Q}}}})$ induced by $\tilde{\sigma}$ preserves the $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 1. Here, we note that since $\tilde{\sigma}$ preserves the Q-conjugacy class of cuspidal inertia subgroups of Q associated to the cusp 0 (respectively, ∞), and the finite étale Galois covering $Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}$ is totally ramified over the cusp 0 (respectively, ∞), it holds that $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}}$ preserves the $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 0 (respectively, ∞). Write

$$\sigma_Y : \Pi^l_{Y_{\overline{\mathbb{Q}}}} \xrightarrow{\sim} \Pi^l_{Y_{\overline{\mathbb{Q}}}}$$

for the outer automorphism determined by $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^l_{Y_{\overline{\mathbb{Q}}}})$. Observe that since the outer action of G_K, together with σ_Y, on $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ preserves the $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 1, it follows from our assumption that $\sigma \in Z_{\mathrm{Out}^{|\mathrm{C}|}(Q)}(\mathrm{Im}(\rho))$ that σ_Y commutes with the outer action of G_K on $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$. Then it follows from the Grothendieck Conjecture [cf. [23], Theorem A] that σ_Y arises from a unique isomorphism $f : Y_{\overline{\mathbb{Q}}} \xrightarrow{\sim} Y_{\overline{\mathbb{Q}}}$ of schemes over $\overline{\mathbb{Q}}$. Note that since $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}}$ induces the identity automorphism on the set of the $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$-conjugacy classes of cuspidal inertia subgroups of $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusps 0, 1, ∞, it holds that f induces the identity automorphism on the subset $\{0, 1, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}}$. In particular, we conclude that f is the identity automorphism, hence that σ_Y is the identity outer automorphism. Recall that the automorphism $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^l_{Y_{\overline{\mathbb{Q}}}})$ is the restriction of $\tilde{\sigma} \in \mathrm{Aut}(Q)$. Thus, since Q is slim [cf. [29], Proposition 1.4], it follows from [21], Lemma 1.6, that $\tilde{\sigma}$ is an inner automorphism, hence that σ is the identity outer automorphism. This completes the proof of Lemma 4.5. ☐

Lemma 4.6. *Let p be a prime number; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset of prime numbers such that $p \notin \Sigma$; k a finite field of characteristic p. In the notation of Definition 3.10, suppose that Z is a hyperbolic curve of genus 0 over k such that all cusps of Z are k-rational. Write $\rho \stackrel{\mathrm{def}}{=} \rho^{\Sigma}_{Z}$. Then the following hold:*

(i) Suppose that $\Sigma = \mathfrak{Primes} \setminus \{p\}$. Then the natural homomorphism $\mathrm{Aut}(Z_{\overline{k}}) \to \mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{k}}})$ determines an isomorphism

$$\mathrm{Aut}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k)).$$

(ii) Let l be a prime number $\neq p$. Suppose that $\Sigma = \{l\}$ or $\Sigma = \mathfrak{Primes} \setminus \{p\}$. Then, if we write $\chi_{\Sigma} : \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}}) \to (\widehat{\mathbb{Z}}^{\Sigma})^{\times}$ for the pro-Σ cyclotomic character [which is obtained by considering the actions on the cuspidal inertia subgroups of $\Pi^{\Sigma}_{Z_{\overline{k}}}$], then the natural composite

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}}) \xrightarrow{\chi_{\Sigma}} (\widehat{\mathbb{Z}}^{\Sigma})^{\times}$$

is injective.

Proof. First, we verify assertion (i). Write $\mathrm{Out}_{G_k}(\Pi^{[p]'}_{Z})$ for the group of $\Pi^{(p)'}_{Z_{\overline{k}}}$-outer automorphisms of $\Pi^{[p]'}_{Z}$ that lie over G_k [cf. Definition 3.10]. Then since $\Pi^{(p)'}_{Z_{\overline{k}}}$ is center-free [cf. Corollary 3.5], it is well-known that the natural homomorphism

$$\mathrm{Out}_{G_k}(\Pi^{[p]'}_{Z}) \to Z_{\mathrm{Out}(\Pi^{(p)'}_{Z_{\overline{k}}})}(\rho(G_k))$$

is an isomorphism [cf. [36], Lemma 7.1]. On the other hand, since G_k is abelian, it follows immediately from [33], Theorem 1, together with the definition of $\mathrm{Out}_{G_k}(\Pi_Z^{[p]'})$, that

$$\mathrm{Aut}(Z_{\overline{k}}/Z) \xrightarrow{\sim} \mathrm{Out}_{G_k}(\Pi_Z^{[p]'}),$$

where $\mathrm{Aut}(Z_{\overline{k}}/Z) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ denotes the subgroup consisting of automorphisms of $Z_{\overline{k}}$ that induce automorphisms of Z compatible with the natural morphism $Z_{\overline{k}} \to Z$.

Next, we verify the following assertion:

Claim 4.6.A: The inclusion $\mathrm{Aut}(Z_{\overline{k}}/Z) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ is bijective.

Indeed, let $\alpha \in \mathrm{Aut}(Z_{\overline{k}})$ be an element; $\sigma \in G_k$ ($\hookrightarrow \mathrm{Aut}(Z_{\overline{k}})$). Then since G_k is abelian, it follows that

$$\gamma \stackrel{\mathrm{def}}{=} \sigma \circ \alpha \circ \sigma^{-1} \circ \alpha^{-1} \in \mathrm{Aut}_{\overline{k}}(Z_{\overline{k}}).$$

Next, we note that γ induces the identity automorphism on the set of cusps of $Z_{\overline{k}}$. Thus, we conclude that $\gamma = 1$, hence that α induces a unique automorphism $\in \mathrm{Aut}(Z)$ compatible with the natural morphism $Z_{\overline{k}} \to Z$. This completes the proof of Claim 4.6.A.

Thus, by applying Claim 4.6.A, we obtain a natural isomorphism

$$\phi : \mathrm{Aut}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{Z_{\overline{k}}}^{(p)'})}(\rho(G_k)).$$

This completes the proof of assertion (i).

Next, we verify assertion (ii). If $\Sigma = \{l\}$, then the desired conclusion follows immediately from the latter half of the proof of [30], Proposition 2.2.4. Thus, we may assume without loss of generality that $\Sigma = \mathfrak{Primes} \setminus \{p\}$. Write $\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ for the subgroup of automorphisms of $Z_{\overline{k}}$ that induce the identity automorphisms on the set of cusps of $Z_{\overline{k}}$; $\chi' \stackrel{\mathrm{def}}{=} \chi_{\mathfrak{Primes}\setminus\{p\}}$. Then ϕ induces a composite

$$\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{k}}}^{(p)'})}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{k}}}^{(p)'}) \xrightarrow{\chi'} (\widehat{\mathbb{Z}}^{(p)'})^{\times}.$$

Observe that this composite factors as the composite of the natural injection $\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \hookrightarrow G_{\mathbb{F}_p}$ with the pro-prime-to-p cyclotomic character $G_{\mathbb{F}_p} \hookrightarrow (\widehat{\mathbb{Z}}^{(p)'})^{\times}$. Thus, we conclude that the natural composite

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{k}}}^{(p)'})}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{k}}}^{(p)'}) \xrightarrow{\chi'} (\widehat{\mathbb{Z}}^{(p)'})^{\times}$$

is injective. This completes the proof of assertion (ii), hence of Lemma 4.6. □

Remark 4.6.1. It is natural to pose the following question:

> Question: In the notation of Lemma 4.6, (i), (ii), can the assumptions on the subset of prime numbers $\Sigma \subseteq \mathfrak{Primes}$ be dropped?

However, at the time of writing the present paper, the authors do not know whether the answer is affirmative or not.

Lemma 4.7. *Let l be a prime number; $K \subseteq \overline{\mathbb{Q}}$ a number field. In the notation of Definition 3.10, suppose that $k = K$, and Z is a hyperbolic curve over K. Write $\rho \stackrel{\text{def}}{=} \rho^l_Z$. Then $\mathrm{Im}(\rho)$ is nonabelian.*

Proof. Let us recall that, since K is l-cyclotomically full, it holds that $\mathrm{Im}(\rho)$ is infinite [cf. [21], Definition 4.1; [21], Lemma 4.2, (iv)]. Suppose that $\mathrm{Im}(\rho)$ is abelian. Then since $\mathrm{Im}(\rho) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\mathrm{Im}(\rho))$, the centralizer $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\mathrm{Im}(\rho))$ is infinite. However, since $\mathrm{Aut}_K(Z)$ is finite, this contradicts the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [23], Theorem A]. Thus, we conclude that $\mathrm{Im}(\rho)$ is nonabelian. This completes the proof of Lemma 4.7. □

Lemma 4.8. *Let l be a prime number; $K \subseteq \overline{\mathbb{Q}}$ a number field. In the notation of Definition 3.10, suppose that $k = K$, and Z is a hyperbolic curve of genus 0 over K. Write*

$$\rho_l : \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}}) \to \mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})$$

for the natural homomorphism. Let

$$G \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}}) \ (\subseteq \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}))$$

be a closed subgroup such that

- *G contains an open subgroup of G_K, where we regard G_K as a subgroup of $\mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}})$ via the natural outer action of G_K on $\Pi_{Z_{\overline{\mathbb{Q}}}}$ [cf. [14], Theorem C];*
- *there exist normal closed subgroups $G_1 \subseteq G$ and $G_2 \subseteq G$ such that $G = G_1 \times G_2$.*

Then $\rho_l(G_1) = \{1\}$ or $\rho_l(G_2) = \{1\}$.

Proof. First, by replacing K by a finite extension of K, we may assume without loss of generality that $G_K \subseteq G$. Let $\mathfrak{p}$ be a maximal ideal of the ring of integers of K such that

- the characteristic of the residue field at $\mathfrak{p}$ is not equal to l, and
- Z has good reduction at $\mathfrak{p}$;

$F \in G_K$ $(\subseteq G)$ a lifting of the Frobenius element at $\mathfrak{p}$. We shall write,

- for each $i = 1, 2$, $\mathrm{pr}_i : G \twoheadrightarrow G_i$ for the natural projection;
- $I \subseteq G_K$ for the closed subgroup topologically generated by F, where we note that I is isomorphic to $\widehat{\mathbb{Z}}$;
- $I_1 \stackrel{\mathrm{def}}{=} \mathrm{pr}_1(I) \times \{1\} \subseteq G_1 \times G_2 = G$, $I_2 \stackrel{\mathrm{def}}{=} \{1\} \times \mathrm{pr}_2(I) \subseteq G_1 \times G_2 = G$.

Here, we note that, since I is abelian, it holds that

$$I \subseteq I_1 \times I_2 \subseteq Z_G(I),$$

hence that

$$\rho_l(I) \subseteq \rho_l(I_1) \cdot \rho_l(I_2) \subseteq Z_{\rho_l(G)}(\rho_l(I)).$$

Thus, since Z has good reduction at $\mathfrak{p}$, it follows immediately from Lemma 4.6, (ii), together with the theory of specialization isomorphism, that we have the composite of natural injections

$$\rho_l(I) \subseteq \rho_l(I_1) \cdot \rho_l(I_2) \subseteq Z_{\rho_l(G)}(\rho_l(I)) \subseteq Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\rho_l(I)) \hookrightarrow \mathbb{Z}_l^{\times}.$$

Note that since $\rho_l(I)$ is infinite [cf. [21], Lemma 4.2, (iv)], it holds that $\rho_l(I_1)$ is infinite, or $\rho_l(I_2)$ is infinite. We may assume without loss of generality that

$$\rho_l(I_1) \text{ is infinite.}$$

Observe that every infinite closed subgroup of $\mathbb{Z}_l^{\times}$ is an open subgroup. In particular, $\rho_l(I_1) \cap \rho_l(I) \subseteq \rho_l(I)$ is an open subgroup. Then since $G_2 \subseteq Z_G(I_1)$, there exists an open subgroup ${}^{\dagger}I \subseteq I$ such that

$$\rho_l(G_2) \subseteq Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\rho_l({}^{\dagger}I)) \hookrightarrow \mathbb{Z}_l^{\times}$$

[cf. Lemma 4.6, (ii)].

Suppose that $\rho_l(G_2)$ is infinite. Then since $\rho_l(I) \subseteq Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\rho_l({}^{\dagger}I))$ ($\hookrightarrow \mathbb{Z}_l^{\times}$), it holds that $\rho_l(G_2) \cap \rho_l(I) \subseteq \rho_l(I)$ is an open subgroup. On the other hand, since $G_1 \subseteq Z_G(G_2)$, there exists an open subgroup ${}^{\ddagger}I \subseteq {}^{\dagger}I$ ($\subseteq I$) such that

$$\rho_l(G_1) \subseteq Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\rho_l({}^{\ddagger}I)) \hookrightarrow \mathbb{Z}_l^{\times}$$

[cf. Lemma 4.6, (ii)]. In particular, the closed subgroups

$$\rho_l(G_K) \subseteq \rho_l(G) = \rho_l(G_1) \cdot \rho_l(G_2) \subseteq Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\rho_l({}^{\ddagger}I)) \hookrightarrow \mathbb{Z}_l^{\times}$$

are abelian. This contradicts Lemma 4.7. Thus, we conclude that $\rho_l(G_2)$ is finite. Then there exists a finite extension L ($\subseteq \overline{\mathbb{Q}}$) of K such that $\rho_l(G_2) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\rho_l(G_L))$. Thus, since $\rho_l(G_2)$ induces the identity automorphism on the set of the conjugacy classes of cuspidal inertia subgroups of $\Pi^l_{Z_{\overline{\mathbb{Q}}}}$, it follows immediately from [23], Theorem A, that $\rho_l(G_2) = \{1\}$. This completes the proof of Lemma 4.8. $\square$

Definition 4.9. Let G be a profinite group; Π a topologically finitely generated profinite group; $G \to \mathrm{Out}(\Pi)$ a continuous homomorphism. Then we shall write

$$\Pi \overset{\mathrm{out}}{\rtimes} G$$

for the profinite group obtained by pulling-back the continuous homomorphism $G \to \mathrm{Out}(\Pi)$ via the natural surjection $\mathrm{Aut}(\Pi) \twoheadrightarrow \mathrm{Out}(\Pi)$.

Theorem 4.10. *Let $K \subseteq \overline{\mathbb{Q}}$ be a number field; Z a hyperbolic curve of genus* 0 *over K;*

$$G \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}}) \ (\subseteq \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}))$$

a closed subgroup such that G contains an open subgroup of G_K, where we regard G_K as a subgroup of $\mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}})$ *via the natural outer action of G_K on $\Pi_{Z_{\overline{\mathbb{Q}}}}$ [cf. [14], Theorem C]. Then G is strongly indecomposable. In particular, the Grothendieck-Teichmüller group* GT *is strongly indecomposable [cf. Remark 4.4.1].*

Proof. First, since every open subgroup of G contains an open subgroup of G_K, it suffices to prove that G is indecomposable. Next, by replacing K by a finite extension of K, we may assume without loss of generality that $G_K \subseteq G$, and all cusps of Z are K-rational. Moreover, we may assume without loss of generality that Z is an open subscheme of $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$ obtained by forming the complement of a finite subset of K-rational points of $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$.

Suppose that there exist normal closed subgroups $G_1 \subseteq G$ and $G_2 \subseteq G$ such that

$$G = G_1 \times G_2.$$

We shall write,

- for each $i = 1, 2$, $\mathrm{pr}_i : G \twoheadrightarrow G_i$ for the natural projection;
- for each $n \in \mathbb{Z}_{\geq 1}$, $(\mathbb{P}^1_{\overline{\mathbb{Q}}} \supseteq)\ {}^nY_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}$ $(\subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}})$ for the finite étale Galois covering of $Z_{\overline{\mathbb{Q}}}$ of degree n determined by $t \mapsto t^n$;
- for each $l \in \mathfrak{Primes}$, $Q_{n,l} \overset{\mathrm{def}}{=} \Pi_{Z_{\overline{\mathbb{Q}}}}/\mathrm{Ker}(\Pi_{{}^nY_{\overline{\mathbb{Q}}}} \to \Pi^l_{{}^nY_{\overline{\mathbb{Q}}}})$;
- $\rho_{n,l} : \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}}) \to \mathrm{Out}^{|\mathrm{C}|}(Q_{n,l})$ for the natural homomorphism [cf. the second bullet in the proof of Lemma 4.5]; $\rho_l \overset{\mathrm{def}}{=} \rho_{1,l}$.

Note that ${}^1Y_{\overline{\mathbb{Q}}} = Z_{\overline{\mathbb{Q}}}$, and $Q_{1,l} = \Pi^l_{Z_{\overline{\mathbb{Q}}}}$.

Next, by applying Lemma 4.8, we have the following assertion:

> Claim 4.10.A: Let $l \in \mathfrak{Primes}$ be an element. Then $\rho_l(G_1) = \{1\}$ or $\rho_l(G_2) = \{1\}$.

Next, we verify the following assertion:

Claim 4.10.B: Let $n \in \mathbb{Z}_{\geq 1}$ be an element; $l \in \mathfrak{Primes}$ such that $\rho_l(G_1) = \{1\}$. Then $\rho_{n,l}(G_1) = \{1\}$.

Indeed, let $H \subseteq G$, $H_1 \subseteq G_1$, and $H_2 \subseteq G_2$ be normal open subgroups such that

- $H = H_1 \times H_2$;
- there exists an injection $H \hookrightarrow \mathrm{Out}^{|\mathrm{C}|}(\Pi_{{}^nY_{\overline{\mathbb{Q}}}})$;
- there exists an injection $\Pi_{{}^nY_{\overline{\mathbb{Q}}}} \overset{\mathrm{out}}{\rtimes} H \hookrightarrow \Pi_{Z_{\overline{\mathbb{Q}}}} \overset{\mathrm{out}}{\rtimes} G$ that is compatible with the inclusions between respective subgroups $\Pi_{{}^nY_{\overline{\mathbb{Q}}}} \subseteq \Pi_{Z_{\overline{\mathbb{Q}}}}$ and quotients $H \subseteq G$.

[Note that the existence of such normal open subgroups $H \subseteq G$, $H_1 \subseteq G_1$, and $H_2 \subseteq G_2$ follows from a similar argument to the argument applied in the proof of [37], Lemma 1.2.] Then it follows immediately from Lemma 4.8, together with [29], Proposition 1.4, that $\rho_{n,l}(H_1) = \{1\}$ or $\rho_{n,l}(H_2) = \{1\}$. Suppose that $\rho_{n,l}(H_2) = \{1\}$. Here, we note that since $Q^l_{n,l} \overset{\sim}{\to} \Pi^l_{Z_{\overline{\mathbb{Q}}}}$, it holds that ρ_l factors as the composite of $\rho_{n,l}$ with the natural homomorphism $\mathrm{Out}^{|\mathrm{C}|}(Q_{n,l}) \to \mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})$. In particular, $\rho_l(H_2) = \{1\}$. Then our assumption that $\rho_l(G_1) = \{1\}$ implies that $\rho_l(G_1 \times H_2) = \{1\}$, hence that $\rho_l(G_K) \subseteq \rho_l(G)$ is finite. This is a contradiction [cf. [21], Lemma 4.2, (iv)]. Thus, we conclude that $\rho_{n,l}(H_1) = \{1\}$, hence that $\rho_{n,l}(G_1)$ is finite. In particular, there exists a finite extension L $(\subseteq \overline{\mathbb{Q}})$ of K such that $\rho_{n,l}(G_1) \subseteq Z_{\mathrm{Out}^{|\mathrm{C}|}(Q_{n,l})}(\rho_{n,l}(G_L))$. Finally, it follows immediately from Lemma 4.5 that $\rho_{n,l}(G_1) = \{1\}$. This completes the proof of Claim 4.10.B.

Write $\chi : \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}}) \to \widehat{\mathbb{Z}}^\times$ for the cyclotomic character [which is obtained by considering the actions on the cuspidal inertia subgroups of $\Pi_{Z_{\overline{\mathbb{Q}}}}$]. Then it follows immediately from Claims 4.10.A, 4.10.B, that $\chi(G_1) = \{1\}$ or $\chi(G_2) = \{1\}$. In particular, we may assume without loss of generality that

$$\chi(G_1) = \{1\}.$$

For each $p \in \mathfrak{Primes}$, write

$$\rho^{(p)'} : \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}) \to \mathrm{Out}(\Pi^{(p)'}_{Z_{\overline{\mathbb{Q}}}})$$

for the natural homomorphism.

Next, we verify the following assertion:

Claim 4.10.C: There exists a finite subset $S \subseteq \mathfrak{Primes}$ such that, for each $p \in \mathfrak{Primes} \setminus S$, it holds that $\rho^{(p)'}(G_1) = \{1\}$.

Indeed, let $\mathfrak{p}$ be a maximal ideal of the ring of integers of K such that Z has good reduction at $\mathfrak{p}$; $F \in G_K \subseteq G$ a lifting of the Frobenius element at $\mathfrak{p}$. Write $p \in \mathfrak{Primes}$ for the characteristic of the residue field at $\mathfrak{p}$; $I \subseteq G_K$ for the closed subgroup topologically generated by F; $I_1 \stackrel{\text{def}}{=} \text{pr}_1(I) \times \{1\}$; $I_2 \stackrel{\text{def}}{=} \{1\} \times \text{pr}_2(I)$. Then since I is abelian, it holds that

$$I \subseteq I_1 \times I_2 \subseteq Z_G(I).$$

Then it follows immediately from Lemma 4.6, (ii), together with the theory of specialization isomorphism, that our assumption that $\chi(I_1) \subseteq \chi(G_1) = \{1\}$ implies that $\rho^{(p)'}(I_1) = \{1\}$. In particular, $\rho^{(p)'}(I) \subseteq \rho^{(p)'}(I_2)$. Thus, since $\chi(G_1) = \{1\}$, and $G_1 \subseteq Z_G(I_2)$, we conclude from Lemma 4.6, (ii), that $\rho^{(p)'}(G_1) = \{1\}$. Observe that there exists a finite subset $S \subseteq \mathfrak{Primes}$ such that Z has good reduction at any maximal ideal of the ring of integers of K that lies over a prime number $\in \mathfrak{Primes} \setminus S$. Thus, we obtain the desired conclusion. This completes the proof of Claim 4.10.C.

Finally, by applying Claim 4.10.C and Lemma 4.3, we conclude that $G_1 = \{1\}$, hence that G is indecomposable. This completes the proof of Theorem 4.10. □

Remark 4.10.1. Let $\iota \in G_{\mathbb{Q}} \subseteq \text{GT}$ be a complex conjugation; $H \subseteq \text{GT}$ a closed subgroup such that H contains a GT-conjugate of ι. Then

H is *indecomposable.*

Indeed, suppose that there exist normal closed subgroups $H_1 \subseteq H$ and $H_2 \subseteq H$ such that

$$H = H_1 \times H_2.$$

By replacing H by a suitable GT-conjugate of H, we may assume without loss of generality that $\iota \in H$. Then there exist 2-torsion elements $\iota_1 \in H_1 \subseteq H$ and $\iota_2 \in H_2 \subseteq H$ such that $\iota = \iota_1 \cdot \iota_2$. Note that ι_1 and ι_2 commute with ι. Recall that

$$\langle \iota \rangle = N_{\text{GT}}(\langle \iota \rangle),$$

where $\langle \iota \rangle$ denotes the closed subgroup generated by ι [cf. [19], Proposition 4, (ii)]. Thus, since $\iota \neq 1$, we conclude that $\iota_1 = \iota$ or $\iota_2 = \iota$. In the case where $\iota_1 = \iota$ (respectively, $\iota_2 = \iota$), since ι_1 (respectively, ι_2) commutes with H_2 (respectively, H_1), and $\langle \iota \rangle = N_{\text{GT}}(\langle \iota \rangle)$, it holds that $H_2 = \{1\}$ (respectively, $H_1 = \{1\}$).

Remark 4.10.2. Let l be a prime number. In light of Lemma 4.6, (ii), it follows from a similar argument to the argument applied in the proof of [21], Theorem 6.1, that the pro-l analogue of Theorem 4.10 also holds. Thus, it is natural to pose the following question:

> Question: More generally, for each nonempty subset of prime numbers $\Sigma \subseteq \mathfrak{Primes}$, does the pro-$\Sigma$ analogue of Theorem 4.10 hold?

However, at the time of writing the present paper, the authors do not know whether the answer is affirmative or not.

Remark 4.10.3. Let us recall that strongly internally indecomposable profinite groups are strongly indecomposable [cf. Remark 1.1.2, (ii)]. On the other hand, since $\mathbb{Q}$ is Hilbertian [cf. Remark 2.7.1, (i)], $G_{\mathbb{Q}}$ is strongly internally indecomposable [cf. Theorem 2.7]. Thus, it is natural to pose the following question:

> Question: Is the Grothendieck-Teichmüller group GT strongly internally indecomposable?

However, at the time of writing the present paper, the authors do not know whether the answer is affirmative or not.

Corollary 4.11. *In the notation of Theorem 4.10,* $\Pi_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G$ *is strongly indecomposable.*

Proof. First, since $\Pi_{Z_{\overline{\mathbb{Q}}}}$ is center-free [cf. [29], Proposition 1.4], we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{Z_{\overline{\mathbb{Q}}}} \longrightarrow \Pi_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G \longrightarrow G \longrightarrow 1.$$

Next, since G contains an open subgroup of G_K, it follows immediately from the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [23], Theorem A; [36], Theorem 0.4] that G ($\subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}})$) is slim. Thus, since G is infinite, we conclude from Theorem 4.10, together with [21], Proposition 1.8, (i); [29], Proposition 1.4; [29], Proposition 3.2, that $\Pi_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G$ is strongly indecomposable. This completes the proof of Corollary 4.11. ☐

Acknowledgements

The authors would like to express deep gratitude to Professor Ivan Fesenko for stimulating discussions on this topic. Part of this work was done during their stay in University of Nottingham. The authors would like to thank their supports and hospitalities. Moreover, the authors would like to thank Professor Shinichi Mochizuki for his constructive comments on the contents of the present paper. On the other hand, the first author would like to thank Doctor Wojciech Porowski for stimulating discussions on various topics surrounding anabelian geometry. Finally, the first author would like to convey his sincere appreciation to Doctors Weronika Czerniawska and Paolo Dolce for their kindness, help, and warm encouragement. The first author was supported by JSPS KAKENHI Grant Number 20K14285, and the second author was supported by JSPS KAKENHI Grant Number 18J10260. This research was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. This research was partially supported by EPSRC programme grant "Symmetries and Correspondences" EP/M024830.

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`http://www.kurims.kyoto-u.ac.jp/~motizuki/`

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