RIMS-1934

# On the structure of normal projective surfaces admitting non-isomorphic surjective endomorphisms

By

Noboru NAKAYAMA

December 2020

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

# ON THE STRUCTURE OF NORMAL PROJECTIVE SURFACES ADMITTING NON-ISOMORPHIC SURJECTIVE ENDOMORPHISMS

NOBORU NAKAYAMA

Abstract. Normal projective surfaces admitting non-isomorphic surjective endomorphisms are classified up to isomorphism except singular rational surfaces with only quotient singularities and with big anti-canonical divisor. A surface in our list admits a finite Galois cover étale in codimension 1 from one of the following surfaces: a toric surface with Galois invariant open torus, an abelian surface, a $\mathbb{P}^1$-bundle over an elliptic curve, a projective cone over an elliptic curve, and the direct product of a non-singular projective curve of genus at least 2 and of a rational or elliptic curve.

## 1. Introduction

As a continuation of [24], we study normal Moishezon surfaces $X$ admitting non-isomorphic surjective endomorphisms $f\colon X \to X$. We know the following by [24, Cors. B and C, and Thm. E]:

- $X$ is projective;
- the Weil–Picard number $\hat{\boldsymbol{\rho}}(X)$ equals the Picard number $\boldsymbol{\rho}(X)$;
- $(X, S)$ is log-canonical for any $f$-completely invariant divisor $S$.

Here, a reduced divisor $S$ is said to be *$f$-completely invariant* if $f^{-1}S = S$ (cf. [24, Def. 2.12]). Moreover, we have a structure theorem [24, Thm. A] on $(X, S, f)$ for an $f$-completely invariant divisor $S$ such that $K_X + S$ is pseudo-effective, but it gives only necessary conditions for the existence of non-isomorphic surjective endomorphisms. We can prove that these conditions are also sufficient for the existence by constructing several examples, and we have a complete version of [24, Thm. A] as Theorem 3.1 below. By studying further cases in which $K_X + S$ is not pseudo-effective, we have the following theorem as the main result in this article:

**Theorem 1.1.** *Let $X$ be a normal projective surface. If $X$ admits a non-isomorphic surjective endomorphism, then either* (I) *or* (II) *below holds. Conversely, if $X$ satisfies* (I)*, then $X$ admits a non-isomorphic surjective endomorphism*:

(I) *There is a finite Galois cover $V \to X$ étale in codimension* 1 *satisfying one of the six conditions below*:

(I-1) $V \simeq \mathbb{P}^1 \times T$ *for a non-singular projective curve $T$ of genus at least* 2;

2020 *Mathematics Subject Classification.* Primary 14J26, 14J27; Secondary 32H50.
*Key words and phrases.* endomorphism, normal surface, ruled surface, elliptic surface.

(I-2) *$V \simeq C \times T$ of an elliptic curve $C$ and a non-singular projective curve $T$ of genus at least* 2;

(I-3) *$V$ is an abelian surface*;

(I-4) *$V$ is a $\mathbb{P}^1$-bundle over an elliptic curve*;

(I-5) *$V$ is a* projective cone *over an elliptic curve* (*cf.* [24, Def. 1.16]);

(I-6) *$V$ is a toric surface and the action of the Galois group* $\mathrm{Gal}(V/X)$ *preserves the open torus.*

(II) *The surface $X$ is rational and singular with only quotient singularities, and the anti-canonical divisor $-K_X$ is big.*

The proof of Theorem 1.1 is given in Section 5.3 below. The first assertion of Theorem 1.1 has been proved in [24] for some special cases. In deed, $X$ satisfies (I) either if $K_X + S$ is pseudo-effective for an $f$-completely invariant divisor $S$ or if $X$ is irrational (cf. [24, Thms. A and 4.16]). For the proof of Theorem 1.1, we need some examples constructed in Section 2, the study of some special cases in which $K_X + S$ is not pseudo-effective in Section 4, and the following theorem on non-quotient singularity:

**Theorem 1.2.** *Let $X$ be a normal projective surface admitting a non-isomorphic surjective endomorphism $f$. If $X$ has a non-quotient singular point, then there exists a finite cyclic cover $V \to X$ étale in codimension 1 from a projective cone $V$ over an elliptic curve such that $f$ lifts to an endomorphism of $V$.*

Theorem 1.2 is proved in Section 5.2 below by applying a result of Favre [6, Prop. 2.1] (cf. [23, Thm. 5.3]). We have also the following result on the number $\boldsymbol{n}(S_f)$ of prime components of the *characteristic completely invariant divisor* $S_f$ (cf. [24, Def. 2.16]):

**Theorem 1.3.** *Let $f$ be a non-isomorphic surjective endomorphism of a normal projective surface $X$. Then $\boldsymbol{n}(S_f) \leq \boldsymbol{\rho}(X) + 2$. If $\boldsymbol{n}(S_f) = \boldsymbol{\rho}(X) + 2$, then $(X, S_f)$ is a toric surface, i.e., $X$ is a toric surface with $S_f$ as the boundary divisor. If $\boldsymbol{n}(S_f) = \boldsymbol{\rho}(X) + 1$, then one of the following holds*:

(1) *$(X, B + S_f)$ is a toric surface for a prime divisor $B \not\subset S_f$*;

(2) *$(X, S_f)$ is a* half-toric surface *in the sense of* [22].

Theorem 1.3 is proved in Section 5.1 below by applying [22, Thm. 1.3] on a variant of Shokurov's criterion for toric surfaces (cf. [29, Thm. 6.4]). This is a generalization of Theorem 4.5(5) below on $\mathcal{L}$-surfaces.

Section 5 is devoted to proving these three theorems. We shall explain results obtained in the other sections along the following topics:

(A) Examples of endomorphisms.

(B) Classification of completely invariant curves with positive arithmetic genus.

(C) Classification of $(X, f)$ in the case where the *refined ramification divisor* $\Delta_f$ (cf. [24, Def. 2.16]) is zero.

(D) Study of $(X, f)$ in the case where $\boldsymbol{\rho}(X) > 2$ and $K_X + S_f$ is not pseudo-effective.

(E) Study of $(X, f)$ in the case where $\boldsymbol{\rho}(X) = 2$, $K_X$ is not pseudo-effective, and $-K_X$ is not big.

For (A), in Section 2, we shall construct several endomorphisms of varieties which are equivariant under actions of finite groups, and as a result, we have endomorphisms of the quotient varieties: Lemmas 2.2, 2.5, and 2.6 below treat the projective space $\mathbb{P}^n$, an abelian variety, and a toric varieties, respectively, as a variety with actions of a finite group. Equivariant endomorphisms of $\mathbb{P}^1$-bundles are studied by the notion of *G-linearizations*. Especially, Propositions 2.15 and 2.16 below give sufficient conditions in terms of $G$-linearization for the existence of equivariant non-isomorphic surjective endomorphisms of a $\mathbb{P}^1$-bundle associated with the direct sum of two invertible sheaves. By these results, we can prove the existence of equivariant non-isomorphic surjective endomorphisms for any $\mathbb{P}^1$-bundle over an elliptic curve and any projective cone over an elliptic curve in Theorems 2.20 and 2.21. Some other examples of endomorphisms are obtained in Examples 2.30, 2.31, 2.32, and 2.33 below which are related to results in [24].

Theorem 3.1, a complete version of [24, Thm. A], is proved in Section 3 by using examples obtained in Section 2. As an application of Theorem 3.1, we have Theorem 3.4 concerning (B), where the structure of a normal projective surface $X$ is determined when it has a non-isomorphic surjective endomorphism $f$ and an $f$-completely invariant curve $C$ of positive arithmetic genus. Moreover, Theorem 3.5 gives a finer description in the case where $\emptyset \neq \operatorname{Sing} C \subset X_{\mathrm{reg}}$. As another application of Theorem 3.1, we have Theorem 3.10 as a classification theorem for (C). Note that $\Delta_f = 0$ if and only if the induced endomorphism of $X \setminus S_f$ is étale in codimension 1. Theorem 3.11 below is a classification theorem on endomorphisms étale in codimension 1, i.e., the case where the ramification divisor $R_f$ is zero. Moreover, Theorem 3.12 below treats the subcase of Theorem 3.10 not considered in [24, Thm. A], i.e., the case where $\Delta_f = 0$ and $K_X + S$ is not pseudo-effective for any $f$-completely invariant divisor.

Topics (D) and (E) are treated in Section 4. For $(X, f)$ in (D), in Proposition 4.3 below, we shall show that $(X, S_f)$ is an *$\mathcal{L}$-surface* in the sense of Definition 4.2. Theorem 4.5 collects basic properties of $\mathcal{L}$-surfaces. Corollary 4.6 proves that $(X, B + S_f)$ is a toric surface or a half-toric surface for a prime divisor $B \not\subset S_f$ if $-(K_X + S_f)$ is not big in addition. Theorem 4.7 below is a structure theorem for $(X, f)$ in (E). We have an additional result as Proposition 4.8 giving a finer description in case $\lambda_f \neq \deg f$.

As a consequence of Theorems 1.1, 4.5, and 4.7, we can determine the structure of a normal projective surface $X$ admitting a non-isomorphic surjective endomorphism except the following cases:

(i) $\boldsymbol{\rho}(X) = 1$, and $X$ is a *log del Pezzo surface* (cf. [1, Def. 1.1]), i.e., $X$ is a rational surface with only quotient singularities and $-K_X$ is ample.

(ii) $X$ is a rational surface with $\boldsymbol{\rho}(X) = 2$, $-K_X$ is big, and $X$ has only quotient singularities.

(iii) $(X, S_f)$ is an $\mathcal{L}$-surface and $\boldsymbol{n}(S_f) = \boldsymbol{\rho}(X) \geq 3$.

Some examples belonging to (ii) are in Examples 2.19 and 2.28 below.

**Background.** As is mentioned in [24, Background], this article contains the remaining part of the non-public preprint [21]: the other parts of [21] have already been included in [22], [23], and [24]. Some results in further modified versions of [21] are also contained in this article. On the other hand, results concerning half-toric surfaces and pseudo-toric surfaces were obtained after [22] was written. Ideas, results, and techniques in [22], [23], and [24] are used frequently in this article.

**Acknowledgement.** The author is grateful to Professors Yoshio Fujimoto and De-Qi Zhang for discussions in seminars at Research Institute for Mathematical Sciences, Kyoto University. He expresses his gratitude to Professor Charles Favre for sending a preprint version of [6] with communication by email. He thanks Professor Shigeru Mukai for informing of a remark on the symmetric product of an elliptic curve (cf. Fact 2.23(C)). The author is partially supported by Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science.

**Notation and conventions.** We use the same notation and conventions as in [24]. In particular, we treat complex analytic spaces rather than schemes over $\mathbb{C}$, and a complex analytic variety is called a variety, for short. Our specific notations are listed in Table 1.

TABLE 1. List of notations

| | |
|---|---|
| $\mathbb{C}^\star$ | 1-dimensional algebraic torus ($= \mathbb{C} \setminus \{0\}$) |
| $\boldsymbol{q}(X)$ | irregularity of a normal projective variety $X$ ($= \dim H^1(X, \mathcal{O}_X)$) |
| $\boldsymbol{\rho}(X)$ | Picard number of a normal projective variety $X$ |
| $\mathsf{N}(X)$ | vector space of numerical classes of $\mathbb{R}$-divisors on a normal projective surface $X$ (cf. [24, §1.1]) |
| $\overline{\mathrm{NE}}(X)$ | pseudo-effective cone in $\mathsf{N}(X)$ (cf. [24, §1.1]) |
| $\mathrm{Nef}(X)$ | nef cone in $\mathsf{N}(X)$ (cf. [24, §1.1]) |
| $\mathrm{cl}(D)$ | numerical class of an $\mathbb{R}$-divisor $D$ |
| $\kappa(D, X)$ | Iitaka's $D$-dimension for a divisor $D$ on $X$ (cf. [13]) |
| $\boldsymbol{\pi}_1(U)$ | fundamental group of a topological space $U$ |
| $\boldsymbol{e}(U)$ | Euler number of a topological space $U$ |
| $\boldsymbol{p_a}(C)$ | arithmetic genus of a projective curve $C$ ($= \dim H^1(C, \mathcal{O}_C)$) |
| $\boldsymbol{g}(C)$ | (geometric) genus of a projective curve $C$ |
| $\boldsymbol{n}(S)$ | number of prime components of a reduced divisor $S$ (cf. Definition 4.1 below) |
| **For an endomorphism $f$:** | |
| $R_f$ | ramification divisor (cf. [23, §1.5]) |
| $S_f$ | characteristic completely invariant divisor (cf. [24, Def. 2.16]) |
| $\Delta_f$ | refined ramification divisor (cf. [24, Def. 2.16]) |
| $\lambda_f$ | the first dynamical degree (cf. [24, Def. 3.1]) |
| $\deg f$ | (mapping) degree |
| $\delta_f$ | $:= (\deg f)^{1/2} > 0$ (cf. [24, Def. 3.2]) |

## 2. Examples of endomorphisms

We shall construct several examples of normal projective surfaces (and varieties of higher dimension) admitting non-isomorphic surjective endomorphisms. In Sections 2.1, 2.2, and 2.3, we shall construct endomorphisms of a variety $X$ equivariant under an action of a finite group $G$ on $X$. The equivariant endomorphisms induce endomorphisms of the quotient variety $G\backslash X$. Section 2.1 treats the case where $X$ is a projective space, an abelian variety, or a toric variety. In Section 2.2, we consider some $\mathbb{P}^1$-bundles over a variety, and in Section 2.3, we shall show that $\mathbb{P}^1$-bundles and projective cones over an elliptic curve have equivariant non-isomorphic surjective endomorphisms. In Section 2.4, we present some examples of endomorphisms related to discussions in [24].

**Convention.** Let $X$ be a complex analytic space and let $G$ be a finite group acting on $X$ from the left.

- The left action of $\sigma \in G$ is denoted by $\sigma_X : X \to X$. Here, $(\sigma\tau)_X = \sigma_X \circ \tau_X$ for any $\sigma$, $\tau \in G$.
- Let $f: X \to Y$ be a morphism to another complex analytic space $Y$ with a left action of $G$. We say that $f$ is *$G$-equivariant*, or equivariant under the action of $G$, if $f \circ \sigma_X = \sigma_Y \circ f$ for any $\sigma \in G$.
- A subset $\mathcal{S}$ of $X$ is said to be *$G$-invariant*, or preserved by the action of $G$, if $\sigma_X \mathcal{S} \subset \mathcal{S}$ for any $\sigma \in G$.
- A closed analytic subspace $Z$ of $X$ is said to be *$G$-invariant* if $\sigma_X : X \to X$ induces an isomorphism $\sigma_X|_Z : Z \to Z$ for any $\sigma \in G$. In this case, $G$ acts on $Z$ and the closed immersion $Z \hookrightarrow X$ is $G$-equivariant.
- When $X$ is normal, a divisor $D$ on $X$ is said to be *$G$-invariant* if $\sigma_X^* D = D$.

*Remark.* The quotient space $G\backslash X$ exists as a complex analytic space, and the quotient map $X \to G\backslash X$ is a morphism of complex analytic spaces (cf. [5, Thm. 1]). If $X$ is normal, then so is $G\backslash X$.

*Remark* 2.1. A $G$-equivariant endomorphism $f: X \to X$ induces an endomorphism $\bar{f}$ of the quotient space $G\backslash X$ such that $\pi \circ f = \bar{f} \circ \pi$ for the quotient morphism $\pi: X \to G\backslash X$. Here, $\deg \bar{f} = \deg f$. In particular, if $f$ is non-isomorphic surjective, then so is $\bar{f}$. Furthermore, when $X$ is a normal variety and $\pi$ is étale in codimension 1, the endomorphism $f$ is étale in codimension 1 if and only if $\bar{f}$ is so. In fact, by $K_X = \pi^* K_{G\backslash X}$, we have $R_f = \pi^* R_{\bar{f}}$ for the ramification divisors $R_f$ and $R_{\bar{f}}$.

### 2.1. Equivariant endomorphisms of $\mathbb{P}^n$, abelian varieties, and toric varieties.

We shall construct $G$-equivariant non-isomorphic surjective endomorphisms of a projective variety $X$, in the case where $X$ is the projective space $\mathbb{P}^n$, an abelian variety, or a toric variety whose open torus is $G$-invariant. The following result on $\mathbb{P}^n$ is shown by Amerik in the first part of [2, §1]. We write the proof for readers' convenience.

**Lemma 2.2.** *Let $G$ be a finite group acting on $\mathbb{P}^n$. Then there exists a $G$-equivariant non-isomorphic surjective endomorphism of $\mathbb{P}^n$ whose degree is coprime to the order of $G$.*

*Proof.* We may assume that $G$ is a subgroup of $\mathrm{GL}(n+1,\mathbb{C})$. The action of $\sigma \in \mathrm{GL}(n+1,\mathbb{C})$ on $\mathbb{P}^n$ is defined by

$$\sigma^*(\mathtt{x}_i) = \sum\nolimits_{j=0}^{n} a_{i,j}(\sigma)\mathtt{x}_j, \tag{II-1}$$

where $\sigma = (a_{i,j}(\sigma))_{0\le i,j\le n}$ as a matrix and where $\sigma^*$ is regarded as an automorphism of the polynomial ring $\mathbb{C}[\mathtt{x}_0,\dots,\mathtt{x}_n]$. Let $S \subset \mathbb{P}^n$ be the pullback of a general ample divisor on the quotient variety $G\backslash\mathbb{P}^n$ by the quotient morphism $\mathbb{P}^n \to G\backslash\mathbb{P}^n$. Then $S$ is non-singular by Bertini's theorem. We may assume that $\deg S$ is divisible by the order of $G$. Let $F \in \mathbb{C}[\mathtt{x}_0,\dots,\mathtt{x}_n]$ be a defining equation of $S$, i.e., a homogeneous polynomial defining $S$ as the zero locus $\{F = 0\}$ in $\mathbb{P}^n$. Then we can define a group homomorphism $\chi\colon G \to \mathbb{C}^\star$ by

$$\sigma^*(F) = F(\sigma^*(\mathtt{x}_0),\dots,\sigma^*(\mathtt{x}_n)) = \chi(\sigma)F \tag{II-2}$$

for $\sigma \in G$. Let $\phi\colon \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism corresponding to an endomorphism $\Phi\colon \mathbb{C}[\mathtt{x}_0,\dots,\mathtt{x}_n] \to \mathbb{C}[\mathtt{x}_0,\dots,\mathtt{x}_n]$ defined by $\Phi(\mathtt{x}_i) = \partial F/\partial\mathtt{x}_i$ for $0 \le i \le n$. Note that $\phi$ is holomorphic by the Jacobian criterion, since $S$ is non-singular, and that $\phi$ is surjective with $\deg\phi = (\deg S - 1)^n > 1$. Taking partial differentials to (II-2), we have

$$\chi(\sigma)\partial F/\partial\mathtt{x}_j = \sum\nolimits_{i=0}^{n} a_{i,j}(\sigma)\sigma^*(\partial F/\partial\mathtt{x}_i) \tag{II-3}$$

for any $0 \le j \le n$. Thus,

$$\sigma^*(\partial F/\partial\mathtt{x}_i) = \chi(\sigma)\sum\nolimits_{j=0}^{n} a_{i,j}(\sigma')\partial F/\partial\mathtt{x}_j \tag{II-4}$$

for the matrix $\sigma' = (a_{i,j}(\sigma')) := {}^{\mathrm{t}}\sigma^{-1} = {}^{\mathrm{t}}(a_{i,j}(\sigma))^{-1}$. Hence, $\phi\circ\sigma_{\mathbb{P}^n} = \sigma'_{\mathbb{P}^n}\circ\phi$ for the automorphism $\sigma'_{\mathbb{P}^n}\colon \mathbb{P}^n \to \mathbb{P}^n$, since we have

$$\begin{aligned}\sigma^*(\Phi(\mathtt{x}_i)) = \sigma^*(\partial F/\partial\mathtt{x}_i) &= \chi(\sigma)\sum\nolimits_{j=0}^{n} a_{i,j}(\sigma')\partial F/\partial\mathtt{x}_j\\ &= \chi(\sigma)\Phi\left(\sum\nolimits_{j=0}^{n} a_{i,j}(\sigma')\mathtt{x}_j\right) = \chi(\sigma)\Phi(\sigma'^*(\mathtt{x}_i))\end{aligned}$$

by (II-1) and (II-4). Let $G'$ be the finite subgroup $\{\sigma' \mid \sigma \in G\} \subset \mathrm{GL}(n+1,\mathbb{C})$ and let $S' \subset \mathbb{P}^n$ be the pullback of a general ample divisor on $G'\backslash\mathbb{P}^n$, where we assume that $\deg S'$ is a multiple of the order of $G$. For a defining equation $F' \in \mathbb{C}[\mathtt{x}_0,\dots,\mathtt{x}_n]$ of $S'$, let $\phi'\colon \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism corresponding to an endomorphism $\Phi'$ of $\mathbb{C}[\mathtt{x}_0,\dots,\mathtt{x}_n]$ defined by $\Phi'(\mathtt{x}_i) = \partial F'/\partial\mathtt{x}_i$ for $0 \le i \le n$. Then $\phi'$ is surjective, $\deg\phi' = (\deg S' - 1)^n > 1$, and $\phi'\circ\sigma'_{\mathbb{P}^n} = \sigma_{\mathbb{P}^n}\circ\phi'$ for any $\sigma \in G$, by the same argument as above. Thus, $\phi'\circ\phi$ is a $G$-equivariant non-isomorphic surjective endomorphism of $\mathbb{P}^n$, and $\deg\phi'\circ\phi = (\deg S-1)^n(\deg S'-1)^n$ is coprime to the order of $G$. □

**Lemma 2.3.** *Let $B$ be a compact normal variety and let $G$ be a finite group acting on both $B$ and $\mathbb{P}^n\times B$ so that the second projection $p_2\colon \mathbb{P}^n\times B \to B$ is $G$-equivariant. Then the action of $G$ on $\mathbb{P}^n\times B$ is* diagonal, *i.e., $G$ acts on $\mathbb{P}^n$ so that $\sigma_{\mathbb{P}^n\times B} = \sigma_{\mathbb{P}^n}\times\sigma_B$ for any $\sigma \in G$, and there exists a $G$-equivariant non-isomorphic finite surjective endomorphism $f$ of $\mathbb{P}^n\times B$ such that $p_2\circ f = p_2$ and that $\deg f$ is coprime to the order of $G$.*

*Proof.* The action of an element $\sigma \in G$ on $\mathbb{P}^n \times B$ is expressed as

$$\mathbb{P}^n \times B \ni (x, b) \mapsto \sigma_{\mathbb{P}^n \times B}(x, b) = (\psi_\sigma(x, b), \sigma_B(b)),$$

where $\psi_\sigma \colon \mathbb{P}^n \times B \to \mathbb{P}^n$ is a holomorphic map inducing an automorphism $x \mapsto \psi_\sigma(x, b)$ of $\mathbb{P}^n$ for any $b \in B$. Here, $\psi_\sigma(x, b)$ does not depend on $b \in B$, since any holomorphic map $B \to \mathrm{Aut}(\mathbb{P}^n) \simeq \mathrm{PGL}(n+1, \mathbb{C})$ is constant. Therefore, for a fixed point $b \in B$, we have an action of $\sigma \in G$ on $\mathbb{P}^n$ by $\sigma_{\mathbb{P}^n}(x) := \psi_\sigma(x, b)$, and the action of $G$ on $\mathbb{P}^n \times B$ is diagonal. There is a $G$-equivariant non-isomorphic finite surjective endomorphism $\phi \colon \mathbb{P}^n \to \mathbb{P}^n$ by Lemma 2.2, in which $\deg \phi$ is coprime to the order of $G$. Then $\phi \times \mathrm{id}_B \colon \mathbb{P}^n \times B \to \mathbb{P}^n \times B$ satisfies required conditions for $f$. □

**Corollary 2.4.** *Let $X$ be a normal projective surface admitting a finite surjective morphism $\nu \colon \mathbb{P}^1 \times B \to X$ étale in codimension 1 for an irrational non-singular projective curve $B$. Then $X$ admits a non-isomorphic surjective endomorphism.*

*Proof.* Let $V \to X$ be the Galois closure of $\nu$. Then the induced Galois cover $V \to \mathbb{P}^1 \times B$ is étale. Let $V \to B' \to B$ be the Stein factorization of the smooth morphism $V \to \mathbb{P}^1 \times B \to B$. Then the induced étale morphism $V \to \mathbb{P}^1 \times B'$ over $B'$ is an isomorphism, since $\mathbb{P}^1$ is simply connected. Therefore, we may assume that $\nu$ is Galois. Since $B$ is irrational, the second projection $p_2 \colon \mathbb{P}^1 \times B \to B$ is the Albanese morphism. Thus, the Galois group $G$ of $\nu$ acts on $B$ so that $p_2$ is $G$-equivariant. Hence, a non-isomorphic surjective endomorphism of $X$ exists by Lemma 2.3 and Remark 2.1. □

The following result on abelian varieties is proved by essentially the same arguments as in the proof of [8, Thm. 2.26] and in [7, App. to §4].

**Lemma 2.5.** *Let $A$ be an abelian variety and $B$ a variety, and let $G$ be a finite group acting on $A \times B$ and $B$ so that the second projection $p_2 \colon A \times B \to B$ is $G$-equivariant. Then there exists a $G$-equivariant non-isomorphic finite surjective étale endomorphism $f \colon A \times B \to A \times B$ such that $p_2 \circ f = p_2$ and that $\deg f$ is coprime to the order of $G$.*

*Proof.* We fix a point $0 \in A$ and an abelian group structure of $A$ with 0 being the zero element. Then the set $\mathrm{Hom}(B, A)$ of morphisms of varieties from $B$ to $A$ is regarded as an abelian group. The action of $\sigma \in G$ on $A \times B$ is given by

$$A \times B \ni (a, b) \mapsto \sigma_{A \times B}(a, b) = (\lambda_\sigma(a + \zeta_\sigma(b)), \sigma_B(b)),$$

for some $\lambda_\sigma \in \mathrm{Aut}(A, 0)$ and $\zeta_\sigma \in \mathrm{Hom}(B, A)$. Here, $\mathrm{Aut}(A, 0)$ denotes the group of automorphisms of $A$ fixing 0, $\sigma \mapsto \lambda_\sigma$ gives rise to a group homomorphism $G \to \mathrm{Aut}(A, 0)$, and $\mathrm{Hom}(B, A)$ has a right $G$-module structure by

$$\varphi^\sigma(b) := \lambda_\sigma^{-1} \varphi(\sigma_B(b))$$

for $\varphi \in \mathrm{Hom}(B, A)$, $\sigma \in G$, and $b \in B$. The collection $\{\zeta_\sigma\}_{\sigma \in G}$ is a 1-cocycle of $\mathrm{Hom}(B, A)$, i.e., the cocycle condition

$$\zeta_{\sigma\sigma'} = \zeta_{\sigma'} + \zeta_\sigma^{\sigma'}$$

is satisfied for any $\sigma$, $\sigma' \in G$. The group cohomology $H^1(G, \mathrm{Hom}(B, A))$ is a torsion module annihilated by the order of $G$. Let $n$ be a positive integer divisible by the order of $G$. Then one can find a morphism $\psi \in \mathrm{Hom}(B, A)$ such that

$$n\zeta_\sigma = \psi - \psi^\sigma \tag{II-5}$$

for any $\sigma \in G$. Let $f$ be an endomorphism of $A \times B$ defined by

$$A \times B \ni (a, b) \mapsto ((n+1)a + \psi(b), b).$$

Then $p_2 \circ f = p_2$, and $\deg f = (n+1)^{2\dim A}$ is coprime to the order of $G$. Moreover, $f$ is a $G$-equivariant non-isomorphic étale surjective endomorphism, since

$$\begin{aligned} f \circ \sigma_{A\times B}(a, b) &= f\left(\lambda_\sigma(a + \zeta_\sigma(b)), \sigma_B(b)\right) \\ &= ((n+1)\lambda_\sigma(a + \zeta_\sigma(b)) + \psi(\sigma_B(b)), \sigma_B(b)) \\ &= (\lambda_\sigma((n+1)a + \zeta_\sigma(b) + n\zeta_\sigma(b) + \psi^\sigma(b)), \sigma_B(b)) \\ &= (\lambda_\sigma((n+1)a + \psi(b) + \zeta_\sigma(b)), \sigma_B(b)) = \sigma_{A\times B} \circ f(a, b) \end{aligned}$$

for any $(a, b) \in A \times B$ and $\sigma \in G$ by (II-5). Thus, we are done. □

The following is on toric varieties.

**Lemma 2.6.** *Let $X$ be a compact toric variety and let $G$ be a finite group acting on $X$ preserving the open torus $\mathbb{T}$. Then there exists a $G$-equivariant non-isomorphic finite surjective endomorphism $f\colon X \to X$ such that $f^{-1}(\mathbb{T}) = \mathbb{T}$ and that $\deg f$ is coprime to the order of $G$.*

*Proof.* Let $\mathsf{N}$ be a free abelian group such that $\mathbb{T} \simeq \mathsf{N} \otimes_{\mathbb{Z}} \mathbb{C}^\star$ as an algebraic group. Then there is a complete fan $\triangle$ of $\mathsf{N}$ such that

$$X = \mathbb{T}_{\mathsf{N}}(\triangle) = \bigcup\nolimits_{\boldsymbol{\sigma}\in\triangle} \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}),$$

where $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is the affine toric variety $\mathbf{Specan}\, \mathbb{C}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$ for $\mathsf{M} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z})$: In [26], $\mathbb{T}_{\mathsf{N}}(\triangle)$ and $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ are written as $T_{\mathsf{N}}\mathrm{emb}(\triangle)$ and $U_\sigma$, respectively. The open torus $\mathbb{T}_{\mathsf{N}}(\{0\})$ is identified with $\mathbb{T}$. For an element $u \in \mathbb{T}$, let $L_u\colon X \to X$ denote the automorphism of action of $u$ on $X$. In other words, $u \mapsto L_u$ gives rise to a group homomorphism $\mathbb{T} \to \mathrm{Aut}(X)$ corresponding to the action of $\mathbb{T}$ on $X$. Note that $u \mapsto L_u$ is injective, i.e., the action of $\mathbb{T}$ on $X$ is faithful. Let $\mathrm{End}(\mathsf{N}, \triangle)$ (resp. $\mathrm{Aut}(\mathsf{N}, \triangle)$) be the set of endomorphisms (resp. automorphisms) $\phi\colon \mathsf{N} \to \mathsf{N}$ which gives rise to a *morphism* $(\mathsf{N}, \triangle) \to (\mathsf{N}, \triangle)$ *of fans*, i.e., for any $\boldsymbol{\sigma} \in \triangle$, there is a cone $\boldsymbol{\tau} \in \triangle$ satisfying $\phi(\boldsymbol{\sigma}) \subset \boldsymbol{\tau}$ (cf. [26, §1.5]). For $\phi \in \mathrm{End}(\mathsf{N}, \triangle)$, let $\mathbb{T}_\phi\colon X \to X$ denote the $\mathbb{T}$-equivariant endomorphism extending $\phi \otimes \mathrm{id}\colon \mathsf{N} \otimes \mathbb{C}^\star \to \mathsf{N} \otimes \mathbb{C}^\star$, whose existence is shown in [26, Thm. 1.13]. Here, we write $\phi(u) := \mathbb{T}_\phi(u) \in \mathbb{T}$ for an element $u$ of the open torus $\mathbb{T}$, for simplicity. Then

$$\mathbb{T}_\phi \circ L_u = L_{\phi(u)} \circ \mathbb{T}_\phi \tag{II-6}$$

as an endomorphism of $X$ for any $u \in \mathbb{T}$. For an integer $m$, we define $\nu_m := \mathbb{T}_{\phi_m}\colon X \to X$ for the multiplication map $\phi_m\colon \mathsf{N} \to \mathsf{N}$ by $m$. Then $\nu_m$ induces the power map $u \mapsto u^m$ as an endomorphism of $\mathbb{T}$, and we have

$$\nu_m \circ L_u = l_{u^m} \circ \nu_m \quad \text{and} \quad \nu_m \circ \mathbb{T}_\phi = \mathbb{T}_\phi \circ \nu_m \tag{II-7}$$

for any $u \in \mathbb{T}$ and any $\phi \in \mathrm{End}(\mathsf{N}, \triangle)$ by (II-6) and by the property that $\phi$ commutes with the multiplication map $\phi_m$.

We note that an automorphism of $X$ preserving the open torus is expressed as $L_u \circ \mathbb{T}_\varphi$ for some $u \in \mathbb{T}$ and $\varphi \in \mathrm{Aut}(\mathsf{N}, \triangle)$: To show it, by composition with $L_u$ for some $u$, it is enough to consider only an automorphism $\Psi \colon X \to X$ preserving the open torus $\mathbb{T}$ and fixing the identity element of $\mathbb{T}$. Then $\Psi|_{\mathbb{T}}$ is a group homomorphism associated with an automorphism $\varphi \colon \mathsf{N} \to \mathsf{N}$. It implies that $\Psi$ is equivariant under the action of $\mathbb{T}$ with respect to the group homomorphism $\Psi|_{\mathbb{T}} \colon \mathbb{T} \to \mathbb{T}$, i.e., $\Psi \circ L_u = L_{\Psi(u)} \circ \Psi$ for any $u \in \mathbb{T}$. Then $\varphi \in \mathrm{Aut}(\mathsf{N}, \triangle)$ and $\Psi = \mathbb{T}_\varphi$ by [26, Thm. 1.13].

By the action of $G$ on $X$, we have a group homomorphism $G \ni \sigma \mapsto \lambda_\sigma \in \mathrm{Aut}(\mathsf{N}, \triangle)$ and elements $u_\sigma \in \mathbb{T}_{\mathsf{N}}$ for $\sigma \in G$ such that

$$\sigma_X = L_{u_\sigma} \circ \mathbb{T}_{\lambda_\sigma}. \tag{II-8}$$

Then, by (II-6) and by the injectivity of $u \mapsto L_u$, we have

$$u_{\sigma_1\sigma_2} = u_{\sigma_1} \lambda_{\sigma_1}(u_{\sigma_2}) \tag{II-9}$$

for any $\sigma_1$, $\sigma_2 \in G$. Hence, $\mathbb{T}$ has a structure of left $G$-module by $u \mapsto \lambda_\sigma(u) = \mathbb{T}_{\lambda_\sigma}(u)$ for $u \in \mathbb{T}$ and $\sigma \in G$, and the collection $\{u_\sigma\}_{\sigma \in G}$ is a 1-cocycle of $\mathbb{T}$ by (II-9). The group cohomology $H^1(G, \mathbb{T})$ is a torsion module annihilated by the order of $G$. Let $n$ be a positive integer divisible by the order of $G$. Then there is an element $v \in \mathbb{T}$ such that

$$u_\sigma^n = \lambda_\sigma(v) v^{-1} \tag{II-10}$$

for any $\sigma \in G$. We set $f := L_v \circ \nu_{n+1}$ as a finite surjective endomorphism of $X$. Then $\deg f = (n+1)^{\dim X}$ is greater than 1 and coprime to the order of $G$, $f^{-1}(\mathbb{T}) \subset \mathbb{T}$ for the open torus $\mathbb{T}$, and moreover,

$$\begin{aligned}
f \circ \sigma_X &= L_v \circ \nu_{n+1} \circ L_{u_\sigma} \circ \mathbb{T}_{\lambda_\sigma} && \text{by} \quad \text{(II-8)} \\
&= L_v \circ L_{u_\sigma^{n+1}} \circ \mathbb{T}_{\lambda_\sigma} \circ \nu_{n+1} && \text{by} \quad \text{(II-7)} \\
&= L_{u_\sigma} \circ L_{\lambda_\sigma(v)} \circ \mathbb{T}_{\lambda_\sigma} \circ \nu_{n+1} && \text{by} \quad \text{(II-10)} \\
&= L_{u_\sigma} \circ \mathbb{T}_{\lambda_\sigma} \circ L_v \circ \nu_{n+1} = \sigma_X \circ f && \text{by} \quad \text{(II-6)}
\end{aligned}$$

for any $\sigma \in G$. Thus, $f$ is $G$-equivariant, and we are done. □

2.2. **$G$-linearizations and equivariant endomorphisms of $\mathbb{P}^1$-bundles.** We shall study equivariant endomorphisms of $\mathbb{P}^1$-bundles by the notion of *$G$-linearizations* (cf. Definition 2.7 below). Especially, in Propositions 2.15 and 2.16 below, we shall give sufficient conditions for the existence of $G$-equivariant non-isomorphic surjective endomorphisms of a $\mathbb{P}^1$-bundle associated with the direct sum of two invertible sheaves. These are applied to $\mathbb{P}^1$-bundles over an elliptic curve in Section 2.3.

**Definition 2.7.** Let $X$ be a complex analytic space with a left action of a finite group $G$. For an $\mathcal{O}_X$-module $\mathcal{F}$, a *$G$-linearization* of $\mathcal{F}$ is a collection $\varepsilon = \{\varepsilon_\sigma\}_{\sigma \in G}$

of isomorphisms $\varepsilon_\sigma\colon \sigma_X^*\mathcal{F} \to \mathcal{F}$ of $\mathcal{O}_X$-modules such that $\varepsilon_{\mathsf{e}} = \mathrm{id}_{\mathcal{F}}$ for the unit element $\mathsf{e} \in G$ and that, for any $\sigma$, $\tau \in G$, the diagram

$$\begin{CD}
\tau_X^*(\sigma_X^*\mathcal{F}) @>{\tau_X^*(\varepsilon_\sigma)}>> \tau_X^*\mathcal{F} \\
@V{\simeq}VV @VV{\varepsilon_\tau}V \\
(\sigma\tau)_X^*\mathcal{F} @>{\varepsilon_{\sigma\tau}}>> \mathcal{F}
\end{CD}$$

is commutative, where the left vertical arrow indicates the canonical isomorphism on composition $(\sigma\tau)_X = \sigma_X \circ \tau_X$. Sometimes, we write

$$\varepsilon_{\sigma\tau} = \varepsilon_\tau \circ \tau_X^*(\varepsilon_\sigma)$$

modulo the canonical isomorphism $(\sigma\tau)_X^*\mathcal{F} \simeq \tau_X^*(\sigma_X^*\mathcal{F})$ for the diagram. For two $G$-linearized $\mathcal{O}_X$-modules $\mathcal{F} = (\mathcal{F}, \varepsilon^{\mathcal{F}})$ and $\mathcal{G} = (\mathcal{G}, \varepsilon^{\mathcal{G}})$, a homomorphism $\phi\colon \mathcal{F} \to \mathcal{G}$ is said to be *$G$-linear* if

$$\begin{CD}
\sigma_X^*\mathcal{F} @>{\sigma_X^*(\phi)}>> \sigma_X^*\mathcal{G} \\
@V{\varepsilon_\sigma^{\mathcal{F}}}VV @VV{\varepsilon_\sigma^{\mathcal{G}}}V \\
\mathcal{F} @>{\phi}>> \mathcal{G}
\end{CD}$$

is commutative for any $\sigma \in G$.

*Remark.* The notion of $G$-linearizations is introduced for invertible sheaves in [17, I, §3]. The category of $G$-linearized $\mathcal{O}_X$-modules with $G$-linear homomorphisms is an abelian category. If $G$ acts on $X$ trivially, then a $G$-linearization of an $\mathcal{O}_X$-module $\mathcal{F}$ is just a right $G$-module structure of $\mathcal{F}$.

*Remark* 2.8. The structure sheaf $\mathcal{O}_X$ has a canonical $G$-linearization. In fact, for a morphism $f\colon U \to V$ of ringed spaces, we have a canonical homomorphism $\mathcal{O}_V \to f_*\mathcal{O}_U$ of sheaves of rings on $V$, and its left adjoint $c_f\colon f^*\mathcal{O}_V \to \mathcal{O}_U$ as an isomorphism of sheaves of rings on $U$. Hence, for any $\sigma \in G$, we have a canonical isomorphism $c_{\sigma_X} : \sigma_X^*\mathcal{O}_X \to \mathcal{O}_X$, and $\{c_{\sigma_X}\}$ is the canonical $G$-linearization of $\mathcal{O}_X$.

*Remark* 2.9. For $G$-linearized $\mathcal{O}_X$-modules $\mathcal{F} = (\mathcal{F}, \varepsilon^{\mathcal{F}})$ and $\mathcal{G} = (\mathcal{G}, \varepsilon^{\mathcal{G}})$, the tensor product $\mathcal{F} \otimes \mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and the hom sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ have canonical $G$-linearizations given by

$$\sigma_X^*(\mathcal{F} \otimes \mathcal{G}) \simeq \sigma_X^*\mathcal{F} \otimes \sigma_X^*\mathcal{G} \xrightarrow{\varepsilon_\sigma^{\mathcal{F}} \otimes \varepsilon_\sigma^{\mathcal{G}}} \mathcal{F} \otimes \mathcal{G} \qquad \text{and}$$
$$\sigma_X^*\,\mathcal{H}om(\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om(\sigma_X^*\mathcal{F}, \sigma_X^*\mathcal{G}) \xrightarrow{(\dagger)} \mathcal{H}om(\mathcal{F}, \mathcal{G})$$

for $\sigma \in G$, where $(\dagger)$ is defined by $(\varepsilon_\sigma^{\mathcal{F}})^{-1}\colon \mathcal{F} \to \sigma_X^*\mathcal{F}$ and $\varepsilon_\sigma^{\mathcal{G}}\colon \sigma_X^*\mathcal{G} \to \mathcal{G}$. In particular, the set $\mathrm{Pic}^G(X)$ of $G$-linearized invertible sheaves on $X$ modulo isomorphisms is an abelian group (cf. [17, I, §3]).

*Remark* 2.10. Let $f\colon X \to Y$ be a $G$-equivariant morphism for complex analytic spaces $X$ and $Y$ with left actions of $G$. For a $G$-linearized $\mathcal{O}_X$-module $\mathcal{F} = (\mathcal{F}, \varepsilon^{\mathcal{F}})$

and a $G$-linearized $\mathcal{O}_Y$-module $\mathcal{G} = (\mathcal{G}, \varepsilon^{\mathcal{G}})$, the direct image $f_*\mathcal{F}$ and the inverse image $f^*\mathcal{G}$ have canonical $G$-linearizations given by isomorphisms

$$\sigma_Y^*(f_*\mathcal{F}) \simeq f_*(\sigma_X^*\mathcal{F}) \xrightarrow{f_*(\varepsilon_\sigma^{\mathcal{F}})} f_*\mathcal{F} \quad \text{and} \quad \sigma_X^*(f^*\mathcal{G}) \simeq f^*(\sigma_Y^*\mathcal{G}) \xrightarrow{f^*(\varepsilon_\sigma^{\mathcal{G}})} f^*\mathcal{G}$$

for $\sigma \in G$. In particular, by considering the case where $Y = \operatorname{Spec}\mathbb{C}$, we have a canonical right $G$-module structure of $H^0(X, \mathcal{F})$ by the $G$-linearization $\varepsilon^{\mathcal{F}}$. Moreover, the canonical bijection

$$\operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \simeq \operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$$

on $\mathcal{O}_Y$-module homomorphisms and $\mathcal{O}_X$-module homomorphisms given by the adjoint pair $(f^*, f_*)$ of functors induces a bijection

$$\operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})^G \simeq \operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})^G$$

on $G$-linear $\mathcal{O}_Y$-module homomorphisms and $G$-linear $\mathcal{O}_X$-module homomorphisms. Since the isomorphism $c_f \colon f^*\mathcal{O}_Y \to \mathcal{O}_X$ in Remark 2.8 is $G$-linear, the canonical morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is also $G$-linear.

*Remark* 2.11. For $X$ and $G$ in Definition 2.7, let $Z$ be a $G$-invariant closed analytic subspace of $X$. Then the canonical surjection $\mathcal{O}_X \to \mathcal{O}_Z$ is $G$-linear. Thus, the ideal sheaf $\mathcal{I}_Z$ of $Z$ is also $G$-linearized as the kernel of $\mathcal{O}_X \to \mathcal{O}_Z$.

**Lemma 2.12.** *Let $Y$ be a complex analytic space with a left action of a finite group $G$ of order $n$ and let $\varpi \colon Y \to \overline{Y} := G\backslash Y$ be the quotient morphism. Let $\mathcal{E}$ be a locally free sheaf of finite rank on $Y$ admitting a $G$-linearization. Then*:

(1) *The $G$-linearization defines a left action of $G$ on the projective bundle $X = \mathbb{P}_Y(\mathcal{E})$ so that the structure morphism $X \to Y$ is $G$-equivariant.*

(2) *Assume that $G$ acts on $Y$ freely. Then $\mathcal{E} \simeq \varpi^*\overline{\mathcal{E}}$ for a locally free sheaf $\overline{\mathcal{E}}$ on $\overline{Y}$. In particular, the quotient morphism $X \to G\backslash X$ by the action in* (1) *is isomorphic to the base change of $\varpi$ by the structure morphism $\mathbb{P}_{\overline{Y}}(\overline{\mathcal{E}}) \to \overline{Y}$.*

(3) *If $\mathcal{E}$ is an invertible sheaf and if $H^0(Y, \mathcal{E}) \neq 0$, then $H^0(Y, \mathcal{E}^{\otimes n})^G \neq 0$.*

(4) *If $\mathcal{E}$ is an invertible sheaf, then there is an invertible sheaf $\mathcal{M}$ on $\overline{Y}$ such that $\varpi^*\mathcal{M} \simeq \mathcal{E}^{\otimes n}$ as a $G$-linearized $\mathcal{O}_Y$-module, where we regard $\mathcal{M}$ as a $\mathcal{O}_{\overline{Y}}$-module with a trivial right action of $G$.*

*Proof.* Let $\pi \colon X = \mathbb{P}_Y(\mathcal{E}) \to Y$ be the structure morphism and let $\{\varepsilon_\sigma \colon \sigma_Y^*\mathcal{E} \to \mathcal{E}\}_{\sigma \in G}$ be the $G$-linearization of $\mathcal{E}$. For $\sigma \in G$, let $\pi_\sigma \colon X_\sigma \to Y$ be the base change of $\pi$ by $\sigma_Y \colon Y \to Y$ and let $p_\sigma \colon X_\sigma \to X$ be the induced morphism, which is just the base change of $\sigma_Y$ by $\pi$. For the isomorphism $e_\sigma := \mathbb{P}_Y(\varepsilon_\sigma) \colon X = \mathbb{P}_Y(\mathcal{E}) \to \mathbb{P}_Y(\sigma_Y^*\mathcal{E})$ over $T$ associated with $\varepsilon_\sigma$, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "e_\sigma"] \arrow[dr, "\pi"'] & X_\sigma \arrow[r, "p_\sigma"] \arrow[d, "\pi_\sigma"] & X \arrow[d, "\pi"] \\
 & Y \arrow[r, "\sigma_Y"] & Y.
\end{tikzcd}$$

We set $\sigma_X := p_\sigma \circ e_\sigma$ as an automorphism of $X$. Then $\pi \circ \sigma_X = \sigma_Y \circ \pi$, and $G \ni \sigma \mapsto \sigma_X \in \operatorname{Aut}(X)$ is a group homomorphism, since $\{\varepsilon_\sigma\}$ is a $G$-linearization of $\mathcal{E}$. Thus, $G$ acts on $X$ by $\sigma \mapsto \sigma_X$ and $\pi$ is $G$-equivariant. This shows (1).

We shall show (2), where $G$ acts on $Y$ freely. Then the quotient morphism $\varpi\colon Y \to \overline{Y} = G\backslash Y$ is a finite Galois étale cover with Galois group $G$, since we have an isomorphism $G \times Y \simeq Y \times_{\overline{Y}} Y$ by $(\sigma, y) \mapsto (\sigma_Y(y), y)$. The direct image $\varpi_*\mathcal{E}$ is a right $G$-module by Remark 2.10. We set $\overline{\mathcal{E}}$ to be the $G$-invariant part of $\varpi_*\mathcal{E}$. It suffices to show that $\mathcal{E} \simeq \varpi^*\overline{\mathcal{E}}$. In fact, by the isomorphism, we have a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathbb{P}_{\overline{Y}}(\overline{\mathcal{E}}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\pi}} \\ Y & \xrightarrow{\varpi} & \overline{Y}, \end{array}$$

in which the upper horizontal arrow $\phi$ is isomorphic to the quotient morphism $X \to G\backslash X$ by the action of $G$ in (1).

In order to prove: $\mathcal{E} \simeq \varpi^*\overline{\mathcal{E}}$, by localizing $\overline{Y}$, we may assume that $Y = G \times \overline{Y}$ over $\overline{Y}$. Then the restriction of $\mathcal{E}$ to the open and closed subset $\{\tau\} \times \overline{Y}$ for $\tau \in G$ is identified with a locally free sheaf $\mathcal{E}_{(\tau)}$ on $\overline{Y}$. The isomorphism $\varepsilon_\sigma\colon \sigma_Y^*\mathcal{E} \to \mathcal{E}$ corresponds to a collection $\{\varepsilon_{\sigma|\tau}\colon \mathcal{E}_{(\sigma\tau)} \to \mathcal{E}_{(\tau)}\}_{\tau \in G}$ of isomorphisms on $\overline{Y}$ and

$$\varepsilon_{\sigma'\sigma|\tau} = \varepsilon_{\sigma|\tau} \circ \varepsilon_{\sigma'|\sigma\tau}$$

as an isomorphism $\mathcal{E}_{(\sigma'\sigma\tau)} \to \mathcal{E}_{(\tau)}$ for any $\sigma$, $\sigma'$, and $\tau \in G$, since $\{\varepsilon_\sigma\}$ is a $G$-linearization. The right $G$-module structure of

$$\varpi_*\mathcal{E} = \prod\nolimits_{\tau \in G} \mathcal{E}_{(\tau)}$$

is given by $\{\varepsilon_{\sigma|\tau}\}$ for $\sigma \in G$. Thus, we have an isomorphism $\overline{\mathcal{E}} \simeq \mathcal{E}_{(\tau)}$ for any $\tau$, and hence, $\mathcal{E} \simeq \varpi^*\overline{\mathcal{E}}$, and (2) has been proved.

Finally, we shall prove (3) and (4), where $\mathcal{E}$ is an invertible sheaf. For a non-zero element $\xi$ of the right $G$-module $H^0(Y, \mathcal{E})$, the product

$$\overline{\xi} = \prod\nolimits_{\sigma \in G} \xi^\sigma \tag{II-11}$$

is regarded as a non-zero $G$-invariant element of $H^0(Y, \mathcal{E}^{\otimes n})$. This shows (3). Let $\mathcal{M}$ be the $G$-invariant part $(\varpi_*(\mathcal{E}^{\otimes n}))^G$ of $\varpi_*(\mathcal{E}^{\otimes n})$. For (4), it is enough to prove that $\mathcal{M}$ is an invertible sheaf and that the canonical composite homomorphism

$$\Phi\colon \varpi^*\mathcal{M} \to \varpi^*(\varpi_*(\mathcal{E}^{\otimes n})) \to \mathcal{E}^{\otimes n}$$

is an isomorphism. Let us take an arbitrary point $Q \in G\backslash Y$ and let $\overline{U}$ be a Stein open neighborhood of $Q$. Then $U = \varpi^{-1}\overline{U}$ is a $G$-invariant Stein open neighborhood of $\varpi^{-1}Q$, and there is a section $\xi \in H^0(U, \mathcal{E})$ such that $\varpi^{-1}Q \subset \{\xi \neq 0\}$. Let $\overline{\xi}$ be the product (II-11) over $U$. Then $\overline{\xi}$ is a $G$-invariant element of $H^0(U, \mathcal{E}^{\otimes n})$, the $G$-invariant open subset $V = \{\overline{\xi} \neq 0\} \subset U$ contains $\varpi^{-1}Q$, and $\overline{\xi}|_V\colon \mathcal{O}_V \to \mathcal{E}^{\otimes n}|_V$ is an isomorphism. Then $\overline{\xi}$ is regarded as an element of $H^0(\overline{U}, \mathcal{M})$, $V = \varpi^{-1}\overline{V}$ for the open subset $\overline{V} = \varpi(V) \subset \overline{U}$, and $\overline{\xi}|_{\overline{V}}\colon \mathcal{O}_{\overline{V}} \to \mathcal{M}|_{\overline{V}}$ is an isomorphism. Since $Q \in \overline{V}$, $\mathcal{M}$ is invertible at $Q$, and $\Phi$ is an isomorphism along $\varpi^{-1}Q$. Thus, (4) has been proved, and we are done. $\square$

*Remark.* When $G$ acts on $Y$ freely, the $G$-linearization $\{\varepsilon_\sigma\}$ corresponds to a *descent datum* of $\mathcal{E}$ relative to $\varpi\colon Y \to \overline{Y}$ and one can find $\overline{\mathcal{E}}$ satisfying $\mathcal{E} \simeq \varpi^*\overline{\mathcal{E}}$ as a consequence of the descent theory (cf. [11, Exp. VIII, Cor. 1.3]) in the case of schemes.

**Lemma 2.13.** *Let $Y$ be a compact variety with a left action of a finite group $G$ of order $n$ and let $f\colon Y \to Y$ be a $G$-equivariant endomorphism. Let $\mathcal{L}$ be a $G$-linearized invertible sheaf on $Y$ with an isomorphism $f^*\mathcal{L} \simeq \mathcal{L}^{\otimes m}$ of $\mathcal{O}_Y$-modules for an integer $m$ coprime to $n$. Then there exists an isomorphism $(f^k)^*\mathcal{L} \simeq \mathcal{L}^{\otimes m^k}$ of $G$-linearized $\mathcal{O}_Y$-modules for a positive integer $k > 0$.*

*Proof.* By Remarks 2.9 and 2.10, we can consider $\mathcal{N} := f^*\mathcal{L} \otimes \mathcal{L}^{\otimes -m}$ as a $G$-linearized $\mathcal{O}_Y$-module. Since $\mathcal{N} \simeq \mathcal{O}_Y$ as an $\mathcal{O}_Y$-module, the $G$-linearization of $\mathcal{N}$ is determined by the right action of $G$ on $H^0(Y, \mathcal{N}) \simeq \mathbb{C}$, which corresponds to a group homomorphism $\chi\colon G \to \mathbb{C}^\star$. Since $\mathbb{C} \simeq H^0(X, \mathcal{N}) \simeq H^0(X, f^*\mathcal{N})$, the $G$-linearization of $f^*\mathcal{N}$ is also determined by $\chi$. By induction, we have a canonical isomorphism

$$\bigotimes\nolimits_{i=0}^{k-1} (f^i)^*\mathcal{N}^{\otimes m^{k-1-i}} \simeq (f^k)^*\mathcal{L} \otimes \mathcal{L}^{\otimes -m^k}$$

of $G$-linearized $\mathcal{O}_Y$-modules for any $k \geq 0$. Hence, the $G$-linearization of the right hand side is determined by $\chi^{l(k)}$ for $l(k) = 1 + m + \cdots + m^{k-1}$. Since $\gcd(m, n) = 1$, $m^a \equiv 1 \bmod n$ for some $a > 0$, and

$$l(an) = \sum\nolimits_{i=0}^{an-1} m^i = l(a) \sum\nolimits_{j=0}^{n-1} m^{aj} \equiv 0 \bmod n.$$

Thus, it is enough to set $k = an$. $\square$

**Lemma 2.14.** *Let $\pi\colon X \to T$ be a $\mathbb{P}^1$-bundle over a compact variety $T$ with a section $\Gamma$. Let $G$ be a finite group acting on $X$ and $T$ such that $\pi$ is $G$-equivariant and that $\Gamma$ is $G$-invariant. If $\Gamma' \cap \Gamma = \emptyset$ for another section $\Gamma'$ of $\pi$, then $\Theta \cap \Gamma = \emptyset$ for a $G$-invariant section $\Theta$ of $\pi$.*

*Proof.* Note that the section $\Gamma$ is a Cartier divisor on $X$. We can regard $0 \to \mathcal{O}_X(-\Gamma) \to \mathcal{O}_X \to \mathcal{O}_\Gamma \to 0$ as an exact sequence of $G$-linearized $\mathcal{O}_X$-modules (cf. Remark 2.11). Hence, $\mathcal{O}_X(\Gamma)$ has an induced $G$-linearization (cf. Remark 2.9) and the defining equation of $\Gamma$ corresponding to $\mathcal{O}_X(-\Gamma) \to \mathcal{O}_X$ is a $G$-invariant section of the right $G$-module $H^0(X, \mathcal{O}_X(\Gamma))$ (cf. Remark 2.10). The invertible sheaves

$$\mathcal{L} := \pi_*(\mathcal{O}_X(-\Gamma) \otimes \mathcal{O}_\Gamma) \quad \text{and} \quad \mathcal{M} := \pi^*\mathcal{L} \otimes \mathcal{O}_X(\Gamma)$$

on $T$ and $X$, respectively, have also induced $G$-linearizations (cf. Remarks 2.9 and 2.10). Then we have an isomorphism $\mathcal{M} \otimes \mathcal{O}_\Gamma \simeq \mathcal{O}_\Gamma$ of $G$-linearized $\mathcal{O}_\Gamma$-modules. Note that $\Gamma' \in |\mathcal{M}|$. In fact, $\mathcal{O}_X(\Gamma') \simeq \mathcal{M} \otimes \pi^*\mathcal{N}$ for an invertible sheaf $\mathcal{N}$ on $T$, but we have $\mathcal{N} \simeq \mathcal{O}_T$ by $\mathcal{O}_X(\Gamma') \otimes \mathcal{O}_\Gamma \simeq \mathcal{O}_\Gamma$. Now, we have an exact sequence

$$0 \to \pi^*\mathcal{L} \to \mathcal{M} \to \mathcal{M} \otimes \mathcal{O}_\Gamma \simeq \mathcal{O}_\Gamma \to 0 \tag{II-12}$$

of $G$-linearized $\mathcal{O}_X$-modules and an exact sequence

$$0 \to \mathcal{L} \to \pi_*\mathcal{M} \to \mathcal{O}_T \to 0 \tag{II-13}$$

of $G$-linearized $\mathcal{O}_T$-modules (cf. Remarks 2.9 and 2.10). The existence of $\Gamma'$ implies that (II-13) is split as an exact sequence of $\mathcal{O}_T$-modules. Thus,

$$0 \to H^0(T, \mathcal{L}) \to H^0(X, \mathcal{M}) \to H^0(\Gamma, \mathcal{O}_\Gamma) \to 0$$

is an exact sequence of $G$-modules. Since $G$ is finite, $H^1(G, H^0(T, \mathcal{L})) = 0$ and we have a $G$-invariant element $\theta$ of $H^0(X, \mathcal{M})$ which goes to $1 \in H^0(\Gamma, \mathcal{O}_\Gamma)$. In particular, (II-13) is also split as an exact sequence of $G$-linearized $\mathcal{O}_X$-modules. Then the effective divisor $\Theta := \operatorname{div}(\theta) \in |\mathcal{M}|$ is $G$-invariant, and $\Theta \cap \Gamma = \emptyset$. This $\Theta$ is a desired section of $\pi$. $\square$

**Proposition 2.15.** *Let $\pi\colon X \to T$ be a $\mathbb{P}^1$-bundle over a compact normal variety $T$, and let $\Theta_1$ and $\Theta_2$ be mutually disjoint sections of $\pi$. Let $G$ be a finite group acting on $X$ and $T$ such that $\pi$ is $G$-equivariant and that $\Theta_1$ and $\Theta_2$ are $G$-invariant. Then:*

(1) *There is a $G$-linearized invertible sheaf $\mathcal{L}$ on $T$ such that $\mathcal{O}_X(\Theta_1 - \Theta_2) \simeq \pi^*\mathcal{L}$ as a $G$-linearized $\mathcal{O}_X$-module.*

*Let $h\colon T \to T$ be a $G$-equivariant surjective endomorphism with an isomorphism $h^*\mathcal{L} \simeq \mathcal{L}^{\otimes m}$ of $G$-linearized $\mathcal{O}_T$-modules for an integer $m > 1$. Then:*

(2) *There exists a $G$-equivariant non-isomorphic surjective endomorphism $f$ of $X$ such that*
   - $\deg f = m \deg h$, $\pi \circ f = h \circ \pi$,
   - $f^*\Theta_1 = m\Theta_1$, $f^*\Theta_2 = m\Theta_2$, $S_f = \pi^*S_h + \Theta_1 + \Theta_2$, *and*
   - $\Gamma \neq f^{-1}f(\Gamma)$ *for any prime divisor $\Gamma$ dominating $T$ except $\Theta_1$ and $\Theta_2$.*

(3) *Assume that $m > 2$ and that $H^0(T, \mathcal{L}^{\otimes j})^G \neq 0$ for some $1 \leq j < m$. Then there exists a $G$-equivariant non-isomorphic surjective endomorphism $f$ of $X$ such that*
   - $\deg f = m \deg h$, $\pi \circ f = h \circ \pi$,
   - $f^*\Theta_2 = m\Theta_2$, $S_f = \pi^*S_h + \Theta_2$, *and*
   - $\Gamma \neq f^{-1}f(\Gamma)$ *for any prime divisor $\Gamma$ dominating $T$ except $\Theta_2$.*

*Here, $S_h$ and $S_f$ stand for the* characteristic completely invariant divisors *of $h$ and $f$, respectively* (*cf.* [24, Def. 2.16]).

*Proof.* Assertion (1) is shown by an argument in the proof of Lemma 2.14. For (2) and (3), let us consider a homomorphism

$$\Psi\colon \mathcal{O}_T \oplus h^*\mathcal{L} \to \operatorname{Sym}^m(\mathcal{O}_T \oplus \mathcal{L}) = \bigoplus\nolimits_{j=0}^{m} \mathcal{L}^{\otimes j}$$

of $\mathcal{O}_T$-modules defined by the following conditions:

(i) The induced homomorphism $\ell_j\colon \mathcal{O}_T \to \mathcal{L}^{\otimes j}$ from the factor $\mathcal{O}_T$ is $G$-linear for any $0 \leq j \leq m$, and $\ell_0$ is the identity morphism $\mathcal{O}_T \to \mathcal{O}_T$.

(ii) The induced homomorphism $h^*\mathcal{L} \to \mathcal{L}^{\otimes j}$ from the factor $h^*\mathcal{L}$ is zero for any $0 \leq j < m$, and $h^*\mathcal{L} \to \mathcal{L}^{\otimes m}$ is a $G$-linear isomorphism.

Note that $\ell_j$ corresponds to a $G$-invariant section of $H^0(T, \mathcal{L}^{\otimes j})$. The homomorphism $\Psi$ is $G$-linear by (i) and (ii), since the diagram

$$\begin{tikzcd}
\sigma_T^*(\mathcal{O}_T \oplus h^*\mathcal{L}) \arrow[r] \arrow[d, "c_{\sigma_T} \oplus \varepsilon_\sigma"'] & \sigma_T^* \operatorname{Sym}^m(\mathcal{O}_T \oplus \mathcal{L}) \arrow[r, "\simeq^\dagger"] & \bigoplus_{j=0}^m \sigma_T^*\mathcal{L}^{\otimes j} \arrow[d, "\bigoplus \varepsilon_\sigma^{\otimes j}"] \\
\mathcal{O}_T \oplus h^*\mathcal{L} \arrow[r] & \operatorname{Sym}^m(\mathcal{O}_T \oplus \mathcal{L}) \arrow[r, "\simeq^\dagger"] & \bigoplus_{j=0}^m \mathcal{L}^{\otimes j}
\end{tikzcd}$$

is commutative for any $\sigma \in G$, where $\{\varepsilon_\sigma : \sigma_T^*\mathcal{L} \to \mathcal{L}\}_{\sigma\in G}$ is the $G$-linearization of $\mathcal{L}$, $\{c_{\sigma_T} : \sigma_T^*\mathcal{O}_T \to \mathcal{O}_T\}_{\sigma\in G}$ is the canonical $G$-linearization of $\mathcal{O}_T$ (cf. Remark 2.8), and $\simeq^\dagger$ stands for canonical isomorphisms. By adjunction for $(\pi^*, \pi_*)$, $\Psi$ corresponds to a surjective homomorphism

$$\widetilde{\Psi} : \pi^*(\mathcal{O}_T \oplus h^*\mathcal{L}) \to \mathcal{O}_X(m\Theta_1)$$

of $\mathcal{O}_X$-modules, since $\pi_*\mathcal{O}_X(\Theta_1) \simeq \mathcal{O}_T \oplus \mathcal{L}$ and since homomorphisms $\mathcal{O}_T \to \mathcal{O}_T$ and $h^*\mathcal{L} \to \mathcal{L}^{\otimes m}$ in (i) and (ii), respectively, are surjective. Let $\pi_h : X_h \to T$ be the base change of $\pi : X \to T$ by $h$. Then we have a morphism

$$\psi : X = \mathbb{P}_X(\mathcal{O}_T \oplus \mathcal{L}) \to X_h \simeq \mathbb{P}_T(\mathcal{O}_T \oplus h^*\mathcal{L})$$

over $T$ of degree $m$ associated with $\widetilde{\Psi}$, and an isomorphism $\psi^*(p_1^*\mathcal{O}_X(\Theta_1)) \simeq \mathcal{O}_X(m\Theta_1)$ for the first projection $p_1 : X_h = X \times_{T,h} T \to X$. In particular, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[dr, "\pi"'] & X_h \arrow[r, "p_1"] \arrow[d, "\pi_h"] & X \arrow[d, "\pi"] \\ & T \arrow[r, "h"] & T. \end{tikzcd}$$

The morphism $p_1$ is $G$-equivariant, since $\pi$ and $h$ are so, and the morphism $\psi$ is also $G$-equivariant, since $\Psi$ is $G$-linear. Hence, $f := p_1 \circ \psi : X \to X$ is a $G$-equivariant endomorphism, and we have $\pi \circ f = h \circ \pi$ and $\deg f = m \deg h > 1$.

For other assertions, we want to determine prime divisors $\Gamma$ on $X$ such that $\Gamma = f^{-1}f(\Gamma)$ and $\pi(\Gamma) = T$ when one of the following conditions for $\ell_j$ is satisfied:

(A) $\ell_j = 0$ for any $1 \le j \le m$;

(B) $m > 2$ and $\ell_j \ne 0$ for some $1 \le j < m$.

Note that, for a point $t \in T$, the morphism

$$\psi_t := \psi|_{\pi^{-1}(t)} : \mathbb{P}^1 = \pi^{-1}(t) \to \mathbb{P}^1 = \pi_h^{-1}(t) \simeq \pi^{-1}(h(t))$$

of fibers over $t$ is expressed as

$$(\mathtt{x} : \mathtt{y}) \mapsto (\mathtt{x}^m + \sum\nolimits_{j=1}^m \ell_j(t)\mathtt{x}^{m-j}\mathtt{y}^j : \mathtt{y}^m)$$

for a suitable homogeneous coordinate $(\mathtt{x} : \mathtt{y})$ by (i) and (ii), where $\Theta_1 \cap \pi^{-1}(t) = \{(0 : 1)\}$ and $\Theta_2 \cap \pi^{-1}(t) = \{(1 : 0)\}$. Hence, $f^*\Theta_2 = \psi^*(p_1^*\Theta_2) = m\Theta_2$. By the same reason, $f^*\Theta_1 = m\Theta_1$ when (A) holds. Suppose that $\psi_t^{-1}(q' : 1) = \{(q : 1)\}$ for some $q$, $q' \in \mathbb{C}$. Then

$$\mathtt{x}^m + \sum\nolimits_{j=1}^m \ell_j(t)\mathtt{x}^{m-j} = q' + (\mathtt{x} - q)^m \in \mathbb{C}[\mathtt{x}].$$

In particular, for any $1 \le j \le m-1$,

$$\text{(II-14)} \qquad \ell_j(t) = (-1)^j \binom{m}{j} q^j.$$

If (A) holds, then $q = q' = 0$, and it implies that $\Gamma \ne f^{-1}f(\Gamma)$ for any prime divisor $\Gamma$ dominating $T$ except $\Theta_1$ and $\Theta_2$. When (B) holds, by replacing $\ell_j$ by constant multiples, we may assume that (II-14) does not hold for any $q \in \mathbb{C}$ for some $t \in T$.

In fact, we may assume that $\ell_j(t) \neq 0$ for some $1 \leq j < m$, and (II-14) implies that $q \neq 0$ and $\ell_j(t) \neq 0$ for any $1 \leq j < m$. Hence, if we replace $\ell_j$ with $c_j\ell_j$ for some $c_j \notin \{0, 1\}$, then (II-14) does not hold for any $q$, since $m > 2$. Under the additional assumption for (B), $\Gamma \neq f^{-1}f(\Gamma)$ for any prime divisor $\Gamma$ dominating $T$ except $\Theta_2$.

For the assertion (2), it is enough to take $\ell_j$ to satisfy (A), and for the assertion (3), it is enough to take $\ell_j$ to satisfy (B) with the additional assumption. In fact, in this situation, we have verified required conditions for $f$ in (2) and (3) except conditions on characteristic completely invariant divisors $S_f$ and $S_h$. Note that $S_f = \pi^*S_h + D$ for the union $D$ of prime divisors $\Gamma$ on $X$ such that $\pi(\Gamma) = T$ and that $(f^k)^*\Gamma = b\Gamma$ for some $k \geq 1$ and $b \geq 2$ (cf. [24, Def. 2.16, Lem. 2.19(2)]). Then $D = \Theta_1 + \Theta_2$ in case (2), and $D = \Theta_2$ in case (3). Thus, we are done. □

The following is a variant of Proposition 2.15.

**Proposition 2.16.** *Let $\pi\colon X \to T$ be a $\mathbb{P}^1$-bundle over a compact normal variety $T$ with mutually disjoint sections $\Theta_1$ and $\Theta_2$. Let $G$ be a finite group acting on $X$ and $T$ such that $\pi$ is $G$-equivariant, $\Theta_1 + \Theta_2$ is $G$-invariant, and $\sigma(\Theta_1) = \Theta_2$ for some $\sigma \in G$. Let $G_0$ be the subgroup $\{\sigma \in G \mid \sigma(\Theta_1) = \Theta_1\}$. Then:*

(1) *There is a $G_0$-linearized invertible sheaf $\mathcal{L}$ on $T$ with an isomorphism $\mathcal{O}_X(\Theta_1 - \Theta_2) \simeq \pi^*\mathcal{L}$ as a $G_0$-linearized $\mathcal{O}_X$-module.*

*Let $h\colon T \to T$ be a $G$-equivariant surjective endomorphism with an isomorphism $h^*\mathcal{L} \simeq \mathcal{L}^{\otimes m}$ of $G_0$-linearized $\mathcal{O}_T$-modules for an integer $m > 1$. Then:*

(2) *there exists a $G$-equivariant surjective endomorphism $f$ of $X$ such that*
   - $\deg f = m \deg h$, $\pi \circ f = h \circ \pi$,
   - $f^*\Theta_1 = m\Theta_1$, $f^*\Theta_2 = m\Theta_2$, $S_f = \pi^*S_h + \Theta_1 + \Theta_2$, *and*
   - $\Gamma \neq f^{-1}f(\Gamma)$ *for any prime divisor* $\Gamma$ *dominating* $T$ *except* $\Theta_1$ *and* $\Theta_2$.

For the proof, we use:

**Notation 2.17.** Let $\varphi\colon \mathcal{L} \to \mathcal{M}$ be an isomorphism of invertible sheaves on a complex analytic space $X$. For a positive integer $m$, $\varphi^{\otimes m}$ denotes the induced isomorphism $\mathcal{L}^{\otimes m} \to \mathcal{M}^{\otimes m}$ as usual. Let $\varphi^{\otimes 0}$ denote the identity morphism $\mathcal{O}_X \to \mathcal{O}_X$. For a negative integer $m$, let $\varphi^{\otimes m}\colon \mathcal{L}^{\otimes m} \to \mathcal{M}^{\otimes m}$ denote the inverse of the dual homomorphism

$$(\varphi^{\otimes -m})^\vee\colon \mathcal{M}^{\otimes m} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^{\otimes -m}, \mathcal{O}_X) \to \mathcal{L}^{\otimes m} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^{\otimes -m}, \mathcal{O}_X).$$

*Remark.* For integers $m$ and $m'$, the isomorphism $(\varphi^{\otimes m})^{\otimes m'}$ is equal to $\varphi^{\otimes mm'}$ under canonical isomorphisms $(\mathcal{L}^{\otimes m})^{\otimes m'} \simeq \mathcal{L}^{\otimes mm'}$ and $(\mathcal{M}^{\otimes m})^{m'} \simeq \mathcal{M}^{\otimes mm'}$.

*Proof of Proposition* 2.16. Assertion (1) is just Proposition 2.15(1) for the action of $G_0$. For $i = 1, 2$, let $\{\varepsilon^{(i)}_\sigma\}_{\sigma \in G_0}$ be the $G_0$-linearization of $\mathcal{O}_X(\Theta_i)$. Then, by Remark 2.11, we have a commutative diagram

$$\begin{tikzcd}
\sigma_X^*\mathcal{O}_X(-\Theta_i) \arrow[r] \arrow[d, "(\varepsilon^{(i)}_\sigma)^{\otimes -1}"'] & \sigma_X^*\mathcal{O}_X \arrow[d, "c_{\sigma_X}"] \\
\mathcal{O}_X(-\Theta_i) \arrow[r] & \mathcal{O}_X
\end{tikzcd}$$

for any $\sigma \in G_0$, where $c_{\sigma_X}$ is the isomorphism associated with the morphism $\sigma_X\colon X \to X$ of ringed spaces (cf. Remark 2.8), and where $(\varepsilon_\sigma^{(i)})^{\otimes -1}$ is the isomorphism in Notation 2.17. For any $\tau \in G \setminus G_0$, we can define isomorphisms

$$\eta_{12}(\tau)\colon \tau^*\mathcal{O}_X(\Theta_1) \to \mathcal{O}_X(\Theta_2) \quad \text{and} \quad \eta_{21}(\tau)\colon \tau^*\mathcal{O}_X(\Theta_2) \to \mathcal{O}_X(\Theta_1)$$

by commutative diagrams

$$\begin{tikzcd} \tau_X^*\mathcal{O}_X(-\Theta_1) \arrow[r] \arrow[d, "\eta_{12}(\tau)^{\otimes -1}"'] & \tau_X^*\mathcal{O}_X \arrow[d, "c_{\tau_X}"] \\ \mathcal{O}_X(-\Theta_2) \arrow[r] & \mathcal{O}_X, \end{tikzcd} \qquad \begin{tikzcd} \tau_X^*\mathcal{O}_X(-\Theta_2) \arrow[r] \arrow[d, "\eta_{21}(\tau)^{\otimes -1}"'] & \tau_X^*\mathcal{O}_X \arrow[d, "c_{\tau_X}"] \\ \mathcal{O}_X(-\Theta_1) \arrow[r] & \mathcal{O}_X. \end{tikzcd}$$

Then we can identify $\eta_{12}(\sigma\tau\sigma')$ with the composite

$$(\sigma\tau\sigma')_X^*\mathcal{O}_X(\Theta_1) \simeq \sigma_X'^*(\tau_X^*(\sigma_X^*\mathcal{O}_X(\Theta_1))) \xrightarrow{\sigma_X'^*(\tau_X^*\varepsilon_\sigma^{(1)})} \sigma_X'^*(\tau_X^*\mathcal{O}_X(\Theta_1))$$
$$\xrightarrow{\sigma_X'^*\eta_{12}(\tau)} \sigma_X'^*\mathcal{O}_X(\Theta_2) \xrightarrow{\varepsilon_{\sigma'}^{(2)}} \mathcal{O}_X(\Theta_2)$$

for any $\sigma$, $\sigma' \in G_0$, and identify $\varepsilon_{\tau\tau'}^{(1)}$ with the composite

$$(\tau\tau')_X^*\mathcal{O}_X(\Theta_1) \simeq \tau_X'^*(\tau_X^*\mathcal{O}_X(\Theta_1)) \xrightarrow{\tau_X'^*\eta_{12}(\tau)} \tau_X'^*\mathcal{O}_X(\Theta_2) \xrightarrow{\eta_{21}(\eta')} \mathcal{O}_X(\Theta_1)$$

for any $\tau' \in G \setminus G_0$. We have similar identifications for $\eta_{21}(\tau)$. Since $\mathcal{L} = \pi_*\mathcal{O}_X(\Theta_1 - \Theta_2)$, an isomorphism

$$\eta_\tau\colon \tau_T^*\mathcal{L} \to \mathcal{L}^{\otimes -1}$$

is induced by $\eta_{12}(\tau) \otimes \eta_{21}(\tau)^{\otimes -1}$. Let $\{\varepsilon_\sigma\colon \sigma_T^*\mathcal{L} \to \mathcal{L}\}_{\sigma\in G_0}$ be the $G_0$-linearization of $\mathcal{L}$. Then $\eta_{\sigma\tau\sigma'}$ is identified with the composite

$$(\sigma\tau\sigma')_T^*\mathcal{L} \simeq \sigma_T'^*(\tau_T^*(\sigma_T^*\mathcal{L})) \xrightarrow{\sigma_T'^*(\tau_T^*\varepsilon_\sigma)} \sigma_T'^*(\tau_T^*\mathcal{L}) \xrightarrow{\sigma_T'^*(\eta_\tau)} \sigma_T'^*\mathcal{L}^{\otimes -1} \xrightarrow{\varepsilon_{\sigma'}^{\otimes -1}} \mathcal{L}^{\otimes -1}$$

for any $\sigma$, $\sigma' \in G_0$, and $\varepsilon_{\tau\tau'}$ is identified with the composite

$$(\tau\tau')_T^*\mathcal{L} \simeq \tau_T'^*(\tau^*\mathcal{L}) \xrightarrow{\tau_T'^*\eta_\tau} \tau_T'^*\mathcal{L}^{\otimes -1} \xrightarrow{\eta_{\tau'}^{\otimes -1}} \mathcal{L}$$

for any $\tau' \in G \setminus G_0$. For any $\tau \in G \setminus G_0$, the automorphism $\tau_X$ of $X = \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ is induced from the isomorphism

$$\tau_T^*(\mathcal{O}_T \oplus \mathcal{L}) \xrightarrow{c_{\tau_T} \oplus \eta_\tau} \mathcal{O}_T \oplus \mathcal{L}^{\otimes -1} \simeq \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_T} (\mathcal{L} \oplus \mathcal{O}_T),$$

since it is identified with the isomorphism

$$\tau_T^*(\pi_*\mathcal{O}_X(\Theta_1)) \simeq \pi_*(\tau_X^*\mathcal{O}_X(\Theta_1)) \xrightarrow{\pi_*(\eta_{12}(\tau))} \pi_*\mathcal{O}_X(\Theta_2) \simeq \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_T} \pi_*\mathcal{O}_X(\Theta_1).$$

Let $\alpha\colon h^*\mathcal{L} \to \mathcal{L}^{\otimes m}$ be an isomorphism of $G_0$-linearized $\mathcal{O}_T$-modules. Then, for any $\tau \in G \setminus G_0$, there is a non-zero constant $c(\tau) \in \mathbb{C}$ such that the diagram

$$\begin{tikzcd} h^*(\tau_T^*\mathcal{L}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "h^*\eta_\tau"'] & \tau_T^*(h^*\mathcal{L}) \arrow[r, "\tau_T^*\alpha"] & \tau_T^*\mathcal{L}^{\otimes m} \arrow[d, "\eta_\tau^{\otimes m}"] \\ h^*\mathcal{L}^{\otimes -1} \arrow[rr, "c(\tau)\alpha^{\otimes -1}"] & & \mathcal{L}^{\otimes -m} \end{tikzcd}$$

is commutative. Since $\alpha$ is $G_0$-linear, we have $c(\sigma\tau\sigma') = c(\tau)$ for any $\sigma$, $\sigma' \in G_0$, and $c(\tau)$ is independent of the choice of $\tau \in G \setminus G_0$. Hence, by replacing $\alpha$ with a constant multiple, we may assume that $c(\tau) = 1$.

Let $\Psi\colon \mathcal{O}_T \oplus h^*\mathcal{L} \to \mathrm{Sym}^m(\mathcal{O}_T \oplus \mathcal{L})$ be the $G_0$-linear homomorphism defined in the proof of Proposition 2.15 for the action of $G_0$, in which $\ell_j = 0$ for any $1 \le j \le m$: this is given by the identity morphism $\mathcal{O}_T \to \mathcal{O}_T$ and the $G_0$-linear isomorphism $\alpha\colon h^*\mathcal{L} \to \mathcal{L}^{\otimes m}$ above. Then $\Psi$ corresponds to a homomorphism $\pi^*(\mathcal{O}_T \oplus h^*\mathcal{L}) \to \mathcal{O}_X(m\Theta_1)$ and we have an associated morphism

$$\psi\colon X = \mathbb{T}(\mathcal{O}_T \oplus \mathcal{L}) \to X_h := \mathbb{P}_T(\mathcal{O}_T \oplus h^*\mathcal{L})$$

over $T$, where $X_h \to T$ is the base change of $\pi\colon X \to T$ by $h$. For the first projection $p_1\colon X_h \simeq X \times_{T,h} T \to X$, we set $f := p_1 \circ \psi$ as an endomorphism of $X$. Then $\psi$ and $f$ are $G_0$-equivariant, and $f$ satisfies the following conditions by the proof of Proposition 2.15:

- $\deg f = m \deg h$, $\pi \circ f = f \circ \pi$,
- $f^*\Theta_1 = m\Theta_1$, $f^*\Theta_2 = m\Theta_2$, $S_f = \pi^*S_h + \Theta_1 + \Theta_2$, and
- $\Gamma \neq f^{-1}f(\Gamma)$ for any prime divisor $\Gamma$ dominating $T$ except $\Theta_1$ and $\Theta_2$.

Thus, it suffices to prove that $\psi$ is $G$-equivariant. Let us consider a commutative diagram

$$\begin{array}{ccc} h^*\mathcal{L}^{\otimes -1} \otimes (\mathcal{O}_T \oplus h^*\mathcal{L}) & \xrightarrow{\alpha^{\otimes -1}\otimes\Psi} & \mathcal{L}^{\otimes -m} \otimes \mathrm{Sym}^m(\mathcal{O}_T \oplus \mathcal{L}) \\ {\scriptstyle\simeq^\dagger}\downarrow & & \downarrow{\scriptstyle\simeq^\dagger} \\ \mathcal{O}_T \oplus h^*\mathcal{L}^{\otimes -1} & \xrightarrow{\Psi'} & \mathrm{Sym}^m(\mathcal{O}_T \oplus \mathcal{L}^{\otimes -1}), \end{array}$$

where $\simeq^\dagger$ are canonical isomorphisms induced by interchange isomorphisms

$$h^*\mathcal{L}^{\otimes -1} \oplus \mathcal{O}_T \simeq \mathcal{O}_T \oplus h^*\mathcal{L}^{\otimes -1} \quad \text{and} \quad \mathcal{L}^{\otimes -1} \oplus \mathcal{O}_T \simeq \mathcal{O}_T \oplus \mathcal{L}^{\otimes -1}.$$

Then the bottom homomorphism $\Psi'$ is a $G_0$-linear, and the diagram

$$\begin{array}{ccccc} \tau_T^*(\mathcal{O}_T \oplus h^*\mathcal{L}) & \xrightarrow{\tau_T^*\Psi} & \tau_T^*\,\mathrm{Sym}^m(\mathcal{O}_T \oplus \mathcal{L}) & \xrightarrow{\simeq^\ddagger} & \bigoplus_{j=0}^m \tau_T^*\mathcal{L}^{\otimes j} \\ {\scriptstyle c_{\tau_T} \oplus h^*(\eta_\tau)}\downarrow & & & & \downarrow{\scriptstyle \bigoplus \eta_\tau^{\otimes j}} \\ \mathcal{O}_T \oplus h^*\mathcal{L}^{\otimes -1} & \xrightarrow{\Psi'} & \mathrm{Sym}^m(\mathcal{O}_T \oplus \mathcal{L}^{\otimes -1}) & \xrightarrow{\simeq^\ddagger} & \bigoplus_{j=0}^m \mathcal{L}^{\otimes -j} \end{array}$$

is commutative for any $\tau \in G \setminus G_0$, where $\simeq^\ddagger$ stands for canonical isomorphisms. The commutativity of the diagram for $\tau$ implies that $\tau_{X_h} \circ \psi = \psi \circ \tau_X$. Thus, $\psi$ is $G$-equivariant, and we are done. □

**Corollary 2.18.** *Let $X = \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{O}_T(e))$ be the Hirzebruch surface of degree $e > 0$, where $T = \mathbb{P}^1$, and let $\pi\colon X \to T$ be the $\mathbb{P}^1$-bundle structure. Let $G$ be a finite group acting on $X$. Then $G$ acts on $T$ so that $\pi$ is $G$-equivariant, and the negative section $\Theta$ of $\pi$ is $G$-invariant. Moreover, there exist $G$-equivariant non-isomorphic surjective endomorphisms $f\colon X \to X$ and $h\colon T \to T$ such that $\deg f = (\deg h)^2$, $\pi \circ f = h \circ \pi$, and $f^*\Theta = (\deg h)\Theta$. Furthermore, one can require*

(1) $S_h = 0$

*for the endomorphism $h$, and require each of the following conditions for $f$:*

(2) *there is another $G$-invariant section $\Theta'$ such that $f^*\Theta' = (\deg h)\Theta'$, $S_f = \pi^* S_h + \Theta + \Theta'$, and $\Gamma \neq f^{-1}f(\Gamma)$ for any prime divisor $\Gamma$ dominating $T$ except $\Theta$ and $\Theta'$;*

(3) *$S_f = \pi^* S_h + \Theta$, and $\Gamma \neq f^{-1}f(\Gamma)$ for any prime divisor $\Gamma$ dominating $T$ except $\Theta$.*

*Proof.* The negative section $\Theta$ is $G$-invariant, since it is a unique negative curve on $X$. On the other hand, $\pi$ is a unique surjective morphism onto a curve up to isomorphism. Thus, $G$ acts on $T$ so that $\pi$ is $G$-equivariant. By Lemma 2.14, there is a $G$-invariant section $\Theta'$ of $\pi$ such that $\Theta \cap \Theta' = \emptyset$. Then $\mathcal{O}_X(\Theta' - \Theta) \simeq \pi^*\mathcal{O}_T(e)$, and $\mathcal{O}_T(e)$ has an induced $G$-linearization. By Lemma 2.2, there is a $G$-equivariant non-isomorphic surjective endomorphism $h\colon T \to T$ such that $m := \deg h$ is coprime to the order of $G$. By replacing $h$ with a power $h^k$ and by Lemma 2.13, we may assume that $h^*\mathcal{O}(e) \simeq \mathcal{O}(e)^{\otimes m}$ as a $G$-linearized $\mathcal{O}_T$-module. Then we have a $G$-equivariant non-isomorphic surjective endomorphism $f$ of $X$ such that $\deg f = m \deg h = m^2$, $\pi \circ f = h \circ \pi$, and $f^*\Theta = m\Theta$ by Proposition 2.15. Moreover, we can require the condition (2) for $f$ by Proposition 2.15(2). By replacing $h$ with a power $h^k$ and by Lemma 2.12(3), we may assume that $m > 2$ and $H^0(T, \mathcal{O}(e)^{\otimes j})^G \neq 0$ for some $1 \leq j < m$. Then we can require (3) for $f$ by Proposition 2.15(3).

It remains to show (1) by replacing $h$. Note that $S_h = S_{h^k}$ for any $k > 0$ and that the equality $K_T + S_h = h^*(K_T + S_h) + \Delta_h$ holds for an effective divisor $\Delta_h$ (cf. [24, Lem. 2.17]). Then $\deg(K_T + S_h) \leq 0$, i.e., $\deg S_h \leq 2$. If $\deg S_h = 2$, then $h$ is isomorphic to the standard cyclic cover $\phi_m\colon (\mathtt{x} : \mathtt{y}) \mapsto (\mathtt{x}^m : \mathtt{y}^m)$ for $m = \deg h$ for a homogeneous coordinate $(\mathtt{x} : \mathtt{y})$ of $\mathbb{P}^1 = T$. Let $\bar{h}\colon \overline{T} \to \overline{T}$ be the induced endomorphism of the quotient curve $\overline{T} = G\backslash T$, where $\vartheta \circ h = \bar{h} \circ \vartheta$ for the quotient morphism $\vartheta\colon T \to \overline{T}$. Then $S_h = \vartheta^{-1}S_{\bar{h}}$ by [24, Lem. 2.19]. Moreover, $\deg S_{\bar{h}} \leq 2$ by the same reason above. Let $\overline{G}$ be the image of $G$ in $\mathrm{Aut}(T)$, which is the Galois group of $\vartheta$. Note that if $G$ fixes a point in $T$, then it is a cyclic group, and $\vartheta$ is isomorphic to $\phi_d$ for $d = \deg \vartheta$. If $\deg S_{\bar{h}} = 2$, then $\vartheta^{-1}(P)$ consists of one point for any $P \in S_{\bar{h}}$, and hence, $\overline{G}$ is a cyclic group. If $\deg S_{\bar{h}} = 1$, then $\deg S_h = 2$, and $\overline{G}$ is a dihedral group by the well-known classification of finite subgroups of $\mathrm{Aut}(\mathbb{P}^1)$. If $S_{\bar{h}} = 0$, then $S_h = 0$ and we have nothing to prove. Consequently, (1) holds for the original $h$ when $\overline{G}$ is not a cyclic group nor a dihedral group.

For an integer $n \geq 2$ and for $\zeta := \exp(2\pi\sqrt{-1}/n)$, let $C_n$ be the cyclic subgroup of $\mathrm{Aut}(T)$ generated by the automorphism $(\mathtt{x} : \mathtt{y}) \mapsto (\zeta\mathtt{x} : \mathtt{y})$ for a homogeneous coordinate $(\mathtt{x} : \mathtt{y})$ of $\mathbb{P}^1$. Let $D_n$ be the subgroup of $\mathrm{Aut}(T)$ generated by $C_n$ and the involution $(\mathtt{x} : \mathtt{y}) \mapsto (\mathtt{y} : \mathtt{x})$. We may assume that $\overline{G} = C_n$ or $D_n$. Let us take a complex number $c \notin \{0, 1, -1\}$ and a positive integer $r$ such that $rn > 2$ and $rn$ is divisible by the order of $G$. Then we can define a $\overline{G}$-equivariant endomorphism $h^\dagger$ of $T$ by

$$h^\dagger\colon (\mathtt{x} : \mathtt{y}) \mapsto (\mathtt{x}(\mathtt{x}^{rn} - c\mathtt{y}^{rn}) : \mathtt{y}(c\mathtt{x}^{rn} - \mathtt{y}^{rn})).$$

Here, $\deg h^\dagger = rn + 1$ is coprime to the order of $G$. We can show that $h^{\dagger -1}(P)$ consists of at least two points for any $P \in \mathbb{P}^1$. In fact, if $h^{\dagger -1}(P) = \{Q\}$ for some

$Q \in T$, then $P = (a : 1)$ and $Q = (b : 1)$ for some $a$, $b \in \mathbb{C} \setminus \{0\}$ with $cb^{rn} \neq 1$, and

$$(\mathtt{x} - b\mathtt{y})^{rn+1} = \mathtt{x}(\mathtt{x}^{rn} - c\mathtt{y}^{rn}) - a\mathtt{y}(c\mathtt{x}^{rn} - \mathtt{y}^{rn}).$$

By comparing coefficients of the monomial $\mathtt{x}^{rn-1}\mathtt{y}^2$, we have $\binom{rn+1}{2}b^2 = 0$, since $rn > 2$: This contradicts $b \neq 0$. Note that a point $P \in T$ is contained in $S_h$ if and only if $(h^l)^{-1}(P) = \{P\}$ for some $l > 0$ (cf. [24, Def. 2.16]). Therefore, $S_{h^\dagger} = 0$. For (1), it is enough to take $h$ as a suitable power of $h^\dagger$ as in the argument above. Thus, we are done. □

*Example* 2.19. As an application of Corollary 2.18, we shall construct some special endomorphisms of the quotient surface $\overline{X} = G \backslash X$ of the Hirzebruch surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(2))$ by a suitable action of a non-commutative polyhedral group $G$ on $X$. We consider $G$ as a finite subgroup of $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$ and set $\widetilde{G}$ to be the inverse image by $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PGL}(2, \mathbb{C})$. By a homogeneous coordinate $(\mathtt{x} : \mathtt{y} : \mathtt{z})$ of $\mathbb{P}^2$, we consider the following action of the binary polyhedral group $\widetilde{G}$ on $\mathbb{P}^2$:

$$\sigma_{\mathbb{P}^2}(\mathtt{x} : \mathtt{y} : \mathtt{z}) := (a\mathtt{x} + b\mathtt{y} : c\mathtt{x} + d\mathtt{y} : \mathtt{z}), \qquad \text{where} \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{G} \subset \mathrm{SL}(2, \mathbb{C}).$$

We set $U := \{\mathtt{z} \neq 0\} \simeq \mathbb{C}^2$. It is well known that $\widetilde{G}$ acts on $U \setminus \{(0 : 0 : 1)\}$ freely, and the image of $(0 : 0 : 1)$ by the quotient morphism $U \to \widetilde{G} \backslash U$ is a rational double point of type $\mathsf{D}_n$ ($n \geq 3$), $\mathsf{E}_6$, $\mathsf{E}_7$, or $\mathsf{E}_8$, depending on $G$. The kernel of $\widetilde{G} \to G$ is generated by the minus $-I_2$ of the unit matrix $I_2 \in \mathrm{SL}(2, \mathbb{C})$. The quotient variety of $\mathbb{P}^2$ by the kernel is just the weighted projective space $\mathbb{P}(1, 1, 2)$, and the quotient morphism is given by $(\mathtt{x} : \mathtt{y} : \mathtt{z}) \mapsto (\mathtt{x} : \mathtt{y} : \mathtt{z}^2)$ for a weighted homogeneous coordinate $(\mathtt{x} : \mathtt{y} : \mathtt{w})$ of degree $(1, 1, 2)$. Thus, we have an induced action of $G$ on $\mathbb{P}(1, 1, 2)$, which lifts to an action on $X$ by the minimal resolution $\mu \colon X \to \mathbb{P}(1, 1, 2)$ of singularities. Here, the following hold:

(1) The $\mathbb{P}^1$-bundle structure $\pi \colon X \to T = \mathbb{P}^1$ is $G$-equivariant, where the action of $G$ on $T$ is given by $(\mathtt{x} : \mathtt{y}) \mapsto (a\mathtt{x} + b\mathtt{y} : c\mathtt{x} + d\mathtt{y})$ for $\sigma \in \widetilde{G}$ above.
(2) The inverse image $\Theta$ of $\{\mathtt{w} = 0\} \subset \mathbb{P}(1, 1, 2)$ by $\mu$ and the unique $(-2)$-curve $\Gamma$ on $X$ are mutually disjoint $G$-invariant sections of $\pi$.
(3) The action of $G$ on $X \setminus (\Gamma \cup \Theta)$ is fixed point free.

We set $\overline{X} := G \backslash X$ and $\overline{T} := G \backslash T$ as quotient varieties and set $\bar{\pi} \colon \overline{X} \to \overline{T}$ to be the morphism induced by $\pi$. Then we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\xi} & \overline{X} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \bar{\pi}} \\ T & \xrightarrow{\tau} & \overline{T}, \end{array}$$

where $\xi$ and $\tau$ are quotient morphisms. Since $\deg \xi = \deg \tau = \# G$, the normalization of $\overline{X} \times_{\overline{T}} T$ is isomorphic to $X$. In particular, $\bar{\pi}$ is a $\mathbb{P}^1$-fibration. The morphism $\xi$ is étale in codimension 1 by (3), since the actions of $G$ on $\Gamma$ and $\Theta$ of $\pi$ are faithful. It is well known that $\tau$ has just three branched points $P_1$, $P_2$, $P_3 \in \overline{T}$. For $1 \leq i \leq 3$, let $m_i$ be the ramification index of $\tau$ at a point over $P_i$, i.e., $\tau^*(P_i) = m_i \tau^{-1}(P_i)$. Since $\xi$ is étale in codimension 1, the $\mathbb{P}^1$-fibration $\bar{\pi}$ is smooth over $\overline{T} \setminus \{P_1, P_2, P_3\}$, and $\bar{\pi}^*(P_i) = m_i \overline{F}_i$ for a prime divisor $\overline{F}_i \simeq \mathbb{P}^1$ for any $1 \le i \le 3$ (cf. [24, Lem. 4.2]). As a consequence, $\overline{X}$ is not a toric surface. In fact, if it is toric, then $\bar{\pi}$ is a toric morphism, since $\boldsymbol{\rho}(\overline{X}) = 2$, but now we have at least three non-smooth fibers of $\bar{\pi}$. The images $\overline{\Gamma} = \xi(\Gamma)$ and $\overline{\Theta} = \xi(\Theta)$ are mutually disjoint sections of $\bar{\pi}$, $\Gamma = \xi^*\overline{\Gamma}$ and $\Theta = \xi^*\overline{\Theta}$, and $-(K_{\overline{X}} + \overline{\Gamma} + \overline{\Theta})$ is nef, since

$$-2F \sim K_X + \Gamma + \Theta = \xi^*(K_{\overline{X}} + \overline{\Gamma} + \overline{\Theta})$$

for a fiber $F$ of $\pi$.

In this situation, by Corollary 2.18, we have two $G$-equivariant non-isomorphic surjective endomorphisms $f$ and $g$ of $X$ such that $S_f = \Gamma + \Theta$ and $S_g = \Gamma$. Therefore, $f$ and $g$, respectively, induce non-isomorphic surjective endomorphisms $\bar{f}$ and $\bar{g}$ of $\overline{X}$ such that $S_{\bar{f}} = \overline{\Gamma} + \overline{\Theta}$ and $S_{\bar{g}} = \overline{\Gamma}$ (cf. [24, Lem. 2.19(3)]). Here, $-(K_{\overline{X}} + S_{\bar{f}})$ is nef and not numerically trivial, and $-(K_{\overline{X}} + S_{\bar{g}})$ is nef and big. Hence, $(\overline{X}, S_{\bar{f}}, \bar{f})$ and $(\overline{X}, S_{\bar{g}}, \bar{g})$ are not treated in [24, Thm. A] nor in Section 4.2 below.

*Remark.* If $G$ is a dihedral group $D_n$ with $(m_1, m_2, m_3) = (2, 2, n)$, then $(\overline{X}, \overline{\Gamma} + \overline{\Theta} + \overline{F}_3)$ is a *half-toric* surface (cf. [22, §7]). In fact, we have

$$0 \sim K_X + \Gamma + \Theta + \pi^{-1}(Q) + \pi^{-1}(Q') = \xi^*(K_{\overline{X}} + \overline{\Gamma} + \overline{\Theta} + \overline{F}_3)$$

for $\tau^{-1}(P_3) = \{Q, Q'\}$, and it implies that $(\overline{X}, \overline{\Gamma} + \overline{\Theta} + \overline{F}_3)$ is log-canonical (cf. [23, Prop. 2.12(1)]) and that $K_{\overline{X}} + \overline{\Gamma} + \overline{\Theta} + \overline{F}_3 \approx 0$. Since $\boldsymbol{\rho}(\overline{X}) = 2$ and since $\overline{\Gamma} + \overline{F}_3 + \overline{\Theta}$ is a linear chain of rational curves, the pair $(\overline{X}, \overline{\Gamma} + \overline{\Theta} + \overline{F}_3)$ is a half-toric surface by [22, Thm. 1.3].

2.3. **Equivariant endomorphisms of $\mathbb{P}^1$-bundles and projective cones over elliptic curves.** Any $\mathbb{P}^1$-bundle over an elliptic curve admits a non-isomorphic surjective endomorphism (cf. [20, Prop. 5]). We shall prove the following equivariant version:

**Theorem 2.20.** *Let $X$ be a $\mathbb{P}^1$-bundle over an elliptic curve and let $G$ be a finite group acting on $X$. Then there is a $G$-equivariant non-isomorphic surjective endomorphism of $X$. As a consequence, the quotient surface $G\backslash X$ admits a non-isomorphic surjective endomorphism.*

The $\mathbb{P}^1$-bundle $X$ is associated with a locally free sheaf $\mathcal{E}$ of rank 2 on the elliptic curve. Here, $\mathcal{E}$ is decomposable or semi-stable (cf. Fact 2.23 below). For constructing a $G$-equivariant endomorphism of $X$, Proposition 2.25 below treats the case where $\mathcal{E}$ is an indecomposable semi-stable sheaf of degree 0, and Corollary 2.27 below treats some cases where $\mathcal{E}$ is decomposable. The proof of Corollary 2.27 uses Propositions 2.15(2) and 2.16(2) in Section 2.2 and Lemma 2.26 below on equivariant endomorphisms of an elliptic curve. We have also the following by applying Theorem 2.20:

**Theorem 2.21.** *Let $X$ be a* projective cone *over an elliptic curve in the sense of* [24, Def. 1.16] *and let $G$ be a finite group acting on $X$. Then there is a $G$-equivariant non-isomorphic surjective endomorphism of $X$. As a consequence, the quotient surface $G\backslash X$ admits a non-isomorphic surjective endomorphism. If $G$*

*preserves a cross section $C$ of $X$, then $C$ is completely invariant under some $G$-equivariant non-isomorphic surjective endomorphism of $X$.*

Proofs of Theorems 2.20 and 2.21 are given at the end of Section 2.3.

*Remark* 2.22. In the situation of Theorem 2.20, the action of $G$ on $X$ descends to $T$ so that $\pi$ is $G$-equivariant, since $\pi$ is the Albanese morphism.

We begin with recalling the following fact on $\mathbb{P}^1$-bundles over an elliptic curve:

*Fact* 2.23. Let $\pi\colon X \to T$ be a $\mathbb{P}^1$-bundle over an elliptic curve $T$. Then $X \simeq \mathbb{P}_T(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$, and we may assume that $\mathcal{E}$ satisfies one of the following conditions (cf. [4], [12, V, Thm. 2.15]):

(A) There is an isomorphism $\mathcal{E} \simeq \mathcal{O}_T \oplus \mathcal{L}$ for an invertible sheaf $\mathcal{L}$.
(B) There is a non-split exact sequence $0 \to \mathcal{O}_T \to \mathcal{E} \to \mathcal{O}_T \to 0$.
(C) There is a non-split exact sequence $0 \to \mathcal{O}_T \to \mathcal{E} \to \mathcal{A} \to 0$ for an invertible sheaf $\mathcal{A}$ of degree 1.

We shall explain known properties of $X$ and $\mathcal{E}$ separately in each case of (A)–(C).

(A): Let $\Gamma_1$ and $\Gamma_2$ be sections of $\pi$ corresponding to projections $\mathcal{O}_T \oplus \mathcal{L} \to \mathcal{L}$ and $\mathcal{O}_T \oplus \mathcal{L} \to \mathcal{O}_T$, respectively. Then $\Gamma_1 \cap \Gamma_2 = \emptyset$, $K_X + \Gamma_1 + \Gamma_2 \sim 0$, $\mathcal{O}_X(\Gamma_1 - \Gamma_2) \simeq \pi^*\mathcal{L}$, and $\Gamma_1^2 = -\Gamma_2^2 = \deg \mathcal{L}$. Here, $\mathcal{E}$ is not stable, and it is semi-stable (resp. unstable) if and only if $\deg \mathcal{L} = 0$ (resp. $\neq 0$). Moreover:

- If $\deg \mathcal{L} > 0$, then $\Gamma_2$ is a unique negative curve on $X$, and any prime divisor on $X$ dominating $T$ except $\Gamma_2$ has positive self-intersection number.
- If $\deg \mathcal{L} = 0$, then $X$ contains no negative curve.
- If $\mathcal{L}$ is a *torsion invertible sheaf*, i.e., a torsion element of $\mathrm{Pic}(T)$, then $\Gamma_1$, $\Gamma_2$, and $-K_X$ are semi-ample with $\kappa(-K_X, X) = \kappa(\Gamma_1, X) = \kappa(\Gamma_2, X) = 1$.
- If $\deg \mathcal{L} = 0$ and if $\mathcal{L}$ is not a torsion invertible sheaf, then any prime divisor on $X$ dominating $T$ except $\Gamma_1$ and $\Gamma_2$ has positive self-intersection number, and $\kappa(-K_X, X) = \kappa(\Gamma_1, X) = \kappa(\Gamma_2, X) = 0$.

(B): The locally free sheaf $\mathcal{E}$ in this case is said to be *normalized and indecomposable of degree* 0 (cf. [12, V, Not. 2.8.1]). The section $\Gamma$ of $\pi$ corresponding to the surjection $\mathcal{E} \to \mathcal{O}_T$ is a unique section of self-intersection number 0, and we have $-K_X \sim 2\Gamma$ and $\kappa(-K_X, X) = \kappa(\Gamma, X) = 0$. Moreover, any prime divisor $\Theta$ on $X$ dominating $T$ has positive self-intersection number if $\Theta \neq \Gamma$. In fact, $\Theta \sim d\Gamma + \pi^*L$ for a divisor $L$ on $T$ and for an integer $d > 0$, where $\Theta^2 = 2d \deg L$ and $\Theta\Gamma = \deg L$. If $\Theta^2 = 0$, then $\Theta \cap \Gamma = \emptyset$ and we have $L \sim 0$ by $(\pi^*L)|_\Gamma \sim \Theta|_\Gamma = 0$; thus, $\Theta \in |d\Gamma|$ contradicting $\kappa(\Gamma, X) = 0$. The automorphism group $\mathrm{Aut}_T(\mathcal{E})$ of $\mathcal{E}$ over $T$ is isomorphic to $\mathbb{C}^\star \times \mathbb{C}$ (cf. Remark 2.24 below).

(C): The $\mathbb{P}^1$-bundle $X = \mathbb{P}_T(\mathcal{E})$ is essentially constructed as $\wp_{-1}$ in [3, Thm. 6.1], or as $A_{-1}$ in the proof of [30, Thm. 5] (cf. [27, p. 100]). Furthermore, $X$ is isomorphic to the symmetric product $\mathrm{Sym}^2(T)$ in which the $\mathbb{P}^1$-bundle structure $\pi\colon X \to T$ is induced by the addition morphism $T \times T \ni (x, y) \mapsto x + y \in T$ with respect to a group structure of $T$ (cf. [4, p. 451, §3]). In particular, there is an elliptic fibration $\psi\colon X \to \mathbb{P}^1$ such that $\mathcal{O}_X(-2K_X) \simeq \psi^*\mathcal{O}_{\mathbb{P}^1}(1)$ and that $\psi$ has three singular fibers of type ${}_2\mathrm{I}_0$ in Kodaira's notation (cf. [15]). Moreover, $X \times_T T_{(2)} \simeq \mathbb{P}^1 \times T_{(2)}$ over $T_{(2)}$ for the multiplication map $\mu_{(2)} \colon T_{(2)} = T \to T$ by 2.

*Remark* 2.24. For a normalized and indecomposable locally free sheaf $\mathcal{E}$ of rank 2 and degree 0 on an elliptic curve $T$, let $i \colon \mathcal{O}_T \to \mathcal{E}$ and $p \colon \mathcal{E} \to \mathcal{O}_T$ be the injection and the surjection in the exact sequence in Fact 2.23(B). For $v \in \mathbb{C}$, we set $\rho(v) := \mathrm{id}_{\mathcal{E}} + v(i \circ p)$ as an automorphism of $\mathcal{E}$. Then $\rho \colon v \mapsto \rho(v)$ is regarded as a group homomorphism $\mathbb{C} \to \mathrm{Aut}_T(\mathcal{E})$, i.e., $\rho(v_1 + v_2) = \rho(v_1) \circ \rho(v_2)$ for any $v_1$, $v_2 \in \mathbb{C}$. Moreover, we have a group isomorphism $\mathbb{C}^\star \times \mathbb{C} \to \mathrm{Aut}_T(\mathcal{E})$ by $(u, v) \mapsto u\rho(v)$. This is shown by the indecomposability of $\mathcal{E}$. In fact, for any automorphism $\Phi \in \mathrm{Aut}_T(\mathcal{E})$, there is a constant $u \in \mathbb{C}^\star$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_T \arrow[r, "i"] \arrow[d, "u\,\mathrm{id}_{\mathcal{O}_T}"'] & \mathcal{E} \arrow[r, "p"] \arrow[d, "\Phi"'] & \mathcal{O}_T \arrow[r] \arrow[d, "u\,\mathrm{id}_{\mathcal{O}_T}"] & 0 \\ 0 \arrow[r] & \mathcal{O}_T \arrow[r, "i"] & \mathcal{E} \arrow[r, "p"] & \mathcal{O}_T \arrow[r] & 0 \end{tikzcd}$$

of exact sequences is commutative. The automorphism $\mathbb{P}_T(\rho(v)) \colon \mathbb{P}_T(\mathcal{E}) \to \mathbb{P}_T(\mathcal{E})$ associated with $\rho(v)$ is the identity if and only if $v = 0$.

**Proposition 2.25.** *Let $\pi \colon X \to T$ be a $\mathbb{P}^1$-bundle over an elliptic curve $T$ associated with an indecomposable locally free sheaf of degree 0. Assume that $X$ admits a left action of a finite group $G$. Then*

(1) *the quotient morphism $X \to G\backslash X$ is étale in codimension 1, and*
(2) *$X$ admits a $G$-equivariant non-isomorphic surjective étale endomorphism.*

*Proof.* By Remark 2.22, $\pi$ is $G$-equivariant for an action of $G$ on $T$. For the action $\sigma_T \colon T \to T$ of $\sigma \in G$, the pullback homomorphism $\sigma_T^* \colon H^1(T, \mathcal{O}_T) \to H^1(T, \mathcal{O}_T)$ is the multiplication map by a constant $\alpha_\sigma \in \mathbb{C}^\star$. The correspondence $\sigma \mapsto \alpha_\sigma$ gives rise to a group homomorphism $G \to \mathbb{C}^\star$. In the discussion below, we regard $\mathbb{C}$ as a right $G$-module by setting $\zeta^\sigma = \alpha_\sigma^{-1}\zeta$ for $\zeta \in \mathbb{C}$.

We may assume that $X = \mathbb{P}_T(\mathcal{E})$ for a normalized indecomposable locally free sheaf $\mathcal{E}$ of rank 2 and degree 0 on $T$ (cf. Fact 2.23(B)). We shall construct a $G$-linearization of $\mathcal{E}$ which induces the original action of $G$ on $X$. As in the proof of Lemma 2.12, for $\sigma \in G$, let $\pi_\sigma \colon X_\sigma \to T$ be the base change of $\pi$ by $\sigma_T \colon T \to T$ and let $p_\sigma \colon X_\sigma \to X$ be the base change of $\sigma_T$ by $\pi \colon X \to T$. Then the automorphism $\sigma_X$ is expressed as $p_\sigma \circ e_\sigma$ for an isomorphism $e_\sigma \colon X = \mathbb{P}_T(\mathcal{E}) \to X_\sigma = \mathbb{P}_T(\sigma_T^*\mathcal{E})$ over $T$, and we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "e_\sigma"] \arrow[rr, bend left, "\sigma_X"] \arrow[dr, "\pi"'] & X_\sigma \arrow[r, "p_\sigma"] \arrow[d, "\pi_\sigma"] & X \arrow[d, "\pi"] \\ & T \arrow[r, "\sigma_T"] & T. \end{tikzcd}$$

Since $\sigma_T^*\mathcal{E}$ is also normalized and indecomposable of degree 0, the isomorphism $e_\sigma$ is induced by an isomorphism $\Phi_\sigma \colon \sigma_T^*\mathcal{E} \to \mathcal{E}$ of $\mathcal{O}_T$-modules, i.e., $e_\sigma = \mathbb{P}_T(\Phi_\sigma)$.

Multiplying by a non-zero constant if necessary, we can normalize $\Phi_\sigma$ so that the diagram

$$\text{(II-15)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \sigma_T^*\mathcal{O}_T & \xrightarrow{\sigma_T^*(i)} & \sigma_T^*\mathcal{E} & \xrightarrow{\sigma_T^*(p)} & \sigma_T^*\mathcal{O}_T & \longrightarrow & 0 \\ & & {\scriptstyle \alpha_\sigma^{-1}c_{\sigma_T}}\Big\downarrow\simeq & & {\scriptstyle \Phi_\sigma}\Big\downarrow\simeq & & {\scriptstyle c_{\sigma_T}}\Big\downarrow\simeq & & \\ 0 & \longrightarrow & \mathcal{O}_T & \xrightarrow{i} & \mathcal{E} & \xrightarrow{p} & \mathcal{O}_T & \longrightarrow & 0 \end{array}$$

of exact sequences of $\mathcal{O}_T$-modules is commutative, where $i$ and $p$ are homomorphisms in Remark 2.24, $c_{\sigma_T} \colon \sigma_T^*\mathcal{O}_T \to \mathcal{O}_T$ is the canonical isomorphism associated with the morphism $\sigma_T$ of ringed spaces (cf. Remark 2.8). By the normalization and by Remark 2.24, for any $\sigma$, $\tau \in G$, we have

$$\Phi_{\sigma\tau} = \Phi_\tau \circ \tau_T^*(\Phi_\sigma)$$

modulo the canonical isomorphism $\tau_T^*(\sigma_T^*\mathcal{E}) \simeq (\sigma\tau)_T^*\mathcal{E}$, since $(\sigma\tau)_X = \sigma_X \circ \tau_X$ and since $\{c_{\sigma_T}\}$ is a $G$-linearization of $\mathcal{O}_T$ (cf. Remark 2.8). This means that $\{\Phi_\sigma\}_{\sigma\in G}$ is a $G$-linearization of $\mathcal{E}$. Thus, the original action of $G$ on $X$ is recovered by the $G$-linearization as in Lemma 2.12(1).

We shall show (1): Let $G_1$ be the kernel of the natural homomorphism $G \to \mathrm{Aut}(H_1(T,\mathbb{Z}))$ induced by $(\sigma_T)_* \colon H_1(T,\mathbb{Z}) \to H_1(T,\mathbb{Z})$ for $\sigma \in G$. Then $G_1\backslash T$ is an elliptic curve. By the existence of the $G$-linearization $\{\Phi_\sigma\}$ and by Lemma 2.12(2), $\mathcal{E}$ is isomorphic to the pullback of a locally free sheaf $\mathcal{E}_1$ by the étale quotient morphism $T \to G_1\backslash T$, and the quotient morphism $X \to G_1\backslash X$ is also étale. It is enough to prove that $G_1\backslash X \to G\backslash X$ is étale in codimension 1. Since $\mathcal{E}_1$ is also indecomposable of degree 0, we may assume that $G \to \mathrm{Aut}(H_1(T,\mathbb{Z}))$ is injective. Then the homomorphism $G \to \mathbb{C}^\star$ given by $\sigma \mapsto \alpha_\sigma$ is also injective, and hence, $G$ is a cyclic group. We may assume that $G \neq \{1\}$. Thus, the stabilizer $G_P := \{\sigma \in G \mid \sigma(P) = P\}$ is non-trivial at a point $P \in T$. The action of $\sigma \in G_P$ on the fiber $\pi^{-1}(P) \simeq \mathbb{P}^1$ is expressed as $(\mathtt{x}:\mathtt{y}) \mapsto (\alpha_\sigma(\mathtt{x} + \beta_\sigma\mathtt{y}):\mathtt{y})$ for some $\beta_\sigma \in \mathbb{C}$ by the description of $\rho$ in Remark 2.24. Hence, $G$ acts on $X$ freely outside finitely many points. As a consequence, $X \to G\backslash X$ is étale in codimension 1. Thus, (1) holds.

We shall show (2): We have a $G$-equivariant non-isomorphic surjective étale endomorphism $h \colon T \to T$ by Lemma 2.5 applied to the case where $A = T$ and $B$ is a point. The pullback endomorphism $h^* \colon H^1(T,\mathcal{O}_T) \to H^1(T,\mathcal{O}_T)$ is the multiplication map by a constant $\lambda \in \mathbb{C}^\star$. We can find an isomorphism $\Psi \colon h^*\mathcal{E} \to \mathcal{E}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^*\mathcal{O}_T & \xrightarrow{h^*(i)} & h^*\mathcal{E} & \xrightarrow{h^*(p)} & h^*\mathcal{O}_T & \longrightarrow & 0 \\ & & {\scriptstyle \lambda^{-1}c_h}\Big\downarrow\simeq & & {\scriptstyle \Psi}\Big\downarrow\simeq & & {\scriptstyle c_h}\Big\downarrow\simeq & & \\ 0 & \longrightarrow & \mathcal{O}_T & \xrightarrow{i} & \mathcal{E} & \xrightarrow{p} & \mathcal{O}_T & \longrightarrow & 0 \end{array}$$

of exact sequences is commutative for the canonical isomorphism $c_h \colon h^*\mathcal{O}_T \to \mathcal{O}_T$. Comparing with (II-15), for any $\sigma \in G$, we can find an element $v_\sigma \in \mathbb{C}$ such that

$$\text{(II-16)}\qquad \Psi \circ h^*(\Phi_\sigma) = \rho(v_\sigma) \circ \Phi_\sigma \circ \sigma_T^*(\Psi)$$

modulo the canonical isomorphism $h^*(\sigma_T^*\mathcal{E}) \simeq \sigma_T^*(h^*\mathcal{E})$, where $\rho\colon \mathbb{C} \to \mathrm{Aut}_T(\mathcal{E})$ is the homomorphism in Remark 2.24. Here, we note that

$$\Phi_\sigma \circ \sigma_T^*(\rho(v)) = \rho(\alpha_\sigma^{-1}v) \circ \Phi_\sigma \tag{II-17}$$

and for any $\sigma \in G$ and $v \in \mathbb{C}$. For, the equivalent equality

$$\Phi_\sigma \circ \sigma_T^*(i \circ p) = \alpha_\sigma^{-1}(i \circ p) \circ \Phi_\sigma$$

is verified by (II-15), since $\Phi_\sigma \circ \sigma_T^*(i) = i \circ \alpha_\sigma^{-1} c_{\sigma_T}$ and $p \circ \Phi_\sigma = c_{\sigma_T} \circ \sigma_T^*(p)$. Then

$$\begin{aligned}
&\Psi \circ h^*(\Phi_{\sigma\tau}) = \Psi \circ h^*(\Phi_\tau \circ \tau_T^*(\Phi_\sigma)) = (\Psi \circ h^*(\Phi_\tau)) \circ \tau_T^*(h^*(\Phi_\sigma)) \\
&\quad = \rho(v_\tau) \circ \Phi_\tau \circ \tau_T^*(\Psi \circ h^*\Phi_\sigma) = \rho(v_\tau) \circ \Phi_\tau \circ \tau_T^*(\rho(v_\sigma) \circ \Phi_\sigma \circ \sigma_T^*(\Psi)) \\
&\quad = \rho(v_\tau) \circ \rho(\alpha_\tau^{-1}v_\sigma) \circ \Phi_\tau \circ \tau_T^*(\Phi_\sigma \circ \sigma_T^*(\Psi)) = \rho(v_\tau + \alpha_\tau^{-1}v_\sigma) \circ \Phi_{\sigma\tau} \circ (\sigma\tau)_T^*(\Psi)
\end{aligned}$$

for any $\sigma$, $\tau \in G$ by (II-16) and (II-17). Therefore, $v_{\sigma\tau} = v_\tau + \alpha_\tau^{-1}v_\sigma$, and $\{v_\sigma\}_{\sigma\in G}$ is a 1-cocycle of the right $G$-module $\mathbb{C}$ mentioned above, where $\zeta^\sigma = \alpha_\sigma^{-1}\zeta$ for any $\zeta \in \mathbb{C}$. Since $H^1(G, \mathbb{C}) = 0$, by replacing $\Psi$ with $\rho(v) \circ \Psi$ for suitable $v \in \mathbb{C}$, we may assume that $v_\sigma = 0$ for any $\sigma \in G$.

Let $\pi_h\colon X_h \to T$ be the base change of $\pi$ by $h\colon T \to T$ and let $p_h\colon X_h \to X$ be the base change of $h$ by $\pi$. Then we have an isomorphism $\psi_h := \mathbb{P}(\Psi)\colon X \to X_h$ over $T$ with a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\psi_h"] \arrow[dr, "\pi"'] & X_h \arrow[r, "p_h"] \arrow[d, "\pi_h"] & X \arrow[d, "\pi"] \\
& T \arrow[r, "h"] & T.
\end{tikzcd}$$

We set $f := p_h \circ \psi_h\colon X \to X$ as an étale endomorphism of $X$. Then $\pi \circ f = h \circ \pi$, $\deg f = \deg h$, and $f \circ \sigma_X = \sigma_X \circ f$ for any $\sigma \in G$ by (II-16) with $v_\sigma = 0$. Therefore, (2) holds, and we are done. $\square$

**Lemma 2.26.** *Let $T$ be an elliptic curve with an invertible sheaf $\mathcal{L}$ of degree $\delta \geq 0$. Let $G$ be a finite group acting on $T$ from the left such that $\sigma_T^*\mathcal{L} \simeq \mathcal{L}$ or $\sigma_T^*\mathcal{L} \simeq \mathcal{L}^{\otimes -1}$ for any $\sigma \in G$. Then there exists a $G$-equivariant non-isomorphic surjective endomorphism $h\colon T \to T$ such that $h^*\mathcal{L} \simeq \mathcal{L}^{\otimes m}$ for an integer $m > 1$ and that $m$ and $\deg h$ are coprime to the order of $G$.*

*Proof.* We fix a group structure of the elliptic curve $T$. The holomorphic automorphism group $\mathrm{Aut}(T, 0)$ preserving the origin 0 is nothing but the automorphism group of the complex Lie group $T$. This is a subgroup of $\mathbb{C}^\star \simeq \mathrm{Aut}(H^0(T, \Theta_T))$ for the tangent sheaf $\Theta_T = (\Omega_T^1)^\vee$. The action of $\sigma \in G$ on $T$ is given by

$$T \ni z \mapsto \sigma_T(z) = \alpha_\sigma(z + w_\sigma)$$

for certain $\alpha_\sigma \in \mathrm{Aut}(T, 0) \subset \mathbb{C}^\star$ and $w_\sigma \in T$. Here, $\sigma \mapsto \alpha_\sigma$ gives rise to a group homomorphism $\alpha\colon G \to \mathrm{Aut}(T, 0)$ and $\{w_\sigma\}$ is a 1-cocycle with respect to the right $G$-module structure of $T$ defined by $T \ni z \mapsto z^\sigma = \alpha_\sigma^{-1}z$, i.e., $w_{\sigma\tau} = w_\tau + w_\sigma^\tau$ for $\sigma$, $\tau \in G$. Since $H^1(G, T)$ is a torsion module annihilated by the order $k$ of $G$, by replacing the origin 0, we may assume that $kw_\sigma = 0$.

We define a point $q \in T$ by $\mathcal{L} \simeq \mathcal{O}_T(L)$ for the divisor

$$L = (\delta - 1)[0] + [q],$$

where $[p]$ denotes the prime divisor corresponding to a point $p \in T$. Let $G_0 \subset G$ be the subgroup of elements $\sigma \in G$ such that $\sigma_T^* L \sim L$. Then:

(1) if $G \neq G_0$, then $\delta = \deg L = 0$;
(2) $(1 - \alpha_\sigma)q = \alpha_\sigma \delta w_\sigma$ for any $\sigma \in G_0$;
(3) if $\delta > 0$ and if $q$ is not a torsion point, then the homomorphism $G \to \mathrm{Aut}(T, 0)$ is trivial.

In fact, (1) is trivial, and (2) is deduced from

$$\begin{aligned}
\sigma_T^* L &= (\delta - 1)[\sigma_T^{-1} 0] + [\sigma_T^{-1} q] = (\delta - 1)[-w_\sigma] + [a_\sigma^{-1} q - w_\sigma] \\
&= \delta[0] + \delta([-w_\sigma] - [0]) + [a_\sigma^{-1} q - w_\sigma] - [-w_\sigma] \\
&\sim \delta[0] + \delta([-w_\sigma] - [0]) + [a_\sigma^{-1} q] - [0] \\
&\sim \delta[0] + [-\delta w_\sigma + a_\sigma^{-1} q] - [0] = (\delta - 1)[0] + [-\delta w_\sigma + a_\sigma^{-1} q].
\end{aligned}$$

If $\alpha_\sigma \neq 1$ for some $\sigma \in G_0$, then (2) implies that $q$ is a torsion point as $w_\sigma$ is so. Thus, we have (3) by (1) and (2).

Let $n$ be a positive integer such that $n$ is divisible by the order of $G$ and that $nq = 0$ in case $q$ is a torsion point. We can choose a point $c \in T$ satisfying the following conditions:

- If $\delta > 0$ and if $q$ is not a torsion point, then $(n + 1)\delta c = nq$.
- If $\delta = 0$ or if $q$ is a torsion point, then $c = 0$.

Note that $(1 - a_\sigma)c = 0$ and $nw_\sigma = 0$ for any $\sigma \in G$ by (3) and by the assumption on $w_\sigma$. Let $h \colon T \to T$ be an étale endomorphism defined by

$$h(z) := (n + 1)(z - c)$$

for $z \in T$. Then $\deg h = (n + 1)^2$, which is coprime to the order of $G$, and $h$ is $G$-equivariant by

$$\begin{aligned}
\sigma_T(h(z)) - h(\sigma_T(z)) &= a_\sigma((n + 1)(z - c) + w_\sigma) - (n + 1)(a_\sigma(z + w_\sigma) - c) \\
&= -n a_\sigma w_\sigma + (n + 1)(1 - a_\sigma)c = 0.
\end{aligned}$$

If $\delta = 0$, then $L = [q] - [0]$, and $h^* L \sim (n + 1)L$, since $h^* \colon \mathrm{Pic}^0(T) \to \mathrm{Pic}^0(T)$ is the multiplication map by $n + 1$. Assume that $\delta > 0$. Then $G = G_0$ by (1), and $(n+1)\delta c = nq$ by the choice of $c$. Moreover, $h^* L \sim (n+1)^2 L$. In fact, $L = \delta[0] + L_0$ for $L_0 := [q] - [0]$, and we have $h^* L_0 \sim (n + 1)L_0$ and

$$\begin{aligned}
h^*[0] &= \sum\nolimits_{\beta \in T_{n+1}} [\beta + c] = (n + 1)^2[c] + \sum\nolimits_{\beta \in T_{n+1}} ([\beta + c] - [c]) \\
&\sim (n + 1)^2[0] + (n + 1)^2([c] - [0]) + \sum\nolimits_{\beta \in T_{n+1}} ([\beta] - [0]) \\
&\sim (n + 1)^2[0] + \Big[(n + 1)^2 c + \sum\nolimits_{\beta \in T_{n+1}} \beta\Big] - [0] \\
&\sim (n + 1)^2[0] + [(n + 1)^2 c] - [0],
\end{aligned}$$

where $T_{n+1}$ denotes the group of $(n+1)$-torsion points of $T$. Hence,

$$\begin{aligned} h^*L = \delta h^*[0] + h^*L_0 &\sim (n+1)^2\delta[0] + [(n+1)^2\delta c] - [0] + (n+1)L_0 \\ &= (n+1)^2\delta[0] + [n(n+1)q] - [0] + (n+1)L_0 \\ &\sim (n+1)^2(\delta[0] + L_0) = (n+1)^2 L. \end{aligned}$$

Thus, $h$ satisfies the required condition. □

**Corollary 2.27.** *Let $\pi\colon X \to T$ be a $\mathbb{P}^1$-bundle over an elliptic curve $T$ with two mutually disjoint sections $\Theta_1$ and $\Theta_2$. Let $G$ be a finite group acting on $X$ preserving $\Theta_1 + \Theta_2$. Then there is a $G$-equivariant non-isomorphic surjective endomorphism $f$ of $X$ such that*

(1) *$f^{-1}\Theta_1 = \Theta_1$, $f^{-1}\Theta_2 = \Theta_2$, $S_f = \Theta_1 + \Theta_2$, and*
(2) *$f^{-1}f(\Gamma) \neq \Gamma$ for any prime divisor $\Gamma$ on $X$ dominating $T$ except $\Theta_1$ and $\Theta_2$.*

*If $\Theta_2$ is a negative curve, then there is a $G$-equivariant non-isomorphic surjective endomorphism $g$ of $X$ such that*

(3) *$g^{-1}\Theta_2 = \Theta_2$, $S_g = \Theta_2$, and*
(4) *$g^{-1}g(\Gamma) \neq \Gamma$ for any prime divisor $\Gamma$ on $X$ dominating $T$ except $\Theta_2$.*

*Proof.* By Remark 2.22, $\pi$ is $G$-equivariant for an action of $G$ on $T$. There is a divisor $L$ on $T$ such that $\Theta_1 \sim \Theta_2 + \pi^*L$. Interchanging $\Theta_1$ and $\Theta_2$ if necessary, we may assume that $\deg L \geq 0$. We set $G_0 := \{\sigma \in G \mid \sigma(\Theta_1) = \Theta_1\}$, which is a subgroup of $G$ of index 1 or 2. For $\sigma \in G$, if $\sigma \in G_0$ (resp. $\notin G_0$), then $\sigma^*L \sim L$ (resp. $\sigma^*L \sim -L$). By Lemma 2.26, there is a $G$-equivariant non-isomorphic surjective endomorphism $h\colon T \to T$ such that $h^*L \sim mL$ for some $m > 1$ and that $m$ and $\deg h$ are coprime to the order of $G$. Note that $S_h = 0$, since $h$ is étale. By replacing $h$ with a power $h^k$ and by Lemma 2.13, we have an isomorphism $h^*\mathcal{O}_T(L) \simeq \mathcal{O}_T(mL)$ of $G$-linearized $\mathcal{O}_T$-modules. Hence, the existence of $f$ follows from Proposition 2.15(2) in case $G = G_0$, and from Proposition 2.16(2) in case $G \neq G_0$. Assume that $\Theta_2$ is a negative section. Then $G = G_0$, $\deg L > 0$, $m = \deg h$, and $H^0(T, \mathcal{O}_T(nL))^G \neq 0$ for the order $n$ of $G$, by Lemma 2.12(3). Thus, by replacing $h$ with a power, we may assume that $m > 2$ and $H^0(T, \mathcal{O}_T(jL))^G \neq 0$ for some $1 \leq j < m$. Hence, the existence of $g$ follows from Proposition 2.15(2). □

In Examples 2.28 and 2.29 below, we shall construct some normal projective rational surfaces $\overline{X}$ with $\mathbb{P}^1$-fibrations $\overline{X} \to \overline{T} = \mathbb{P}^1$ and construct some non-isomorphic surjective endomorphisms of $\overline{X}$ by applying Corollary 2.27.

*Example* 2.28. Let $D$ be an effective $\mathbb{Q}$-divisor on the rational curve $\overline{T} = \mathbb{P}^1$ of the form

$$D = \sum\nolimits_{j=1}^{l} (1 - m_j^{-1}) P_j$$

for distinct points $P_1, \ldots, P_l$ and integers $2 \leq m_1 \leq m_2 \leq \ldots \leq m_l$. Assume that $\deg D = 2$. Then $(m_1, m_2, \ldots, m_l)$ is one of $(2,2,2,2)$, $(2,3,6)$, $(2,4,4)$, and $(3,3,3)$, and there is a cyclic cover $\varpi\colon T \to \overline{T}$ from an elliptic curve $T$ such that $\varpi$ is étale over $\overline{T} \setminus \operatorname{Supp} D$ and $K_T = \varpi^*(K_{\overline{T}} + D)$. In other words, $\varpi$ is an *index*

1-*cover* with respect to $K_{\overline{T}} + D \sim_{\mathbb{Q}} 0$ (cf. [23, Def. 4.18(2)]). Let $G$ be the cyclic Galois group of $\varpi$. Then the order $n$ of $G$ equals $\operatorname{lcm}\{m_j\}$. Let $\chi\colon G \to \mathbb{C}^\star$ be an injective group homomorphism. The image of $\chi$ is the group of $n$-th roots of unity. Let $\mathcal{L}$ be the invertible sheaf $\varpi^*\mathcal{O}_{\mathbb{P}^1}(1)$. We can define a $G$-linearization $\{\varepsilon_\sigma\colon \sigma_T^*\mathcal{L} \to \mathcal{L}\}_{\sigma\in G}$ of $\mathcal{L}$ by the composite

$$\sigma_T^*\mathcal{L} = \sigma_T(\varpi^*\mathcal{O}(1)) \xrightarrow{\simeq^\dagger} \varpi^*\mathcal{O}(1) \xrightarrow{\times\chi(\sigma)} \varpi^*\mathcal{O}(1) = \mathcal{L},$$

where $\simeq^\dagger$ is the canonical isomorphism on the composite $\varpi = \varpi \circ \sigma_T$. Then an action of $G$ on $\mathbb{P}^1$-bundle $X = \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ is induced by the $G$-linearized $\mathcal{O}_T$-module $\mathcal{O}_T \oplus \mathcal{L}$, and the structure morphism $\pi\colon X \to T$ is $G$-equivariant (cf. Lemma 2.12(1)). By the $G$-linearization $\{\varepsilon_\sigma\}$, we see that the action of $G$ on $X$ is free outside finitely many points. Thus, the quotient morphism $\xi\colon X \to \overline{X} = G\backslash X$ is étale in codimension 1. Then the induced $\mathbb{P}^1$-fibration $\bar{\pi}\colon \overline{X} \to \overline{T}$ is smooth over $\overline{T} \setminus \operatorname{Supp} D$, and $\bar{\pi}^*P_j = m_j\overline{F}_j$ for a prime divisor $\overline{F}_j \simeq \mathbb{P}^1$ for any $1 \le j \le l$ (cf. [24, Lem. 4.2]). Here, $\overline{X}$ is a rational surface with $\boldsymbol{\rho}(\overline{X}) = 2$, and $-K_{\overline{X}}$ is big, since $-K_X = \xi^*(-K_{\overline{X}})$ is so. Let $\Theta_1$ (resp. $\Theta_2$) be the $G$-invariant section of $\pi$ corresponding to the projection $\mathcal{O}_T \oplus \mathcal{L} \to \mathcal{L}$ (resp. $\mathcal{O}_T \oplus \mathcal{L} \to \mathcal{O}_T$). Then the image $\overline{\Theta}_i := \xi(\Theta_i)$ is a section of $\bar{\pi}$ and $\Theta_i = \xi^*\overline{\Theta}_i$ for any $i = 1$ and 2. In particular, $\overline{\Theta}_2$ is a negative section of $\bar{\pi}$.

By Corollary 2.27, we have two $G$-equivariant non-isomorphic surjective endomorphisms $f$ and $g$ of $X$ satisfying conditions (1)–(4) of Corollary 2.27. Let $\bar{f}$ and $\bar{g}$ be endomorphisms of $\overline{X}$ induced by $f$ and $g$, respectively, i.e., $\xi \circ f = \bar{f} \circ \xi$ and $\xi \circ g = \bar{g} \circ \xi$. Here, $\deg \bar{f} = \deg f > 1$, $\deg \bar{g} = \deg g > 1$, $S_{\bar{f}} = \overline{\Theta}_1 + \overline{\Theta}_2$, and $S_{\bar{g}} = \overline{\Theta}_2$ by [24, Lem. 2.19(3)]. Since $\xi$ is étale in codimension 1, we have

$$K_X + S_f = \xi^*(K_{\overline{X}} + S_{\bar{f}}) \quad \text{and} \quad K_X + S_g = \xi^*(K_{\overline{X}} + S_{\bar{g}}).$$

In particular, $K_{\overline{X}} + S_{\bar{f}} \sim_{\mathbb{Q}} 0$, and $(\overline{X}, S_{\bar{f}}, \bar{f})$ is an example satisfying [24, Thm. A(4)]. On the other hand, $-(K_{\overline{X}} + S_{\bar{g}}) \sim_{\mathbb{Q}} \overline{\Theta}_1$ is nef and big, and the example $(\overline{X}, S_{\bar{g}}, \bar{g})$ is not considered in [24, Thm. A] nor in Section 4.2 below.

*Example* 2.29. Let $\iota$ be an involution of an elliptic curve $T$ with a fixed point and let $P \in T$ be a non-fixed point of $\iota$. The invertible sheaf $\mathcal{L} = \mathcal{O}_T(P - \iota(P))$ on $T$ is of degree 0, and $\mathcal{L}$ is not a torsion invertible sheaf for general $P$. We have an isomorphism $\eta\colon \iota^*\mathcal{L} \to \mathcal{L}^{\otimes -1}$ such that the composite

$$\mathcal{L} \xrightarrow{\simeq^\dagger} \iota^*(\iota^*\mathcal{L}) \xrightarrow{\iota^*\eta} \iota^*\mathcal{L}^{\otimes -1} \xrightarrow{\eta^{\otimes -1}} \mathcal{L}$$

is the identity morphism of $\mathcal{L}$ for the canonical isomorphism $\simeq^\dagger$ on the composite $\iota \circ \iota = \mathrm{id}_T$. Then we have an isomorphism

$$\iota^*(\mathcal{O}_T \oplus \mathcal{L}) \xrightarrow{c_\iota \oplus \eta} \mathcal{O}_T \oplus \mathcal{L}^{\otimes -1} \simeq \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_T} (\mathcal{L} \oplus \mathcal{O}_T)$$

for the canonical isomorphism $c_\iota$ (cf. Remark 2.8), and it defines an involution $\iota_X$ of the $\mathbb{P}^1$-bundle $X = \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ over $T$ as a lift of $\iota$ (cf. the proof of Proposition 2.16). Hence, $G = \mathbb{Z}/2\mathbb{Z}$ acts on $X$ and $T$, and the structure morphism $\pi\colon X \to T$ is $G$-equivariant. Let $\xi\colon X \to \overline{X} = G\backslash X$ and $\tau\colon T \to \overline{T} = G\backslash T$ be quotient morphisms, and let $\bar{\pi}\colon \overline{X} \to \overline{T}$ be the induced $\mathbb{P}^1$-fibration. Since $\iota$ has a fixed point, $\overline{T} \simeq \mathbb{P}^1$

and the double cover $\tau$ has 4 branched points. Let $\Theta_1$ (resp. $\Theta_2$) be the section of $\pi$ corresponding to the projection $\mathcal{O}_T \oplus \mathcal{L} \to \mathcal{L}$ (resp. $\mathcal{O}_T \oplus \mathcal{L} \to \mathcal{O}_T$). Then $\iota_X(\Theta_1) = \Theta_2$ and $\iota_X(\Theta_2) = \Theta_1$. Hence, $\iota_X$ has only finitely many fixed points, $\xi$ is étale in codimension 1, and there are 4 non-reduced fibers of $\bar{\pi}$ over the 4 branched points of the double cover $T \to \overline{T}$ (cf. [24, Lem. 4.2]). The image $\overline{\Theta}$ of $\Theta_1$ by $X \to \overline{X}$ coincides with the image of $\Theta_2$, and $\overline{\Theta}$ is a double section of $\bar{\pi}$. In particular, $K_{\overline{X}} + \overline{\Theta} \sim_{\mathbb{Q}} 0$ by $K_X + \Theta_1 + \Theta_2 \sim 0$. Consequently, $\overline{X}$ is rational and $-K_{\overline{X}}$ is nef with $(-K_{\overline{X}})^2 = 0$ but is not numerically trivial.

By Corollary 2.27, we have a $G$-equivariant non-isomorphic surjective endomorphism $f$ of $X$ satisfying conditions (1) and (2) of Corollary 2.27. Then the endomorphism $\bar{f}$ of $\overline{X}$ induced by $f$ is also non-isomorphic and surjective, and we have $\bar{f}^{-1}\overline{\Theta} = \overline{\Theta}$ and $S_{\bar{f}} = \overline{\Theta}$. In particular, $(\overline{X}, S_{\bar{f}}, \bar{f})$ is an example of [24, Thm. A(4)].

Now, we are ready to prove Theorem 2.20.

*Proof of Theorem* 2.20. By Remark 2.1, it suffices to prove the existence of a $G$-equivariant non-isomorphic surjective endomorphism of $X$. Now, $\pi$ is $G$-equivariant for an action of $G$ on $T$ by Remark 2.22. Let $\mathcal{E}$ be a locally free sheaf of rank 2 on $T$ such that $X \simeq \mathbb{P}_T(\mathcal{E})$. We may assume that one of conditions in Fact 2.23 is satisfied for $\mathcal{E}$. If $\mathcal{E}$ is normalized indecomposable of degree 0, then the assertion holds by Proposition 2.25(2).

Assume that $\mathcal{E}$ is decomposable, i.e., $\mathcal{E} \simeq \mathcal{O}_T \oplus \mathcal{L}$ for an invertible sheaf $\mathcal{L}$. If $\mathcal{L} \simeq \mathcal{O}_T$, then $X \simeq \mathbb{P}^1 \times T$, and the assertion holds by Lemma 2.3. Thus, we may assume that $\mathcal{L} \not\simeq \mathcal{O}_T$. Then $\pi$ has two mutually disjoint sections whose sum is $G$-invariant. In fact, if $\deg \mathcal{L} = 0$, then $\pi$ has exactly two sections of self-intersection number 0, and the sum of them is $G$-invariant. If $\deg \mathcal{L} \neq 0$, then $X$ has a unique negative curve $\Gamma$ as a section of $\pi$, and by Lemma 2.14, we can find another $G$-invariant section $\Theta$ of $\pi$ such that $\Theta \cap \Gamma = \emptyset$. Therefore, $X$ admits a non-isomorphic surjective endomorphism by Corollary 2.27. Thus, the assertion holds when $\mathcal{E}$ is decomposable.

It remains the case where $\mathcal{E}$ is stable, and we may assume that $\mathcal{E}$ is as in Fact 2.23(C). We fix an abelian group structure of $T$. Let $\mu_{(2)} \colon T = T_{(2)} \to T$ be the multiplication map by 2, and let $\pi_{(2)} \colon X_{(2)} := X \times_T T_{(2)} \to T_{(2)}$ be the base change of $\pi$ by $\mu_{(2)}$. Then $X_{(2)} \simeq \mathbb{P}^1 \times T_{(2)}$ over $T_{(2)}$ (cf. Fact 2.23(C)). Let $G_{(2)}$ be the group of pairs $(\alpha, \sigma)$ for $\alpha \in \mathrm{Aut}(T_{(2)})$ and $\sigma \in G$ such that $\mu_{(2)} \circ \alpha = \sigma_T \circ \mu_{(2)}$, i.e., the diagram

$$\begin{array}{ccc} T_{(2)} & \xrightarrow{\mu_{(2)}} & T \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \sigma_T} \\ T_{(2)} & \xrightarrow{\mu_{(2)}} & T \end{array}$$

is commutative. Note that, for any $\beta \in \mathrm{Aut}(T)$, there is an automorphism $\alpha \in \mathrm{Aut}(T_{(2)})$ such that $\mu_{(2)} \circ \alpha = \beta \circ \mu_{(2)}$. In fact, if $\beta$ is given by $T \ni z \mapsto a(z) + b$ for some $a \in \mathrm{Aut}(T, 0)$ and $b \in T$, then the automorphism $\alpha \colon z \mapsto a(z) + c$ satisfies $\mu_{(2)} \circ \alpha = \beta \circ \mu_{(2)}$ when $b = 2c$. Thus, the homomorphism $G_{(2)} \to G$ defined by $(\alpha, \sigma) \mapsto \sigma$ is surjective, and its kernel is identified with the Galois group $\mathrm{Gal}(\mu_{(2)})$

of $\mu_{(2)}$. The group $G_{(2)}$ acts on $T_{(2)}$ by $G_{(2)} \ni (\alpha, \sigma) \mapsto \alpha \in \mathrm{Aut}(T_{(2)})$. We can define an action of $G_{(2)}$ on $X_{(2)} \subset X \times T_{(2)}$ by

$$(\alpha, \sigma)\colon X \times T_{(2)} \ni (x, t) \mapsto (\sigma_X(x), \alpha(t)).$$

In fact, if $\pi(x) = \mu_{(2)}(t)$, then $\pi(\sigma_X(x)) = \sigma_T(\pi(x)) = \sigma_T(\mu_{(2)}(t)) = \mu_{(2)}(\alpha(t))$. Hence, $G_{(2)}$ acts on $X_{(2)}$ and $T_{(2)}$, and $\pi_{(2)}\colon X_{(2)} \to T_{(2)}$ is $G_{(2)}$-equivariant. We have a $G_{(2)}$-equivariant non-isomorphic surjective endomorphism $f_{(2)}$ of $X_{(2)} \simeq \mathbb{P}^1 \times T_{(2)}$ by Lemma 2.3. It induces a $G$-equivariant non-isomorphic surjective endomorphism of $X$, since $X \simeq \mathrm{Gal}(\mu_{(2)}) \backslash X_{(2)}$ and $G \simeq G_{(2)} / \mathrm{Gal}(\mu_{(2)})$. Thus, the assertion holds for any condition of Fact 2.23, and we are done. □

Finally in Section 2.3, we shall prove Theorem 2.21:

*Proof of Theorem* 2.21. Let $\mu\colon M \to X$ be the minimal resolution of singularity. Then $\mu$ is $G$-equivariant for an action of $G$ on $M$. Now $M$ has a structure of a $\mathbb{P}^1$-bundle $\pi\colon M \to T$ over an elliptic curve $T$, in which the $\mu$-exceptional curve $\Gamma$ is a section of $\pi$. By Theorem 2.20, we have a $G$-equivariant non-isomorphic surjective endomorphism $\tilde{f}\colon M \to M$. Here, $\Gamma$ is completely invariant under $\tilde{f}$. Then we have a $G$-equivariant non-isomorphic surjective endomorphism $f\colon X \to X$ such that $\mu \circ \tilde{f} = f \circ \mu$ by [24, Lem. 3.14]. This also induces a non-isomorphic surjective endomorphism of $G \backslash X$ by Remark 2.1.

The last assertion remains to be proved. Let $C$ be a cross section of $X$ preserved by the action of $G$. Then $\mu^*C$ is a $G$-invariant section of $\pi$ and $\Gamma \cap \mu^*C = \emptyset$. By Corollary 2.27, we have a $G$-equivariant non-isomorphic surjective endomorphism $\tilde{f}\colon M \to M$ such that $\tilde{f}^{-1}(\mu^*C) = \mu^*C$. Hence, it descends to a $G$-equivariant non-isomorphic surjective endomorphism $f$ of $X$, and $C$ is completely invariant under $f$. Thus, we are done. □

2.4. **Examples of endomorphisms related to** [24]**.** We shall give examples of the following normal surfaces $X$ with non-isomorphic surjective endomorphisms which are related to results in [24]:

(I) A normal compact complex analytic surface $X$ having cusp singularities. This $X$ is not a Moishezon surface by [24, Cor. B and Thm. 6.1]. We shall give an example in Example 2.30 with a remark.

(II) A rational surface $X$ with a non-zero completely invariant divisor $S$ under the endomorphism such that $K_X + S \sim 0$. For the surface $X$, we have a finite cover $\nu\colon V \to X$ étale in codimension 1 satisfying conditions in [24, Lem. 5.6]. In Example 2.31 below, we shall give two examples of $X$ with finite covers $V \to X$.

(III) A rational surface $X$ having a finite cover $V \to X$ étale in codimension 1 from an abelian surface $V$ (cf. [24, Thm. A(3)]). We shall give an example of $X$ with $\boldsymbol{\rho}(X) = 4$ in Example 2.32 below.

(IV) A rational surface $X$ having a finite cover $V \to X$ étale in codimension 1 from the product $V = C \times T$ for an elliptic curve $C$ and a curve $T$ of genus $\geq 2$ (cf. [24, Thm. A(2)]). We shall give an example in Example 2.33 below.

*Example* 2.30. We shall give an example of (I). Let $M$ be a *hyperbolic Inoue surface* (cf. [14], [25], [18], [19]). This is a non-singular compact complex surface of algebraic dimension 0 without $(-1)$-curves and with $\dim H_1(M, \mathbb{C}) = 1$ such that the set of prime divisors on $M$ forms a disjoint union $D$ of two *cyclic chains of rational curves* (cf. [22, Def. 4.3]). Here, $K_M + D \sim 0$, and $D$ can be contracted to two points by a bimeromorphic morphism $\mu\colon M \to X$ to a normal surface $X$. Then $K_X \sim 0$, and $X$ has two cusp singularities. There is a non-isomorphic surjective endomorphism $f_M\colon M \to M$ by [7, Prop. 9.2]. Since $f_M^{-1}(D) = D$, it induces a non-isomorphic surjective endomorphism $f\colon X \to X$ such that $\mu \circ f_M = f \circ \mu$.

*Remark.* In [6, Prop. 2.2], Favre has constructed a remarkable example of an endomorphism of $X$ for a certain hyperbolic Inoue surface $M$, in which the endomorphism of $X$ does not lift to a holomorphic endomorphism of any non-singular model of $X$.

*Example* 2.31. We shall give examples of (II). More precisely, for a rational (resp. elliptic) curve $T$, we shall construct a normal projective rational surface $X$ with a non-isomorphic surjective endomorphism $f$, a non-zero $f$-completely invariant divisor $S$, and a finite Galois cover $\nu\colon V \to X$ from $V = \mathbb{P}^1 \times T$ satisfying the following conditions (cf. [24, Lem. 5.6]):

(1) $\nu$ is étale in codimension 1;
(2) $K_X + S \sim 0$ and $S \subset X_{\mathrm{reg}}$;
(3) the Euler number $\boldsymbol{e}(V \setminus \nu^{-1}S) = 0$;
(4) $f^k \circ \nu\colon V \to X$ is a Galois closure of $f^k\colon X \to X$ for $k \gg 0$.

First, we consider the case: $T = \mathbb{P}^1$. For a homogeneous coordinate $(\mathtt{x}:\mathtt{y})$ of $\mathbb{P}^1$, let $\iota$ be an involution of $V = \mathbb{P}^1 \times \mathbb{P}^1$ defined by $((\mathtt{x}:\mathtt{y}), (\mathtt{u}:\mathtt{v})) \mapsto ((\mathtt{y}:\mathtt{x}), (\mathtt{v}:\mathtt{u}))$, and set $X$ to be the quotient surface of $V$ by $\iota$. Then the quotient morphism $\nu\colon V \to X$ is étale outside the fixed point locus of $\iota$, which consists of four points $((1:(-1)^i), (1:(-1)^j))$ for $0 \le i, j \le 1$. For odd integers $a > 1$ and $b > 1$, let $f'$ be an endomorphism of $V$ defined by

$$((\mathtt{x}:\mathtt{y}), (\mathtt{u}:\mathtt{v})) \mapsto ((\mathtt{x}^a:\mathtt{y}^a), (\mathtt{u}^b:\mathtt{v}^b)).$$

Then $\iota \circ f' = f' \circ \iota$, and $f'$ induces an endomorphism $f$ of $X$ of degree $ab \ge 2$. By construction, $f \circ \nu = \nu \circ f'\colon V \to X$ is a Galois cover and its Galois group is isomorphic to the semi-direct product $\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z})$. Since $a$ and $b$ are odd, $f \circ \nu$ is a Galois closure of $f$. By replacing $(a, b)$ with $(a^k, b^k)$, we see that the composite $f^k \circ \nu\colon V \to X$ is the Galois closure of $f^k$ for any $k \ge 1$. For prime divisors

$$C_{1,0} = p_1^*(1:0), \quad C_{0,1} = p_1^*(0:1), \quad D_{1,0} = p_2^*(1:0), \quad D_{0,1} = p_2^*(0:1)$$

on $V$, where $p_i$ is the $i$-th projection $V \to \mathbb{P}^1$ for $i = 1, 2$, we set

$$C := \nu(C_{1,0}) = \nu(C_{0,1}), \quad D := \nu(D_{1,0}) = \nu(D_{0,1}), \quad \text{and} \quad S := C + D.$$

Then $C_{1,0} + C_{0,1} + D_{1,0} + D_{0,1} = \nu^{-1}S$ and $V \setminus \nu^{-1}S \simeq \mathbb{C}^\star \times \mathbb{C}^\star$. In particular, $\boldsymbol{e}(V \setminus \nu^{-1}S) = 0$. Moreover, any fixed point of $\iota$ is not contained in $\nu^{-1}S$. Thus,

$S \subset X_{\mathrm{reg}}$. We have $K_V + \nu^{-1}S = \nu^*(K_X + S) \sim 0$, and furthermore, $K_X + S \sim 0$ by

$$\iota^*(s^{-1}ds \wedge t^{-1}dt) = s^{-1}ds \wedge t^{-1}dt$$

for $s = \mathtt{x}/\mathtt{y}$ and $t = \mathtt{u}/\mathtt{v}$. Since $f'^*(C_{i,j}) = aC_{i,j}$ and $f'^*D_{i,j} = bD_{i,j}$ for any $(i,j) \in \{(1,0),(0,1)\}$, $\nu^{-1}S$ is completely invariant under $f'$, and $S$ is completely invariant under $f$. Thus, $(X, f, S, \nu)$ satisfies the required condition.

Second, we consider the case where $T$ is an elliptic curve. For a group structure of $T$, let $\mu_{(m)} \colon T \to T$ stand for the multiplication map by $m \in \mathbb{Z}$. Let $\theta \colon \mathbb{P}^1 \to \mathbb{P}^1$ be an involution defined by $(\mathtt{x}:\mathtt{y}) \mapsto (\mathtt{y}:\mathtt{x})$ for a homogeneous coordinate $(\mathtt{x}:\mathtt{y})$ of $\mathbb{P}^1$. Let $X$ be the quotient surface of $V$ by the involution $\iota = \theta \times \mu_{(-1)}$ of $V = \mathbb{P}^1 \times T$. Then the quotient morphism $\nu \colon V \to X$ is étale in codimension 1 and is étale along the disjoint union $S_V$ of $p_1^*(1:0)$ and $p_1^*(0:1)$ for the first projection $p_1 \colon V \to \mathbb{P}^1$. In particular, the image $S = \nu(S_V)$ is an elliptic curve isomorphic to $T$, and we have $S \subset X_{\mathrm{reg}}$, $\nu^*(S) = S_V$, and $\nu^*(K_X + S) = K_V + S_V \sim 0$. Moreover, $\boldsymbol{e}(V \setminus S_V) = 0$ by $V \setminus S_V \simeq \mathbb{C}^\star \times T$.

We shall show that $X$ is rational and $K_X + S \sim 0$. The vector space $H^0(V, \Omega^1_{V/\mathbb{C}})$ of global holomorphic 1-forms on $V$ is 1-dimensional and is generated by $p_2^*(d\zeta)$ for a non-zero global holomorphic 1-form $d\zeta$ on $T$ for the second projection $p_2 \colon V \to T$. Since $(\mu_{(-1)})^*(d\zeta) = -d\zeta$, the irregularity $\boldsymbol{q}(M) = 0$ for the minimal resolution $M$ of $X$. The logarithmic 2-form $\omega := (\mathtt{y}/\mathtt{x})d(\mathtt{x}/\mathtt{y}) \wedge d\zeta$ gives a nowhere vanishing section of $\mathcal{O}_V(K_V + S_V)$ which is invariant under $\iota^*$, i.e., $\iota^*(\omega) = \omega$. Thus, we have $K_X + S \sim 0$. As a consequence, $X$ is rational, since $K_X$ is not pseudo-effective and $\boldsymbol{q}(X) = \boldsymbol{q}(M) = 0$.

Let $\varphi_n \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the endomorphism defined by $(\mathtt{x}:\mathtt{y}) \mapsto (\mathtt{x}^n:\mathtt{y}^n)$ for an integer $n > 0$. We consider the endomorphism $\tilde{f} = \varphi_a \times \mu_{(b)}$ of $V = \mathbb{P}^1 \times T$ for odd integers $a > 1$ and $b > 1$. Then $\iota \circ \tilde{f} = \tilde{f} \circ \iota$ and there is an endomorphism $f$ of $X$ satisfying $\nu \circ \tilde{f} = f \circ \nu$. Here, $\deg f = \deg \tilde{f} = ab^2 > 1$. By definition, $S_V$ is completely invariant under $\tilde{f}$, and hence, $S$ is completely invariant under $f$. By construction, $f \circ \nu = \nu \circ \tilde{f} \colon V \to X$ is a Galois cover and its Galois group is isomorphic to the semi-direct product $\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/a\mathbb{Z} \oplus T_b)$ for the group $T_b$ of $b$-torsion points of $T$, which is isomorphic to $(\mathbb{Z}/b\mathbb{Z})^{\oplus 2}$. Since $a$ and $b$ are odd, $f \circ \nu$ is a Galois closure of $f \colon X \to X$. By replacing $(a, b)$ with $(a^k, b^k)$, we see that $f^k \circ \nu$ is a Galois closure of $f^k$ for any $k > 0$. Thus, $(X, f, S, \nu)$ satisfies the required condition.

*Example* 2.32. We shall give an example of (III). Let $E$ be the elliptic curve $\mathbb{C}/(\mathbb{Z}\sqrt{-1} + \mathbb{Z})$ and set $V = E \times E$. Let $\sigma$ be an automorphism of $V$ of order 4 given by $E \times E \ni (x, y) \mapsto (\sqrt{-1}x, \sqrt{-1}y)$, and set $X$ to be the quotient surface of $V$ by the action of $\sigma$. Then the quotient morphism $\nu \colon V \to X$ is étale in codimension 1, since the action of $\sigma$ is free outside a finite subset of $V$. Thus, $K_X \sim_{\mathbb{Q}} 0$ by $K_V = \nu^* K_X$, but $K_X \not\sim 0$ as $\operatorname{Sing} X$ contains a non-Gorenstein cyclic quotient singularity of order 4. On the other hand,

$$H^0(M, \Omega^1_M) = H^0(E \times E, \Omega^1_{E \times E})^{\langle \sigma \rangle} = 0$$

for the minimal resolution $M$ of singularity of $X$, where the superscript $\langle\sigma\rangle$ stands for the $\sigma$-invariant part. As a consequence, $X$ is rational.

We shall show that the Picard number $\boldsymbol{\rho}(X)$ is equal to 4. The fibers $F_1 = \{0\}\times E$ and $F_2 = E \times \{0\}$ over 0 of the first and second projections $V \to E$, respectively, are $\sigma$-invariant. Similarly, the diagonal locus $\Delta = \{(x, x) \mid x \in E\}$ and the locus $\Sigma = \{(x, \sqrt{-1}x) \mid x \in E\}$ are also $\sigma$-invariant. The numerical classes $\mathrm{cl}(F_1)$, $\mathrm{cl}(F_2)$, $\mathrm{cl}(\Delta)$, and $\mathrm{cl}(\Sigma)$ form a basis of the 4-dimensional vector space $\mathsf{N}(V)$. Therefore, $\boldsymbol{\rho}(X) = 4$.

By Lemma 2.5, there is a $\sigma$-equivariant non-isomorphic surjective endomorphism of $V$: As a simple example, the endomorphism of $V$ defined by $(x, y) \mapsto (mx, my)$ for an integer $m > 1$ is $\sigma$-invariant. It descends to a non-isomorphic surjective endomorphism of $X$, and it gives an example of [24, Thm. A(3)].

*Example* 2.33. We shall give an example of (IV). Let $T$ be the Fermat quartic curve $\{\mathtt{X}^4 + \mathtt{Y}^4 + \mathtt{Z}^4 = 0\} \subset \mathbb{P}^2$, where $(\mathtt{X} : \mathtt{Y} : \mathtt{Z})$ is a homogeneous coordinate. Let $\sigma$ be an automorphism of $T$ defined by $(\mathtt{X}{:}\mathtt{Y}{:}\mathtt{Z}) \mapsto (\mathtt{X}{:}\mathtt{Y}{:}\sqrt{-1}\mathtt{Z})$. Then the projection $\tau\colon T \to \mathbb{P}^1$ defined by $(\mathtt{X}{:}\mathtt{Y}{:}\mathtt{Z}) \mapsto (\mathtt{X}{:}\mathtt{Y})$ is the quotient morphism of $T$ by the action of $\sigma$, and it is a cyclic cover of degree 4. The intersection $T \cap \{\mathtt{Z} = 0\}$ consists of four points $P_1, \ldots, P_4$. Let $Q_i$ be the image $\tau(P_i) \in \mathbb{P}^1$ for $1 \le i \le 4$. Then $\tau^{-1}(Q_i) = \{P_i\}$ for any $1 \le i \le 4$ and $\tau$ is étale over $\mathbb{P}^1 \setminus \{Q_1, \ldots, Q_4\}$. Let $C$ be the elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$ and let $\sigma'$ be an automorphism of $C$ defined by $z \mapsto \sqrt{-1}z$ for $z \in \mathbb{C}$. Then the automorphism $\hat{\sigma} := \sigma' \times \sigma$ of $C \times T$ has order 4 and it acts freely outside a finite set. Thus, the quotient morphism $\nu\colon C \times T \to X$ by $\hat{\sigma}$ is étale in codimension 1 and is a cyclic cover of degree 4. Moreover, the second projection $C \times T \to T$ induces an elliptic fibration $\pi\colon X \to \mathbb{P}^1$ which is smooth over $\mathbb{P}^1 \setminus \{Q_1, \ldots, Q_4\}$. Note that $K_X$ is semi-ample with $\kappa(K_X) = 1$ by $K_{C\times T} = \nu^* K_X$.

Let $\psi\colon Y \to \mathbb{P}^1$ be the relatively minimal elliptic surface birational to $\pi$. Then the singular fiber over $Q_i$ is of type III in Kodaira's notation (cf. [15]). In fact, $\mathtt{Z}/\mathtt{X}$ is a local parameter of $T$ at $P_i$ with $\sigma^*(\mathtt{Z}/\mathtt{X}) = \sqrt{-1}\mathtt{Z}/\mathtt{X}$, and $z \in \mathbb{C}$ gives rise to a local parameter of $C$ with $\sigma'^*(z) = \sqrt{-1}z$. Thus, the singular fiber type is III (cf. [15, §8 (iv) Case $3_1$]). By the canonical bundle formula of elliptic surfaces (cf. [16, Thm. 12], [31, App.]), we have

$$K_Y \sim_{\mathbb{Q}} \psi^*\Big(K_{\mathbb{P}^1} + \sum\nolimits_{i=1}^{4} (1/4)Q_i\Big) \sim_{\mathbb{Q}} \psi^*\mathcal{O}_{\mathbb{P}^1}(-1).$$

In particular, $K_Y$ is not pseudo-effective. On the other hand,

$$H^1(Y, \mathcal{O}_Y) \simeq H^1(C \times T, \mathcal{O}_{C\times T})^{\langle\hat{\sigma}\rangle} = H^1(C, \mathcal{O}_C)^{\langle\sigma'\rangle} \oplus H^1(T, \mathcal{O}_T)^{\langle\sigma\rangle} = 0.$$

Therefore, $Y$ is a rational surface, and as a consequence, $X$ is also rational.

By Lemma 2.5, there is a $\hat{\sigma}$-equivariant non-isomorphic surjective endomorphism of $C \times T$: As a simple example, we have an endomorphism defined by $C \times T \ni (z, t) \mapsto (mz, t)$ for $m > 1$. It descends to a non-isomorphic surjective endomorphism of $X$ and it gives an example $(X, S, f)$ in [24, Thm. (2)] with $S = 0$.

## 3. Generalizations and applications of Theorem A in [24]

We shall present generalizations and applications of [24, Thm. A] using results in Section 2. A complete version of [24, Thm. A] is given as Theorem 3.1 in Section 3.1. Theorem 3.4 below determines the structure of a completely invariant curve with positive arithmetic genus, which is proved in Section 3.2 by applying [24, Thm. A]. Section 3.3 concerns Theorem 3.10 on a normal projective surface $X$ admitting a non-isomorphic surjective morphism $f$ such that the *refined ramification divisor* $\Delta_f$ is zero (cf. [24, Def. 2.16]).

### 3.1. **A complete version of Theorem A in** [24].

**Theorem 3.1.** *Let $X$ be a normal Moishezon surface with a reduced divisor $S$ such that $K_X + S$ is pseudo-effective. Then there exists a non-isomorphic surjective endomorphism $f: X \to X$ satisfying $f^{-1}S = S$ if and only if $X$ is projective and there exists a finite surjective morphism $\nu: V \to X$ étale in codimension* 1 *satisfying one of the conditions* (1)–(6) *below*:

1. *$V = \mathbb{P}^1 \times T$ and $\nu^*(S) = \mathrm{pr}_1^*(P_1 + P_2) + \mathrm{pr}_2^*(D)$ for a non-singular projective curve $T$, two points $P_1$ $P_2 \in \mathbb{P}^1$, and a reduced divisor $D \subset T$ such that* $\deg(K_T + D) > 0$, *where $\mathrm{pr}_i$ denotes the $i$-th projection for $i = 1, 2$;*
2. *$V = C \times T$ and $\nu^*(S) = \mathrm{pr}_2^*(D)$ for an elliptic curve $C$, a non-singular projective curve $T$, and a reduced divisor $D \subset T$ such that* $\deg(K_T + D) > 0$;
3. *$V$ is an abelian surface and $S = 0$;*
4. *$V$ is a $\mathbb{P}^1$-bundle over an elliptic curve and $\nu^*(S)$ is a disjoint union of two sections*;
5. *$V$ is a* projective cone *over an elliptic curve and $\nu^*(S)$ is a* cross section (*cf.* [24, Def. 1.16]);
6. *$V$ is a toric surface with $\nu^*(S)$ as the boundary divisor.*

*Remark* 3.2. We allow 0 as a reduced divisor. The "only if" part has been proved by [24, Thm. A]. We may assume that $X$ is projective by [24, Cor. B]. For the *characteristic completely invariant divisor* $S_f$ of $f$ (cf. [24, Def. 2.16]), we have $S \geq S_f$ by [24, Thm. 2.24]. Moreover, $\nu^* S_f = \mathrm{pr}_1^*(P_1 + P_2)$ in (1), and $S_f = 0$ in (2), by [24, Lem. 5.1].

We shall prove Theorem 3.1 using results in Section 2 and the following:

**Lemma 3.3.** *Let $\nu: V \to X$ be a finite surjective morphism étale in codimension* 1 *satisfying one of the six conditions in Theorem* 3.1*. Then there is a finite Galois cover $\tilde{\nu}: \widetilde{V} \to X$ satisfying the same condition.*

*Proof.* Let $\hat{\nu}: \widehat{V} \to X$ be the Galois closure of $\nu$ and let $\theta: \widehat{V} \to V$ be the induced Galois cover such that $\hat{\nu} = \nu \circ \theta$. Then $\hat{\nu}$ is étale in codimension 1. We shall show that $\hat{\nu}$ satisfies the same condition as $\nu$ except the case (2). Note that $V$ is non-singular and $\theta$ is étale except the cases (5) and (6).

In the case (1) (resp. (4)), the étale cover $\widehat{V}$ of $V$ is obtained as the base change of $V \to T$ by an étale cover $\widehat{T} \to T$; thus, $\hat{\nu}$ satisfies (1) (resp. (4)). In the case (3),

$\widehat{V}$ is also an abelian surface, and hence, $\hat{\nu}$ satisfies (3). For the cases (5) and (6), we need more arguments:

(5): Let $W$ be the minimal resolution of singularities of $V$. Then $W$ is a $\mathbb{P}^1$-bundle over an elliptic curve $T$, and the exceptional divisor $\Theta$ lying over the vertex is a negative section of the $\mathbb{P}^1$-bundle. Hence, $W' := \widehat{V} \times_V W \to W$ is étale over $W \setminus \Theta$. Since $\boldsymbol{\pi}_1(W \setminus \Theta) \simeq \boldsymbol{\pi}_1(T)$, there is a finite étale cover $T' \to T$ such that $(W \setminus \Theta) \times_W W' \simeq (W \setminus \Theta) \times_T T'$ over $W \setminus \Theta$. Therefore, $W \times_T T'$ is the normalization of $W'$, and $\widehat{V}$ is a projective cone over $T'$. The pullback of $\nu^* S$ by $\widehat{V} \to V$ is also a cross section, since it is isomorphic to $S \times_T T'$. Thus, $\hat{\nu} \colon \widehat{V} \to X$ satisfies (5).

(6): The Galois cover $\theta \colon \widehat{V} \to V$ is étale over the open torus $U := V \setminus \nu^{-1} S$, where $V$ is expressed as the toric surface $\mathbb{T}_{\mathsf{N}}(\triangle)$ associated with the abelian group $\mathsf{N} = \boldsymbol{\pi}_1(U) \simeq \mathbb{Z}^{\oplus 2}$ and a fan $\triangle$ of $\mathsf{N}$. For the subgroup $\mathsf{N}' = \boldsymbol{\pi}_1(\theta^{-1} U) \subset \mathsf{N}$, let $p \colon \mathbb{T}_{\mathsf{N}'}(\triangle) \to \mathbb{T}_{\mathsf{N}}(\triangle)$ be the associated morphism of toric surfaces. Then $\theta^{-1}(U) \simeq p^{-1}(U)$ as a complex analytic space over $U$. This extends to an isomorphism $\widehat{V} \simeq \mathbb{T}_{\mathsf{N}'}(\triangle)$ of normal projective surfaces over $V$ by a theorem of Grauert–Remmert (cf. [10], [11, XII, Thm. 5.4]). Thus, $\widehat{V}$ satisfies (6).

Finally, we consider the case (2). Here, we shall find another finite Galois cover étale in codimension 1 satisfying (2). Let $\widehat{V} \to \widehat{T} \to T$ be the Stein factorization of $\mathrm{pr}_2 \circ \theta \colon \widehat{V} \to V = C \times T \to T$. Then induced finite morphisms $\widehat{T} \to T$ and $\widehat{V} \to V \times_T \widehat{T} \simeq C \times \widehat{T}$ are both étale. We may replace $T$ with $\widehat{T}$, since $C \times \widehat{T} \to X$ satisfies (2). Thus, we may assume that $\pi := \mathrm{pr}_2 \circ \theta \colon \widehat{V} \to T$ is a fibration. For the ample divisor $K_T + D$, the linear equivalence class of $\pi^*(K_T + D)$ is preserved by the Galois group $G = \mathrm{Gal}(\hat{\nu})$ of $\hat{\nu}$, since $\nu^*(K_X + S) \sim \mathrm{pr}_2^*(K_T + D)$. Thus, $\pi$ is $G$-equivariant for an action of $G$ on $T$, and we have an induced fibration $X = G \backslash \widehat{V} \to \overline{T} := G \backslash T$. Let $\tau \colon T \to \overline{T}$ be the quotient morphism and let $\overline{V}$ be the normalization of the fiber product $X \times_{\overline{T}} T$. Then we have a commutative diagram

$$\begin{tikzcd}
\widehat{V} \arrow[r, "\theta"] \arrow[rrd, "\pi"'] & V = C \times T \arrow[r, "\lambda"] \arrow[rd, "\mathrm{pr}_2"] \arrow[rr, bend left, "\nu"] & \overline{V} \arrow[r, "\bar{\nu}"] \arrow[d, "\bar{\pi}"] & X \arrow[d] \\
 & & T \arrow[r, "\tau"] & \overline{T},
\end{tikzcd}$$

where the induced finite covers $\lambda$ and $\bar{\nu}$ are also étale in codimension 1 and $\bar{\nu}$ is a Galois cover with the same Galois group as that of $\tau$. Here, $\bar{\pi}^* D = \bar{\nu}^* S$ by $\mathrm{pr}_2^* D = \nu^* S$.

It is enough to prove that $\overline{V} \simeq \overline{C} \times T$ over $T$ for an elliptic curve $\overline{C}$. For, the Galois cover $\overline{C} \times T \simeq \overline{V} \to X$ satisfies (2). Note that $\bar{\pi}$ is an elliptic fibration, since $\lambda$ is étale in codimension 1. Let $T^\star$ be a Zariski-open dense subset of $T$ over which $\bar{\pi}$ is smooth. Then the smooth elliptic fibration $\bar{\pi}|_{T^\star} \colon \bar{\pi}^{-1}(T^\star) \to T^\star$ admits a section, the associated period map is constant, and the associated monodromy transformation is also trivial, since these hold for the trivial elliptic fibration $C \times T^\star \to T^\star$ and since $C \times T^\star$ is étale over $\bar{\pi}^{-1} T^\star$. By [15, Thm. 10.2], $\overline{V}$ is birational over $T$ to the product $\overline{C} \times T$ for an elliptic curve $\overline{C}$. Since $\overline{V}$ has only rational singularities, every rational map from $\overline{V}$ to the elliptic curve $\overline{C}$ is holomorphic. Therefore, we have a birational morphism $\overline{V} \to \overline{C} \times T$ over $T$, and it is an isomorphism, since every fiber of $\bar{\pi}$ is irreducible. Thus, we are done. □

*Proof of Theorem* 3.1. It suffices to prove the "if" part, i.e., the existence of a non-isomorphic surjective endomorphism $f\colon X \to X$ such that $f^{-1}S = S$ in each case of Theorem 3.1. Here, we may assume that $\nu\colon V \to X$ is a Galois cover by Lemma 3.3. We set $G = \mathrm{Gal}(\nu)$. For cases (4), (5), and (6), the existence of $f$ follows from Theorems 2.20 and 2.21, and Lemma 2.6, respectively, on $G$-equivariant endomorphisms. For the case (3), the existence of $f$ follows from Lemma 2.5 applied to the case where $B$ is a point. For (1) and (2), we need more arguments:

(1): The second projection $\mathrm{pr}_2\colon V = \mathbb{P}^1 \times T \to T$ is $G$-equivariant for an action of $G$ on $T$, since $K_T + D$ is ample and $\nu^*(K_X + S) \sim \mathrm{pr}_2^*(K_T + D)$ is preserved by $G$. Here, the action of $G$ on $V$ is diagonal by Lemma 2.3, and we have an action on $\mathbb{P}^1$. Since $\nu^*(S) = \mathrm{pr}_1^*(P_1 + P_2) + \mathrm{pr}_2^*(D)$ is $G$-invariant, the divisor $P_1 + P_2$ on $\mathbb{P}^1$ is also $G$-invariant. We may assume that $P_1 = (1:0)$ and $P_2 = (0:1)$ for a homogeneous coordinate $(\mathtt{x}:\mathtt{y})$ of $\mathbb{P}^1$. Then the image of $G \to \mathrm{Aut}(\mathbb{P}^1)$ is either a cyclic group generated by the automorphism $e(\zeta)\colon (\mathtt{x}:\mathtt{y}) \mapsto (\zeta\mathtt{x}:\mathtt{y})$ for a root $\zeta$ of unity or a dihedral group generated by $e(\zeta)$ above and by the involution $(\mathtt{x}:\mathtt{y}) \mapsto (\mathtt{y}:\mathtt{x})$. Then, for any integer $m > 0$ such that $\zeta^m = 1$, the endomorphism $f_{m+1}\colon (\mathtt{x}:\mathtt{y}) \mapsto (\mathtt{x}^{m+1}:\mathtt{y}^{m+1})$ of $\mathbb{P}^1$ is $G$-equivariant and satisfies $f_{m+1}^{-1}(P_1+P_2) = P_1+P_2$. Let $f$ be the endomorphism of $X = G\backslash V$ induced by the $G$-equivariant endomorphism $f_{m+1} \times \mathrm{id}_T\colon \mathbb{P}^1 \times T \to \mathbb{P}^1 \times T$. Then $\deg f = m+1 > 1$ and $f^{-1}(S) = S$ by construction.

(2): The second projection $\mathrm{pr}_2\colon V = C \times T \to T$ is $G$-equivariant for an action of $G$ on $T$, since $K_T + D$ is ample and $\nu^*(K_X + S) \sim \mathrm{pr}_2^*(K_T + D)$ is preserved by $G$. Then there is a $G$-equivariant non-isomorphic surjective endomorphism $f_V\colon V \to V$ such that $\mathrm{pr}_2 = \mathrm{pr}_2 \circ f_V$, by Lemma 2.5. Here, $f_V^*(\nu^*S) = \nu^*S$ by $\mathrm{pr}_2^* D = \nu^*S$. Hence, $f_V$ induces a non-isomorphic surjective endomorphism $f$ of $X = G\backslash V$ such that $f^{-1}S = S$. Thus, we are done. □

3.2. **Completely invariant curves of positive arithmetic genus.** We shall prove Theorems 3.4 and 3.5 below on normal projective surfaces with curves of positive arithmetic genus which are completely invariant under non-isomorphic surjective endomorphisms.

**Theorem 3.4.** *Let $X$ be a normal projective surface with a non-isomorphic surjective endomorphism $f$ and let $C$ be an $f$-completely invariant curve such that $\boldsymbol{p_a}(C) > 0$. Then one of the following holds*:

(1) *The curve $C$ is non-singular and there is a $\mathbb{P}^1$-fibration $\pi\colon X \to T$ to a non-singular projective curve $T$ such that $\pi(C) = T$ and $\deg C/T \le 2$. Moreover, there is a finite cover $T' \to T$ from a non-singular projective irrational curve $T'$ such that*
   - *the normalization $X'$ of $X \times_T T'$ is isomorphic to $\mathbb{P}^1 \times T'$ over $T'$,*
   - *the induced morphism $\nu\colon X' \to X$ is étale in codimension 1,*
   - *$\nu^*C$ is a fiber or a union of two fibers of the first projection $X' \simeq \mathbb{P}^1 \times T' \to \mathbb{P}^1$.*

(2) *The curve $C$ is an elliptic curve contained in $X_{\mathrm{reg}}$, and $C$ is a set-theoretic fiber of an elliptic fibration $\pi\colon X \to T$ to a non-singular projective curve $T$. Moreover, there exist an elliptic curve $C'$ and a finite cover $T' \to T$ from a non-singular projective curve $T'$ such that*
   - *the normalization $X'$ of $X \times_T T'$ is isomorphic to $C' \times T'$ over $T'$, and*
   - *the induced cover $X' \to X$ is étale in codimension* 1.

(3) *The surface $X$ is a projective cone over an elliptic curve and $C$ is a cross section.*

(4) *The surface $X$ is a $\mathbb{P}^1$-bundle over an elliptic curve and $C$ is section.*

(5) *The surface $X$ is a $\mathbb{P}^1$-bundle over an elliptic curve and $C$ is an elliptic curve such that $C + K_X \sim_{\mathbb{Q}} 0$. In particular, $C$ is a double section with $C^2 = 0$.*

(6) *The surface $X$ is rational, $C$ is an elliptic curve contained in $X_{\mathrm{reg}}$, $K_X + C \sim 0$, and there is a double cover $\nu\colon V \to X$ étale in codimension* 1 *from a $\mathbb{P}^1$-bundle $V$ over an elliptic curve such that $\nu^*C$ is a disjoint union of two sections.*

(7) *The curve $C$ is rational with exactly one node $P$, $C \cap \operatorname{Sing} X \subset \{P\}$, $K_X + C \sim 0$, and there is a finite Galois cover $\nu\colon V \to X$ étale in codimension* 1 *from a toric surface $V$ with $\nu^*C$ as the boundary divisor.*

By the list above, $C$ is singular if and only if (7) holds. We have the following finer description of $(X, C)$ in a special case of (7):

**Theorem 3.5.** *Let $f$ be a non-isomorphic surjective endomorphism of a normal projective surface $X$ and let $C$ be a singular $f$-completely invariant curve. Suppose that* $\operatorname{Sing} C \subset X_{\mathrm{reg}}$. *Then $C$ is a rational curve with exactly one node, $C \subset X_{\mathrm{reg}}$, $X$ is a* log del Pezzo surface (*cf.* [1, Def. 1.1]) *of Picard number* 1, *and there is a finite cyclic cover $\nu\colon V \to X$ étale in codimension* 1 *from a non-singular toric surface $V$ with $\nu^*C$ as the boundary divisor, where $(V, \nu^*C, \deg \nu)$ is one of the following*:

(1) *$V \simeq \mathbb{P}^2$, $\nu^*C$ is a union of three lines without triple points, and* $\deg \nu = 3$;

(2) *$V \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\nu^*C \simeq p_1^*(D_1) + p_2^*(D_2)$ for reduced divisors $D_1$, $D_2 \subset \mathbb{P}^1$ of degree* 2, *where $p_i$ denotes the $i$-the projection for $i = 1$, 2, and* $\deg \nu = 4$;

(3) *$V$ is a del Pezzo surface of degree* 6, *$\nu^*C = \sum_{i=1}^{6} C_i$ for $(-1)$-curves $C_i$, and* $\deg \nu = 6$.

The proof of Theorem 3.4 is divided into three cases where $K_X + C$ is not pseudo-effective, nef but not numerically trivial, and numerically trivial: these are treated in Lemmas 3.7, 3.8, and 3.9, below, respectively. The proofs of Theorems 3.4 and 3.5 are given at the end of Section 3.2. We fix $(X, f, C)$ in Theorem 3.4 throughout Section 3.2. We begin with:

**Lemma 3.6.** *Assume that $H^0(X, \mathcal{O}_X(K_X + C)) = 0$. Then $C$ is non-singular, and there is a fibration $\pi\colon X \to T$ to a non-singular projective irrational curve $T$ such that $\pi(C) = T$, $\boldsymbol{g}(C) = \boldsymbol{g}(T)$, and $\pi \circ f = h \circ \pi$ for an étale endomorphism $h$ of $T$. Moreover, $f^*C = bC$ for the integer $b := \deg f / \deg h$, and the following hold*:

(1) *If* $b = 1$, *then* $C$ *and* $T$ *are elliptic curves,* $f$ *is étale, and* $\pi$ *is a smooth morphism isomorphic to the Albanese morphism of* $X$.
(2) *If* $1 < b < \deg f$, *then* $C$ *and* $T$ *are elliptic curves and* $\pi$ *is a* $\mathbb{P}^1$*-bundle with* $\deg C/T \leq 2$.
(3) *If* $b = \deg f$, *then* $\pi$ *is a* $\mathbb{P}^1$*-fibration with* $\deg C/T \leq 2$, *and there is a finite cover* $T' \to T$ *from a non-singular projective curve* $T'$ *satisfying the same condition as in Theorem* 3.4(1).

*Proof.* We first show that $X$ has only rational singularities. Assume the contrary. Then $X$ is a projective cone over an elliptic curve $E$ by [24, Prop. 6.2]. Now, $X$ has only quotient singularities along $C$, since $(X, C)$ is log-canonical (cf. [24, Thm. E]). Thus, $C \subset X_{\mathrm{reg}}$. Since $\boldsymbol{\rho}(X) = 1$, $f^*C = \delta_f C$ for the positive square root $\delta_f = (\deg f)^{1/2} > 1$. Then $\boldsymbol{p_a}(C) = 1$ by [24, Lem. 3.11], and we have $K_X + C \approxeq 0$ by $(K_X + C)C = 2\boldsymbol{p_a}(C) - 2 = 0$. For the minimal resolution $\mu\colon M \to X$ of singularities, the Albanese morphism of $M$ is a $\mathbb{P}^1$-bundle $\varpi\colon M \to E$, and the $\mu$-exceptional locus $\Theta$ is a section of $\varpi$ (cf. [24, Rem. 1.17]). Here, $K_M + \Theta + \mu^*C \sim \mu^*(K_X + C) \approxeq 0$. Thus, $\mu^*C$ is also a section of $\varpi$, and hence, $C$ is an elliptic curve. Then $C$ is a cross section of the projective cone $X$ and $K_X + C \sim 0$ by [24, Lem. 1.18]. This contradicts $H^0(X, \mathcal{O}_X(K_X + C)) = 0$. Therefore, $X$ has only rational singularities.

By a property of rational singularities, the Albanese morphism of the minimal resolution of singularities of $X$ descends to the Albanese morphism of $X$. Let $\pi\colon X \to T$ be the fibration obtained by the Stein factorization of the Albanese morphism. Then $\pi^*\colon H^1(T, \mathcal{O}_T) \to H^1(X, \mathcal{O}_X)$ is an isomorphism and $\pi^*\colon H^2(T, \mathcal{O}_T) \to H^2(X, \mathcal{O}_X)$ is injective. By our assumption, $H^0(X, \mathcal{O}_X(K_X + C)) \simeq H^2(X, \mathcal{O}_X(-C))^\vee = 0$. Hence, $H^2(X, \mathcal{O}_X) = 0$ and the restriction homomorphism $H^1(X, \mathcal{O}_X) \to H^1(C, \mathcal{O}_C) \neq 0$ is surjective. As a consequence, $\dim T = 1$, and the pullback homomorphism $(\pi|_C)^*\colon H^1(T, \mathcal{O}_T) \to H^1(C, \mathcal{O}_C)$ is an isomorphism. In particular, $\pi(C) = T$. Then $C$ is non-singular by [24, Prop. 3.13], and $\boldsymbol{g}(C) = \boldsymbol{g}(T) = \boldsymbol{q}(X)$. By the universality of the Albanese morphism, there is an étale endomorphism $h$ of $T$ satisfying $\pi \circ f = h \circ \pi$. Then $b := \deg f / \deg h \in \mathbb{Z}$ and $f^*C = bC$ by [24, Props. 2.20(1) and 3.17], since $FC > 0$ for a general fiber $F$ of $\pi$ and since $\pi^*F \approxeq (\deg h)F$.

If $b = 1$, then $C$ and $T$ are elliptic curves, $f$ is étale, and $\pi$ is smooth by [24, Lem. 4.4(3)]. Thus, (1) holds. Assume that $b > 1$. Then $C \subset S_f$, and hence, $\pi$ is a $\mathbb{P}^1$-fibration with $\deg C/T \leq 2$ by [24, Lem. 4.4(5)]. If $1 < b < \deg f$, then $T$ is an elliptic curve and $\pi$ is a $\mathbb{P}^1$-bundle by [24, Cor. 4.7]. This shows (2). If $b = \deg f$, i.e., $\deg h = 1$, then the conclusion of (3) holds by [24, Thm. 4.9]. $\square$

**Lemma 3.7.** *Suppose that* $K_X + C$ *is not nef. Then either* (1) *or* (4) *of Theorem* 3.4 *holds.*

*Proof.* By [24, Thm. 2.24], $K_X + C$ is not pseudo-effective. Hence, we can apply Lemma 3.6. As a consequence, $C$ is non-singular. Let $\pi\colon X \to T$ be the fibration in Lemma 3.6 with the étale endomorphism $h$ of $T$.

There is an extremal ray $\mathsf{R} \subset \overline{\mathrm{NE}}(X)$ with $(K_X + C)\mathsf{R} < 0$ by [24, Thm. 1.9]. We shall show that the contraction morphism of $\mathsf{R}$ is not birational. Assume the contrary, i.e., $\mathrm{cl}(\Gamma) \in \mathsf{R}$ for a negative curve $\Gamma$. Then $\deg f = (\deg h)^2$, $T$ is an elliptic curve, $\pi$ is a $\mathbb{P}^1$-bundle, and $\Gamma$ is the unique negative section by [24, Cor. 4.8]. Since $(K_X+\Gamma)\Gamma = 2\boldsymbol{g}(\Gamma) - 2 = 0$, if $C \neq \Gamma$, then $(K_X+C)\Gamma \geq -\Gamma^2 > 0$ contradicting $(K_X + C)\Gamma < 0$.

Therefore, the contraction morphism of $\mathsf{R}$ is a fibration $\varphi\colon X \to B$ to a non-singular projective curve $B$ by [24, Thm. 1.10], since $\boldsymbol{\rho}(X) \geq \boldsymbol{\rho}(T) + 1 = 2$. Every set-theoretic fiber of $\varphi$ is rational and is contracted to a point by $\pi\colon X \to T$, since $T$ is irrational. Therefore, $\varphi \simeq \pi$. In particular, $(K_X+C)F < 0$ for a general fiber $F$ of $\pi$. This implies that $C$ is a section of $\pi$. By Lemma 3.6, we see that Theorem 3.4(4) holds in case $\deg h > 1$, and Theorem 3.4(1) holds in case $\deg h = 1$. □

**Lemma 3.8.** *Suppose that $K_X + C$ is nef but not numerically trivial. Then $C$ is non-singular. If $S_f \neq 0$, then $S_f = C$, and Theorem* 3.4(1) *holds. If $S_f = 0$, then Theorem* 3.4(2) *holds.*

*Proof.* By [24, Thm. 2.24], $C \geq S_f$, $K_X + C$ is semi-ample, $(K_X + C)^2 = 0$, and $K_X + C = f^*(K_X + C)$. Then we have a fibration $\pi\colon X \to T$ to a non-singular projective curve $T$ such that $K_X + C$ is $\mathbb{Q}$-linearly equivalent to the pullback of an ample $\mathbb{Q}$-divisor on $T$. As in the argument in [24, §5.1], there is an automorphism $h\colon T \to T$ such that $\pi \circ f = h \circ \pi$ by [24, Lem. 3.16]. Hence, $S_f$ is non-singular by [24, Lem. 4.4(1)]. If $C$ is singular, then $f^*C = \delta_f C$ by [24, Prop. 3.13], and $C \subset S_f$; this is a contradiction. Hence, $C$ is non-singular.

Assume that $\pi(C) = T$. Since $\deg h = 1$, $f^*F \approx F$ for a general fiber $F$ of $\pi$, and we have $f^*C = (\deg f)C$ by $FC > 0$ (cf. [24, Prop. 2.20(1)]). Hence, $C = S_f$. By $(K_X + C)F = 0$, we see that $\pi$ is a $\mathbb{P}^1$-fibration and $C$ is a double section of $\pi$. By [24, Thm. 4.9], there exists a finite Galois cover $\tau\colon T' \to T$ such that

- the normalization $X'$ of $X \times_T T'$ is isomorphic to $\mathbb{P}^1 \times T'$ over $T'$,
- the induced morphism $\nu\colon X' \to X$ is étale in codimension 1, and
- $\nu^*C$ is a disjoint union of two fibers of the first projection $X' \simeq \mathbb{P}^1 \times T' \to \mathbb{P}^1$.

Here, $\boldsymbol{g}(T') \geq 2$, since $K_{X'} + \nu^*C = \nu^*(K_X + C)$ is linearly equivalent to the pullback of an ample divisor on $T'$. Thus, Theorem 3.4(1) holds in this case.

Assume next that $\pi(C) \neq T$. Then $\pi$ is an elliptic fibration, since $\deg(K_F) = (K_X + C)F = 0$ for a general fiber $F$ of $\pi$. By [24, Lem. 4.4(2), (5)], $C$ is a set-theoretic fiber of $\pi$, and $S_f = 0$. Since $(X, C)$ is log-canonical (cf. [24, Thm. E]) with $(K_X + C)C = 0$, $C$ is an elliptic curve contained in $X_{\mathrm{reg}}$ by [22, Prop. 3.29]. Therefore, Theorem 3.4(2) holds by [24, Thm. 4.9]. □

**Lemma 3.9.** *Suppose that $K_X + C$ is numerically trivial. Then $\boldsymbol{p_a}(C) = 1$ and $C_{\mathrm{reg}} \subset X_{\mathrm{reg}}$. If $C$ is singular, then Theorem* 3.4(7) *holds. If $C$ is non-singular, then one of* (3), (5), *and* (6) *of Theorem* 3.4 *holds.*

*Proof.* By [24, Thm. 2.24], we have $C \geq S_f$ and $K_X + C \sim_{\mathbb{Q}} 0$. Then $C$ is either an elliptic curve or a nodal rational curve with one node, $C \cap \mathrm{Sing}\, X \subset \mathrm{Sing}\, C$, and

$K_X + C$ is Cartier along $C$ with $\mathcal{O}_X(K_X + C)|_C \simeq \mathcal{O}_C$, by [22, Prop. 3.29] applied to the log-canonical pair $(X, C)$.

Assume first that $C$ is a nodal rational curve. Then $f^*C = \delta_f C$ by [24, Prop. 3.13], and we have $\lambda_f = \delta_f$ by [24, Lem. 3.7 and Thm. 3.22]. Moreover, $K_X + C \sim 0$ by Lemma 3.6. Thus, we have a finite Galois cover $\nu\colon V \to X$ satisfying Theorem 3.4(7) by [24, Lems. 5.4, 5.7, and 5.8].

Thus, it is enough to consider the case where $C$ is an elliptic curve. Assume that $X$ is irrational. If $K_X + C \sim 0$, then Theorem 3.4(3) holds by [24, Lem. 5.4]. On the other hand, $X$ satisfies one of three conditions of [24, Thm. 4.16]. We shall show that, if $K_X + C \not\sim 0$, then $X$ satisfies only [24, Thm. 4.16(2)]: If [24, Thm. 4.16(1)] holds, then there is a finite morphism $\mathbb{P}^1 \times T' \to X$ étale in codimension 1 from a non-singular projective curve $T'$ of genus $> 1$. Here, the inverse image $C'$ of $C$ is étale over $C$ by $C \subset X_{\mathrm{reg}}$, thus, every component of $C'$ is an elliptic curve: this is a contradiction, since an elliptic curve does not dominate $T'$ and is not contained in a fiber of $\mathbb{P}^1 \times T' \to T'$. If [24, Thm. 4.16(3)] holds, then $C$ is a cross section of $X$ by [24, Lem. 1.18], and hence, $K_X + C \sim 0$. Therefore, if $K_X + C \not\sim 0$, then [24, Thm. 4.16(2)] occurs, where $K_X + C \sim \pi^*N$ for a non-zero divisor $N \sim_{\mathbb{Q}} 0$ on $T$ for the $\mathbb{P}^1$-bundle $X \to T$ over an elliptic curve $T$. Here, $C \to T$ is an étale double cover and $C^2 = K_X^2 = 0$. Thus, Theorem 3.4(5) holds when $X$ is irrational and $K_X + C \not\sim 0$.

Assume next that $X$ is rational. Then $H^1(X, \mathcal{O}_X) = 0$, and the sequence

$$H^0(X, \mathcal{O}_X(K_X + C)) \to H^0(C, \mathcal{O}_C) \to H^1(X, \mathcal{O}_X(K_X)) \simeq H^1(X, \mathcal{O}_X)^\vee = 0$$

induced by $\mathcal{O}_X(K_X + C)|_C \simeq \mathcal{O}_C$ is exact. Thus, $K_X + C \sim 0$ by $K_X + C \sim_{\mathbb{Q}} 0$. Let $\nu^\sharp\colon V^\sharp \to X$ be the morphism $\nu = \tau_k\colon V_k \to X$ in [24, Def. 5.5] for $k \gg 0$ which satisfies conditions in [24, Lem. 5.6] for $S = C$. If $V^\sharp$ is rational, then $\nu^{\sharp-1}C$ is a union of rational curves by [24, Lem. 5.7], but $C$ is an elliptic curve; this is a contradiction. Hence, $V^\sharp$ is irrational, and Theorem 3.4(6) holds by [24, Lem. 5.8 and Cor. 5.9]. Thus, we are done. □

Theorem 3.4 is proved by Lemmas 3.7, 3.8, and 3.9, as follows:

*Proof of Theorem* 3.4. If $K_X + C$ is not nef, then either (1) or (4) of Theorem 3.4 holds, by Lemma 3.7. If $K_X + C$ is nef and not numerically trivial, then either (1) or (2) of Theorem 3.4 holds, by Lemma 3.8. If $K_X + C$ is numerically trivial, then one of (3), (5), (6), and (7), of Theorem 3.4 holds, by Lemma 3.9. Thus, we are done. □

Finally in Section 3.2, we shall prove Theorem 3.5:

*Proof of Theorem* 3.5. Now Theorem 3.4(7) holds for $(X, C)$. We have $C \subset X_{\mathrm{reg}}$ by the assumption $\operatorname{Sing} C \subset X_{\mathrm{reg}}$ and by the property $C \cap \operatorname{Sing} X \subset \{P\}$ in Theorem 3.4(7) for the node $P$ of $C$. Hence, the finite Galois cover $\nu\colon V \to X$ in Theorem 3.4(7) is étale along $\nu^{-1}C$. It implies that the toric surface $V$ is non-singular, since the open torus $V \setminus \nu^{-1}C$ is non-singular. Since $C$ is a rational curve with one node $P$, the number of prime components of $\nu^{-1}C$ equals $\deg \nu$, and the

Galois group $G = \mathrm{Gal}(\nu)$ of $\nu$ is a cyclic group, i.e., $\nu$ is a cyclic cover. For the toric surface $V$, $\mathrm{Pic}(V)$ is generated by invertible sheaves $\mathcal{O}_V(\Gamma_i)$ associated with prime components $\Gamma_i$ of the boundary divisor $\nu^{-1}C$. Since $G$ acts transitively on the set of prime components of $\nu^{-1}C$, we see that $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(C) \simeq \mathcal{O}_X(-K_X)$. As a consequence, $X$ is a Gorenstein log del Pezzo surface of Picard number 1, i.e., $X$ has only rational double points as singularities, $-K_X$ is ample, and $\boldsymbol{\rho}(X) = 1$. In particular, $-K_V = \nu^*(-K_X)$ is also ample. So, $V$ is a toric del Pezzo surface. Now,

$$\boldsymbol{e}(V) = \boldsymbol{e}(V \setminus \nu^{-1}C) + \boldsymbol{e}(\nu^{-1}C) = \boldsymbol{e}(\nu^{-1}C) = \deg \nu$$

as $\nu^{-1}C$ is a cyclic chain of rational curves with $\deg \nu$ prime components. We set $b := \Gamma_i^2$ for a prime component $\Gamma_i$ of $\nu^{-1}C$; this is independent of the choice of $\Gamma_i$ as $G$ acts transitively on the set of prime components of $\nu^{-1}C$. Then $(\nu^*C)^2 = (\deg \nu)(b+2)$, and $C^2 = b + 2 > 0$. Since the number $\deg \nu$ of prime components of the boundary divisor $\nu^{-1}C$ is greater than 2 and since

$$12 = K_V^2 + \boldsymbol{e}(V) = (\nu^*C)^2 + \boldsymbol{e}(V) = (\deg \nu)(b+3),$$

the pair $(b, \deg \nu)$ is one of $(1,3)$, $(0,4)$, and $(-1,6)$. Then $(V, \nu^*C, \deg \nu)$ satisfies conditions (1), (2), and (3) of Theorem 3.5, respectively in the cases $(1,3)$, $(0,4)$, and $(-1,6)$. Thus, we are done. $\square$

3.3. **Classification theorem in the case: $\Delta_f = 0$.** We consider normal projective surfaces $X$ admitting non-isomorphic surjective morphisms $f$ such that $\Delta_f = 0$. Theorem 3.10 below classifies such $X$. Note that $\Delta_f = 0$ if and only if $f|_{X \setminus S} \colon X \setminus S \to X \setminus S$ is étale in codimension 1 for an $f$-completely invariant divisor $S$ (cf. [24, Prop. 2.21]). If $K_X + S$ is pseudo-effective for an $f$-completely invariant divisor $S$, then $\Delta_f = 0$ by [24, Thm. 2.24]. Thus, Theorem 3.10 is considered as a partial generalization of Theorem 3.1.

**Theorem 3.10.** *For a normal projective surface $X$, it has a non-isomorphic surjective endomorphism $f$ satisfying $\Delta_f = 0$ if and only if there exists a finite Galois cover $V \to X$ étale in codimension 1 such that $V$ and its Galois group $G = \mathrm{Gal}(V/X)$ satisfy one of the following conditions*:

(1) *$V$ is a toric surface and $G$ preserves the open torus of $V$*;
(2) *$V = \mathbb{P}^1 \times \mathbb{P}^1$ and $G$ preserves a union of two fibers of the first projection $V \to \mathbb{P}^1$ and a union of at least three fibers of the second projection $V \to \mathbb{P}^1$*;
(3) *$V$ is an abelian surface*;
(4) *$V$ is a projective cone over an elliptic curve and $G$ preserves a cross section*;
(5) *$V$ is a $\mathbb{P}^1$-bundle over an elliptic curve and $G$ preserves a disjoint union of two sections*;
(6) *$V$ is a $\mathbb{P}^1$-bundle over an elliptic curve associated with an indecomposable locally free sheaf of degree* 0;
(7) *$V = \mathbb{P}^1 \times T$ for a non-singular projective curve $T$ of genus at least* 2, *and $G$ preserves a disjoint union of two fibers of the first projection $V \to \mathbb{P}^1$*;
(8) *$V = C \times T$ for an elliptic curve $C$ and a non-singular projective curve $T$ of genus at least* 2.

*Remark.* There exist no examples of $V \to X$ satisfying two mutually different conditions in (1)–(8), except for one pair: (1) and (2).

Theorems 3.11 and 3.12 below are considered as special cases of Theorem 3.10.

**Theorem 3.11.** *For a normal projective surface $X$, it admits a non-isomorphic surjective endomorphism $f\colon X \to X$ étale in codimension 1 if and only if there exists a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying one of the following conditions*:

1. (1) *$V$ is an abelian surface*;
2. (2) *$V = C \times T$ for an elliptic curve $C$ and a curve $T$ of genus at least 2*;
3. (3) *$V = \mathbb{P}^1 \times T$ for an elliptic curve $T$*;
4. (4) *$V$ is a $\mathbb{P}^1$-bundle over an elliptic curve associated with an indecomposable locally free sheaf of degree 0.*

**Theorem 3.12.** *Let $X$ be a normal projective surface with a non-isomorphic surjective endomorphism $f$. Assume that*

- (i) *the refined ramification divisor $\Delta_f = 0$, and*
- (ii) *$K_X + S$ is not pseudo-effective for any $f$-completely invariant divisor $S$.*

*Then one of the following holds*:

1. (1) *$X \simeq \mathbb{P}^1 \times T$ for an elliptic curve $T$*;
2. (2) *$X \simeq \mathbb{P}^1 \times \mathbb{P}^1$*;
3. (3) *there is a finite cyclic cover $V \to X$ étale in codimension 1 from a $\mathbb{P}^1$-bundle $V$ over an elliptic curve $T$ associated with an indecomposable locally free sheaf of degree 0.*

First, we shall prove Theorem 3.12 using Lemma 3.13 below, and next prove Theorems 3.10 and 3.11 using Theorems 3.1 and 3.12.

**Lemma 3.13.** *In the situation of Theorem 3.12, the following hold*:

1. (1) *There is no negative curve on $X$, $\boldsymbol{\rho}(X) = 2$, and $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X) = \mathsf{R} + \mathsf{R}'$ for two rays $\mathsf{R}$ and $\mathsf{R}'$ in which $\mathsf{R}$ (resp. $\mathsf{R}'$) is generated by an eigenvector of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ of eigenvalue $\deg f$ (resp. 1).*
2. (2) *There is a $\mathbb{P}^1$-fibration $\pi\colon X \to T$ to a non-singular projective curve $T$ with an endomorphism $h\colon T \to T$ such that $\pi \circ f = h \circ \pi$, $\deg f = \deg h$, and $\mathrm{cl}(F) \in \mathsf{R}$ for any fiber $F$ of $\pi$.*

*Moreover, the following hold for any $f$-completely invariant divisor $S$ satisfying $S \geq S_f$*:

3. (3) *$R_f = f^*S - S$, $\mathrm{Supp}\, R_f \subset S_f$, and $S_f = \pi^{-1}S_h$*;
4. (4) *$-(K_X + S)$ is nef, $(K_X + S)^2 = 0$, $(K_X + S)\mathsf{R} < 0$, and $\mathrm{cl}(-(K_X + S)) \in \mathsf{R}'$*;
5. (5) *if $S \neq S_f$, then $S - S_f$ is a section of $\pi$, and $\mathrm{cl}(S - S_f) \in \mathsf{R}'$.*

*Proof.* Let $S$ be an $f$-completely invariant divisor such that $S \geq S_f$. Then $R_f = f^*S - S$ by [24, Lem. 2.18] with Theorem 3.12(i), and $-(K_X + S)$ is nef with $(K_X + S)^2 = 0$ and $K_X + S \not\approx 0$ by [24, Cor. 2.25] with Theorem 3.12(ii). In particular, $\mathrm{cl}(K_X + S)$ is an eigenvector of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ with eigenvalue 1.

Hence, $\lambda_f = \deg f$. Thus, $\boldsymbol{\rho}(X) = 2$ and $X$ has no negative curve by [24, Thm. 3.22 and Prop. 3.24]. There is an extremal ray $\mathsf{R}$ such that $(K_X + S)\mathsf{R} < 0$, and the contraction morphism of $\mathsf{R}$ is a $\mathbb{P}^1$-fibration $\pi\colon X \to T$ to a non-singular projective curve $T$ (cf. [24, Thms. 1.9 and 1.10]). By [24, Lem. 3.7], $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X) = \mathsf{R}+\mathsf{R}'$ for $\mathsf{R}' := \mathbb{R}_{\geq 0}\,\mathrm{cl}(-(K_X + S))$, where $\mathsf{R}$ (resp. $\mathsf{R}'$) is generated by an eigenvector of $f^*$ with eigenvalue $\deg f$ (resp. 1). In particular, $\mathsf{R}$ and $\mathsf{R}'$ are independent of the choice of $S$. Since $f^*$ preserves $\mathsf{R}$, there is an endomorphism $h\colon T \to T$ such that $\pi \circ f = h \circ \pi$ and $\deg h = \deg f$, by [24, Lem. 3.16]. Thus, (1), (2), and (4) have been proved. We shall prove the rest of (3) and (5).

(3): We have $\mathrm{Supp}\, R_f \subset S_f \cup \mathrm{Supp}\, \Delta_f = S_f$ (cf. [24, Lem. 2.17(4)]). For (3), it suffices to show the equality $S_f = \pi^{-1}S_h$, and by [24, Lem. 2.19(2)], we are reduced to proving that any prime component of $S_f$ does not dominate $T$. Let $k$ be a positive integer such that $f^k$ is *sufficiently iterated* (cf. [24, Def. 2.16]). For any prime component $\Gamma$ of $S_f$, we have $(f^k)^*\Gamma = (\deg f)^k\Gamma$, since $(f^k)^* = (f^*)^k\colon \mathsf{N}(X) \to \mathsf{N}(X)$ has only two eigenvalues 1 and $(\deg f)^k$. This implies that $\mathrm{cl}(\Gamma) \in \mathsf{R}$ and that $\Gamma$ is a fiber of $\pi$. Thus, any prime component of $S_f$ does not dominate $T$, and we have proved (3).

(5): Assume that $S - S_f \neq 0$. For a prime component $C$ of $S - S_f$, we have $(f^m)^*C = C$ for some $m > 0$. Thus, $\mathrm{cl}(C) \in \mathsf{R}'$ and $CF > 0$ for a general fiber $F$ of $\pi$. Here, $(S - S_f)F = SF = 1$ by $0 > (K_X + S)F = -2 + SF$. Therefore, $S - S_f = C$ and it is a section of $\pi$. Thus, (5) has been proved, and we are done. $\square$

We shall prove Theorem 3.12 applying Lemma 3.13.

*Proof of Theorem* 3.12. We know that $-(K_X + S_f)$ is nef by Lemma 3.13(4). First assume that $-(K_X + S_f)$ is not semi-ample. Then $(K_X + S_f)K_X \leq 0$ by [24, Lem. 1.4]. By Lemma 3.13(4), we have $(K_X + S_f)K_X = (K_X + S_f)S_f = 0$ and $(K_X + S_f)F < 0$ for a general fiber $F$ of $\pi$. Thus, $S_f = R_f = 0$ and $S_h = 0$ by Lemma 3.13(3). In particular, $-K_X$ is nef but not semi-ample. By [24, Prop. 4.3], we have a finite cyclic cover $\tau\colon T' \to T$ from an elliptic curve $T'$ with an endomorphism $h'\colon T' \to T'$ such that $\tau \circ h' = h \circ \tau$, in which the following conditions are satisfied for the normalization $X'$ of $X \times_T T'$:

- The induced morphism $\nu\colon X' \to X$ is étale in codimension 1.
- The induced $\mathbb{P}^1$-fibration $\pi'\colon X' \to T'$ has only reduced fibers.
- There is an endomorphism $f'\colon X' \to X'$ such that $\pi' \circ f' = h' \circ \pi'$ and $\nu \circ f' = f \circ \nu$.

Since $h'$ is étale, $\pi'\colon X' \to T'$ is a $\mathbb{P}^1$-bundle by [24, Lem. 4.6(5)]. Note that $\deg f' = \deg f = \deg h = \deg h'$. In particular, $f'$ is an étale endomorphism isomorphic to the base change of $\pi'$ by $h'$. Since $-K_{X'} = \nu^*(-K_X)$ is nef but not semi-ample, either

- $X' \simeq \mathbb{P}_{T'}(\mathcal{F})$ for an indecomposable locally free sheaf of degree 0, or
- $X' \simeq \mathbb{P}_{T'}(\mathcal{O}_{T'} \oplus \mathcal{L})$ for a non-torsion invertible sheaf $\mathcal{L}$ of degree 0

by Fact 2.23. The latter case does not occur. In fact, in this case, we have just two sections $\Gamma'_1$ and $\Gamma'_2$ of $\pi'$ of self-intersection number 0, and $\Gamma'_1 + \Gamma'_2$ are completely invariant under $f$: This is a contradiction to Lemma 3.13(5), since the image $\nu(\Gamma'_1+\Gamma'_2)$ is $f$-completely invariant but is not a section of $\pi$. Hence, the first case occurs, and as a consequence, Theorem 3.12(3) holds.

Next, assume that $-(K_X + S_f)$ is semi-ample. Since $(K_X + S_f)F < 0$ for a general fiber $F \simeq \mathbb{P}^1$ of $\pi$ (cf. Lemma 3.13(4)), we have a fibration $\psi\colon X \to B \simeq \mathbb{P}^1$ such that $\mathcal{O}_X(-m(K_X + S_f)) \simeq \psi^*\mathcal{O}_B(l)$ for some $m$, $l > 0$. Moreover, there is an automorphism $h_B\colon B \to B$ such that $\psi \circ f = h_B \circ \psi$ by [24, Lem. 3.16], since $K_X + S_f = f^*(K_X + S_f)$. Since $S_f = \pi^{-1}S_h$ (cf. Lemma 3.13(3)), a general fiber of $\psi$ is rational (resp. elliptic) if $S_f \neq 0$ (resp. $= 0$). Let $\Sigma_\psi$ be the set of points $b \in B$ such that $\psi^*(b)$ is not reduced. Then, $h_B^{-1}(\Sigma_\psi) = \Sigma_\psi$ by [24, Lem. 4.6(1)]. If $\Sigma_\psi \neq \emptyset$, then $\#\Sigma_\psi \geq 2$ by [24, Prop. 4.14, Lem. 4.15], and hence, the reduced divisor $\psi^{-1}(\Sigma_\psi)$ is $f$-completely invariant and it has at least two prime components dominating $T$ by $\pi$: This is a contradiction to Lemma 3.13(5). Therefore, $\Sigma_\psi = \emptyset$ and $\psi\colon X \to B$ is a $\mathbb{P}^1$-bundle or a smooth elliptic fibration by [24, Cor. 4.7]. If $\psi$ is an elliptic fibration, then $X$ is the product of an elliptic curve and $B \simeq \mathbb{P}^1$, i.e., Theorem 3.12(1) holds. If $\psi$ is a $\mathbb{P}^1$-bundle, then $X \simeq \mathbb{P}^1 \times B$, since $X$ has no negative section (cf. Lemma 3.13(1)), and hence, Theorem 3.12(2) holds. Thus, we are done. □

We shall prove Theorem 3.10 applying Theorems 3.1 and 3.12.

*Proof of Theorem* 3.10. First, we shall prove the "if" part. Namely, we shall prove the existence of a non-isomorphic surjective endomorphism $f$ of $X$ such that $\Delta_f = 0$ assuming that there is a finite Galois cover $\nu\colon V \to X$ satisfying one of conditions (1)–(8) of Theorem 3.10. If $\nu$ satisfies Theorem 3.10(6), then the endomorphism $f$ exists by Proposition 2.25 (cf. Remark 2.1). For other conditions in (1)–(8) of Theorem 3.10, we shall construct a reduced divisor $S$ on $X$ such that $K_X + S$ is pseudo-effective and that $(V, \nu^*S)$ satisfies one of conditions in Theorem 3.1: This implies the existence of $f$ by Theorem 3.1 and [24, Thm. 2.24].

If $\nu$ satisfies Theorem 3.10(1), then we have a reduced divisor $S$ on $X$ such that $\nu^*S$ is the boundary divisor; thus, $(V, \nu^*S)$ satisfies Theorem 3.1(6), where $K_X + S \sim_{\mathbb{Q}} 0$ by $K_V + \nu^*S = \nu^*(K_X + S)$.

Assume that $\nu$ satisfies Theorem 3.10(2). Then we have a reduced divisor $S$ on $X$ such that $\nu^*S = \mathrm{pr}_1^*(P_1 + P_2) + \mathrm{pr}_2^* D$ for two points $P_1 \neq P_2$ of $\mathbb{P}^1$ and for a reduced divisor $D$ on $\mathbb{P}^1$ with $\deg D \geq 3$, where $\mathrm{pr}_i\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ stands for the $i$-th projection for $i = 1$, 2. Thus, $(V, \nu^*S)$ satisfies Theorem 3.1(1) for $T = \mathbb{P}^1$, where $K_X + S$ is semi-ample by $\nu^*(K_X + S) = K_V + \nu^*S = \mathrm{pr}_2^*(K_T + D)$.

If $\nu$ satisfies (3) (resp. (8)) of Theorem 3.10, then $(V, S = 0)$ satisfies Theorem 3.1(3) (resp. 3.1(2)), and $K_X$ is semi-ample by $K_V = \nu^*K_X$.

If $\nu$ satisfies Theorem 3.10(4), then we have a prime divisor $S$ on $X$ such that $\nu^*S$ is a cross section of the projective cone $V$; thus, $(V, \nu^*S)$ satisfies Theorem 3.1(5), where $K_X + S \sim_{\mathbb{Q}} 0$ by $K_V + \nu^*S = \nu^*(K_X + S) \sim 0$.

If $\nu$ satisfies Theorem 3.10(5), then we have a reduced divisor $S$ on $X$ such that $\nu^*S$ is a disjoint union of two sections of the $\mathbb{P}^1$-bundle $V$ over an elliptic curve; thus, $(V, \nu^*S)$ satisfies Theorem 3.1(4), where $K_X + S \sim_{\mathbb{Q}} 0$ by $K_V + \nu^*S = \nu^*(K_X + S) \sim 0$.

Assume that $\nu$ satisfies Theorem 3.10(7). Then we have a reduced divisor $S$ on $X$ such that $\nu^*S = \mathrm{pr}_1^*(P_1 + P_2)$ for two points $P_1 \neq P_2$ of $\mathbb{P}^1$ for the first projection $\mathrm{pr}_1\colon V = \mathbb{P}^1 \times T \to \mathbb{P}^1$; thus, $(V, \nu^*S)$ satisfies Theorem 3.1(1). Here, $K_X + S$ is semi-ample, since $\nu^*(K_X + S) = K_V + \nu^*S = \mathrm{pr}_2^* K_T$ for the second projection $\mathrm{pr}_2\colon V = \mathbb{P}^1 \times T \to T$. Thus, we have completed the proof of 'if' part.

Second, we shall prove the 'only if' part. Now, $X$ is assumed to have a non-isomorphic surjective endomorphism $f$ such that $\Delta_f = 0$, and we shall find a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying one of conditions (1)–(8) of Theorem 3.10. We see that $X$ is one of surfaces listed in Theorem 3.1 or in Theorem 3.12. If $X$ is listed in Theorem 3.12, then one of conditions (1), (2), (5), and (6) of Theorem 3.10 is satisfied for such a Galois cover $V \to X$. In fact:

- If Theorem 3.12(1) holds, then the identity morphism $V = X \to X$ satisfies Theorem 3.10(5).
- If Theorem 3.12(2) holds, then the identity morphism $V = X \to X$ satisfies (1) and (2) of Theorem 3.10.
- The Galois cover $V \to X$ in Theorem 3.12(3) satisfies Theorem 3.10(6).

Thus, we may assume that $X$ is listed in Theorem 3.1. Then there exist an $f$-completely invariant divisor $S$ with $K_X + S$ being semi-ample and a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying one of conditions (1)–(6) of Theorem 3.1. We can verify that $V \to X$ satisfies one of conditions of Theorem 3.10. In fact:

- If Theorem 3.1(1) holds, then (2) (resp. (5), resp. (7)) of Theorem 3.10 is satisfied when $\boldsymbol{g}(T) = 0$ (resp. $= 1$, resp. $\geq 2$).
- If Theorem 3.1(2) holds, then (5) (resp. (3), resp. (8)) of Theorem 3.10 is satisfied when $\boldsymbol{g}(T) = 0$ (resp. $= 1$, resp. $\geq 2$).

Moreover, we have the following implications for conditions for $V \to X$ and $S$:

Theorem 3.1(3) $\Rightarrow$ Theorem 3.10(3), Theorem 3.1(4) $\Rightarrow$ Theorem 3.10(5),
Theorem 3.1(5) $\Rightarrow$ Theorem 3.10(4), Theorem 3.1(6) $\Rightarrow$ Theorem 3.10(1).

Thus, we are done. $\square$

Finally, we shall prove Theorem 3.11 applying Theorems 3.1, 3.10, and 3.12.

*Proof of Theorem* 3.11. First, we shall prove the 'if' part. Namely, we shall prove the existence of a non-isomorphic surjective endomorphism $f$ of $X$ such that $R_f = 0$ assuming that there is a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying one of conditions (1)–(4) of Theorem 3.11. If $\nu$ satisfies (1) (resp. (2)) of Theorem 3.11, then $(V, S = 0)$ satisfies (3) (resp. (2)) of Theorem 3.1: In this case, $K_X$ is nef by $K_V = \nu^*K_X$, and hence, $X$ has an expected endomorphism $f$ by Theorem 3.1 and [24, Lem. 2.22]. If $\nu$ satisfies (3) (resp. (4)) of Theorem 3.11, then $X$ has such an endomorphism by Lemma 2.5 for $V = A \times B$ with $(A, B) = (T, \mathbb{P}^1)$ (resp. by Proposition 2.25) and by Remark 2.1. Thus, we have proved the 'if' part.

Second, we shall prove the 'only if' part. Now, $X$ is assumed to have a non-isomorphic surjective endomorphism $f$ such that $R_f = 0$, and we shall find a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying one of conditions (1)–(4) of Theorem 3.11. Hence, $X$ is one of the surfaces listed in Theorem 3.1 or in Theorem 3.12 in which $R_f = 0$. If $X$ is listed in Theorem 3.12, then either (3) or (4) of Theorem 3.11 is satisfied. In fact:

- If Theorem 3.12(1) holds, then the identity morphism $V = X \to X$ satisfies Theorem 3.11(3).
- The case Theorem 3.12(2) does not occur, since $\mathbb{P}^1 \times \mathbb{P}^1$ is simply connected.
- If Theorem 3.12(3) holds, then the cover $V \to X$ satisfies Theorem 3.11(4).

Thus, we may assume that $X$ is listed in Theorem 3.1. Then there exist an $f$-completely invariant divisor $S$ with $K_X + S$ being semi-ample and a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying one of (1)–(6) of Theorem 3.1. We shall finish the proof by showing the following:

(a) If one of (1), (5), and (6) of Theorem 3.1 holds, then $R_f \neq 0$.
(b) If (3) (resp. (2)) of Theorem 3.1 holds, then $\nu$ satisfies (1) (resp. one of (1), (2), and (3)) of Theorem 3.11.
(c) If Theorem 3.1(4) holds and if $R_f = 0$, then there is another finite Galois cover $V'' \to X$ étale in codimension 1 satisfying Theorem 3.11(3).

Assertion (b) holds trivially. We shall prove (a). If Theorem 3.1(1) holds, then $S_f \neq 0$ by Remark 3.2, and it implies: $R_f \neq 0$ (cf. [24, Lem. 2.17(4)]). If Theorem 3.1(5) holds, then $f^*S = \delta_f S$ for the cross section $S$, since $\boldsymbol{\rho}(X) = 1$ and $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is the multiplication map by $\delta_f > 1$; thus, $0 \neq S \leq S_f$ and $R_f \neq 0$. We shall show that $R_f \neq 0$ when Theorem 3.1(6) holds. In this case, the non-singular part $V_{\mathrm{reg}}$ is also a toric variety and its fundamental group is finite, since the 1-dimensional cones in the fan generate a finite index subgroup of the group of 1-parameter subgroups (cf. [25, Prop. 10.2], [9, §3.2]). Now, $f$ lifts to an endomorphism $f_V$ of $V$ under which the boundary divisor $\nu^*S$ is completely invariant, by [24, Thm. A]. If $R_f = 0$, then $R_{f_V} = 0$ and the $m$-th power $f_V^m\colon V \to V$ induces a finite étale cover $(f_V^m)^{-1}V_{\mathrm{reg}} \to V_{\mathrm{reg}}$ of degree $\gg 1$ as $m \gg 1$. This is a contradiction. This shows (a).

We shall show (c). Here, we assume that $R_f = 0$ and that Theorem 3.1(4) holds for $\nu\colon V \to X$. By [24, Thm. A], we may assume that there is an endomorphism $f_V\colon V \to V$ satisfying $\nu \circ f_V = f \circ \nu$. Then $f_V$ is étale and $\nu^*S$ is completely invariant under $f_V$. Here, $X$ and $V$ have no negative curve by [24, Prop. 2.20(3)]. Since $\nu^*S$ is a disjoint union of two sections, $V \simeq \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ for an elliptic curve $T$ and an invertible sheaf $\mathcal{L}$ on $T$ with $\deg \mathcal{L} = 0$. Since $\boldsymbol{\pi}_1(V) \simeq \boldsymbol{\pi}_1(T)$, the étale endomorphism $f_V$ descends to an étale endomorphism $h\colon T \to T$ such that $\deg h = \deg f$ and that $\pi \circ f_V = h \circ \pi$ for the structure morphism $\pi\colon V \to T$. In particular, $f_V$ induces an isomorphism $V \simeq V \times_{T,h} T$. Therefore, $h^*\mathcal{L} \simeq \mathcal{L}$ or $h^*\mathcal{L} \simeq \mathcal{L}^{\otimes -1}$, which implies that $\mathcal{L}$ is a torsion invertible sheaf, since $h^* \pm \mathrm{id}\colon \operatorname{Pic}^0(T) \to \operatorname{Pic}^0(T)$ is surjective. Thus, we can find a finite étale cover $\tau\colon T' \to T$ such that $\tau^*\mathcal{L} \simeq \mathcal{O}_{T'}$. Then there is an isomorphism $V' := V \times_T T' \simeq \mathbb{P}^1 \times T'$ over $T'$. Let $V'' \to X$ be the

Galois closure of the composite $V' \to V \to X$. Then $V'' \to V'$ is étale and $V'' \simeq V' \times_{T'} T'' \simeq \mathbb{P}^1 \times T''$ for a finite étale cover $T'' \to T'$, since $\boldsymbol{\pi}_1(V') \simeq \boldsymbol{\pi}_1(T')$. The finite Galois cover $V'' \to X$ is étale in codimension 1 and satisfies Theorem 3.11(3). Thus, (c) has been proved, and the proof of Theorem 3.11 has been completed. □

## 4. Non pseudo-effective case of Picard number greater than 1

Let $X$ be a normal projective surface with a non-isomorphic surjective endomorphism $f$. By Theorem 3.1 (cf. [24, Thm. A]), the structure of $X$ has been determined when $K_X + S$ is pseudo-effective for an $f$-completely invariant divisor $S$. For non-pseudo-effective $K_X + S$, in Section 4, we shall study the following two cases:

- $\boldsymbol{\rho}(X) \geq 3$ and $K_X + S_f$ is not pseudo-effective;
- $\boldsymbol{\rho}(X) = 2$, $K_X$ is not pseudo-effective, and $-K_X$ is not big.

These cases are treated in Sections 4.1 and 4.2, respectively.

4.1. **The case where Picard number is greater than 2.** We introduce the notion of an *$\mathcal{L}$-surface* in Definition 4.2 below, and prove in Proposition 4.3 below that $(X, S_f)$ is an $\mathcal{L}$-surface when $K_X + S_f$ is not pseudo-effective and $\boldsymbol{\rho}(X) \geq 3$. Theorem 4.5 is a basic structure theorem for $\mathcal{L}$-surfaces. In Corollary 4.6, we shall show that $(X, B + S_f)$ is a *toric surface* or a *half-toric surface* for a prime divisor $B \not\subset S_f$ in the sense of [22] provided that $-(K_X + S_f)$ is not big in addition.

**Definition 4.1.** Let $X$ be a normal projective surface. The number of negative curves is denoted by $\operatorname{neg}(X) \leq \infty$. For a reduced divisor $D$, the number of prime components of $D$ is denoted by $\boldsymbol{n}(D)$ (cf. [22]).

**Definition 4.2.** Let $X$ be a normal projective surface and $S$ a reduced divisor on $X$. If the following conditions are satisfied, then $(X, S)$ is called an *$\mathcal{L}$-surface*:

(i) $X$ is rational, $\boldsymbol{\rho}(X) \geq 3$, and $(X, S)$ is log-canonical;
(ii) $-(K_X + S)$ is nef but not numerically trivial;
(iii) $S$ contains all the negative curves on $X$.

*Remark.* The prefix "$\mathcal{L}$-" comes from a property that $S$ is a linear chain of rational curves (see Theorem 4.5(6) below).

**Proposition 4.3.** *Let $f$ be a non-isomorphic surjective endomorphism of a normal projective surface $X$ such that $\boldsymbol{\rho}(X) \geq 3$ and that $K_X + S_f$ is not pseudo-effective. Then $(X, S_f)$ is an $\mathcal{L}$-surface.*

*Proof.* By [24, Thm. E and Prop. 2.20(3)], we know that $(X, S_f)$ is log-canonical and that $S_f$ contains all the negative curves on $X$. Now $X$ is ruled, since $K_X$ is not pseudo-effective. Then $X$ is rational by [24, Thm. 4.16] and by $\boldsymbol{\rho}(X) \geq 3$. Since $S_f = S_{f^k}$ for any $k > 0$ (cf. [24, Lem. 2.17(3)]), we may assume that $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is the multiplication map by $\delta_f = (\deg f)^{1/2} > 1$, by [24, Thm. 3.22]. Then

$$\Delta_f = K_X + S_f - f^*(K_X + S_f) = -(\delta_f - 1)(K_X + S_f).$$

Thus, $-(K_X + S_f)$ is nef. This is not numerically trivial as $K_X + S_f$ is not pseudo-effective. Therefore, $(X, S_f)$ is an $\mathcal{L}$-surface. □

**Lemma 4.4.** *Let $X$ be a normal projective surface. For a nef $\mathbb{Q}$-divisor $D$ and a prime divisor $C$ on $X$, if $DC > 0$, then there is a positive rational number $\alpha$ such that $tC + D$ is nef and big for any $0 < t < \alpha$.*

*Proof.* It is enough to take a positive rational number $\alpha$ such that $(tC + D)C > 0$ for any $0 < t < \alpha$. For, $tC + D$ is nef and $(tC + D)^2 \geq t(tC + D)C > 0$ for any $0 < t < \alpha$. ☐

**Theorem 4.5.** *The following hold for any $\mathcal{L}$-surface $(X, S)$:*

(1) *The surface $X$ has only rational singularities. In particular, $X$ is $\mathbb{Q}$-factorial, the* Weil–Picard number $\hat{\boldsymbol{\rho}}(X)$ (*cf.* [24, §1.1]) *equals the Picard number $\boldsymbol{\rho}(X)$, and the numerical equivalence $\approx$ coincides with the $\mathbb{Q}$-linear equivalence $\sim_{\mathbb{Q}}$ for $\mathbb{Q}$-divisors on $X$.*

(2) *The pseudo-effective cone $\overline{\mathrm{NE}}(X)$ is polyhedral and generated by the numerical classes of negative curves on $X$.*

(3) *The divisors $-K_X$ and $S$ are big, and $-(K_X + S)$ is semi-ample.*

(4) *One has inequalities*

$$\boldsymbol{\rho}(X) \leq \operatorname{neg}(X) \leq \boldsymbol{n}(S) \leq \boldsymbol{\rho}(X) + 1.$$

(5) *If $\boldsymbol{n}(S) = \boldsymbol{\rho}(X) + 1$, then $(X, B + S)$ is a toric surface for a prime divisor $B \not\subset S$.*

(6) *The divisor $S$ is a* linear chain of rational curves (*cf.* [22, Def. 4.1])*, and one end component $C$ of $S$ is a negative curve satisfying $(K_X + S)C < 0$.*

(7) *Let $S^\natural$ be the union of non-end components of $S$. Then $K_X + S$ is Cartier along $S^\natural$ and $\mathcal{O}_X(K_X + S) \otimes \mathcal{O}_{S^\natural} \simeq \mathcal{O}_{S^\natural}$.*

(8) *If the intersection matrix of $S^\natural$ is not negative definite, then there exist a prime divisor $B \not\subset S$ and a $\mathbb{P}^1$-fibration $\pi\colon X \to \mathbb{P}^1$ such that*
- *$(X, B + S)$ is a toric surface,*
- *$S^\natural$ is a set-theoretic fiber of $\pi$, the other fibers are all irreducible,*
- *$B$ is a fiber of $\pi$, and end components of $S$ are sections of $\pi$.*

*Proof.* (1): This is a consequence of [22, Lem. 2.31], since $-K_X$ is not pseudo-effective and $H^2(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X(K_X))^\vee = 0$.

(2) and (3): The cone $\overline{\mathrm{NE}}(X)$ is defined in $\mathsf{N}(X)$ (cf. [24, §1.1]), but now $\mathsf{N}(X) = \mathrm{NS}(X) \otimes \mathbb{R}$ by (1) for the Néron–Severi group $\mathrm{NS}(X)$. By the cone and contraction theorems (cf. [24, Thm. 1.9 and 1.10 and Cor. 1.11(2)]) and by $\boldsymbol{\rho}(X) \geq 3$, there is a rational curve $C$ on $X$ such that $(K_X + S)C < 0$ and $C^2 < 0$. Then $tC - (K_X + S)$ is nef and big for $0 < t \ll 1$ by Lemma 4.4. Hence, $-K_X$ is big by $C \subset S$ (cf. Definition 4.2(iii)). Therefore, $-(K_X + S)$ is semi-ample by [24, Prop. 1.5], and (2) holds by [24, Thm. 1.13]. Then $S$ is big by (2) and Definition 4.2(iii). Thus, (3) has been shown.

(4) and (5): We have $\operatorname{neg}(X) \leq \boldsymbol{n}(S)$ by Definition 4.2(iii), and $\boldsymbol{\rho}(X) \leq \operatorname{neg}(X)$ by (2). On the other hand, $\boldsymbol{n}(S) \leq \boldsymbol{\rho}(X) + 1$ by Shokurov's criterion for toric surfaces [29, Thm. 6.4] (cf. [22, Thm. 1.1]), since $(X, S)$ is log-canonical and $-(K_X + S)$ is nef. Thus, we have inequalities in (4). Moreover, (5) holds by [22, Thm. 1.3], since $K_X + S \not\approx 0$.

(6): Since $(X, S)$ is log-canonical and $-(K_X + S)$ is nef, if $S$ is connected, then, by [22, Lem. 4.4], $S$ is a linear chain of rational curves and the negative curve $C$ in the proof of (2) and (3) is an end component. To prove the connectedness of $S$, we first show the following weaker assertion:

($6'$) *$S - \Gamma$ is connected for any prime component $\Gamma$ of $S$ such that* $(K_X + S)\Gamma < 0$.

By Lemma 4.4, $t\Gamma - (K_X + S)$ is nef and big for $0 < t \ll 1$. Then $H^1(X, \mathcal{O}_X(\Gamma - S)) = 0$ by a version of Kawamata–Viehweg vanishing's theorem [28, Thm. (5.1)] (cf. [23, Prop. 2.15]), since

$$\Gamma - S = K_X + \ulcorner t\Gamma - (K_X + S) \urcorner.$$

In particular, $\mathbb{C} \simeq H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{S-\Gamma})$ is surjective, and ($6'$) has been shown. The connectedness of $S$ is proved as follows: Assume the contrary. Then $S$ has two connected components $S - C$ and $C$ by ($6'$). Here, $(K_X + S)\Gamma = 0$ for any prime component $\Gamma$ of $S - C$. For, otherwise, $S = C \sqcup \Gamma$ by ($6'$), contradicting $\boldsymbol{n}(S) \geq \operatorname{neg}(X) \geq \boldsymbol{\rho}(X) \geq 3$ (cf. (4)). Then $(tC - (K_X + S))(S - C) = 0$ and it implies that the intersection matrix of $S - C$ is negative definite. This contradicts the bigness of $S = (S - C) + C$. Therefore, $S$ is connected, and we have proved (6).

(7): This follows from (6) and [22, Lem. 4.4(3)], since $(X, S)$ is log-canonical and $-(K_X + S)$ is nef.

(8): Assume that the intersection matrix of $S^\natural$ is not negative definite. Then the positive $P$ of the *Zariski-decomposition* (cf. [23, Lem.-Def. 2.16]) of $S^\natural$ is not zero, and $P \approxeq -c(K_X + S)$ for a rational number $c > 0$ by the Hodge index theorem, since $-(K_X + S)$ is nef and $(K_X + S)P = 0$ (cf. (7)). Let $\pi \colon X \to T \simeq \mathbb{P}^1$ be a fibration defined by the semi-ample divisor $-(K_X + S)$, i.e., $-(K_X + S) \sim_{\mathbb{Q}} \pi^* H$ for an ample $\mathbb{Q}$-divisor $H$ on $T$. Then $P \approxeq \pi^* D$ for a $\mathbb{Q}$-divisor $D$ on $T$, and $S^\natural$ is contained in a fiber of $\pi$. Thus, $\operatorname{Supp} P = S^\natural$ is a set-theoretic fiber of $\pi$. End components of $S$ dominate $T$, since $S^\natural$ intersects them. Hence, $\pi$ is a $\mathbb{P}^1$-fibration by $K_X F = -SF < 0$ for a general fiber $F$ of $\pi$. In particular, the end components are sections of $\pi$. The fiber of $\pi$ different from $S^\natural$ is irreducible, since any negative curve is contained in $S$. Thus, $\boldsymbol{n}(X) = 2 + \boldsymbol{n}(S^\natural) - 1 = \boldsymbol{n}(S) - 1$ by [22, Prop. 2.33(7)]. Then $(X, B + S)$ is a toric surface for a prime divisor $B \not\subset S$ by (5). Here, $B$ is a fiber of $\pi$ by $B \cap S^\natural = \emptyset$. This shows (8). Thus, we are done. □

**Corollary 4.6.** *Let $X$ be a normal projective surface admitting a non-isomorphic surjective endomorphism $f$ such that $\boldsymbol{\rho}(X) \geq 3$, $K_X + S_f$ is not pseudo-effective, and $-(K_X + S_f)$ is not big. Then $(X, B + S_f)$ is a toric surface or a* half-toric surface *(cf.* [22, Def. 7.1]*) for a prime divisor $B \not\subset S_f$.*

*Proof.* The pair $(X, S_f)$ is an $\mathcal{L}$-surface by Proposition 4.3, and $-(K_X + S_f)$ is semi-ample by Theorem 4.5(3). Since $X$ is rational, we have a fibration $\pi \colon X \to T \simeq \mathbb{P}^1$ such that $-(K_X + S_f) \sim_{\mathbb{Q}} \pi^* A$ of an ample $\mathbb{Q}$-divisor $A$ on $T$. By (6) and (7) of Theorem 4.5, $S_f$ is a linear chain of rational curves and the union $(S_f)^\natural$ of non-end components is contained in a fiber $\pi^{-1}(t)$ for some point $t \in T$. Since negative curves of $X$ are all contained in $S_f$ (cf. [24, Prop. 2.20(3)]), we see that $\pi^{-1}(t) \subset S_f$ and that every fiber of $\pi$ different from $\pi^{-1}(t)$ is irreducible. In particular, $\boldsymbol{n}(\pi^{-1}(t)) + 1 = \boldsymbol{\rho}(X)$ by [22, Prop. 2.33(7)]. Thus, $(S_f)^\natural = \pi^{-1}(t)$ if and only if $\boldsymbol{n}(S_f) = \boldsymbol{\rho}(X)+1$; in this case, $(X, B+S_f)$ is a toric surface for a prime divisor $B \subset S_f$ by Theorem 4.5(5).

Therefore, we may assume that $(S_f)^\natural \neq \pi^{-1}(t)$ and $\boldsymbol{n}(S_f) = \boldsymbol{\rho}(X)$. In this case, we can write $\pi^{-1}(t) = S_f - C_1 = (S_f)^\natural + C_2$ for end components $C_1$ and $C_2$ of $S_f$, since $S_f$ is big (cf. Theorem 4.5(3)). The curve $C_1$ is a double section of $\pi$ by $(K_X + S_f)F = (K_X + C_1)F = -2 + C_1F = 0$ for a general fiber $F$ of $\pi$. Here, $C_1 \cap \pi^{-1}(t) = C_1 \cap (S_f)^\natural$ consists of one point. Thus, $t$ is a branched point of the double cover $\pi|_{C_1} \colon C_1 \to T$. Let $t' \in T$ be the other branched point of $\pi|_{C_1}$. Then $S_f \cap \pi^{-1}(t') = C_1 \cap \pi^{-1}(t')$ consists of one point $x'$, and $(X, S_f + \pi^{-1}(t'))$ is log-canonical by [24, Prop. 3.17(6)]. Let $m$ be the multiplicity of $\pi^*(t')$, i.e., $\pi^*(t') = m\pi^{-1}(t')$. Then $S_f\pi^{-1}(t') = C_1\pi^{-1}(t') = 2/m$. Hence, $m$ is even, and if $m > 2$ (resp. $m = 2$), then $(X, x')$ is a cyclic quotient singularity of order $m/2$ (resp. $x' \in X_{\mathrm{reg}}$). Since $\#(S_f - C_1) \cap C_1 = \#\pi^{-1}(t) \cap C_1 = 1$ and $(K_X + S_f)C_1 = -(\pi^*A)C_1 = -2 \deg A < 0$, we have $(K_X + S_f)C_1 = -2/m$ by [22, Prop. 3.29]. Therefore, $K_X + S_f + \pi^{-1}(t') \approx 0$. Now $(X, S_f + \pi^{-1}(t'))$ is log-canonical and $S_f + \pi^{-1}(t')$ is a linear chain of rational curves. Since $\boldsymbol{n}(S_f + \pi^{-1}(t)) = \boldsymbol{\rho}(X) + 1$ and since $\boldsymbol{\rho}(X) = \hat{\boldsymbol{\rho}}(X)$ (cf. Theorem 4.5(1) or [24, Prop. C]), $(X, S_f + \pi^{-1}(t'))$ is a half-toric surface by [22, Thm. 1.3]. Thus, we are done. □

4.2. **The case where the Picard number equals 2 and $-K_X$ is not big.** We shall prove Theorem 4.7 and Proposition 4.8 below:

**Theorem 4.7.** *Let $f \colon X \to X$ be a non-isomorphic surjective endomorphism of a normal projective surface $X$ with Picard number $\boldsymbol{\rho}(X) = 2$. Assume that $K_X$ is not pseudo-effective and that $-K_X$ is not big. Then there exist a finite Galois cover $\nu \colon V \to X$ étale in codimension 1 and an endomorphism $f_V \colon V \to V$ such that $\nu \circ f_V = f^k \circ \nu$ for some $k > 0$ and that $V$ is one of the following surfaces:*

(1) *The direct product $\mathbb{P}^1 \times T$ for a non-singular projective curve $T$ of genus at least 2.*
(2) *A $\mathbb{P}^1$-bundle over an elliptic curve associated with a semi-stable locally free sheaf of rank 2.*

*Remark.* When $X$ is irrational, Theorem 4.7 is deduced from [24, Thm. 4.16], since $X$ is ruled, $\boldsymbol{\rho}(X) = 2$, and $-K_X$ is not big.

**Proposition 4.8.** *In Theorem 4.7, assume in addition that $\deg f \neq \lambda_f$. Then the surface $V$ can be taken as the $\mathbb{P}^1$-bundle $\mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ over an elliptic curve $T$ for an invertible sheaf $\mathcal{L}$ of degree 0.*

For the proof of Theorem 4.7, we begin with the following:

**Lemma 4.9.** *The following hold in the situation of Theorem 4.7:*

(1) *The cone $\overline{\mathrm{NE}}(X)$ is the sum $\mathsf{R} + \mathsf{R}'$ of two extremal rays $\mathsf{R}$ and $\mathsf{R}'$ such that $K_X\mathsf{R} < 0$ and $K_X\mathsf{R}' \geq 0$.*
(2) *The pullback homomorphism $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ preserves $\mathsf{R}$ and $\mathsf{R}'$, i.e., $f^*\mathsf{R} = \mathsf{R}$ and $f^*\mathsf{R}' = \mathsf{R}'$.*

(3) *There is no negative curve on* $X$.
(4) *The contraction morphism of the extremal ray* $\mathsf{R}$ *is a fibration* $\pi\colon X \to T$ *to a non-singular projective curve* $T$.
(5) *There is an endomorphism* $h\colon T \to T$ *such that* $\pi \circ f = h \circ \pi$.
(6) *The eigenvalues of* $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ *are* $\deg h$ *and* $\deg f/\deg h$, *where* $\deg f/\deg h$ *is also an integer.*

*Proof.* Since $\boldsymbol{\rho}(X) = 2$, $\overline{\mathrm{NE}}(X)$ is spanned by two extremal rays $\mathsf{R}$ and $\mathsf{R}'$. Either $K_X\mathsf{R}$ or $K_X\mathsf{R}'$ is negative, since $K_X$ is not pseudo-effective (cf. [24, Thm. 1.9]). If both $K_X\mathsf{R}$ and $K_X\mathsf{R}'$ are negative, then $-K_X$ is ample, contradicting the assumption. Thus, we may assume that $K_X\mathsf{R} < 0$ and $K_X\mathsf{R}' \geq 0$. This shows (1).

We have (3) $\Rightarrow$ (4) by the contraction theorem (cf. [24, Thm. 1.10]), since $\boldsymbol{\rho}(X) = 2$. Moreover, we have (2) + (4) $\Rightarrow$ (5) by [24, Lem. 3.16]. The implication (4) + (5) $\Rightarrow$ (6) is shown by [24, Prop. 3.17] and by $\boldsymbol{\rho}(X) = 2$, since $\mathsf{R}$ (resp. $\mathsf{R}'$) is generated by an eigenvector of $f^*$ with eigenvalue $\deg h$ (resp. $\deg f/\deg h$).

If $\lambda_f \neq \delta_f$, then (2) and (3) hold by [24, Lem. 3.7]. Thus, we may assume that $\lambda_f = \delta_f$ and it is enough to prove (2) and (3). Then $(f^2)^* = (f^*)^2\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is the multiplication map by $\deg f$, by [24, Lem. 3.7]. Hence, $-K_X$ is numerically equivalent to an effective $\mathbb{Q}$-divisor by

$$R_{f^2} = K_X - (f^2)^*(K_X) \approx (\deg f - 1)(-K_X).$$

Since $-K_X$ is not big, $\mathrm{cl}(-K_X)$ is contained in an extremal ray of $\overline{\mathrm{NE}}(X)$. If $\mathrm{cl}(-K_X) \in \mathsf{R}$, then $(-K_X)^2 > 0$ by $K_X\mathsf{R} < 0$; it implies that $-K_X$ is big by [22, Lem. 2.16(2)], and this is a contradiction. Therefore, $\mathrm{cl}(-K_X) \in \mathsf{R}'$, and $K_X^2 \leq 0$ by $K_X\mathsf{R}' \geq 0$. Note that $\mathrm{cl}(\Gamma) \in \mathsf{R}'$ for any prime component $\Gamma$ of $R_{f^2} \approx (\deg f - 1)(-K_X)$ as $\mathsf{R}'$ is an extremal ray.

We shall show that $\mathsf{R}'$ is nef, i.e., it is generated by the numerical class of a nef $\mathbb{Q}$-divisor. Assume the contrary. Then $\mathsf{R}' = \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Gamma)$ for a negative curve $\Gamma$. Let $\phi\colon X \to \overline{X}$ be the contraction morphism of $\Gamma$. Then there is an endomorphism $\bar{f}\colon \overline{X} \to \overline{X}$ such that $\phi \circ f = \bar{f} \circ \phi$ by [24, Lem. 3.14], and we have $K_{\overline{X}} \approx 0$ by $\mathrm{cl}(-K_X) \in \mathsf{R}'$. By [24, Thm. A], there exist a finite Galois cover $A \to \overline{X}$ étale in codimension 1 from an abelian surface $A$ and an endomorphism $f_A\colon A \to A$ as a lift of $\bar{f}$. Let $A'$ be the normalization of $X \times_{\overline{X}} A$. Then $f \times f_A$ induces a non-isomorphic surjective endomorphism $f'\colon A' \to A'$. Any prime component of the pullback of $\Gamma$ to $A'$ is a negative curve. Hence $R_{f'} \neq 0$. This contradicts [24, Lem. 2.22], since $K_{A'}$ is pseudo-effective.

Any prime component of $R_{f^2}$ and $-K_X$ are nef, since $\mathsf{R}'$ is so. Any negative curve on $X$ is contained in $S_f$ (cf. [24, Prop. 2.20(3)]), but $S_f = S_{f^k} \leq R_{f^k}$ for an integer $k > 0$ such that $f^k$ is sufficiently iterated (cf. [24, Def. 2.16 and Lem. 2.17(4)]). Hence, there is no negative curve on $X$, i.e., (3) holds. If $f^*\mathsf{R}' = \mathsf{R}$, then $K_X f^*(-K_X) < 0$ and $R_f f^*(-K_X) < 0$; this is a contradiction, since $f^*(-K_X)$ is nef. Hence, $f^*\mathsf{R} = \mathsf{R}$ and $f^*\mathsf{R}' = \mathsf{R}'$, i.e., (2) holds. Thus, we are done. $\square$

**Corollary 4.10.** *In the situation of Lemma* 4.9, $\mathrm{cl}(R_f) \in \mathsf{R}'$. *In particular,* $\pi(\Gamma) = T$ *for any prime component* $\Gamma$ *of* $R_f$. *If* $\deg h = 1$, *then* $R_f \neq 0$. *If* $\deg h > 1$, *then* $\mathrm{cl}(-K_X) \in \mathsf{R}'$, *in particular,* $-K_X$ *is nef, and* $K_X^2 = 0$.

*Proof.* Let $F$ be a general fiber of $\pi$. Then $0 \neq \mathrm{cl}(F) \in \mathsf{R}$. Let $D$ be a $\mathbb{Q}$-divisor such that $0 \neq \mathrm{cl}(D) \in \mathsf{R}'$. Then $F^2 = D^2 = 0$ and $FD > 0$. There exist rational numbers $\alpha$ and $\beta$ such that $-K_X \approx \alpha F + \beta D$. Here $\beta > 0$ by $2 = -K_X F = \beta FD$. We have

$$R_f \approx \alpha(\deg h - 1)F + \beta(\deg f / \deg h - 1)D$$

by $K_X = f^*(K_X) + R_f$, $f^*F \approx (\deg h)F$, and $f^*D \approx (\deg f/\deg h)D$. Since $\mathrm{cl}(R_f) \in \overline{\mathrm{NE}}(X) = \mathsf{R} + \mathsf{R}'$, we have

$$\alpha(\deg h - 1) \geq 0.$$

If $\deg h = 1$, then $R_f \approx \beta(\deg f - 1)D$, and hence, $R_f \neq 0$ and $\mathrm{cl}(R_f) \in \mathsf{R}'$. Assume that $\deg h > 1$. Then $\alpha \geq 0$ and $-K_X$ is nef. Since $-K_X$ is not big and $K_X F < 0$, we have $\mathrm{cl}(-K_X) \in \mathsf{R}'$. It implies that $\alpha = 0$, $K_X^2 = 0$, and $R_f \approx \beta(\deg f/\deg h - 1)D$. In particular, $\mathrm{cl}(R_f) \in \mathsf{R}'$ even when $\deg h > 1$. For any prime component $\Gamma$ of $R_f$, we have $\mathrm{cl}(\Gamma) \in \mathsf{R}'$ as $\mathsf{R}'$ is an extremal ray, and hence, $F\Gamma > 0$ and $\pi(\Gamma) = T$. □

**Lemma 4.11.** *Let* $X$ *be a* $\mathbb{P}^1$*-bundle over an elliptic curve* $T$ *associated with an indecomposable locally free sheaf of rank* 2 *and degree* 0. *Then* $\lambda_f = \deg f$ *for any surjective endomorphism* $f\colon X \to X$.

*Proof.* We may assume that $\deg f > 1$. The structure morphism $\pi\colon X \to T$ of the $\mathbb{P}^1$-bundle is just the Albanese morphism of $X$. Thus, there is an endomorphism $h\colon T \to T$ such that $\pi \circ f = h \circ \pi$. Here, $\deg h \mid \deg f$ by [24, Prop. 3.17(1)]. If $\lambda_f = \delta_f$, then $\deg h = \delta_f$ by [24, Lem. 3.7(1), Prop. 3.17(2)]. Hence, if $\deg h = \deg f$, then $\lambda_f > \delta_f$, and $\lambda_f = \deg f$ by [24, Prop. 3.25]. Hence, for the proof, it suffices to derive a contradiction assuming that $\deg h < \deg f$.

We may assume that $X = \mathbb{P}_T(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ with a non-split exact sequence $0 \to \mathcal{O}_T \to \mathcal{E} \to \mathcal{O}_T \to 0$ (cf. Fact 2.23(B)). Then $h^*\mathcal{E} \simeq \mathcal{E}$. Thus, the base change $\pi_h\colon X_h = X \times_T T \to T$ of $\pi$ by $h\colon T \to T$ is isomorphic to $\pi$. There is a surjective morphism $g\colon X \to X_h$ over $T$ such that $f = p_1 \circ g$ for the first projection $p_1\colon X_h \to X$. Since $\pi_h \simeq \pi$, $g$ is regarded as a non-isomorphic surjective endomorphism of $X$ over $T$, where $\deg g = \deg f/\deg h$. Therefore, we may assume that $h = \mathrm{id}_T$.

Let $\Gamma$ be a section of $\pi$ corresponding to the surjection $\mathcal{E} \to \mathcal{O}_T$. Then $K_X \sim -2\Gamma$, $\kappa(\Gamma, X) = 0$, and $\Gamma$ is a unique prime divisor of self-intersection number zero which dominates $T$ (cf. Fact 2.23(B)). As a consequence, $f^*\Gamma = m\Gamma$ for a positive integer $m$, where $m = \deg f$, since $(f^*\Gamma)F = \Gamma(f_*F) = (\deg f)\Gamma F$ for a fiber $F$ of $\pi$. There is an effective divisor $\Delta$ on $X$ such that $\Gamma \not\subset \mathrm{Supp}\,\Delta$ and that $K_X + \Gamma = f^*(K_X + \Gamma) + \Delta$ (cf. [23, Lem. 1.39]). Then $\Delta \sim (\deg f - 1)\Gamma$ by $-K_X \sim 2\Gamma$, and hence, $\Delta = (\deg f - 1)\Gamma$ by $\kappa(\Gamma, X) = 0$, contradicting: $\Gamma \not\subset \mathrm{Supp}\,\Delta$. Thus, we are done. □

*Proof of Theorem* 4.7. Assume that $\deg h = 1$ for the endomorphism $h\colon T \to T$ in Lemma 4.9. Then, by [24, Thm. 4.9], we have a finite Galois cover $\nu\colon \mathbb{P}^1 \times T' \to X$ étale in codimension 1 for a non-singular projective curve $T'$ and an endomorphism $f'$ of $\mathbb{P}^1 \times T'$ such that $\nu \circ f' = f^k \circ \nu$ for some $k > 0$. Here, $T'$ is not rational, since $-K_{\mathbb{P}^1 \times T'} = \nu^*(-K_X)$ is not big. Thus, Theorem 4.7 holds true when $\deg h = 1$.

For the rest of the proof, we assume that $\deg h > 1$. Note that every fiber of $\pi$ is irreducible as $\pi$ is the contraction morphism of an extremal ray. Let $\Sigma$ be the set of points $t \in T$ such that $\pi^*(t)$ is not reduced. If $\Sigma = \emptyset$, then $X$ is a $\mathbb{P}^1$-bundle over $T$ by [22, Prop. 2.33(4)], and $T$ is an elliptic curve, since $K_X^2 = 0$ (cf. Lemma 4.10). This $\mathbb{P}^1$-bundle is associated with a semi-stable locally free sheaf by Lemma 4.9(3).

Thus, we may assume that $\Sigma \neq \emptyset$. By Corollary 4.10, any prime component of $R_f$ dominates $T$. Then, by applying [24, Lems. 4.1 and 4.2 and Prop. 4.3] to morphisms $\pi\colon X \to T$ and $h\colon T \to T$ in Lemma 4.9, we have a finite cyclic cover $\tau\colon T' \to T$ from an elliptic curve $T'$ with an endomorphism $h'\colon T' \to T'$ such that $\tau \circ h' = h^k \circ \tau$ for some $k > 0$ and that the normalization $X'$ of $X \times_T T'$ satisfies the following conditions:

- The induced finite cyclic cover $\nu\colon X' \to X$ is étale in codimension 1.
- The induced $\mathbb{P}^1$-fibration $\pi'\colon X' \to T'$ has only reduced fibers.
- There is an endomorphism $f'\colon X' \to X'$ such that $\pi' \circ f' = h' \circ \pi'$ and $\nu \circ f' = f^k \circ \nu$.

Since $h'$ is étale with $\deg h' = (\deg h)^k > 1$, $X'$ is a $\mathbb{P}^1$-bundle over $T'$ by [24, Cor. 4.7]. This $\mathbb{P}^1$-bundle is associated with a semi-stable locally free sheaf, since $-K_{X'} = \nu^*(-K_X)$ and $-K_X$ is nef (cf. Corollary 4.10). Thus, Theorem 4.7 holds true when $\deg h > 1$, and we are done. □

*Proof of Proposition* 4.8. By the assumption: $\deg f \neq \lambda_f$ and by Lemma 4.9(6), we have $1 < \deg h < \deg f$. Note that

$$\deg h' = \deg h, \quad \deg f' = \deg f, \quad \text{and} \quad \lambda_{f'} = \lambda_f = \max\{\deg h, \deg f/\deg h\}$$

for endomorphisms $h'$ and $f'$ in the proof of Theorem 4.7 considered in the case where $\Sigma \neq \emptyset$. By the proof of Theorem 4.7 and by Lemma 4.11, we may assume that $T$ is an elliptic curve and $\pi\colon X \to T$ is a $\mathbb{P}^1$-bundle associated with a stable locally free sheaf $\mathcal{E}$ of degree 1. There is a point $o \in T$ such that $h(o) = o$, since $\deg h > 1$. Then $h$ is a group homomorphism with respect a Lie group structure on $T$ in which $o$ is the zero element. In particular, $\mu_{(2)} \circ h = h \circ \mu_{(2)}$ for the multiplication map $\mu_{(2)}\colon T = T_{(2)} \to T$ by 2. Here, $X_{(2)} = X \times_T T_{(2)} \simeq \mathbb{P}^1 \times T_{(2)}$ by a property of stable locally free sheaf of degree 1 (cf. Fact 2.23(C)). The first projection $p_1\colon X_{(2)} \to X$ is étale, and there is an endomorphism $f_{(2)}\colon X_{(2)} \to X_{(2)}$ such that $p_1 \circ f_{(2)} = f \circ p_1$ by [24, Lem. 4.1]. Hence, we can take $\mathbb{P}^1 \times X \simeq X_{(2)}$ as $V$. Thus, we are done. □

## 5. Proofs of Theorems in the introduction

5.1. **The number of prime components of $S_f$: Proof of Theorem 1.3.** We shall prove Theorem 1.3 in the introduction. Let $f$ be a non-isomorphic surjective endomorphism of a normal projective surface $X$. Theorem 1.3 announces the inequality $\boldsymbol{n}(S_f) \leq \boldsymbol{\rho}(X) + 2$ with characterization of $X$ in case $\boldsymbol{n}(S_f) \geq \boldsymbol{\rho}(X) + 1$. Here, $\boldsymbol{n}(S_f)$ stands for the number of prime components of $S_f$ (cf. Definition 4.1).

**Lemma 5.1.** *Let $f$ be a non-isomorphic surjective endomorphism of a normal projective surface $X$. Then $(X, S_f)$ is not a* pseudo-toric surface of defect 1 *in the sense of* [22, Def. 6.1].

*Proof.* Assume that $(X, S_f)$ is a pseudo-toric surface of defect 1. Then $S_f$ is a cyclic chain of rational curves and $\boldsymbol{n}(S_f) = \boldsymbol{\rho}(X) + 1$.

First, we consider the case where $\lambda_f > \delta_f$. Then $\boldsymbol{\rho}(X) = 2$ and $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ has two eigenvalues $\lambda_f > \lambda_f^\dagger = \deg f/\lambda_f$ by [24, Prop. 3.25], since $K_X \sim -S_f$ is not pseudo-effective. In particular, $\boldsymbol{n}(S_f) = 3$, and hence, $S_f$ is a cyclic chain of rational curves consisting of three prime components $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$. By replacing $f$ with a power $f^k$, we may assume that $f^*\Gamma_i = m_i\Gamma_i$ for any $1 \leq i \leq 3$ (cf. [24, Lem. 2.17(1)]), where $m_i = \lambda_f$ or $m_i = \lambda_f^\dagger$. If $i \neq j$, then $m_i m_j = \deg f$ by $\Gamma_i\Gamma_j > 0$ (cf. [24, Prop. 2.20(1)]). This is impossible.

Therefore, $\lambda_f = \delta_f$. By replacing $f$ with some power $f^k$, we may assume that $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map by [24, Cor. 3.23], and moreover, we may assume that $f^*\Gamma = \delta_f\Gamma$ for any prime component $\Gamma$ of $S_f$ (cf. [24, Def. 2.16]). By [22, Thm. 6.5], there is a *toroidal blowing up $\mu \colon X' \to X$ with respect to $(X, S_f)$* (cf. [22, Def. 4.19]) such that

- $(X', S')$ is a pseudo-toric surface of defect 1 for $S' = \mu^{-1}(S_f)$,
- $K_{X'} + S' = \mu^*(K_X + S_f) \sim 0$,
- there is a negative curve on $X'$ not contained in $S'$ (cf. [22, Def. 6.7]).

Moreover, there is an endomorphism $f' \colon X' \to X'$ such that $\mu \circ f' = f \circ \mu$ by [23, Prop. 5.6]. Then $S_{f'} = \mu^{-1}(S_f) = S'$ by [24, Lem. 3.15(3)]. This is a contradiction, since $S_{f'}$ contains all the negative curves on $X$ (cf. [24, Prop. 2.20(3)]). Therefore, $(X, S_f)$ is not a pseudo-toric surface of defect 1. $\square$

**Lemma 5.2.** *Let $f$ be a non-isomorphic surjective endomorphism of a normal projective surface $X$ such that $K_X + S_f$ is not pseudo-effective. If one of the three conditions below is satisfied, then $-(K_X + S_f)$ is nef*:

- $\lambda_f = \delta_f$;
- $\boldsymbol{n}(S_f) \geq \boldsymbol{\rho}(X)$;
- $-K_X$ *is pseudo-effective.*

*In particular, if $-K_X$ is big, then $-(K_X + S_f)$ is semi-ample.*

*Proof.* The last assertion follows from the previous one by [24, Prop. 1.5]. Assume first that $\lambda_f = \delta_f$. By [24, Cor. 3.23] and by replacing $f$ with a power $f^k$, we may assume that $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map. Then $-(K_X + S_f)$ is nef by

$$-(\delta_f - 1)(K_X + S_f) \approx K_X + S_f - f^*(K_X + S_f) = \Delta_f,$$

where the refined ramification divisor $\Delta_f$ is nef (cf. [24, Prop. 2.20(4)]). Thus, we may assume that $\lambda_f > \delta_f$.

By applying [24, Prop. 3.25] to $K_X+S_f$, we see that $\boldsymbol{\rho}(X)=2$, $X$ has no negative curves, and there exist a $\mathbb{P}^1$-fibration $\pi\colon X\to T$ to a non-singular projective curve $T$ and an endomorphism $h\colon T\to T$ such that

- $\pi\circ f=h\circ\pi$, $\deg h \mid \deg f$, $\lambda_f=\max\{\deg h, \deg f/\deg h\}$,
- $(K_X+S_f)F<0$ and $F_{\mathrm{red}}\simeq\mathbb{P}^1$ for any fiber $F$ of $\pi$.

Here, $\mathsf{R}=\mathbb{R}_{\geq 0}\,\mathrm{cl}(F)$ is an extremal ray of $\overline{\mathrm{NE}}(X)=\mathrm{Nef}(X)$ and $\pi$ is the contraction morphism of $\mathsf{R}$. Let $\mathsf{R}'$ be the other extremal ray. This is generated by the numerical class of a nef $\mathbb{Q}$-divisor $L$ such that $f^*L\approx(\deg f/\deg h)L$ and $L^2=0$. There exist rational numbers $\alpha$ and $\beta$ such that

$$-(K_X+S_f)\approx\alpha F+\beta L.$$

Here, $\beta>0$ by $(K_X+S_f)F<0$. Moreover, $\alpha(\deg h-1)\geq 0$, since

$$\Delta_f=K_X+S-f^*(K_X+S_f)\approx\alpha(\deg h-1)F+\beta(\deg f/\deg h-1)L$$

is nef. If $\deg h>1$, then $\alpha\geq 0$, and $-(K_X+S_f)$ is nef. Hence, we may assume that $\deg h=1$. Then $\lambda_f=\deg f$, and there is a positive integer $m$ such that $(f^m)^*\Gamma=(\deg f)^m\Gamma$ for any prime component $\Gamma$ of $S_f$ (cf. [24, Lem. 2.17(1)]). Consequently, $\mathrm{cl}(S_f)\in\mathsf{R}'=\mathbb{R}_{\geq 0}\,\mathrm{cl}(L)$ and $S_fL=0$. In particular, every prime component of $S_f$ dominates $T$. Since $0>(K_X+S_f)F=-2+S_fF$, we have $\boldsymbol{n}(S_f)\leq 1$ and $\boldsymbol{n}(S_f)<2=\boldsymbol{\rho}(X)$; by contraposition, if $\boldsymbol{n}(S_f)\geq\boldsymbol{\rho}(X)$, then $\deg h>1$ and $-(K_X+S_f)$ is nef. If $-K_X$ is pseudo-effective, then $K_XL\leq 0$, and we have

$$\alpha FL=-(K_X+S_f)L=-K_XL\geq 0$$

by $S_fL=0$; hence, $\alpha\geq 0$, and $-(K_X+S_f)$ is nef. Thus, we are done. □

**Lemma 5.3.** *Let $f$ be a non-isomorphic surjective endomorphism of a normal projective surface $X$. If $-(K_X+S_f)$ is not nef, then $\boldsymbol{n}(S_f)\leq\boldsymbol{\rho}(X)$.*

*Proof.* We shall derive a contradiction assuming that $-(K_X+S_f)$ is not nef and $\boldsymbol{n}(S_f)>\boldsymbol{\rho}(X)$. Then $K_X+S_f$ is pseudo-effective by Lemma 5.2. Since $K_X+S_f\not\approx 0$ and $S_f\neq 0$, by Theorem 3.1 and Remark 3.2, we have a finite surjective morphism $\nu\colon\mathbb{P}^1\times T\to X$ étale in codimension 1 for a non-singular projective curve $T$ with $\boldsymbol{g}(T)\geq 2$, and $\nu^{-1}(S_f)=\{P_1,P_2\}\times T$ for two points $P_1$ and $P_2$ of $\mathbb{P}^1$; in other words, the case (1) only occurs in Theorem 3.1, where $S=S_f$. Thus, $\boldsymbol{\rho}(X)<\boldsymbol{n}(S_f)\leq\boldsymbol{n}(\nu^*S_f)=2$. On the other hand $\boldsymbol{\rho}(X)>1$, since the semi-ample $\mathbb{Q}$-divisor $K_X+S_f$ defines a fibration from $X$ to a non-singular projective curve (cf. [24, §5.1]). This is a contradiction. □

We shall prove Theorem 1.3.

*Proof of Theorem* 1.3. We may assume that $\boldsymbol{n}(S_f)>\boldsymbol{\rho}(X)$. Then $-(K_X+S_f)$ is nef by Lemma 5.3. Since $(X,S_f)$ is log-canonical (cf. [23, Cor. 3.6], [24, Thm. E]), we can apply Shokurov's criterion [29, Thm. 6.4] for toric surfaces (cf. [22, Thm. 1.1]) to $(X,S_f)$. Then $\boldsymbol{n}(S_f)\leq\boldsymbol{\rho}(X)+2$, where the equality holds if and only if $(X,S_f)$ is a toric surface. If $\boldsymbol{n}(S_f)=\boldsymbol{\rho}(X)+1$, then one of the following holds by [22, Thm. 1.3]:

- $(X, B + S_f)$ is a toric surface for a prime divisor $B \not\subset S_f$;
- $(X, S_f)$ is a pseudo-toric surface of defect 1;
- $(X, S_f)$ is a half-toric surface.

Here, the second case does not occur by Lemma 5.1. Thus, we are done. □

5.2. **On non-quotient singularities: Proof of Theorem 1.2.** We shall prove Theorem 1.2 after proving Proposition 5.4 and Lemma 5.5 below.

**Proposition 5.4.** *Let $X$ be a normal projective surface admitting a non-isomorphic surjective endomorphism $f$. Assume that $X$ has only rational singularities and that $X$ has a non-quotient singular point $P$. Then*:

(1) *$X$ is a rational surface with $\boldsymbol{\rho}(X) = 1$ and $-K_X$ is ample*;
(2) *$P$ is a unique non-quotient singular point of $X$ and $f^{-1}(P) = \{P\}$*;
(3) *$f$ is étale in codimension 1 on an open neighborhood of $P$*;
(4) *the* index 1 cover *of $(X, P)$ with respect to $K_X$ (cf.* [23, Def. 4.18(4)]*) is a simple elliptic singularity.*

*Let $\varphi\colon Y \to X$ be a birational morphism from a normal projective surface $Y$ such that $\varphi$ is an isomorphism outside $\varphi^{-1}(P)$ and that $\varphi$ gives a* standard partial resolution *of the log-canonical pair $(X, 0)$ at $P$ (cf.* [23, Def. 3.24]*). Then*;

(5) *$Y$ has only quotient singularities with $\boldsymbol{\rho}(Y) = 2$*;
(6) *$K_Y + E = \varphi^* K_X$ for $E = \varphi^{-1}(P) \simeq \mathbb{P}^1$*;
(7) *$E$ is a unique negative curve on $Y$*;
(8) *there is an endomorphism $f_Y\colon Y \to Y$ such that $\varphi \circ f_Y = f \circ \varphi$.*

*Proof.* The surface $X$ is rational and $K_X$ is not pseudo-effective by [24, Thm. 6.1]. In fact, surfaces satisfying conditions in [24, Thm. 6.1] have only quotient singularities or simple elliptic singularities. Let $\Lambda$ be the set of non-quotient singular points of $X$. If $P \in \Lambda$, then $f^{-1}(P) \subset \Lambda$. Hence, there is an integer $k > 0$ such that $(f^k)^{-1}(P) = \{P\}$ for any $P \in \Lambda$ (cf. [24, Lem. 2.2]). Then $f^k$ induces a non-isomorphic finite surjective endomorphism $(X, P) \to (X, P)$ of the germ $(X, P)$ of a normal complex analytic surface for any $P \in \Lambda$. Hence, $\Lambda \cap \operatorname{Supp} R_{f^k} = \emptyset$ by [6, Thm. B(3)] (cf. [23, Cor. 3.7]). In particular, $\Lambda \cap \operatorname{Supp} R_f = \emptyset$, and hence, $f$ is étale in codimension 1 on an open neighborhood of $P$.

Let $\varphi\colon Y \to X$ be a birational morphism from a normal projective surface $Y$ such that $\varphi$ is an isomorphism outside $\varphi^{-1}\Lambda$ and that $\varphi$ gives standard partial resolutions of the log-canonical pair $(X, 0)$ at all $P \in \Lambda$. See [23, Exam. 4.29] for a detailed description of the standard partial resolution. Then the following hold for the $\varphi$-exceptional locus $E = \varphi^{-1}\Lambda$ by properties of essential blowings up (cf. [23, Def. 4.24]):

- $(Y, E)$ is log-canonical with $K_Y + E = \varphi^* K_X$;
- $(Y, E)$ is 1-*log-terminal* outside $\operatorname{Sing} E$ (cf. [23, Def. 2.1]);
- $Y$ has only quotient singularities.

Since $(X, P)$ is not a cusp singularity for any $P \in \Lambda$, by applying [23, Thm 5.3], we have an endomorphism $g\colon Y \to Y$ such that $\varphi \circ g = f^{2k} \circ \varphi$. Then $\mu_* S_g = S_{f^{2k}} = S_f$

(cf. [24, Lem. 3.15(3)]), and moreover, $E \leq S_g$, since $E$ consists of negative curves (cf. [24, Prop. 2.20(3)]).

We shall show that $\varphi^{-1}(P)$ is a connected component of $S_g$ for any $P \in \Lambda$: Let $\Gamma$ be a prime component of $S_g$ not contained in $\varphi^{-1}(P)$. Then $\Gamma$ is completely invariant under $g^l$ for some $l$, and $\mu(\Gamma)$ is also completely invariant under $f^{2l}$. Hence, $(X, \mu(\Gamma))$ is log-canonical (cf. [23, Cor. 3.6], [24, Thm. E]), and $X$ has only quotient singularities along $\mu(\Gamma)$ (cf. [23, Fact 2.5]). Thus, $P \notin \mu(\Gamma)$ and $\varphi^{-1}(P) \cap \Gamma = \emptyset$. Therefore, $\varphi^{-1}(P)$ is a connected component of $S_g$.

We shall show that $\boldsymbol{\rho}(Y) = 2$. Assume the contrary. Then $\boldsymbol{\rho}(Y) \geq 3$. If $K_Y + S_g$ is pseudo-effective, then there is a finite Galois cover $\nu\colon V \to Y$ from a toric surface $V$ with $\nu^{-1}S_g$ as the boundary divisor, by Theorem 3.1 and Remark 3.2, since $S_g \neq 0$. In this case, $S_g$ is connected and big, but the connected component $\varphi^{-1}(P)$ is not big for $P \in \Lambda$: This is a contradiction. Therefore, $K_Y + S_g$ is not pseudo-effective. However, even in this case, we have a contradiction, since $S_g$ is connected and big by Proposition 4.3 and Theorem 4.5. Hence, $\boldsymbol{\rho}(Y) = 2$.

Consequently, $\boldsymbol{\rho}(X) = 1$, $-K_X$ is ample, $\Lambda$ consists of one point $P$, and $E = \varphi^{-1}(P)$ is irreducible. In particular, $f^{-1}(P) = \{P\}$. By [23, Exam. 4.29], we see that the index 1 cover of $(X, P)$ is a simple elliptic singularity and $E \simeq \mathbb{P}^1$. Thus, we have shown all the assertions except (7) and (8). Here, (8) follows from [23, Lem. 5.23]. The remaining assertion (7) is shown as follows: If $E'$ is a negative curve on $Y$ different from $E$, then $\varphi(E')$ is not a negative curve, since $\boldsymbol{\rho}(X) = 1$, and we have $E \cap E' \neq \emptyset$. On the other hand, $S_g$ contains all the negative curves on $Y$, and $E$ is a connected component of $S_g$. Thus, such $E'$ does not exist. This proves (7), and we are done. ☐

On the index 1 cover of $(X, P)$, we note the following:

**Lemma 5.5.** *In the situation of Proposition* 5.4, *let $\mathcal{X}$ and $\mathcal{X}^\circ$ be connected open neighborhoods of $P$ in $X$ such that*

$$\operatorname{Sing} \mathcal{X} = \{P\}, \quad \mathcal{X}^\circ \subset \mathcal{X} \cap f^{-1}(\mathcal{X}), \quad \textit{and} \quad rK_{\mathcal{X}} \sim 0$$

*for the* local Cartier index *$r$ of $K_X$ at $P$ (cf.* [23, Def. 4.18(1)]). *Let $\xi\colon \mathcal{V} \to \mathcal{X}$ be an* index 1 cover *with respect to $K_{\mathcal{X}}$ (cf.* [23, Def. 4.18(2)]). *Then*:

(1) *The morphism $\xi$ is étale in codimension* 1, *and $\xi^{-1}(P)$ consists of one point $Q$ at which $\mathcal{V}$ has a simple elliptic singularity.*

(2) *The action of the Galois group $G$ of $\xi$ on $\mathcal{V}$ lifts to the minimal resolution $\mathcal{W}$ of $\mathcal{V}$, and the quotient variety $G\backslash\mathcal{W}$ is isomorphic to $\mathcal{Y} := \varphi^{-1}\mathcal{X}$ over $\mathcal{X}$. In particular, $(\varphi|_{\mathcal{Y}}) \circ \eta = \xi \circ \phi$ for the minimal resolution $\phi\colon \mathcal{W} \to \mathcal{V}$ and the quotient morphism $\eta\colon \mathcal{W} \to G\backslash\mathcal{W} \simeq \mathcal{Y}$.*

(3) *By replacing $\mathcal{X}^\circ$ with an open neighborhood of $P$, one can find morphisms $f_{\mathcal{V}}\colon \mathcal{V}^\circ \to \mathcal{V}$ and $f_{\mathcal{W}}\colon \mathcal{W}^\circ \to \mathcal{W}$ satisfying equalities*

$$\xi \circ f_{\mathcal{V}} = (f|_{\mathcal{X}^\circ}) \circ \xi|_{\mathcal{V}^\circ}, \quad \phi \circ f_{\mathcal{W}} = f_{\mathcal{V}} \circ \phi|_{\mathcal{W}^\circ}, \quad \eta \circ f_{\mathcal{W}} = (f_Y|_{\mathcal{Y}^\circ}) \circ \eta|_{\mathcal{W}^\circ},$$

*where $\mathcal{V}^\circ := \xi^{-1}(\mathcal{X}^\circ)$, $\mathcal{Y}^\circ := \varphi^{-1}(\mathcal{X}^\circ)$, and $\mathcal{W}^\circ := \phi^{-1}(\mathcal{V}^\circ) = \eta^{-1}(\mathcal{Y}^\circ)$. In particular, the cubic diagram in Figure* 1 *is commutative.*

$$
\begin{tikzcd}
\mathcal{W}^\circ \arrow[rrr, "\phi|_{\mathcal{W}^\circ}"] \arrow[dr, "f_{\mathcal{W}}"'] \arrow[ddd, "\eta|_{\mathcal{W}^\circ}"'] & & & \mathcal{V}^\circ \arrow[dl, "f_{\mathcal{V}}"'] \arrow[ddd, "\xi|_{\mathcal{V}^\circ}"] \\
& \mathcal{W} \arrow[r, "\phi"] \arrow[d, "\eta"] & \mathcal{V} \arrow[d, "\xi"'] & \\
& \mathcal{Y} \arrow[r, "\varphi|_{\mathcal{Y}}"'] & \mathcal{X} & \\
\mathcal{Y}^\circ \arrow[ur, "f_Y|_{\mathcal{Y}^\circ}"] \arrow[rrr, "\varphi|_{\mathcal{Y}^\circ}"'] & & & \mathcal{X}^\circ \arrow[ul, "f|_{\mathcal{X}^\circ}"']
\end{tikzcd}
$$

FIGURE 1. A cubic diagram

(4) *The elliptic curve* $C = \phi^{-1}(Q) = \phi^{-1}(\xi^{-1}(P))$ *admits an endomorphism* $f_C$ *such that* $(\eta|_C) \circ f_C = (f_Y|_E) \circ \eta|_C$ *for the induced morphism* $\eta|_C \colon C \to E = \varphi^{-1}(P)$.

*Proof.* We have (1) by Proposition 5.4(4). The minimal resolution of the singularity $(X, P)$ is known to be obtained as the minimal resolution of singularities of $G\backslash\mathcal{W}$. Hence, $G\backslash\mathcal{W} \to \mathcal{X}$ gives the standard partial resolution of $(X, P)$ (cf. [23, Exam. 4.29(4)]). Thus, $G\backslash\mathcal{W} \simeq \mathcal{Y} = \varphi^{-1}\mathcal{X}$, and we have (2). By replacing $\mathcal{X}^\circ$ with an open neighborhood of $P$, we have a morphism $f_{\mathcal{V}} \colon \mathcal{V}^\circ \to \mathcal{V}$ such that

$$
\xi \circ f_{\mathcal{V}} = (f|_{\mathcal{X}^\circ}) \circ \xi|_{\mathcal{V}^\circ} \tag{V-1}
$$

by [23, Lem. 4.21(2)], since $K_{\mathcal{X}^\circ} = f^*K_{\mathcal{X}}$ (cf. Proposition 5.4(3)). On the other hand, the restriction of $f_Y$ to the open subset $\mathcal{Y}^\circ = \varphi^{-1}(\mathcal{X}^\circ)$ is regarded as a morphism $f_Y|_{\mathcal{Y}^\circ} \colon \mathcal{Y}^\circ \to \mathcal{Y} = \varphi^{-1}\mathcal{X}$ such that

$$
(\varphi|_{\mathcal{Y}}) \circ f_Y|_{\mathcal{Y}^\circ} = f \circ \varphi|_{\mathcal{Y}^\circ} \tag{V-2}
$$

by Proposition 5.4(8). Since $\mathcal{W}$ is isomorphic to the normalization of $\mathcal{V} \times_{\mathcal{X}} \mathcal{Y}$ and since $\mathcal{W}^\circ = \phi^{-1}\mathcal{V}^\circ = \eta^{-1}\mathcal{Y}^\circ$ is isomorphic to the normalization of $\mathcal{V}^\circ \times_{\mathcal{X}^\circ} \mathcal{Y}^\circ$, the morphism $f_{\mathcal{V}} \times f_Y|_{\mathcal{Y}^\circ} \colon \mathcal{V}^\circ \times \mathcal{Y}^\circ \to \mathcal{V} \times \mathcal{Y}$ defines a morphism $f_{\mathcal{W}} \colon \mathcal{W}^\circ \to \mathcal{W}$ such that

$$
\phi \circ f_{\mathcal{W}} = f_{\mathcal{V}} \circ \phi|_{\mathcal{W}^\circ} \quad \text{and} \quad \eta \circ f_{\mathcal{W}} = (f_Y|_{\mathcal{Y}^\circ}) \circ \eta|_{\mathcal{W}^\circ},
$$

by (V-1) and (V-2). This shows (3). For (4), it is enough to set $f_C$ to be $f_{\mathcal{W}}|_C$. $\square$

We shall prove Theorem 1.2.

*Proof of Theorem* 1.2. Now $X$ has a non-quotient singular point. If $X$ has an irrational singularity, then $X$ is a projective cone over an elliptic curve by [24, Prop. 6.2]. Thus, we may assume that $X$ has only rational singularities, and we can apply Proposition 5.4. We proceed the arguments in the proof of Proposition 5.4. By replacing $f$ with a power $f^k$, we may assume that $f_Y^* \colon \mathsf{N}(Y) \to \mathsf{N}(Y)$ is a scalar map by [24, Lem. 3.7], since $\boldsymbol{\rho}(Y) = 2$ and $Y$ has a negative curve.

Since $-K_X$ is ample, $K_Y + E = \varphi^* K_X$ is not nef (cf. Proposition 5.4(6)), and there is an extremal ray $\mathsf{R}$ of $\overline{\mathrm{NE}}(Y)$ such that $(K_Y + E)\mathsf{R} < 0$ by the cone theorem (cf. [24, Thm. 1.9]). Hence, $\overline{\mathrm{NE}}(Y) = \mathsf{R} + \mathbb{R}_{\geq 0}\,\mathrm{cl}(E)$ as $(K_Y + E)E = 0$, and the contraction morphism $\pi\colon Y \to T$ of the extremal ray $\mathsf{R}$ is a fibration to a non-singular projective curve $T$, since $E$ is a unique negative curve on $X$ (cf. Proposition 5.4(7)). Here, $T \simeq \mathbb{P}^1$ as $X$ is rational (cf. Proposition 5.4(1)). Since $f_Y^*$ is a scalar map preserving $\mathsf{R}$, there is an endomorphism $h\colon T \to T$ such that $\pi \circ f_Y = h \circ \pi$ and $\deg h = \delta_f$ (cf. [24, Lem. 3.16]). Let $F$ be a general fiber of $\pi$. Then $0 > (K_Y + E)F = -2 + EF$. Thus, $E$ is a section of $\pi$, and $f_Y|_E\colon E \to E$ corresponds to $h\colon T \to T$ by the isomorphism $\pi|_E\colon E \to T$.

Let $\eta|_C\colon C \to E$ be the finite surjective morphism in Lemma 5.5(4), where $C = \phi^{-1}(\xi^{-1}P)$ is an elliptic curve. Then $\eta|_C$ is a cyclic cover, since the morphism $\eta\colon \mathcal{W} \to \mathcal{Y}$ in Lemma 5.5(2) is so. Let $\tau\colon C \to T$ be the composite $(\pi|_E) \circ \eta|_C$. Then, for the endomorphism $f_C\colon C \to C$ in Lemma 5.5(4), we have

$$\tau \circ f_C = (\pi|_E) \circ (f_Y|_E) \circ \eta|_C = h \circ (\pi|_E) \circ \eta|_C = h \circ \tau$$

by $\pi \circ f_Y = h \circ \pi$. Let $W$ be the normalization of $Y \times_T C$, which induces a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\vartheta} & Y \\ {\scriptstyle\varpi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ C & \xrightarrow{\tau} & T. \end{array}$$

We have an endomorphism $f_W : W \to W$ such that $\varpi \circ f_W = f_C \circ \varpi$ and $\vartheta \circ f_W = f_Y \circ \vartheta$ (cf. [24, Lem. 4.1]). Then $\varpi$ is a $\mathbb{P}^1$-bundle by [24, Cor. 4.8], since $C$ is an elliptic curve, $f_C$ is étale, $\varpi$ is a $\mathbb{P}^1$-fibration, and $\vartheta^* E$ is an elliptic curve being a negative section of $\varpi$. We shall show that $\vartheta$ is étale in codimension 1: The ramification divisor $R_\vartheta$ is supported on a union of fibers of $\varpi$, and we have $K_W + \vartheta^* E = \vartheta^*(K_Y + E) + R_\vartheta$. Then

$$R_\vartheta(\vartheta^* E) = (K_W + \vartheta^* E)\vartheta^* E - (\vartheta^*(K_Y + E))\vartheta^* E = -(\deg \vartheta)(K_Y + E)E = 0.$$

Hence, $R_\vartheta = 0$. This means that $\vartheta$ is étale in codimension 1.

Let $W \to V \xrightarrow{\nu} X$ be the Stein factorization of $\varphi \circ \vartheta$. Then $W \to V$ is the contraction morphism of $\vartheta^* E$, and $V$ is isomorphic to a projective cone over the elliptic curve $C \simeq \vartheta^* E$. The finite morphism $\nu\colon V \to X$ is also a cyclic cover and is étale in codimension 1 as $\vartheta$ is so. The endomorphism $f_W$ descends to an endomorphism $f_V : V \to V$ satisfying $\nu \circ f_V = f \circ \nu$, since we have $\varphi \circ \vartheta \circ f_W = \varphi \circ f_Y \circ \vartheta = f \circ \varphi \circ \vartheta$. Thus, we have finished the proof of Theorem 1.2. □

5.3. **Proof of Theorem 1.1.** Finally, we shall prove Theorem 1.1.

*Proof of Theorem* 1.1. If $X$ satisfies (I) of Theorem 1.1, then it admits a non-isomorphic surjective endomorphism by Lemmas 2.5 and 2.6, Corollary 2.4, and Theorems 2.20 and 2.21. Hence, we are reduced to proving the following four assertions for any normal projective surface $X$ having non-isomorphic surjective endomorphism $f$:

(a) If $K_X$ is pseudo-effective, then either (I-2) or (I-3) of Theorem 1.1 holds.

(b) If $K_X$ is not pseudo-effective and $\boldsymbol{\rho}(X) \geq 3$, then either (I-6) or (II) of Theorem 1.1 holds.

(c) If $K_X$ is not pseudo-effective, $\boldsymbol{\rho}(X) \leq 2$, and $-K_X$ is big, then one of (I-4), (I-5), (I-6), and (II) of Theorem 1.1 holds.

(d) If $K_X$ is not pseudo-effective, $\boldsymbol{\rho}(X) = 2$, and $-K_X$ is not big, then either (I-1) or (I-4) of Theorem 1.1 holds.

Here, "(I-j) of Theorem 1.1 holds" for $1 \leq j \leq 6$ means that the condition (I-j) is satisfied for a finite Galois cover $V \to X$ étale in codimension 1. Note that if $K_X$ is not pseudo-effective and $\boldsymbol{\rho}(X) = 1$, then $-K_X$ is big; this case is treated in (c).

Assertions (a) and (d) are consequences of [24, Thm. A] and Theorem 4.7, respectively. We shall show (b). If $K_X + S_f$ is pseudo-effective, then (I-6) of Theorem 1.1 holds by [24, Thm. A], since $\boldsymbol{\rho}(X) \geq 3$ and $S_f \neq 0$. Suppose that $K_X + S_f$ is not pseudo-effective. Then $(X, S_f)$ is an $\mathcal{L}$-surface by Proposition 4.3, since $\boldsymbol{\rho}(X) \geq 3$. Hence, $X$ is rational, and $-K_X$ is big by (1) and (3) of Theorem 4.5. Moreover, $X$ has only quotient singularities by Theorem 1.2. Thus, $X$ satisfies (II) of Theorem 1.1, and we have proved (b).

Finally, we shall prove (c). If $X$ is irrational, then cases (2) and (3) of [24, Thm. 4.16] occurs, since $-K_X$ is big. Thus, in this case, (I-4) or (I-5) of Theorem 1.1 is satisfied for $V = X$. If $X$ is rational and non-singular, then $X$ is a toric surface by [20, Thm. 3]; thus, it satisfies (I-6) of Theorem 1.1 for $V = X$. If $X$ is rational and singular and if there is no finite Galois cover $V \to X$ étale in codimension 1 satisfying (I-5) of Theorem 1.1, then $X$ has only quotient singularities by Theorem 1.2, and hence, it satisfies (II) of Theorem 1.1. This proves (c). Thus, the proof of Theorem 1.1 has been completed. □

## References


[1] V. Alexeev and V. V. Nikulin, *Del Pezzo and K3 surfaces*, MSJ Memoirs vol. **15**, Math. Soc. Japan, 2006.

[2] E. Amerik, On endomorphisms of projective bundles, Manuscripta Math. **111** (2003), 17–28.

[3] M. F. Atiyah, Complex fibre bundles and ruled surfaces, Proc. London Math. Soc. **5** (1955), 407–434.

[4] M. F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc. **7** (1957), 414–452.

[5] H. Cartan, Quotient d'une variété analytique par un groupe discret d'automorphismes, Séminaire Henri Cartan, **6** (1953-1954), exp. n°12, 1–13.

[6] C. Favre, Holomorphic self-maps of singular rational surfaces, Publ. Mat. **54** (2010), 389–432.

[7] Y. Fujimoto and N. Nakayama, Compact complex surfaces admitting non-trivial surjective endomorphisms, Tohoku Math. J. **57** (2005), 395–426.

[8] Y. Fujimoto and N. Nakayama, Endomorphisms of smooth projective 3-folds with nonnegative Kodaira dimension, II, J. Math. Kyoto Univ. **47** (2007), 79–114.

[9] W. Fulton, *Introduction to Toric Varieties*, Ann. of Math. Studies **131**, Princeton Univ. Press, 1993.

[10] H. Grauert and R. Remmert, Komplexe Räume, Math. Ann. **136** (1958), 245–318.

[11] A. Grothendieck and Mme M. Raynaud, *Revêtements Étales et Groupe Fondamental* (SGA1), Lecture Notes in Math. **224**, Springer-Verlag, 1971; A new updated edition: Documents Math. **3**, Soc. Math. France, 2003.

[12] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer-Verlag, 1977.

[13] S. Iitaka, On $D$-dimensions of algebraic varieties, J. Math. Soc. Japan **23** (1971), 356–373.

[14] M. Inoue, New surfaces with no meromorphic functions II, *Complex Analysis and Algebraic Geometry*, (eds. W. L. Baily, Jr. and T. Shioda), pp. 280–292, Iwanami Shoten Publishers and Cambridge Univ. Press 1977.

[15] K. Kodaira, On complex analytic surfaces, II, III, Ann. of Math. **77** (1963), 563–626; ibid. **78** (1963), 1–40.

[16] K. Kodaira, On the structure of compact complex analytic surfaces, I, Amer. J. Math. **86** (1964), 751–798.

[17] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, Third ed. Ergeb. Math. Grenzgeb. (2), **34**, Springer-Verlag, 1994.

[18] I. Nakamura, On surfaces of class $\mathrm{VII}_0$ with curves, Invent. Math. **78** (1984), 393–443.

[19] I. Nakamura, On surfaces of class $\mathrm{VII}_0$ with curves, II, Tohoku Math. J. **42** (1990), 475–516.

[20] N. Nakayama, Ruled surfaces with non-trivial surjective endomorphisms, Kyushu J. Math. **56** (2002), 433–446.

[21] N. Nakayama, On complex normal projective surfaces admitting non-isomorphic surjective endomorphisms, unpublished preprint, 2008.

[22] N. Nakayama, A variant of Shokurov's criterion of toric surface, *Algebraic Varieties and Automorphism Groups* (eds. K. Masuda, et. al.), pp. 287–392, Adv. Stud. in Pure Math., **75**, Math. Soc. Japan, 2017.

[23] N. Nakayama, Singularity of normal complex analytic surfaces admitting non-isomorphic finite surjective endomorphisms, preprint RIMS-1920, Kyoto Univ., 2020.

[24] N. Nakayama, On normal Moishezon surfaces admitting non-isomorphic surjective endomorphisms, preprint RIMS-1923, Kyoto Univ., 2020.

[25] T. Oda, *Torus embeddings and applications*, Tata Inst. Fund. Res. **58**, Springer-Verlag, 1978.

[26] T. Oda, *Convex Bodies and Algebraic Geometry*, Ergeb. Math. Grenzgeb. (3), **15**, Springer-Verlag, 1988.

[27] F. Sakai, Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps, Math. Ann. **254** (1980), 89–120.

[28] F. Sakai, Weil divisors on normal surfaces, Duke Math. J. **51** (1984), 877–888.

[29] V. V. Shokurov, Complements on surfaces, J. Math. Sci. New York **102**, No. 2, (2000), 3876–3932.

[30] T. Suwa, On ruled surfaces of genus 1, J. Math. Soc. Japan **21** (1969), 291–311.

[31] K. Ueno, Kodaira dimensions for certain fibre spaces, *Complex Analysis and Algebraic Geometry* (eds. W. L. Baily, Jr. and T. Shioda), pp. 280–292, Iwanami Shoten Publishers and Cambridge Univ. Press, 1977.



Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606-8502 Japan
*Email address*: `nakayama@kurims.kyoto-u.ac.jp`