



TITLE:

Mono-anabelian reconstruction of generalized fiber subgroups from a configuration space group equipped with its collection of log-full subgroups

AUTHOR(S):

HIGASHIYAMA, Kazumi

CITATION:

HIGASHIYAMA, Kazumi. Mono-anabelian reconstruction of generalized fiber subgroups from a configuration space group equipped with its collection of log-full subgroups. 2021: 1-29: RIMS-1942.

ISSUE DATE:

2021-03

URL:

<http://hdl.handle.net/2433/262135>

RIGHT:

RIMS-1942

**Mono-anabelian reconstruction of
generalized fiber subgroups
from a configuration space group
equipped with its collection of log-full subgroups**

By

Kazumi HIGASHIYAMA

March 2021



京都大学 数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Mono-anabelian reconstruction of generalized fiber subgroups from a configuration space group equipped with its collection of log-full subgroups

Kazumi HIGASHIYAMA

ABSTRACT. In the present paper, we study combinatorial anabelian geometry. The goal is to reconstruct group-theoretically the set of generalized fiber subgroups from the associated configuration space group equipped with its collection of log-full subgroups.

0. Introduction

Mochizuki and Tamagawa gave bi-anabelian algorithm to reconstruct fiber subgroups (cf. [MzTa], Definition 2.3, (iii)):

THEOREM 1 ([MzTa], Corollary 6.3). *Let $n \in \mathbb{Z}_{>0}$; p a prime number; $\square \in \{\dagger, \ddagger\}$; $(\square g, \square r)$ a pair of nonnegative integers such that $2\square g - 2 + \square r > 1$; $\square k$ an algebraic closed field of characteristic 0; $\square X^{\log}$ a smooth log curve over $\square k$ of type $(\square g, \square r)$ (cf. Definition 4, (iv)). Write $\pi_1^p(\square X_n^{\log})$ for the maximal pro- p quotient of the fundamental group of n -th configuration space (cf. Definition 5; Definition 10, (i), (ii)). Let $\alpha: \pi_1^p(\dagger X_n^{\log}) \xrightarrow{\sim} \pi_1^p(\ddagger X_n^{\log})$ be an isomorphism of profinite groups. Then α induces a bijection between the set of fiber subgroups of $\pi_1^p(\dagger X_n^{\log})$ and the set of fiber subgroups of $\pi_1^p(\ddagger X_n^{\log})$.*

After that, Hoshi, Minamide, and Mochizuki gave mono-anabelian algorithm to reconstruct generalized fiber subgroups (cf. Definition 10, (iv); [HMM], Definition 2.1, (ii)):

THEOREM 2 ([HMM], Theorem A, (i), (ii)). *Let $n \in \mathbb{Z}_{>1}$; p a prime number; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; k an algebraic closed field of characteristic 0; X^{\log} a smooth log curve over k of type (g, r) ; $\Delta^p(g, r, n)$ a profinite group which is isomorphic to $\pi_1^p(X_n^{\log})$. Then the following hold:*

- (i) *One may construct (g, r, n) associated to the intrinsic structure of $\Delta^p(g, r, n)$, i.e.,*

$$\Delta^p(g, r, n) \rightsquigarrow (g, r, n).$$

2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H10.

Key words and phrases. anabelian geometry, configuration space.

(ii) One may construct a set GFS (cf. Definition 3.1, (v)) associated to the intrinsic structure of $\Delta^P(g, r, n)$, i.e.,

$$\Delta^P(g, r, n) \rightsquigarrow \text{GFS}.$$

At the same time, the author gave bi-anabelian algorithm to reconstruct generalized fiber subgroups.

THEOREM 3 ([Hgsh], **Theorem 0.1, (v)**). *Let $n \in \mathbb{Z}_{>1}$; $\square \in \{\dagger, \ddagger\}$; $\square p$ a prime number; $(\square g, \square r)$ a pair of nonnegative integers such that $2\square g - 2 + \square r > 0$ and “ $r > 0$ ”; $\square k$ an algebraic closed field of characteristic $\neq \square p$; $\square X^{\log}$ a smooth log curve over $\square k$ of type $(\square g, \square r)$; $\alpha: \pi_1^{\dagger p}(\dagger X_n^{\log}) \xrightarrow{\sim} \pi_1^{\ddagger p}(\ddagger X_n^{\log})$ an isomorphism of profinite groups such that α induces a bijection between the set of log-full subgroups of $\pi_1^{\dagger p}(\dagger X_n^{\log})$ (cf. Definition 10, (iii)) and the set of log-full subgroups of $\pi_1^{\ddagger p}(\ddagger X_n^{\log})$. Then α induces bijection between the set of generalized fiber subgroups of $\pi_1^{\dagger p}(\dagger X_n^{\log})$ and the set of generalized fiber subgroups of $\pi_1^{\ddagger p}(\ddagger X_n^{\log})$.*

In the present paper, we give mono-anabelian algorithm to reconstruct (g, r, n) if $r > 0$ (cf. Theorem A, (ii)), and we give mono-anabelian algorithm to reconstruct generalized fiber subgroups, if $r > 0$ (cf. Theorem A, (v)), i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow (g, r, n) \quad (\text{if } r > 0),$$

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \text{GFS} \quad (\text{if } r > 0).$$

Our main result is as follows:

THEOREM A. *Let $n \in \mathbb{Z}_{>1}$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$ and “ $r > 0$ ”; p a prime number; k an algebraic closed field of characteristic $\neq p$; X^{\log} a smooth log curve over k of type (g, r) ; $\Delta^P(g, r, n)$ a profinite group which is isomorphic to $\pi_1^p(X_n^{\log})$. Write LFS (resp. LD, TD, DD, GFS) for the set of subgroups of $\Delta^P(g, r, n)$ such that any isomorphism $\Delta^P(g, r, n) \xrightarrow{\sim} \pi_1^p(X_n^{\log})$ induces a bijection*

$$\text{LFS} \xrightarrow{\sim} \{\text{log-full subgroups of } \pi_1^p(X_n^{\log})\}$$

$$(\text{resp. LD} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to log divisors}\},$$

$$\text{TD} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to tripodal divisors}\},$$

$$\text{DD} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to drift diagonals}\},$$

$$\text{GFS} \xrightarrow{\sim} \{\text{generalized fiber subgroups of } \pi_1^p(X_n^{\log})\})$$

(cf. Definition 6, (iv); Definition 7, (ii), (iv)). Write DC for the set of subsets of DD such that any isomorphism $\Delta^P(g, r, n) \xrightarrow{\sim} \pi_1^p(X_n^{\log})$ induces a bijection

$$\text{DC} \xrightarrow{\sim} \{\{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in \Lambda\}$$

$$| \Lambda: \text{a drift collection}\}$$

(cf. Definition 7, (v)). Then the following hold:

- (i) One may construct a set LD associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS (cf. Proposition 32, (i)), i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \text{LD}.$$

- (ii) One may construct (g, r, n) associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS (cf. Proposition 32, (v)), i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow (g, r, n).$$

- (iii) One may construct a set TD associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LD (cf. Proposition 34, (viii)), i.e.,

$$(\Delta^P(g, r, n), \text{LD}) \rightsquigarrow \text{TD}.$$

- (iv) One may construct a set DD associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS (cf. Proposition 35, (iii)), i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \text{DD}.$$

- (v) One may construct a set GFS associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS (cf. Proposition 37, (iii)), i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \text{GFS}.$$

REMARK 1. Note that one verifies easily that Theorem 2, (i), (ii), imply Theorem A, (ii), (v). In the present paper, we do not apply Theorem 1; Theorem 2, (i), (ii), to prove Theorem A.

This paper is organized as follows: In §1, we explain some notations. In §2, we introduce various type of log divisors and we calculate the number of various type of log divisors. In §3, we give mono-anabelian algorithm to reconstruct (g, r, n) if $r > 0$, and we give mono-anabelian algorithm to reconstruct a set GFS if $r > 0$.

1. Notation

DEFINITION 1. Let a, b be nonnegative integers. Then

$$\binom{b}{a} \stackrel{\text{def}}{=} \begin{cases} \frac{b!}{a!(b-a)!} & (a \leq b) \\ 0 & (a > b), \end{cases}$$

where $n! \stackrel{\text{def}}{=}} n \times (n-1) \times \cdots \times 2 \times 1$ for $n \in \mathbb{Z}_{>0}$, and $0! \stackrel{\text{def}}{=} 1$.

DEFINITION 2. Let p be a prime number, and \mathcal{G} a semi-graph of anabelioids of pro- p PSC-type (cf. [CmbGC], Definition 1.1, (i)) and \mathbb{G} the underlying semi-graph of \mathcal{G} . Write

$$\text{Cusp}(\mathbb{G}) \text{ (resp. Node}(\mathbb{G}), \text{Vert}(\mathbb{G}), \text{Edge}(\mathbb{G}))$$

for the set of cusps (resp. nodes, vertices, edges) of \mathbb{G} and

$$\text{Cusp}(\mathcal{G}) \stackrel{\text{def}}{=} \text{Cusp}(\mathbb{G}), \text{Node}(\mathcal{G}) \stackrel{\text{def}}{=} \text{Node}(\mathbb{G}),$$

$$\text{Vert}(\mathcal{G}) \stackrel{\text{def}}{=} \text{Vert}(\mathbb{G}), \quad \text{Edge}(\mathcal{G}) \stackrel{\text{def}}{=} \text{Edge}(\mathbb{G}).$$

DEFINITION 3. Let S^{log} be an fs log scheme (cf. [Nky], Definition 1.7).

- (i) Write S for the underlying scheme of S^{log} .
- (ii) Write \mathcal{M}_S for the sheaf of monoids that defines the log structure of S^{log} .
- (iii) Let \bar{s} be a geometric point of S . Then we shall denote by $I(\bar{s}, \mathcal{M}_S)$ the ideal of $\mathcal{O}_{S, \bar{s}}$ generated by the image of $\mathcal{M}_{S, \bar{s}} \setminus \mathcal{O}_{S, \bar{s}}^\times$ via the homomorphism of monoids $\mathcal{M}_{S, \bar{s}} \rightarrow \mathcal{O}_{S, \bar{s}}$ induced by the morphism $\mathcal{M}_S \rightarrow \mathcal{O}_S$ which defines the log structure of S^{log} .
- (iv) Let $s \in S$ and \bar{s} a geometric point of S which lies over s . Write $(\mathcal{M}_{S, \bar{s}}/\mathcal{O}_{S, \bar{s}}^\times)^{\text{gp}}$ for the groupification of $\mathcal{M}_{S, \bar{s}}/\mathcal{O}_{S, \bar{s}}^\times$. Then we shall refer to the rank of the finitely generated free abelian group $(\mathcal{M}_{S, \bar{s}}/\mathcal{O}_{S, \bar{s}}^\times)^{\text{gp}}$ as the *log rank* at s . Note that one verifies easily that this rank is independent of the choice of \bar{s} , i.e., depends only on s .
- (v) Let $m \in \mathbb{Z}$. Then we shall write

$$S^{\text{log} \leq m} \stackrel{\text{def}}{=} \{s \in S \mid \text{the log rank at } s \text{ is } \leq m\}.$$

Note that since $S^{\text{log} \leq m}$ is open in S (cf. [MzTa], Proposition 5.2, (i)), we shall also regard (by abuse of notation) $S^{\text{log} \leq m}$ as an open subscheme of S .

- (vi) We shall write $U_S \stackrel{\text{def}}{=} S^{\text{log} \leq 0}$ and refer to U_S as the *interior* of S^{log} . When $U_S = S$, we shall often use the notation S to denote the log scheme S^{log} .

DEFINITION 4. Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$ and k a field.

- (i) Write $\overline{\mathcal{M}}_{g, r}$ for the moduli stack (over k) of pointed stable curves of type (g, r) , and $\mathcal{M}_{g, r} \subseteq \overline{\mathcal{M}}_{g, r}$ for the open substack corresponding to the smooth curves (cf. [Knu]). Here, we assume the marked points to be ordered.
- (ii) Write

$$\overline{\mathcal{C}}_{g, r} \rightarrow \overline{\mathcal{M}}_{g, r}$$

for the tautological curve over $\overline{\mathcal{M}}_{g, r}$; $\overline{\mathcal{D}}_{g, r} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g, r} \setminus \mathcal{M}_{g, r}$ for the divisor at infinity.

- (iii) Write $\overline{\mathcal{M}}_{g, r}^{\text{log}}$ for the log stack obtained by equipping the moduli stack $\overline{\mathcal{M}}_{g, r}$ with the log structure determined by the divisors with normal crossings $\overline{\mathcal{D}}_{g, r}$.
- (iv) The divisor of $\overline{\mathcal{C}}_{g, r}$ given by the union of $\overline{\mathcal{C}}_{g, r} \times_{\overline{\mathcal{M}}_{g, r}} \overline{\mathcal{D}}_{g, r}$ with the divisor of $\overline{\mathcal{C}}_{g, r}$ determined by the marked points determines a log structure on $\overline{\mathcal{C}}_{g, r}$; we denote the resulting log stack by $\overline{\mathcal{C}}_{g, r}^{\text{log}}$. Thus, we obtain a morphism of log stacks

$$\overline{\mathcal{C}}_{g, r}^{\text{log}} \rightarrow \overline{\mathcal{M}}_{g, r}^{\text{log}},$$

which we refer to as the *tautological log curve* over $\overline{\mathcal{M}}_{g,r}^{\log}$. If S^{\log} is an arbitrary log scheme, then we shall refer to a morphism

$$C^{\log} \rightarrow S^{\log}$$

whose pull-back to some finite étale covering $T \rightarrow S$ is isomorphic to the pull-back of the tautological log curve via some morphism $T^{\log} \stackrel{\text{def}}{=} S^{\log} \times_S T \rightarrow \overline{\mathcal{M}}_{g,r}^{\log}$ as a *stable log curve* (of type (g, r)). If $C \rightarrow S$ is smooth, i.e., every geometric fiber of $C \rightarrow S$ is free of nodes, then we shall refer to $C^{\log} \rightarrow S^{\log}$ as a *smooth log curve* (of type (g, r)).

DEFINITION 5. Let k be a field; $S \stackrel{\text{def}}{=} \text{Spec}(k)$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$;

$$X^{\log} \rightarrow S$$

(cf. Definition 3, (vi)) a smooth log curve of type (g, r) ; $n \in \mathbb{Z}_{>0}$. Suppose the marked points of X^{\log} are equipped with an ordering. Then the smooth log curve X^{\log} over S determines a *classifying morphism* $S \rightarrow \overline{\mathcal{M}}_{g,r}^{\log}$. Thus, by pulling back via this morphism $S \rightarrow \overline{\mathcal{M}}_{g,r}^{\log}$ the morphism $\overline{\mathcal{M}}_{g,r+n}^{\log} \rightarrow \overline{\mathcal{M}}_{g,r}^{\log}$ given by forgetting the last n marked points, we obtain a morphism of log schemes

$$X_n^{\log} \rightarrow S.$$

We shall refer to X_n^{\log} as the *n -th log configuration space associated to $X^{\log} \rightarrow S$* . Note that $X_1^{\log} = X^{\log}$. Write $X_0^{\log} \stackrel{\text{def}}{=} S$.

DEFINITION 6. Let “ $n \in \mathbb{Z}_{>0}$ ”; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; p a prime number; k an algebraic closed field of characteristic $\neq p$; X^{\log} a smooth log curve over k of type (g, r) ; P a point of X_n .

- (i) By abuse of notation, we shall use the notation “ P ” both for the corresponding point of the scheme X_n and for the reduced closed subscheme of X_n determined by this point. Then we shall say that P is a *log-full point* of X_n^{\log} if

$$\dim(\mathcal{O}_{X_n, P}/I(P, \mathcal{M}_{X_n})) = 0$$

(cf. Definition 3, (iii)).

- (ii) P parametrizes a pointed stable curve of type $(g, r + n)$ over k . Thus, P determines a semi-graph of anabelioids of pro- p PSC-type (cf. [CmbGC], Definition 1.1, (i)), which is in fact easily verified to be independent of the choice of geometric point lying over P . We shall write \mathcal{G}_P for this semi-graph of anabelioids of pro- p PSC-type.
- (iii) Let us fix an ordered set

$$\mathcal{C}_{r,n} \stackrel{\text{def}}{=} \{c_1, \dots, c_{r+n}\}.$$

Thus, by definition, we have a natural bijection $\mathcal{C}_{r,n} \xrightarrow{\sim} \text{Cusp}(\mathcal{G}_P)$ that determines a bijection between the subset $\{c_1, \dots, c_r\}$ and the set of cusps

- of X^{\log} (cf. [Hgsh], Definition 2.2, (v)). In the following, let us identify the set $\text{Cusp}(\mathcal{G}_P)$ with $\mathcal{C}_{r,n}$. Write $x_i \stackrel{\text{def}}{=} c_{r+i}$ for each $i \in \{1, \dots, n\}$.
- (iv) We shall refer to an irreducible divisor of X_n contained in the complement $X_n \setminus U_{X_n}$ of the interior U_{X_n} of X_n as a *log divisor* of X_n^{\log} . That is to say, a log divisor of X_n^{\log} is an irreducible divisor of X_n whose generic point parametrizes a pointed stable curve with precisely two irreducible components (cf. [Hgsh], Definition 2.2, (vi)).
 - (v) Let V be a log divisor of X_n^{\log} . Then we shall write \mathcal{G}_V for “ \mathcal{G}_P ” in the case where we take “ P ” to be the generic point of V .
 - (vi) Let $m \in \mathbb{Z}_{>1}$; $y_1, \dots, y_m \in \mathcal{C}_{r,n}$ distinct elements such that $\#(\{y_1, \dots, y_m\} \cap \{c_1, \dots, c_r\}) \leq 1$. Then one verifies immediately — by considering *clutching morphisms* (cf. [Knu], Definition 3.8) — that there exists a unique log divisor V of X_n^{\log} , which we shall denote by $V(y_1, \dots, y_m)$, that satisfies the following condition: the semi-graph of anabelioids \mathcal{G}_V has precisely two vertices v_1, v_2 such that v_1 is of type $(0, m+1)$, v_2 is of type $(g, n+r-m+1)$, and y_1, \dots, y_m are cusps of $\mathcal{G}_V|_{v_1}$ (cf. [CbTpI], Definition 2.1, (iii)).
 - (vii) For each $i \in \{1, \dots, n\}$, write $p_i: X_n^{\log} \rightarrow X^{\log}$ for the projection morphism of co-profile $\{i\}$ (cf. [MzTa], Definition 2.1, (ii)). Write

$$\iota \stackrel{\text{def}}{=} (p_1, \dots, p_n): X_n^{\log} \rightarrow X^{\log} \times_k \cdots \times_k X^{\log}.$$

REMARK 2. *Let V be a log divisor of X_n^{\log} . Then let us observe that there exists a unique collection of distinct elements $y_1, \dots, y_m \in \mathcal{C}_{r,n}$ such that $\#(\{y_1, \dots, y_m\} \cap \{c_1, \dots, c_r\}) \leq 1$ and $V = V(y_1, \dots, y_m)$. (Note that uniqueness holds even in the case where $g = 0$ (in which case $r \geq 3$), as a consequence of the condition that $\#(\{y_1, \dots, y_m\} \cap \{c_1, \dots, c_r\}) \leq 1$.)*

2. Geometric description of log divisors

In the present §2, let “ $n \in \mathbb{Z}_{>1}$ ”; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; k an algebraic closed field; X^{\log} a smooth log curve over k of type (g, r) . In the present §2, we introduce various type of log divisors and we calculate the number of various type of log divisors.

DEFINITION 7. (i) For positive integers $i \in \{1, \dots, n-1\}$, $j \in \{i+1, \dots, n\}$, write

$$\pi_{i,j}: X^n \stackrel{\text{def}}{=} X \times_k \cdots \times_k X \rightarrow X^2 \stackrel{\text{def}}{=} X \times_k X$$

for the projection of the fiber product of n copies of $X \rightarrow \text{Spec}(k)$ to the i -th and j -th factors. Write $\delta'_{i,j}$ for the inverse image via $\pi_{i,j}$ of the image of the diagonal embedding $X \hookrightarrow X^2$. Write $\delta_{i,j}$ for the uniquely determined log divisor of X_n^{\log} whose generic point maps to the generic point of $\delta'_{i,j}$ via the natural morphism $X_n \rightarrow X^n$ (cf. Definition 6, (vii)). We shall refer to the log divisor $\delta_{i,j}$ as a *naive diagonal* of X_n^{\log} .

- (ii) Let V be a log divisor of X_n^{\log} . We shall say that V is a *tripodal divisor* if one of the vertices of \mathcal{G}_V (cf. Definition 6, (vi)) is of type $(0, 3)$ (cf. Definition 6, (vii); [CbTpI], Definition 2.3, (iii)).
- (iii) Let V be a log divisor of X_n^{\log} . We shall say that V is a (g, r) -*divisor* if one of the vertices of \mathcal{G}_V is of type (g, r) .
- (iv) Let V be a log divisor of X_n^{\log} . We shall say that V is a *drift diagonal* if there exist a naive diagonal δ and an automorphism α of X_n^{\log} over S such that $V = \alpha(\delta)$.
- (v) Let Λ be a set of drift diagonals of X_n^{\log} . Then we shall say that Λ is a *drift collection* of X_n^{\log} if there exists an automorphism α of X_n^{\log} over S such that $\Lambda = \{\alpha(V) \mid V \text{ is a naive diagonal}\}$.

PROPOSITION 1. *The following hold:*

- (i) $\{\text{log divisors of } X^{\log}\} = \{\text{log-full points of } X^{\log}\}$.
- (ii) $\#\{\text{log-full points of } X^{\log}\} = r$.

PROOF. Assertions (i), (ii) follow from Definition 6, (i), (iv). \square

PROPOSITION 2.

$$\{\text{naive diagonals}\} \subseteq \{\text{drift diagonals}\} \subseteq \{\text{tripodal divisors}\} \subseteq \{\text{log divisors}\},$$

$$\{(g, r)\text{-divisors}\} \subseteq \{\text{log divisors}\}.$$

PROOF. It follows from Definition 6, (iv); Definition 7, (i), (ii), (iii), (iv); [Hgsh], Proposition 3.4, (i). \square

PROPOSITION 3. *Let $m \in \{2, \dots, n+1\}$. Write*

$$V_{[m]}^{\text{vertical}} \stackrel{\text{def}}{=} \{V(y_1, \dots, y_m) \mid y_1, \dots, y_m \in C_{r,n} \text{ distinct elements} \\ \text{such that } \#\(\{y_1, \dots, y_m\} \cap \{c_1, \dots, c_r\}) = 1\},$$

$$V_{[m]}^{\text{naive}} \stackrel{\text{def}}{=} \{V(y_1, \dots, y_m) \mid y_1, \dots, y_m \in C_{r,n} \text{ distinct elements} \\ \text{such that } \#\(\{y_1, \dots, y_m\} \cap \{c_1, \dots, c_r\}) = 0\},$$

$$V_{[m]} \stackrel{\text{def}}{=} \begin{cases} V_{[m]}^{\text{vertical}} \sqcup V_{[m]}^{\text{naive}} & (2 \leq m \leq n) \\ V_{[n+1]}^{\text{vertical}} & (m = n+1) \end{cases}$$

(cf. Remark 2). Note that $V_{[n+1]} = V_{[m]}^{\text{vertical}} = \emptyset$ if $r = 0$. Then

$$\#V_{[m]}^{\text{vertical}} = \binom{n}{m-1} r, \quad \#V_{[m]}^{\text{naive}} = \binom{n}{m}.$$

PROOF. It follows from Definition 6, (vi); Remark 2. \square

PROPOSITION 4. *Let V be a log divisor of X_n^{\log} . Write V^{\log} for the log scheme obtained by equipping V with the log structure induced by the log structure of X_n^{\log} . Let $T^{\log} \rightarrow \text{Spec}(k)$ be a smooth log curve of type $(0, 3)$. For $m \in \mathbb{Z}_{>0}$, write T_m^{\log} for the m -th log configuration space associated to $T^{\log} \rightarrow \text{Spec}(k)$.*

- (i) Let $V \in V_{[2]}$, then $V^{\log \leq 1}$ is isomorphic to $U_{X_{n-1}}$.
- (ii) Let $V \in V_{[n+1]}$, then $V^{\log \leq 1}$ is isomorphic to $U_{T_{n-1}}$.
- (iii) Let $m \in \{3, \dots, n\}$ and $V \in V_{[m]}$. Then $V^{\log \leq 1}$ is isomorphic to $U_{T_{m-2}} \times_k U_{X_{n-m+1}}$.

PROOF. Assertions (i), (ii), (iii) follow from Definition 3, (v); [Hgsh], Lemma 6.1, (i), (ii), (iii). \square

PROPOSITION 5.

$$\begin{aligned} \{\log \text{ divisors}\} &= \prod_{m=2}^{n+1} V_{[m]} \\ &= \prod_{m=2}^{n+1} V_{[m]}^{\text{vertical}} \sqcup \prod_{m=2}^n V_{[m]}^{\text{naive}}, \end{aligned}$$

$$\begin{aligned} \#\{\log \text{ divisors}\} &= \left(\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} \right) r + \left(\binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} \right) \\ &= (2^n - 1)r + (2^n - 1 - n). \end{aligned}$$

PROOF. Note that

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n, \quad \binom{n}{0} = 1, \quad \binom{n}{1} = n.$$

Then it follows from Remark 2; Proposition 3. \square

PROPOSITION 6. *The following hold:*

- (i) If $(g, r) \neq (0, 3)$, then

$$\begin{aligned} \{\text{tripodal divisors}\} &= V_{[2]} = V_{[2]}^{\text{vertical}} \sqcup V_{[2]}^{\text{naive}}, \\ \#\{\text{tripodal divisors}\} &= \binom{n}{1} r + \binom{n}{2}. \end{aligned}$$

- (ii) If $(g, r) = (0, 3)$, then

$$\begin{aligned} \{\text{tripodal divisors}\} &= V_{[2]} \sqcup V_{[n+1]} \\ &= V_{[2]}^{\text{vertical}} \sqcup V_{[n+1]}^{\text{vertical}} \sqcup V_{[2]}^{\text{naive}}, \\ \#\{\text{tripodal divisors}\} &= \left(\binom{n}{1} + \binom{n}{n} \right) r + \binom{n}{2}. \end{aligned}$$

PROOF. Assertions (i), (ii) follow from Proposition 3; [Hgsh], Proposition 3.3, (ii), (iii). \square

PROPOSITION 7. (i) *If $(g, r) \neq (0, 3), (1, 1)$, then there exists an isomorphism*

$$\begin{aligned} \text{Aut}_k(X_n^{\log}) &\xrightarrow{\sim} \{\beta \in \text{Aut}(C_{r,n}) \mid \beta(c_i) = c_i \text{ for } i \in \{1, \dots, r\}\} \\ \alpha &\mapsto \beta \end{aligned}$$

such that

$$\alpha(V(y_1, \dots, y_m)) = V(\beta(y_1), \dots, \beta(y_m))$$

for each log divisor $V(y_1, \dots, y_m)$ (cf. Remark 2). In particular, $\text{Aut}_k(X_n^{\log})$ is isomorphic to the symmetric group on n letters S_n .

(ii) If $(g, r) = (0, 3)$ or $(1, 1)$, then there exists an isomorphism

$$\begin{aligned} \text{Aut}_k(X_n^{\log}) &\xrightarrow{\sim} \text{Aut}(C_{r,n}) \\ \alpha &\mapsto \beta \end{aligned}$$

such that

$$\alpha(V(y_1, \dots, y_m)) = V(\beta(y_1), \dots, \beta(y_m))$$

for each log divisor $V(y_1, \dots, y_m)$ (cf. Remark 2). In particular, $\text{Aut}_k(X_n^{\log})$ is isomorphic to the symmetric group on $r+n$ letters S_{r+n} .

PROOF. Assertions (i), (ii) follow from the proof of [Hgsh], Proposition 3.4, (ii), (iii); \square

PROPOSITION 8. Let Λ be a drift collection of X_n^{\log} (cf. Definition 7, (v)). Then the following hold:

(i)

$$\#\{\text{naive diagonals}\} = \#\Lambda = \#V_{[2]}^{\text{naive}} = \binom{n}{2}.$$

(ii) If $(g, r) \neq (0, 3), (1, 1)$, then

$$\Lambda = \{\text{drift diagonals}\} = \{\text{naive diagonals}\} = V_{[2]}^{\text{naive}},$$

$$\#\{\text{drift diagonals}\} = \binom{n}{2},$$

$$\#\{\text{drift collections}\} = \binom{n}{n} = 1.$$

(iii) If $(g, r) = (0, 3)$, then there exist distinct elements $y_1, \dots, y_n \in C_{3,n}$ such that

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\},$$

$$\{\text{drift diagonals}\} = \{\text{tripodal divisors}\},$$

$$\#\{\text{drift diagonals}\} = \left(\binom{n}{1} + \binom{n}{n} \right) r + \binom{n}{2},$$

$$\#\{\text{drift collections}\} = \binom{n+3}{n}.$$

(iv) If $(g, r) = (1, 1)$, then there exist distinct elements $y_1, \dots, y_n \in C_{1,n}$ such that

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\},$$

$$\{\text{drift diagonals}\} = \{\text{tripodal divisors}\},$$

$$\#\{\text{drift diagonals}\} = \binom{n}{1} r + \binom{n}{2},$$

$$\#\{\text{drift collections}\} = \binom{n+1}{n}.$$

PROOF. Assertion (i) follows from Proposition 3; [Hgsh], Proposition 3.3, (i). Assertions (ii), (iii), (iv) follow from Proposition 6, (i), (ii); Proposition 7; [Hgsh], Proposition 3.4, (ii), (iii); the proof of [Hgsh], Lemma 8.10. \square

PROPOSITION 9. *The following hold:*

(i) *If $r > 0$ and $(g, r) \neq (0, 3)$, then*

$$\{(g, r)\text{-divisor}\} = V_{[n+1]}$$

and

$$\#\{(g, r)\text{-divisor}\} = \binom{n}{n}r.$$

(ii) *If $(g, r) = (0, 3)$, then*

$$\{(g, r)\text{-divisor}\} = \{\text{tripodal divisors}\}$$

and

$$\#\{(g, r)\text{-divisor}\} = \left(\binom{n}{1} + \binom{n}{n}\right)r + \binom{n}{2}.$$

(iii) *If $r = 0$, then*

$$\{(g, r)\text{-divisor}\} = \emptyset.$$

PROOF. Assertions (i), (ii), (iii) follow from Definition 7, (iii); Proposition 3; Proposition 6, (ii). \square

PROPOSITION 10. *Let $V_1 = V(y_1, \dots, y_s)$, $V_2 = V(z_1, \dots, z_t)$ be log divisors of X_n^{\log} (cf. Remark 2). Then the following conditions are equivalent:*

- (i) $V_1 \cap V_2 \neq \emptyset$.
- (ii) *there exists a log-full point contained in $V_1 \cap V_2$.*
- (iii) $\{y_1, \dots, y_s\} \cap \{z_1, \dots, z_t\} = \emptyset$ or $\{y_1, \dots, y_s\} \subseteq \{z_1, \dots, z_t\}$ or $\{y_1, \dots, y_s\} \supseteq \{z_1, \dots, z_t\}$ in $\mathcal{C}_{r,n}$.

PROOF. The equivalence (i) \iff (ii) follows from [Hgsh], Lemma 8.4. The implication (i) \implies (iii) follows immediately (cf. the proof of [Hgsh], Lemma 8.6). The implication (iii) \implies (i) follows immediately (cf. the proof of [Hgsh], Lemma 8.5). \square

PROPOSITION 11. *Let P be a log-full point of X_n^{\log} and V a log divisor of X_n^{\log} .*

- (i) $P \in V \iff \mathcal{G}_V$ is obtained from \mathcal{G}_P by generalization (cf. [CbTpI], Definition 2.8).
- (ii) $\text{Cusp}(\mathcal{G}_P) = \text{Cusp}(\mathcal{G}_V) = \mathcal{C}_{r,n}$ (cf. Definition 2). In particular, $\#\text{Cusp}(\mathcal{G}_P) = r + n$.
- (iii) $\#\text{Node}(\mathcal{G}_V) = 1$ (cf. Definition 2).
- (iv) *If $r > 0$, then $\text{Node}(\mathcal{G}_P) = n$.*

- (v) If $r > 0$, then there exist distinct log divisors V_1, \dots, V_n of X_n^{\log} such that $P = V_1 \cap \dots \cap V_n$.
- (vi) If $r = 0$, then $\sharp \text{Node}(\mathcal{G}_P) = n - 1$.
- (vii) If $r = 0$, then there exist distinct log divisors V_1, \dots, V_{n-1} of X_n^{\log} such that $P \in V_1 \cap \dots \cap V_{n-1}$.

PROOF. Assertion (i) follows from [Hgsh], Proposition 2.9. Assertion (ii) follows from Definition 6, (iii). Assertion (iii) follows from Definition 6, (iv). Assertions (iv), (vi) follow immediately from Definition 6, (i), together with the well-known modular interpretation of the log moduli stack that appear in the definition of X_n^{\log} (where we recall that the log structure of this log stack arises from a divisor with normal crossings) (cf. [Hgsh], Proposition 3.6). Assertions (v), (vii) follow from [Hgsh], Proposition 3.7, (iii); the proof of [Hgsh], Proposition 3.7, (iii). \square

PROPOSITION 12. Let $p: X_n^{\log} \rightarrow X_{n-1}^{\log}$ be a projection and $m \in \{2, \dots, n+1\}$. Write ${}^\dagger V_{[m]}$ for the set $V_{[m]} \subseteq 2^{X_n^{\log}}$ (cf. Proposition 3) and ${}^\ddagger V_{[m]}$ for the set $V_{[m]} \subseteq 2^{X_{n-1}^{\log}}$. Then the following hold:

- (i) Let V be a log divisor of X_n^{\log} . Then $p(V)$ is a log divisor of X_{n-1}^{\log} or $p(V) = X_{n-1}$. Moreover, suppose that $p: X_n^{\log} \rightarrow X_{n-1}^{\log}$ is a projection of co-profile $\{n\}$ (cf. [MzTa], Definition 2.1, (ii)). Then

$$p(V) = X_{n-1} \iff \text{there exists } y \in \mathcal{C}_{r,n} \setminus \{x_n\} \text{ such that } V = V(y, x_n).$$

- (ii) $p({}^\dagger V_{[m]}) \subseteq {}^\ddagger V_{[m]} \cup {}^\ddagger V_{[m-1]}$ for $m \in \{3, \dots, n\}$.
- (iii) Let $V \in {}^\dagger V_{[2]}$. Then $p(V) = X_{n-1}$ or $p(V) \in {}^\ddagger V_{[2]}$. In particular,

$$p({}^\dagger V_{[2]}) = {}^\ddagger V_{[2]} \sqcup \{X_{n-1}\}.$$

- (iv) $p({}^\dagger V_{[n+1]}) = {}^\ddagger V_{[n]}$. In particular,

$$\begin{aligned} & p(\{(g, r)\text{-divisor of } X_n^{\log}\}) \\ &= \begin{cases} \{(g, r)\text{-divisor of } X_{n-1}^{\log}\} & (\text{if } (g, r) \neq (0, 3)) \\ \{(g, r)\text{-divisor of } X_{n-1}^{\log}\} \sqcup \{X_{n-1}\} & (\text{if } (g, r) = (0, 3)). \end{cases} \end{aligned}$$

PROOF. Assertions (i), (ii), (iii), (iv) follow immediately from the latter portion of Definition 6 (iv), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of X_n^{\log} and X_{n-1}^{\log} (cf. [Hgsh], Proposition 4.1, (i), (ii)). \square

PROPOSITION 13. Let $p: X_n^{\log} \rightarrow X_{n-1}^{\log}$ be a projection. Then the following hold:

- (i) Let P be a log-full point of X_n^{\log} . Then $p(P)$ is a log-full point of X_{n-1}^{\log} .
- (ii) Let V be a log divisor of X_n^{\log} . Then there exist distinct log divisors W_1, W_2 of X_n^{\log} such that $W_1 \cup W_2 = p^{-1}(V)$. Moreover, suppose that

$p: X_n^{\log} \rightarrow X_{n-1}^{\log}$ is a projection of co-profile $\{n\}$ and $V = V(y_1, \dots, y_s)$, where $\{y_1, \dots, y_s\} \subseteq \mathcal{C}_{r, n-1}$. Then

$$\{W_1, W_2\} = \{V(y_1, \dots, y_s), V(y_1, \dots, y_s, x_n)\},$$

where $\{y_1, \dots, y_s, x_n\} \subseteq \mathcal{C}_{r, n}$.

(iii) Suppose that $r > 0$. Let P be a log-full point of X_{n-1}^{\log} . Then there exist log divisors $V_1, \dots, V_{n(n-1)}$ of X_n^{\log} such that

$$\#\{V_1, \dots, V_{n(n-1)}\} = 2n - 2,$$

$V_{1+i(n-1)}, \dots, V_{n-1+i(n-1)}$ are distinct log divisors,

$$p^{-1}(P) = \bigcup_{i=0}^{n-1} (V_{1+i(n-1)} \cap \dots \cap V_{n-1+i(n-1)}).$$

PROOF. Assertion (i) follows immediately from the latter portion of Definition 6 (iv), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of X_n^{\log} and X_{n-1}^{\log} (cf. [Hgsh], Proposition 4.1, (i), (ii)). Since $\#\text{Vert}(\mathcal{G}_V) = 2$ (cf. Definition 2; Proposition 11, (iii)), assertion (ii) follows immediately. Next, we consider assertion (iii). Let W be an irreducible component of $p^{-1}(P)$. Since $\#\text{Node}(\mathcal{G}_P) = n - 1$ (cf. Proposition 11, (iv)), it holds that $\#\text{Node}(\mathcal{G}_W) = n - 1$. In particular, there exists distinct log divisors $V_{1+i(n-1)}, \dots, V_{n-1+i(n-1)}$ such that $W = V_{1+i(n-1)} \cap \dots \cap V_{n-1+i(n-1)}$. Since $\#\text{Vert}(\mathcal{G}_P) = \#\text{Node}(\mathcal{G}_P) + 1 = n$, it holds that

$$\#\{\text{irreducible components of } p^{-1}(P)\} = n.$$

Since $\#\text{Node}(\mathcal{G}_P) = n - 1$, it holds that

$$\#\{V_1, \dots, V_{n(n-1)}\} = 2n - 2.$$

This completes the proof of assertion (iii). \square

PROPOSITION 14. Let $p: X_n^{\log} \rightarrow X_{n-1}^{\log}$ be a projection and P a log-full point of X_{n-1}^{\log} .

(i) If $r > 0$, then

$$\#\{\text{log-full points of } X_n^{\log} \text{ contained in } p^{-1}(P)\} = r + 2(n - 1).$$

In particular,

$$\#\{\text{log-full points of } X_n^{\log}\} = \prod_{i=0}^{n-1} (r + 2i).$$

(ii) If $r = 0$, then

$$\#\{\text{log-full points of } X_n^{\log} \text{ contained in } p^{-1}(P)\} = 2n - 3.$$

PROOF. First, suppose that $r > 0$. Note that by [Hgsh], Proposition 3.7, (i), it holds that $\sharp\text{Node}(\mathcal{G}_P) = n - 1$ and $\sharp\text{Cusp}(\mathcal{G}_P) = \sharp\mathcal{C}_{r,n-1} = r + n - 1$. Thus, it follows immediately from Proposition 1, (ii). Next, suppose that $r = 0$. Then it holds that $\sharp\text{Node}(\mathcal{G}_P) = n - 2$ and $\sharp\text{Cusp}(\mathcal{G}_P) = \sharp\mathcal{C}_{r,n-1} = n - 1$. Thus, assertions (i), (ii) follow immediately from the various definitions involved. \square

DEFINITION 8. Suppose that $r = 0$. Then we shall say that an irreducible subset W of X_n^{\log} is *log-full curve* if each element of W is a log-full point.

PROPOSITION 15. *Suppose that $r = 0$. Let W be a log-full curve of X_n^{\log} and $p: X_n^{\log} \rightarrow X_{n-1}^{\log}$ a projection.*

- (i) *There exists a projection $q: X_n^{\log} \rightarrow X^{\log}$ such that q induces a bijection $W \rightarrow X$.*
- (ii) *If $n > 2$, then $p(W)$ is a log-full curve of X_{n-1}^{\log} .*
- (iii) *There exist distinct log divisors V_1, \dots, V_{n-1} of X_n^{\log} such that $W = V_1 \cap \dots \cap V_{n-1}$.*
- (iv) *Suppose that $n > 2$. Let Z be a log-full curve of X_{n-1}^{\log} . Then there exist log divisors $V_1, \dots, V_{(n-1)(n-2)}$ of X_n^{\log} such that*

$$\sharp\{V_1, \dots, V_{(n-1)(n-2)}\} = 2n - 4,$$

$V_{1+i(n-2)}, \dots, V_{n-2+i(n-2)}$ are distinct log divisors,

$$p^{-1}(Z) = \bigcup_{i=0}^{n-2} (V_{1+i(n-2)} \cap \dots \cap V_{n-2+i(n-2)}).$$

- (v) *Suppose that $n > 2$. Let Z be a log-full curve of X_{n-1}^{\log} . Then*

$$\sharp\{\text{log-full curves of } X_n^{\log} \text{ contained in } p^{-1}(Z)\} = 2n - 3.$$

(vi)

$$\sharp\{\text{log-full curves of } X_n^{\log}\} = \prod_{i=0}^{n-2} (2i + 1).$$

PROOF. Assertions (i), (ii) follow immediately from the latter portion of Definition 6 (iv), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of X_n^{\log} , X_{n-1}^{\log} , and X^{\log} . Assertion (iii) follows from Proposition 11, (vii). Next, we consider assertions (iv), (v). Let $P \in W$ be a log-full point of X_n^{\log} . Since $\text{Node}(\mathcal{G}_P) = n - 2$, $\text{Cusp}(\mathcal{G}_P) = n - 1$, assertions (iv), (v) follow immediately (cf. the proof of Proposition 13, (iii); the proof of Proposition 14, (i)). Assertion (vi) follows from assertion (v); Proposition 14, (ii). \square

PROPOSITION 16. *Let $m \in \{2, \dots, n + 1\}$ and $V = V(y_1, \dots, y_m) \in \mathcal{V}_{[m]}$ a log divisor of X_n^{\log} , where $y_1, \dots, y_m \in \mathcal{C}_{r,n}$ are distinct elements. Then the following hold:*

(i)

$$\begin{aligned}
& \#\{W \subseteq X_n^{\log}: \text{log divisors} \mid V \cap W \neq \emptyset\} \\
&= \begin{cases} \#\{\text{log divisors of } T_{m-2}^{\log}\} + \#\{\text{log divisors of } X_{n-m+1}^{\log}\} & (3 \leq m \leq n) \\ \#\{\text{log divisors of } X_{n-1}^{\log}\} & (m = 2) \\ \#\{\text{log divisors of } T_{n-1}^{\log}\} & (m = n + 1) \end{cases} \\
&= 2^m + 2^{n-m+1}(r+1) - r - n - 4 \quad (2 \leq m \leq n+1).
\end{aligned}$$

(ii) If $r > 0$, then

$$\begin{aligned}
& \#\{\text{log-full points of } X_n^{\log} \text{ contained in } V\} \\
&= \begin{cases} \#\{\text{log-full points of } T_{m-2}^{\log}\} \cdot \#\{\text{log-full points of } X_{n-m+1}^{\log}\} & (3 \leq m \leq n) \\ \#\{\text{log-full points of } X_{n-1}^{\log}\} & (m = 2) \\ \#\{\text{log-full points of } T_{n-1}^{\log}\} & (m = n + 1). \end{cases}
\end{aligned}$$

(iii) If $r = 0$, then

$$\begin{aligned}
& \#\{\text{log-full curves of } X_n^{\log} \text{ contained in } V\} \\
&= \begin{cases} \#\{\text{log-full points of } T_{m-2}^{\log}\} \cdot \#\{\text{log-full curves of } X_{n-m+1}^{\log}\} & (3 \leq m \leq n-1) \\ \#\{\text{log-full curves of } X_{n-1}^{\log}\} & (m = 2) \\ \#\{\text{log-full points of } T_{n-2}^{\log}\} & (m = n). \end{cases}
\end{aligned}$$

(iv) If $(g, r) \neq (0, 3)$ and $V \in V_{[m]}^{\text{naive}}$. Then

$$\begin{aligned}
& \{W \subseteq X_n^{\log}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\
&= (\{V(y_i, y_j) \mid 1 \leq i < j \leq m\} \setminus \{V\}) \\
&\quad \sqcup \{V(z_1, z_2) \mid z_1, z_2 \in \{x_1, \dots, x_n\} \setminus \{y_1, \dots, y_m\}: \text{distinct}\} \\
&\quad \sqcup \{V(x, c) \mid x \in \{x_1, \dots, x_n\} \setminus \{y_1, \dots, y_m\}, c \in \{c_1, \dots, c_r\}\}.
\end{aligned}$$

In particular,

$$\begin{aligned}
& \#\{W \subseteq X_n^{\log}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\
&= \begin{cases} \binom{m}{2} + \binom{n-m}{2} + (n-m)r & (3 \leq m \leq n) \\ \binom{n-2}{2} + (n-2)r & (m = 2). \end{cases}
\end{aligned}$$

(v) If $(g, r) \neq (0, 3)$ and $V \in V_{[m]}^{\text{vertical}}$. We may assume that $y_m \in \{c_1, \dots, c_r\}$.
Then

$$\begin{aligned} & \{W \subseteq X_n^{\text{log}}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= \{V(y_i, y_j) \mid 1 \leq i < j \leq m-1\} \\ & \quad \sqcup \{V(z_1, z_2) \mid z_1, z_2 \in \{x_1, \dots, x_n\} \setminus \{y_1, \dots, y_m\}: \text{distinct}\} \\ & \quad \sqcup \{V(y_k, y_m) \mid 1 \leq k \leq m-1\} \\ & \quad \sqcup \{V(x, c) \mid x \in \{x_1, \dots, x_n\} \setminus \{y_1, \dots, y_m\}, c \in \{c_1, \dots, c_r\} \setminus \{y_m\}\}. \end{aligned}$$

In particular,

$$\begin{aligned} & \#\{W \subseteq X_n^{\text{log}}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= \begin{cases} \binom{m-1}{2} + \binom{n-m+1}{2} + (m-1) + (n-m+1)(r-1) & (3 \leq m \leq n+1) \\ \binom{n-1}{2} + (n-1)(r-1) & (m=2). \end{cases} \end{aligned}$$

(vi) For each $2 \leq m \leq n$, it holds that

$$\begin{aligned} \binom{m}{2} + \binom{n-m}{2} + (n-m)r &= \binom{m-1}{2} + \binom{n-m+1}{2} + (m-1) + (n-m+1)(r-1) \\ &\iff \binom{n-2}{2} + (n-2)r = \binom{n-1}{2} + (n-1)(r-1) \\ &\iff r = 1 \end{aligned}$$

(cf. (iv), (v)).

(vii) If $(g, r) = (0, 3)$. Then

$$\begin{aligned} & \{W \subseteq X_n^{\text{log}}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= (\{V(y_i, y_j) \mid 1 \leq i < j \leq m\} \setminus \{V\}) \\ & \quad \sqcup (\{V(z_1, z_2) \mid z_1, z_2 \in \mathcal{C}_{r,n} \setminus \{y_1, \dots, y_m\}: \text{distinct}\} \setminus \{V\}). \end{aligned}$$

In particular,

$$\begin{aligned} & \#\{W \subseteq X_n^{\text{log}}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= \begin{cases} \binom{m}{2} + \binom{n+3-m}{2} & (3 \leq m \leq n) \\ \binom{n+1}{2} & (m=2 \text{ or } n+1). \end{cases} \end{aligned}$$

(viii) If $m = n+1$, i.e., V is a (g, r) -divisor. Then

$$\#\{W \subseteq X_n^{\text{log}}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} = \binom{n+1}{2}.$$

PROOF. Assertions (i), (ii), (iii) follow from Proposition 4, (i), (ii), (iii); Proposition 10. Assertions (iv), (v), (vii), (viii) follow from Proposition 10. Assertion (vi) follows immediately. \square

PROPOSITION 17. Let $m \in \{2, \dots, n+1\}$. Write $a_m \stackrel{\text{def}}{=} 2^m + 2^{n-m+1}(r+1) - r - n - 4$ (cf. Proposition 16, (i)). Then the following hold:

(i) $a_{m+1} - a_m = 2^m - 2^{n-m}(r+1)$.

(ii) If $16 > 2^n(r+1)$, then a_m is a monotonically increasing sequence i.e.,

$$a_2 < a_3 < \dots < a_{n+1}.$$

(iii) If $2^n < r+1$, then a_m is a monotonically decreasing sequence i.e.,

$$a_2 > a_3 > \dots > a_{n+1}.$$

(iv) Let $m, m' \in \{2, \dots, n+1\}$ be distinct elements. Then $a_m = a_{m'}$ if and only if $r+1$ is a power of 2 and $m+m' = \log_2(2^{n+1}(r+1))$.

PROOF. Assertions (i), (ii), (iii), (iv) follow immediately. \square

PROPOSITION 18. Suppose that $r > 0$. Let $m \in \{2, \dots, n\}$. Then

$$(V_{[2]} \sqcup V_{[n+1]}) \cap \{V \mid V \cap W \neq \emptyset \text{ for each } W \in V_{[m]}^{\text{naive}}\} = V_{[n+1]}.$$

PROOF. Assertion follows immediately from Proposition 10. \square

PROPOSITION 19. Suppose that $r > 0$. Then the following hold:

(i) Let $V \in V_{[n+1]}$. Then

$$\#\{W \in V_{[2]} \sqcup V_{[n+1]} \mid V \cap W \neq \emptyset, V \neq W\} = \frac{n^2 + n}{2}.$$

(ii) Let $V \in V_{[2]}$. Then

$$\#\{W \in V_{[2]} \sqcup V_{[n+1]} \mid V \cap W \neq \emptyset, V \neq W\} = \frac{n^2 + 2nr - 2r - 5n + 6}{2}.$$

(iii)

$$\frac{n^2 + n}{2} = \frac{n^2 + 2nr - 2r - 5n + 6}{2} \text{ if and only if } r = 3$$

(cf. (i), (ii)).

PROOF. Assertions (i), (ii), (iii) follow immediately from Proposition 10. \square

PROPOSITION 20. Suppose that $g \neq 0$ and $r = 3$. Then the following hold:

(i) Let $V \in V_{[n+1]} \sqcup V_{[2]}^{\text{vertical}}$. Then

$$\#\{W \in V_{[n]} \mid V \cap W \neq \emptyset, V \neq W\} = n + 1.$$

(ii) Let $V \in V_{[2]}^{\text{naive}}$. Then

$$\#\{W \in V_{[n]} \mid V \cap W \neq \emptyset, V \neq W\} = 3n - 5.$$

(iii)

$$n + 1 = 3n - 5 \text{ if and only if } n = 3$$

(cf. (i), (ii)).

PROOF. Assertions (i), (ii), (iii) follow immediately from Proposition 10. \square

PROPOSITION 21. *Suppose that $(g, r) \neq (0, 3), (1, 1)$ and $r > 0$. Let $m \in \{2, \dots, n\}$ and $V \in V_{[m]}$. Then the following hold:*

(i) *If $V \in V_{[m]}^{\text{naive}}$, then*

$$\#\{\sigma(V) \mid \sigma \in \text{Aut}_k(X_n^{\log})\} = \binom{n}{m}.$$

(ii) *If $V \in V_{[m]}^{\text{vertical}}$, then*

$$\#\{\sigma(V) \mid \sigma \in \text{Aut}_k(X_n^{\log})\} = \binom{n}{m-1}.$$

(iii)

$$\binom{n}{m} = \binom{n}{m-1} \iff n + 1 = 2m$$

(cf. (i), (ii)).

(iv) *$V \in V_{[m]}^{\text{naive}} \iff$ there exists $W \in V_{[m+1]}$ such that $W \cap \sigma(V) \neq \emptyset$ for each $\sigma \in \text{Aut}_k(X_n^{\log})$.*PROOF. Assertions (i), (ii), (iii) follow immediately from Proposition 7, (i). Assertion (iv) follows immediately (cf. Remark 2). \square

PROPOSITION 22. *Suppose that $(g, r) \neq (0, 3), (1, 1)$, $r > 0$, and $n > 2$. Let $V \in V_{[2]} \sqcup V_{[n+1]}$. If $\#\{\sigma(V) \mid \sigma \in \text{Aut}_k(X_n^{\log})\} = 1$, then $V \in V_{[n+1]}$.*

PROOF. Assertion follows immediately from Proposition 7, (i); Proposition 21, (i), (ii). \square

DEFINITION 9. Suppose that $r > 0$. Let \mathcal{N} be a set of tripodal diagonals of X_n^{\log} such that $\#\mathcal{N} = n$. Then we shall say that \mathcal{N} is a *new vertical collection* of X_n^{\log} if there exist an automorphism of X_n^{\log} over k and $c \in \{c_1, \dots, c_r\} \subseteq \mathcal{C}_{r,n}$ such that $\mathcal{N} = \alpha(\{V(c, x_i)\}_{i=1}^n)$.

PROPOSITION 23. *Suppose that $r > 0$ and $(g, r) \neq (0, 3), (1, 1)$. Let \mathcal{N} be a set of tripodal diagonals of X_n^{\log} such that $\#\mathcal{N} = n$. Then the following conditions are equivalent:*

- (i) \mathcal{N} is a new vertical collection of X_n^{\log} .
- (ii) There exists $c \in \{c_1, \dots, c_r\} \subseteq \mathcal{C}_{r,n}$ such that $\mathcal{N} = \{V(x_i, c)\}_{i=1}^n$.
- (iii) There exists a (g, r) -divisor V such that

$$\mathcal{N} = \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \cap W \neq \emptyset\} \setminus \{\text{naive diagonals}\}.$$

(iv) There exists a (g, r) -divisor V such that

$$\begin{aligned} \mathcal{N} &= \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \cap W \neq \emptyset\} \\ &\quad \setminus \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V(x_1, \dots, x_n) \cap W \neq \emptyset\}. \end{aligned}$$

PROOF. The implication (i) \implies (ii) follows from Proposition 7, (i); Definition 9. The implication (ii) \implies (i) follows immediately from Definition 9. Next, we consider the implication (ii) \implies (iii). Write $V \stackrel{\text{def}}{=} V(x_1, \dots, x_n, c)$. Then by Proposition 10, V is a (g, r) -divisor, and

$$\begin{aligned} &\{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= \{V(y_1, y_2) \mid y_1, y_2 \in \{x_1, \dots, x_n, c\} \text{ are distinct elements}\} \\ &= \mathcal{N} \sqcup \{\text{naive diagonals}\}. \end{aligned}$$

This completes the proof of the implication (ii) \implies (iii). Next, we consider the implication (iii) \implies (ii). Let V be a (g, r) -divisor. Then by Proposition 9; Remark 2, there exist distinct elements $y_1, \dots, y_{n+1} \in \mathcal{C}_{r,n}$ such that $V = V(y_1, \dots, y_{n+1})$, and $\sharp(\{y_1, \dots, y_{n+1}\} \cap \{c_1, \dots, c_r\}) \leq 1$. Note that $n+1 \geq \sharp\{x_1, \dots, x_n\}$. Thus, $\sharp(\{y_1, \dots, y_{n+1}\} \cap \{c_1, \dots, c_r\}) = 1$, and $\{y_1, \dots, y_n\} = \{x_1, \dots, x_n\}$. We may assume $y_{n+1} \in \{c_1, \dots, c_r\}$. By Proposition 10, it holds that

$$\begin{aligned} &\{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= \{V(x_i, y_{n+1})\}_{i=1}^n \sqcup \{V(x_i, x_j) \mid 1 \leq i < j \leq n\} \\ &= \{V(x_i, y_{n+1})\}_{i=1}^n \sqcup \{\text{naive diagonals}\}. \end{aligned}$$

In particular, $\mathcal{N} = \{V(x_i, y_{n+1})\}_{i=1}^n$. This completes the proof of the implication (iii) \implies (ii). Next, we consider the equivalence (iii) \iff (iv). Let W is a tripodal divisor. Since $(g, r) \neq (0, 3)$, it follows immediately from Proposition 8, (ii); Proposition 10, that $V(x_1, \dots, x_n) \cap W \neq \emptyset \iff W$ is a naive diagonal. This completes the proof of the equivalence (iii) \iff (iv). \square

PROPOSITION 24. Suppose that $(g, r) = (0, 3)$. Let \mathcal{N} be a set of tripodal diagonals of X_n^{\log} such that $\sharp\mathcal{V} = n$. Then the following conditions are equivalent:

- (i) \mathcal{N} is a new vertical collection of X_n^{\log} .
- (ii) There exist distinct elements $y_1, \dots, y_{n+1} \in \mathcal{C}_{r,n}$ such that

$$\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^n.$$

- (iii) There exist a (g, r) -divisor V and a drift collection Λ such that

$$\mathcal{N} = \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \setminus \Lambda.$$

- (iv) There exist a (g, r) -divisor ${}^\dagger V$ and ${}^\ddagger V \in V_{[3]} \sqcup V_{[n]}$ such that ${}^\dagger V \neq {}^\ddagger V$, ${}^\dagger V \cap {}^\ddagger V \neq \emptyset$, and

$$\begin{aligned} \mathcal{N} &= \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid {}^\dagger V \neq W, {}^\dagger V \cap W \neq \emptyset\} \\ &\quad \setminus \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid {}^\ddagger V \cap W \neq \emptyset\}. \end{aligned}$$

PROOF. The equivalence (i) \iff (ii) follows from Proposition 7, (ii); Definition 9. Next, we consider the implication (ii) \implies (iii). Write

$$V \stackrel{\text{def}}{=} V(y_1, \dots, y_{n+1}), \text{ and } \Lambda \stackrel{\text{def}}{=} \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}.$$

Then by Proposition 8, (iii); Proposition 10, it holds that V is a (g, r) -divisor, Λ is a drift collection, and

$$\begin{aligned} & \{W \subseteq X_n^{\text{log}} : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= \{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\} = \mathcal{N} \sqcup \Lambda. \end{aligned}$$

This completes the proof of the implication (ii) \implies (iii). Next, we consider the implication (iii) \implies (ii). By Proposition 9, (ii), we may assume that the (g, r) -divisor V is equal to $V(y_1, \dots, y_{n+1})$. Then it follows from Proposition 10 that

$$\{W : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} = \{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\}.$$

Since

$$\begin{aligned} \#\{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\} &= \frac{(n+1)n}{2}, \\ \#\Lambda &= \frac{n(n-1)}{2}, \quad \#\mathcal{N} = n, \end{aligned}$$

and

$$n = \frac{(n+1)n}{2} - \frac{n(n-1)}{2},$$

it holds that

$$\Lambda \subseteq \{W : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\}.$$

So we may assume

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$$

and

$$\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^n.$$

This completes the proof of the implication (iii) \implies (ii). Next, we consider the implication (ii) \implies (iv). Write ${}^\dagger V \stackrel{\text{def}}{=} V(y_1, \dots, y_{n+1})$, ${}^\ddagger V \stackrel{\text{def}}{=} V(y_1, \dots, y_n)$, and $\Lambda \stackrel{\text{def}}{=} \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$. Then by Proposition 8, (iii); Proposition 10, it holds that ${}^\dagger V$ is a (g, r) -divisor, ${}^\ddagger V \in V_{[3]} \sqcup V_{[n]}$, Λ is a drift collection, and

$$\begin{aligned} & \{W \subseteq X_n^{\text{log}} : \text{tripodal divisors} \mid {}^\dagger V \neq W, {}^\dagger V \cap W \neq \emptyset\} \\ &= \{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\} \\ &= \mathcal{N} \sqcup \{W \subseteq X_n^{\text{log}} : \text{tripodal divisors} \mid {}^\ddagger V \cap W \neq \emptyset\}. \end{aligned}$$

This completes the proof of the implication (ii) \implies (iv).

Next, we consider the implication (iv) \implies (ii). By Proposition 9, (ii), we may assume that the (g, r) -divisor ${}^\dagger V$ is equal to $V(y_1, \dots, y_{n+1})$, and ${}^\ddagger V$ is equal to $V(z_1, \dots, z_n)$. Since ${}^\dagger V \neq {}^\ddagger V$, and ${}^\dagger V \cap {}^\ddagger V \neq \emptyset$, it holds that

$\{z_1, \dots, z_n\} \subseteq \{y_1, \dots, y_{n+1}\}$ (cf. Proposition 10). Thus, we may assume that $z_i = y_i$ for $i \in \{1, \dots, n\}$. Then it follows from Proposition 10 that

$$\{W: \text{tripodal divisors} \mid {}^\dagger V \neq W, {}^\dagger V \cap W \neq \emptyset\} = \{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\},$$

$$\begin{aligned} \{W: \text{tripodal divisors} \mid {}^\dagger V \cap W \neq \emptyset\} \cap \{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\} \\ = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}. \end{aligned}$$

Thus,

$$\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^n.$$

This completes the proof of the implication (iv) \implies (ii). \square

PROPOSITION 25. *Suppose that $(g, r) = (1, 1)$. Let \mathcal{N} be a collection of tripodal diagonals of X_n^{\log} such that $\sharp \mathcal{N} = n$. Then the following conditions are equivalent:*

(i) \mathcal{N} is a new vertical collection of X_n^{\log} .

(ii) There exist distinct elements $y_1, \dots, y_{n+1} \in \mathcal{C}_{r,n}$ such that

$$\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^n.$$

(iii) There exist a (g, r) -divisor V and a drift collection Λ such that

$$\mathcal{N} = \{W \subseteq X_n^{\log}: \text{tripodal divisors} \mid V \cap W \neq \emptyset\} \setminus \Lambda.$$

(iv) There exist a (g, r) -divisor ${}^\dagger V \in V_{[n+1]}$ and ${}^\ddagger V \in V_{[n]}$ such that

$$\begin{aligned} \mathcal{N} = \{W \subseteq X_n^{\log}: \text{tripodal divisors} \mid {}^\dagger V \cap W \neq \emptyset\} \\ \setminus \{W \subseteq X_n^{\log}: \text{tripodal divisors} \mid {}^\ddagger V \cap W \neq \emptyset\}. \end{aligned}$$

PROOF. This follows from the same proof of Proposition 24. \square

PROPOSITION 26. *The following hold:*

(i) *Let \mathcal{N} be a new vertical collection. Then there exist a unique (g, r) -divisor V and a unique drift collection Λ such that*

$$\mathcal{N} = \{W \subseteq X_n^{\log}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \setminus \Lambda.$$

In particular, if $\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^n$, then

$$V = V(y_1, \dots, y_{n+1}),$$

and

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}.$$

(ii) *Let*

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$$

be a drift collection. Then $\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^n$ is a new vertical collection such that

$$\mathcal{N} = \{W \subseteq X_n^{\log}: \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \setminus \Lambda,$$

where $y_{n+1} \in \mathcal{C}_{r,n} \setminus \{y_1, \dots, y_n\}$, and $V = V(y_1, \dots, y_{n+1})$.

PROOF. First, we consider assertion (i). The existence and final portion follows from the proof of the implication (ii) \implies (iii) of Proposition 23; the proof of the implication (ii) \implies (iii) of Proposition 24; the proof of the implication (ii) \implies (iii) of Proposition 24. Next, we consider the uniqueness of assertion (i). By the implication (i) \implies (ii) of Proposition 23; the implication (i) \implies (ii) of Proposition 24; the implication (i) \implies (ii) of Proposition 25, we may assume the new vertical collection \mathcal{N} is equal to $\{V(y_{n+1}, y_i)\}_{i=1}^n$. Since

$$\mathcal{N} \subseteq \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\},$$

by Proposition 10, it holds that V is equal to $V(y_1, \dots, y_{n+1})$. By the proof of the implication (iii) \implies (ii) of Proposition 23; the proof of the implication (iii) \implies (ii) of Proposition 24; Proposition 10, it holds that

$$\Lambda \subseteq \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\}$$

and Λ is equal to $\{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$. This completes the proof of assertion (i). Assertion (ii) follows immediately. \square

PROPOSITION 27.

$$\#\{\text{new vertical collections}\} = r\#\{\text{drift collections}\}.$$

PROOF. Since $\#\{C_{r,n} \setminus \{y_1, \dots, y_n\}\} = r$, this follows from Proposition 26, (i), (ii). \square

3. Mono-anabelian reconstruction

In the present §3, let $n \in \mathbb{Z}_{>0}$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$ and “ $r > 0$ ”; p a prime number; k an algebraic closed field of characteristic $\neq p$; X^{\log} a smooth log curve over k of type (g, r) . In the present §3, we give mono-anabelian algorithm to reconstruct (g, r, n) associated to intrinsic structure of the profinite group $\Delta^p(g, r, n)$ which is isomorphic to $\pi_1^p(X_n^{\log})$ (cf. Proposition 32, (v), below), and we give mono-anabelian algorithm to reconstruct a set GFS associated to intrinsic structure of $\Delta^p(g, r, n)$ and LFS (cf. Proposition 37, (iii), below).

DEFINITION 10. (i) Write $\pi_1(X_n^{\log})$ for the fundamental group of the log scheme X_n^{\log} (for a suitable choice of basepoint). We refer to [Hsh], Theorem B.1, B.2, for more details on fundamental groups of log schemes.

(ii) Write $\pi_1^p(X_n^{\log})$ for the maximal pro- p quotient of $\pi_1(X_n^{\log})$.

(iii) Let P be a log-full point of X_n^{\log} (cf. Definition 6, (i)) and P^{\log} the log scheme obtained by restricting the log structure of X_n^{\log} to the reduced closed subscheme of X_n determined by P . Then we obtain an outer homomorphism $\pi_1(P^{\log}) \rightarrow \pi_1^p(X_n^{\log})$ (for suitable choices of basepoints). We shall refer to the subgroup $\text{Im}(\pi_1(P^{\log}) \rightarrow \pi_1^p(X_n^{\log}))$, which is well-defined up to $\pi_1^p(X_n^{\log})$ -conjugation, as a *log-full subgroup* at P .

- (iv) Let H be a closed subgroup of $\pi_1^p(X_n^{\log})$. We shall say that H is a *generalized fiber subgroup* if there exist an automorphism α of X_n^{\log} over k and a fiber subgroup $F \subseteq \pi_1^p(X_n^{\log})$ (cf. [MzTa], Definition 2.3, (iii)) such that $H = \beta(F)$, where β is an automorphism of $\pi_1^p(X_n^{\log})$ which arises from α (cf. [Hgsh], Definition 9.1; [HMM], Definition 2.1, (ii)).
- (v) Let $\Delta^p(g, r, n)$ be a profinite group which is isomorphic to $\pi_1^p(X_n^{\log})$ (cf. Definition 10, (i), (ii)). Write LFS (resp. LD, $\mathcal{V}_{[m]}$, $\mathcal{V}_{[m]}^{\text{naive}}$, $\mathcal{V}_{[m]}^{\text{vertical}}$, TD, DD, GFS) for the set of subgroups of $\Delta^p(g, r, n)$ such that any isomorphism $\Delta^p(g, r, n) \xrightarrow{\sim} \pi_1^p(X_n^{\log})$ induces a bijection

$$\text{LFS} \xrightarrow{\sim} \{\text{log-full subgroups of } \pi_1^p(X_n^{\log})\}$$

(resp. LD $\xrightarrow{\sim}$ {inertia subgroups $\subseteq \pi_1^p(X_n^{\log})$ associated to log divisors},

$$\mathcal{V}_{[m]} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in V_{[m]}\},$$

$$\mathcal{V}_{[m]}^{\text{naive}} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in V_{[m]}^{\text{naive}}\},$$

$$\mathcal{V}_{[m]}^{\text{vertical}} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in V_{[m]}^{\text{vertical}}\},$$

$$\text{TD} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to tripodal divisors}\},$$

$$\text{DD} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to drift diagonals}\},$$

$$\text{GFS} \xrightarrow{\sim} \{\text{generalized fiber subgroups of } \pi_1^p(X_n^{\log})\}.$$

Write DC for the set of subsets of DD such that any isomorphism

$$\Delta^p(g, r, n) \xrightarrow{\sim} \pi_1^p(X_n^{\log})$$

induces a bijection

$$\text{DC} \xrightarrow{\sim} \{ \{ \text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in \Lambda \} \\ | \Lambda: \text{a drift collection} \}.$$

PROPOSITION 28. *The following hold:*

- (i) $\pi_1^p(X_1^{\log})$ is elastic, i.e., every topologically finitely generated closed normal subgroup $N \subseteq H$ of an open subgroup $H \subseteq \pi_1^p(X_1^{\log})$ is either trivial or of finite index in $\pi_1^p(X_1^{\log})$.
- (ii) If $n > 1$, then $\pi_1^p(X_n^{\log})$ is not elastic.
- (iii) Let V be a log divisor of X_n^{\log} . Then the inertia group associated to V is isomorphic to \mathbb{Z}_p .
- (iv) Let P be a log-full point of X_n^{\log} . Then the log-full subgroup at P is isomorphic to $\mathbb{Z}_p^{\oplus n}$.
- (v)

$$\#\{\text{conjugacy class of log-full subgroups } \subseteq \pi_1^p(X_n^{\log})\} = \prod_{i=0}^{n-1} (r + 2i).$$

(vi)

$$\begin{aligned} & \#\{\text{conjugacy class of inertia groups associated to log divisors}\} \\ &= (2^n - 1)r + (2^n - 1 - n). \end{aligned}$$

PROOF. Assertion (i) follows from [MzTa], Theorem 1.5. Next, we consider assertion (ii). Let $F \subseteq \pi_1^p(X_n^{\text{log}})$ be a generalized fiber subgroup (cf. Definition 10, (iv)). Since F is topologically finitely generated closed normal subgroup of $\pi_1^p(X_n^{\text{log}})$ and F is of infinite index in $\pi_1^p(X_n^{\text{log}})$ (cf. [MzTa], Remark 2.4.1), $\pi_1^p(X_n^{\text{log}})$ is not elastic. This completes the proof of assertion (ii). Assertions (iii), (iv) follow from [Hgsh], Proposition 3.7, (iii). Assertion (v) follows from Proposition 14, (i). Assertion (vi) follows from Proposition 5. \square

PROPOSITION 29. *Suppose that $n > 1$. Then the following hold:*

(i) *One may construct a p associated to the intrinsic structure of $\Delta^p(g, r, n)$, i.e.,*

$$\Delta^p(g, r, n) \rightsquigarrow p.$$

(ii) *One may construct an n associated to the intrinsic structure of $\Delta^p(g, r, n)$ and LFS, i.e.,*

$$(\Delta^p(g, r, n), \text{LFS}) \rightsquigarrow n.$$

(iii) *One may construct an r associated to the intrinsic structure of $\Delta^p(g, r, n)$ and LFS, i.e.,*

$$(\Delta^p(g, r, n), \text{LFS}) \rightsquigarrow r.$$

PROOF. Since $\Delta^p(g, r, n)$ is a pro- p group, assertion (i) follows immediate. Assertion (ii) follows from Proposition 28, (iv). Assertion (iii) follows from assertion (ii); Proposition 28, (v). \square

DEFINITION 11. Let $m \in \mathbb{Z}_{>0}$ and G a profinite group. Then we shall say that G is *unique factorization-like* if G satisfies the following properties:

- (i) There exist nontrivial profinite subgroups $G_1, \dots, G_m \subseteq G$ which are slim (cf. [MzTa], §0) and strongly indecomposable (cf. [MzTa], Definition 3.1) such that $G = G_1 \times \dots \times G_m$.
- (ii) Let $H_1, \dots, H_m \subseteq G$ be nontrivial profinite subgroups which are slim and strongly indecomposable. If $G = H_1 \times \dots \times H_m$, then there exists $\sigma \in S_m$ such that $G_i = H_{\sigma(i)}$ for each $i \in \{1, \dots, m\}$.

PROPOSITION 30. *The following hold:*

(i) *Let G be a unique factorization-like profinite group. Then one may construct a set $\{G_1, \dots, G_m\}$ (cf. Definition 11) associated to the intrinsic structure of G , i.e.,*

$$G \rightsquigarrow \{G_1, \dots, G_m\}.$$

(ii) *Let G_1, \dots, G_m are nontrivial profinite groups which are slim and strongly indecomposable. Then $G_1 \times \dots \times G_m$ is unique factorization-like, and one*

may construct a set $\{G_1, \dots, G_m\}$ associated to the intrinsic structure of $G_1 \times \dots \times G_m$, i.e.,

$$G_1 \times \dots \times G_m \rightsquigarrow \{G_1, \dots, G_m\}.$$

(iii) $\pi_1^p(X_n^{\log})$ is slim and strongly indecomposable.

PROOF. Assertion (i) follows immediately. Assertion (ii) follows from assertion (i); [MzTa], Corollary 3.4. Assertion (iii) follows from [MzTa], Proposition 1.4; [MzTa], Proposition 3.2; [Ind], Theorem C, (i). \square

PROPOSITION 31. *Suppose that $n > 1$. Let $m \in \{2, \dots, n+1\}$; $V \in V_{[m]}$ a log divisor of X_n^{\log} ; $I_V \subseteq \pi_1^p(X_n^{\log})$ an inertia group associated to V ; T^{\log} a smooth log curve over k of type $(0, 3)$. Then the following hold:*

- (i) *If $m = 2$, then $Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V$ is isomorphic to $\pi_1^p(X_{n-1}^{\log})$.*
- (ii) *If $m = n+1$, then $Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V$ is isomorphic to $\pi_1^p(T_{n-1}^{\log})$.*
- (iii) *If $m \in \{3, \dots, n\}$, then $Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V$ is isomorphic to*

$$\pi_1^p(T_{m-2}^{\log}) \times \pi_1^p(X_{n-m+1}^{\log}).$$

(iv) $Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V$ is unique factorization-like.

(v)

$$\begin{aligned} & \{I_V \subseteq \pi_1^p(X_n^{\log}) \mid \text{inertia subgroup associated to some log divisor } V \\ & \quad \text{such that } Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V \text{ is strongly indecomposable}\} \\ &= \{I_V \subseteq \pi_1^p(X_n^{\log}) \mid \text{inertia subgroup associated to } V \in V_{[2]} \sqcup V_{[n+1]}\}. \end{aligned}$$

PROOF. Assertion (i), (ii), (iii) follow from [Hgsh], Lemma 6.1, (i), (ii), (iii); [Hgsh], Remark 6.4. Assertion (iv) follows from assertions (i), (ii), (iii); Proposition 30, (ii), (iii). Assertion (v) follows from assertion (i), (ii), (iii); Proposition 30, (iii). \square

PROPOSITION 32. *Suppose that $n > 1$. Then the following hold:*

(i) *One may construct a set LD associated to the intrinsic structure of $\Delta^p(g, r, n)$ and LFS, i.e.,*

$$(\Delta^p(g, r, n), \text{LFS}) \rightsquigarrow \text{LD}.$$

(ii) *One may construct a set $\{\Delta^p(g, r, m), \Delta^p(0, 3, m) \mid 1 \leq m \leq n-1\}$ associated to the intrinsic structure of $\Delta^p(g, r, n)$ and LD, i.e.,*

$$(\Delta^p(g, r, n), \text{LD}) \rightsquigarrow \{\Delta^p(g, r, m), \Delta^p(0, 3, m) \mid 1 \leq m \leq n-1\}.$$

(iii) *One may construct a set $\{\Delta^p(g, r, 1), \Delta^p(0, 3, 1)\}$ associated to the intrinsic structure of $\Delta^p(g, r, n)$ and $\{\Delta^p(g, r, m), \Delta^p(0, 3, m) \mid 1 \leq m \leq n-1\}$, i.e.,*

$$(\Delta^p(g, r, n), \{\Delta^p(g, r, m), \Delta^p(0, 3, m) \mid 1 \leq m \leq n-1\}) \rightsquigarrow \{\Delta^p(g, r, 1), \Delta^p(0, 3, 1)\}.$$

(iv) *One may construct a g associated to the intrinsic structure of $\Delta^p(g, r, n)$, $\{\Delta^p(g, r, 1), \Delta^p(0, 3, 1)\}$, and r , i.e.,*

$$(\Delta^p(g, r, n), \{\Delta^p(g, r, 1), \Delta^p(0, 3, 1)\}, r) \rightsquigarrow g.$$

- (v) One may construct (g, r, n) associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS, i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow (g, r, n).$$

PROOF. Assertion (i) follows from [Hgsh], Theorem 4.7; [Hgsh], Lemma 5.1 (i), (ii), (iii). Assertion (ii) follows from Proposition 30, (ii), (iii); Proposition 31, (i), (ii), (iii), (iv). Assertion (iii) follows from Proposition 28, (i), (ii). Next, we consider assertion (iv). Write $N(g, r)$ for the number of generators of $\Delta^P(g, r, 1)$. Since $2g - 2 + r > 0$, it holds that $N(g, r) = 2g + r - 1 \geq N(0, 3) = 2$ (cf. [MzTa], Remark 1.2.2). Thus,

$$\frac{\max(N(g, r), N(0, 3)) - r + 1}{2} = g.$$

This completes the proof of assertion (iv). Assertion (v) follows from assertions (i), (ii), (iii), (iv); Proposition 29, (ii), (iii). \square

Now, we consider a conjecture.

CONJECTURE 1. *Suppose that $n > 1$. Let $m \in \{2, \dots, n + 1\}$. Then the following hold:*

- (i) *One may construct a set $\mathcal{V}_{[m]}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS, i.e.,*

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \mathcal{V}_{[m]}.$$

- (ii) *One may construct a set $\mathcal{V}_{[m]}^{\text{naive}}, \mathcal{V}_{[m]}^{\text{vertical}}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS, i.e.,*

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \mathcal{V}_{[m]}^{\text{naive}} \text{ and } \mathcal{V}_{[m]}^{\text{vertical}}.$$

REMARK 3. *Suppose that $n > 1$.*

- (i) *If $(g, r) \neq (0, 3)$, then Conjecture 1, (i), follows immediately from Theorem 2, (i); Proposition 31, (i), (ii), (iii); [Hgsh], Lemma 6.5, (iii), (iv).*
(ii) *If $(g, r) \neq (0, 3), (1, 1)$, then Conjecture 1, (ii), follows immediately from Conjecture 1, (i); Proposition 21, (iv).*

In this paper, we do not apply Theorem 2; Remark 3, (i), (ii).

PROPOSITION 33. *Suppose that $n > 1$. Let V_1, V_2 be log divisors of X_n^{log} . Then the following conditions are equivalent:*

- (i) $V_1 \cap V_2 \neq \emptyset$.
(ii) *There exists a log-full subgroup $A \subseteq \pi_1^p(X_n^{\text{log}})$ which contains inertia groups I_{V_1}, I_{V_2} associated to V_1, V_2 .*

PROOF. It follows immediately from Proposition 10; [Hgsh], Proposition 4.3; [Hgsh], Lemma 8.4. \square

PROPOSITION 34. *Suppose that $n > 1$. Then the following hold:*

- (i) One may construct a set $\mathcal{V}_{[2]} \sqcup \mathcal{V}_{[n+1]}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LD, i.e.,

$$(\Delta^P(g, r, n), \text{LD}) \rightsquigarrow \mathcal{V}_{[2]} \sqcup \mathcal{V}_{[n+1]}.$$

- (ii) One may construct a set $\mathcal{V}_{[3]} \sqcup \mathcal{V}_{[n]}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LD, i.e.,

$$(\Delta^P(g, r, n), \text{LD}) \rightsquigarrow \mathcal{V}_{[3]} \sqcup \mathcal{V}_{[n]}.$$

- (iii) Suppose that $(g, r) \neq (0, 3), (1, 1)$. Then one may construct sets $\mathcal{V}_{[3]}, \mathcal{V}_{[n]}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and $\mathcal{V}_{[3]} \sqcup \mathcal{V}_{[n]}$, i.e.,

$$(\Delta^P(g, r, n), \mathcal{V}_{[3]} \sqcup \mathcal{V}_{[n]}) \rightsquigarrow \mathcal{V}_{[3]}, \mathcal{V}_{[n]}.$$

- (iv) Suppose that $r \neq 3$. Then one may construct sets $\mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS, i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}.$$

- (v) Suppose that $(g, r) \neq (0, 3), (1, 1)$ and $n > 2$. Then one may construct sets $\mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS, i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}.$$

- (vi) Suppose that $g \neq 0$, $r = 3$, and $n \neq 3$. Then one may construct sets $\mathcal{V}_{[2]}^{\text{naive}}, \mathcal{V}_{[2]}^{\text{vertical}}, \mathcal{V}_{[n+1]}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS, i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \mathcal{V}_{[2]}^{\text{naive}}, \mathcal{V}_{[2]}^{\text{vertical}}, \mathcal{V}_{[n+1]}.$$

- (vii) Suppose that $(g, r) \neq (0, 3)$. Then one may construct sets $\mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}$ associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS, i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}.$$

- (viii) One may construct a set TD associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LD, i.e.,

$$(\Delta^P(g, r, n), \text{LD}) \rightsquigarrow \text{TD}.$$

PROOF. Assertion (i) follows from Proposition 31, (v). Assertion (ii) follows from Proposition 28, (i), (ii); Proposition 30, (ii); Proposition 31, (iii) (cf. also Proposition 32, (iii)). Next, we consider assertion (iii). Since $(g, r) \neq (0, 3), (1, 1)$, it holds that $N(g, r) > N(0, 3) = N(1, 1) = 2$ (cf. the proof of Proposition 32, (iv)). Thus, assertion (iii) follows immediately. Assertion (iv) follows from assertion (i); Proposition 19, (i), (ii), (iii); Proposition 32, (i); Proposition 33. Assertion (v) follows from assertions (i); Proposition 7, (i); Proposition 22; Proposition 32, (i). Assertion (vi) follows from assertions (i), (ii), (iii); Proposition 18; Proposition 20, (i), (ii), (iii); Proposition 32, (i); Proposition 33. Assertion (vii) follows from assertions (iv), (v), (vi). Assertion (viii) follows from assertions (i), (vii); Proposition 6, (i), (ii). \square

PROPOSITION 35. *Suppose that $n > 1$. Then the following hold:*

- (i) Suppose that $(g, r) = (0, 3), (1, 1)$. Then one may construct a set DD associated to the intrinsic structure of $\Delta^p(g, r, n)$ and LFS, i.e.,

$$(\Delta^p(g, r, n), \text{LFS}) \rightsquigarrow \text{DD}.$$

- (ii) Suppose that $(g, r) \neq (0, 3)$ and $r > 1$. Let $m \in \{2, \dots, n\}$. Then one may construct sets $\mathcal{V}_{[m]}^{\text{naive}}, \mathcal{V}_{[m]}^{\text{vertical}}$ associated to the intrinsic structure of $\Delta^p(g, r, n)$, LFS, and $\mathcal{V}_{[m]}$, i.e.,

$$(\Delta^p(g, r, n), \text{LFS}, \mathcal{V}_{[m]}) \rightsquigarrow \mathcal{V}_{[m]}^{\text{naive}}, \mathcal{V}_{[m]}^{\text{vertical}}.$$

- (iii) Suppose that $(g, r) \neq (0, 3), (1, 1)$. Let $m \in \{2, \dots, n\}$ such that $n+1 \neq m$. Then one may construct sets $\mathcal{V}_{[m]}^{\text{naive}}, \mathcal{V}_{[m]}^{\text{vertical}}$ associated to the intrinsic structure of $\Delta^p(g, r, n)$, LFS, and $\mathcal{V}_{[m]}$, i.e.,

$$(\Delta^p(g, r, n), \text{LFS}, \mathcal{V}_{[m]}) \rightsquigarrow \mathcal{V}_{[m]}^{\text{naive}}, \mathcal{V}_{[m]}^{\text{vertical}}.$$

- (iv) Suppose that $(g, r) \neq (0, 3), (1, 1)$. Let $m \in \{2, \dots, n\}$. Then one may construct sets $\mathcal{V}_{[m]}^{\text{naive}}, \mathcal{V}_{[m]}^{\text{vertical}}$ associated to the intrinsic structure of $\Delta^p(g, r, n)$, LFS, $\mathcal{V}_{[m]}$, and $\mathcal{V}_{[m+1]}$, i.e.,

$$(\Delta^p(g, r, n), \text{LFS}, \mathcal{V}_{[m]}, \mathcal{V}_{[m+1]}) \rightsquigarrow \mathcal{V}_{[m]}^{\text{naive}}, \mathcal{V}_{[m]}^{\text{vertical}}.$$

- (v) One may construct a set DD associated to the intrinsic structure of $\Delta^p(g, r, n)$ and LFS, i.e.,

$$(\Delta^p(g, r, n), \text{LFS}) \rightsquigarrow \text{DD}.$$

PROOF. Assertion (i) follows from Proposition 8, (iii), (iv); Proposition 34, (viii). Assertion (ii) follows from Proposition 16, (iv), (v), (vi); Proposition 33; Proposition 34, (viii). Assertion (iii) follows from Proposition 21, (i), (ii), (iii). Assertion (iv) follows from Proposition 21, (iv). Assertion (v) follows from assertion (i), (ii), (iii), (iv); Proposition 8, (ii) Proposition 34, (iii), (vii). \square

PROPOSITION 36. Suppose that $n > 1$. Then the following hold:

- (i) $\iota: X_n^{\log} \rightarrow X^{\log} \times_k \cdots \times_k X^{\log}$ (cf. Definition 6, (vii)) induces the outer surjective homomorphism

$$\iota_{\Delta}: \pi_1^p(X_n^{\log}) \rightarrow \pi_1^p(X^{\log}) \times \cdots \times \pi_1^p(X^{\log}).$$

- (ii) $\text{Ker} \iota_{\Delta}$ is topologically generated by the inertia groups associated to the naive diagonals.
- (iii) Let V be a log divisor of X_n^{\log} then there exists an inertia group I_V associated to V which is contained in $\text{Ker} \iota_{\Delta}$ if and only if

$$V \in \prod_{m=2}^n \mathcal{V}_{[m]}^{\text{naive}}.$$

- (iv) Let Λ be a drift collection. Write I_{Λ} for the subgroup of $\pi_1^p(X_n^{\log})$ which is topologically generated by the inertia groups associated to $V \in \Lambda$. Then $\pi_1^p(X_n^{\log})/I_{\Lambda}$ is isomorphic to $\pi_1^p(X^{\log}) \times \cdots \times \pi_1^p(X^{\log})$.

- (v) One may construct a surjection $\Delta^P(g, r, 1) \times \cdots \times \Delta^P(g, r, 1) \twoheadrightarrow \Delta^P(g, r, 1)$ associated to the intrinsic structure of $\Delta^P(g, r, 1) \times \cdots \times \Delta^P(g, r, 1)$, i.e.,
 $\Delta^P(g, r, 1) \times \cdots \times \Delta^P(g, r, 1) \rightsquigarrow \Delta^P(g, r, 1) \times \cdots \times \Delta^P(g, r, 1) \twoheadrightarrow \Delta^P(g, r, 1)$.

PROOF. Assertion (i) follows immediately. Assertion (ii) follows from [Hgsh], Lemma 7.1. Assertion (iii) follows from [Hgsh], Lemma 7.2. Assertion (iv) follows from assertion (i), (ii). Assertion (v) follows from Proposition 30, (ii), (iii). \square

PROPOSITION 37. *Suppose that $n > 1$. Then the following hold:*

- (i) One may construct a set DC associated to the intrinsic structure of $\Delta^P(g, r, n)$, LFS, and DD, i.e.,

$$(\Delta^P(g, r, n), \text{LFS}, \text{DD}) \rightsquigarrow \text{DC}.$$

- (ii) One may construct a set GFS associated to the intrinsic structure of $\Delta^P(g, r, n)$ and DC, i.e.,

$$(\Delta^P(g, r, n), \text{DC}) \rightsquigarrow \text{GFS}.$$

- (iii) One may construct a set GFS associated to the intrinsic structure of $\Delta^P(g, r, n)$ and LFS, i.e.,

$$(\Delta^P(g, r, n), \text{LFS}) \rightsquigarrow \text{GFS}.$$

PROOF. Assertion (i) follows from [Hgsh], Proposition 8.12. Assertion (i) follows from Proposition 36, (iv), (v). Assertion (iii) follows from assertions (i), (ii); Proposition 35, (v). \square

References

- [Hgsh] K. Higashiyama, Reconstruction of inertia groups associated to log divisors from a configuration space group equipped with its collection of log-full subgroups, *Math. J. Okayama Univ.* **61** (2019), 37-73.
- [Hsh] Y. Hoshi, The exactness of the log homotopy sequence, *Hiroshima Math. J.* **39** (2009), no. 1, 61-121.
- [CbTpI] Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: Inertia groups and profinite Dehn twists, *Galois-Teichmüller Theory and Arithmetic Geometry, Adv. Stud. Pure Math.* **63**, Math. Soc. Japan, Tokyo (2012), 659-811.
- [Knu] F. F. Knudsen, The projectivity of the moduli space of stable curves II, *Math. Scand.* **52** (1983), 161-199.
- [Ind] A. Minamide, Indecomposability of Anabelian Profinite Groups, *Math. J. Okayama Univ.* **60** (2018), 175-208.
- [HMM] Y. Hoshi, A. Minamide and S. Mochizuki, *Group-theoreticity of numerical invariants and distinguished subgroups of configuration space groups*, RIMS Preprint **1870** (March 2017).
- [CmbGC] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math J.* **59** (2007), 455-479.
- [MzTa] S. Mochizuki and A. Tamagawa, The Algebraic and Anabelian Geometry of Configuration Spaces, *Hokkaido Math. J.* **37** (2008), 75-131.
- [Nky] C. Nakayama, Logarithmic étale cohomology, *Math. Ann.* **308** (1997), 365-404.

Kazumi Higashiyama
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502 JAPAN
E-mail: higashi@kurims.kyoto-u.ac.jp
URL: <http://www.kurims.kyoto-u.ac.jp/higashi/>