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Langevin equation with Coulomb friction - Simulation and theory of granular particles under the external vibration

Hisao Hayakawa and Atsushi Kawarada
Department of Physics, Yoshida-South Campus, Kyoto University, Kyoto 606-8501, Japan

The simulation of granular particles in a quasi two-dimensional container under the vertical vibration as an experimental accessible model for granular gases is performed. The velocity distribution function obeys an exponential-like function during the vibration and deviates from the exponential function in free-cooling states. It is confirmed that this exponential distribution function is produced by Coulomb friction force. A Langevin equation with Coulomb's friction is proposed to describe the motion of such the system. Through the analysis of the corresponding Fokker-Planck equation, we have obtained the steady velocity distribution function under the influence of the external field.

I. INTRODUCTION

Granular physics has been a challenging subject of statistical physics since the rediscovery of significant nature in granular materials in the late 80's or the early 90's.[1] An assembly of grains has so strong fluctuations in the configuration, the force and the motion that mean-field theories cannot be used in most of situations. A typical example of the strong fluctuation appears in the force distribution for a static granular assembly piled by the gravity.[2] The force propagates along force-chains and the distribution function of the magnitude of the force on the bottom of a container does not obey Gaussian but an exponential function.[3]

Such the strong fluctuation coming from non-Gaussian properties should be relevant even in the dynamics of granular assemblies. However, there are not so many systematic researches to focus on the statistical distribution functions in the steady state, because real and numerical experiments report no unified results, *i.e.*, velocity distribution functions (VDF) obey Gaussian-like with the exponential tail[5–12], functions from Gaussian to the exponential depending on the density[13, 14], the stretched exponential[15–17] and even power-law functions[18, 19] depending on situations.

As in the standard statistical mechanics, an assembly of grains in a gas phase is an idealistic situation to study what the proper statistical weight is. Approximate granular gases can be obtained by rapid granular flow on an inclined surface[22, 23], gas-particles mixtures[25] and the external vibration.[11, 14–16, 18] However, these systems have defects, because (i) the boundary and the gravity effects are so strong in the rapid granular flows, (ii) the hydrodynamic interaction between particles are so complicated, and (iii) a dense cluster appears in the vibrating experiment. Therefore, it is difficult to achieve free-cooling gases without the effect of gravity in experiments.[15]

In this report, to remove such the difficulties, first, we propose an experimental accessible situation to produce granular gases. Second, we demonstrate that VDF in both dense and dilute granular gases under the vertical vibration can obey an exponential-like function when Coulomb friction is important through our simulation based on the distinct element method (DEM). This exponential VDF disappears immediately after vibration is stopped, *i.e.* the grains are in a free-cooling process. To explain these results, third, we introduce a phenomenological Langevin equation to describe the motion of particles and explain the mechanism to appear the exponential VDF. We also investigate mathematical properties of Langevin equation with Coulomb friction in details.

The organization of this paper is as follows. In the next section, we introduce our setup in the simulation. In section 3, we summarize the results of our simulation. In section 4, we obtain the steady VDF under the influence of the steady external field based on Langevin equation with Coulomb friction. In section 5 we discuss our result through the comparison of our result with the simulation of granular particles in a quasi-two-dimensional box. In the final section, we conclude our results. We note that this report is basically joint one from two papers[26, 27]

II. THE SETUP OF OUR SIMULATION

We use a three-dimensional DEM for monodispersed spherical particles.[28] Instead of using the Hertzian contact force, we adopt the linear spring model to represent repulsion of contacted spheres. We also include the rotation of spheres and the Coulomb slip for tangential contact. We simulate a monodisperse system with particle diameter d. Because of the limitation of the space we skip the detailed description of model including the choice of parameters. We can see the parameters and the details of model in ref.[26].

We focus on the following situation: Particles are confined in a quasi two-dimensional container in which the height is $1.8d$ and the horizontal plane is a square (Fig.1). In this system, particles cannot compose a multilayer configuration in the vertical direction, but the particles have vertical velocities in their motion. The mobile particles are randomly

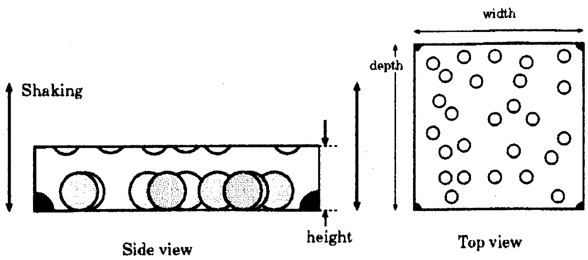

FIG. 1: The schematic side view (left) and the top view (right) of our setup: The random scatters are fixed spherical particles on the top board. The diameter of fixed particles on the top plate (white ones) is uniformly random between $0.6d$ and $0.8d$, and their centers are located $0.15d$ above the top board. To avoid stack of particles at the corners we introduce four fixed particles (black ones in the figure) at the corners.

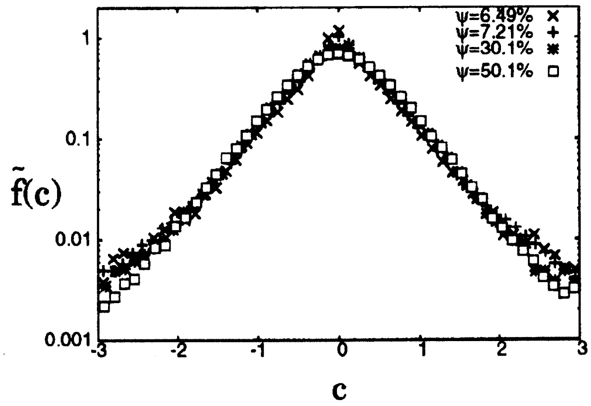

FIG. 2: The scaled VDF $\tilde{f}(c)$ in steady states under the vertical vibration. Here Ψ represents the area fraction. For $\Psi = 7.21\%$ we use 10,000 particles, and use 1,000 particles for other cases. It should be noted that Fig.2 in ref.[26] use the data of 3,000 particles.

scattered by fixed particles on the top of the container if the container is vertically vibrated. The number of fixed scatters is 3.5 times of that of the mobile particles. The vibration is driven by a sinusoidal force whose amplitude and the frequency are given by $A = 1.2d$ and $f = 0.5\sqrt{g/d}$, respectively. Thus, the acceleration amplitude becomes $\Gamma = A(2\pi f)^2/g = 11.5$. If the vibration is stopped, the particles are moving with the rotation on the bottom plate, then the system can simulate a two-dimensional free-cooling granular gas.

The simulation starts from a stationary state of particles on the bottom plate. The particles gain the kinetic energy from the external oscillation and the system reaches a steady state in the balance between collisional dissipations and the gain of the energy from the external force. Typically, the system reaches the statistical steady state after 25 cycles of the oscillation. We should note that the kinetic energy of particles is a little oscillated depending on the phase of the external oscillation after the system reaches the statistical steady state.

III. THE RESULTS OF OUR SIMULATION

In this section, we summarize the result of our simulation.

In the steady state, we confirm that the density is uniform and there is no long-range correlation in contrast to granular gases in free-cooling states. Thus, the system does not have any systematic flows and any definite clusters. We also check that the effects of side boundaries can be neglected. This is because particles always hit the top wall or the bottom wall during the vibration. After we stop the vertical oscillation, the correlation grows with time as the free-cooling process proceeds.

The most characteristic quantity for this gas system is VDF. In the steady state, the vertical component of VDF has double peaks where each peak corresponds to a lifting or a falling process, while the horizontal VDF has a single peak. For later discussion, we only use the horizontal VDFs for the analysis. We plot the scaled horizontal VDF $\tilde{f}(c)$

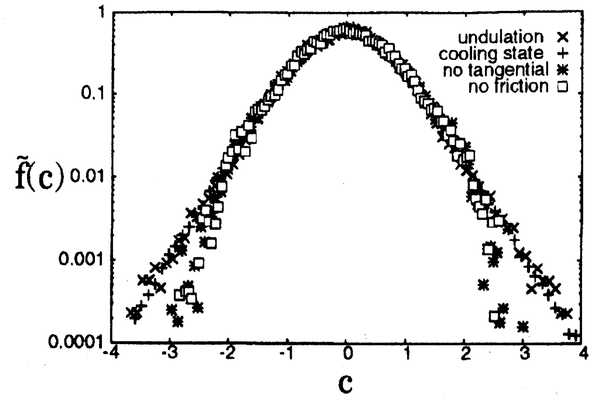

FIG. 3: The scaled VDF for several situations. In the inset, 'no tangential' and 'no friction' correspond to the results of simulations without the tangential force and the friction force, respectively, where the systems are in the steady state and include 1,000 particles with the area fraction 0.0649. Here, 'cooling state' and 'undulation' indicate the results of simulation of the cooling process of 10,000 particles and the undulation, respectively. Note that Fig.3 in ref.[?] contains the data for 3,000 particles.

in Fig.2

$$f(\mathbf{v},t) = nv_0(t)^{-2}\tilde{f}(\mathbf{v}/v_0) \tag{1}$$

with the density n, the average speed $v_0 = \sqrt{2T/m}$ with the granular temperature T, and $\int d\mathbf{c}\tilde{f}(c) = \int d\mathbf{c}c^2\tilde{f}(c) = 1$, where $\tilde{f}(c)$ is averaged over the cycles. As in Fig.2, the scaled VDF can be approximated by an exponential function. In fact, the flatness defined by $< c^4 > / < c^2 >^2 = < c^4 >$ with $< c^n > = \int d\mathbf{c}c^n\tilde{f}(c)$ is not far from 6. It should be noted that the flatness with the Gaussian VDF is 3 and with the exponential VDF is 6. In our simulation, the flatnesses are 8.00, 7.63, 6.89, 6.26, 5.67 and 5.35 for 1000 particles' simulation for corresponding area fractions projected into the horizontal plane 0.0649, 0.100, 0.200, 0.301, 0.400 and 0.501. More details results for 3,000 particles can be seen in Table I in ref.[26]. In addition, the dependence of the flatness on the density is relatively weaker in our situation. For a large system with 10,000 particles with the area fraction 0.0721 has smaller flatness as 6.85. If the external oscillation stops, the flatness decreases quickly and is saturated at 4.20 for 10,000 particles. As can be seen in Fig.3, VDF in the cooling process is nearly Gaussian for low energy particles but has an exponential tail for high energy particles as in the usual gas systems.[5] Note that the area fraction 0.0721 corresponds to the volume fraction 0.0271.

Let us consider the origin of the exponential-like VDF. It is easy to verify that the exponential VDF cannot be reproduced without the existence of Coulomb friction in DEM. In fact, if all component of tangential contact force is omitted, the flatness becomes 3.43 for 3,000 particles with the area fraction 0.0649. While if we only neglect the effect of Coulomb friction in eq.(??), the flatness becomes 3.43 in the same situation. In our system, particles feel the strong shear force when particles hit the fixed scatters on the top wall, because the colliding particles contact the side of fixed scatters, in general, and the directions of their motion are changed drastically. Thus, the tangential slips between particles and the fixed scatters are the dominant dissipative processes in the steady state. Since we specify the origin of exponential-like VDF, we can understand the weak dependence of VDF on the density. Namely, particles directly hit the fixed scatters for dilute case, while dense particles collide with each other and rotate without slips besides the collisions with the walls.

Therefore, the essence to produce the large flatness in VDF is apparently Coulomb's friction. Thus, we propose the following Langevin equation to describe horizontal motion of particles:

$$\frac{d\mathbf{v}}{dt} = -\zeta\frac{\mathbf{v}}{v} + \boldsymbol{\eta}, \tag{2}$$

where $\mathbf{v}$, $u = |\mathbf{v}|$ are respectively the velocity, and the speed. The friction ζ may be proportional to μg. The α component of $\boldsymbol{\eta}$ which is the random force satisfies the fluctuation-dissipation relation:

$$< \eta_\alpha(t) > = 0, \quad < \eta_\alpha(t)\eta_\beta(t') > = 2D\delta_{\alpha\beta}\delta(t-t'), \tag{3}$$

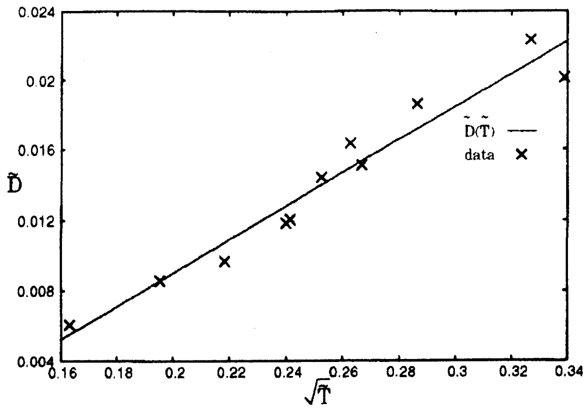

FIG. 4: The relation between $\tilde{D}$ and $\tilde{T}$ for 1,000 particles. The solid line represents $\tilde{D} = 0.0945\sqrt{\tilde{T}} - 0.0099$. The figure including the data of 3,000 particles which contains until $\sqrt{\tilde{T}} \simeq 0.5$ can be seen in Fig.4 in ref.[26].

where $D = \gamma\sqrt{T/3m}$ is the diffusion coefficient. The Langevin equation (2) with (3) can be converted into the Fokker-Planck or Kramers equation for the probability distribution function $P(\mathbf{x}, \mathbf{u}, t)$:

$$\frac{\partial P(\mathbf{u},t)}{\partial t} = \{\zeta\frac{\partial}{\partial \mathbf{v}}\cdot\frac{\mathbf{v}}{v} + D\frac{\partial^2}{\partial v^2}\}P(\mathbf{v},t) \tag{4}$$

For steady states, the equation for $P \to P_{st}$ is reduced to $(\mathbf{u}/u)P_{st} + \sqrt{T/3m}(d/d\mathbf{v})P_{st} = 0$ for $\mathbf{u} \neq 0$. Its solution can be obtained easily

$$P_{st}(\mathbf{v}) = 2\sqrt{\frac{m}{3T}}\exp[-\sqrt{\frac{3m}{T}}v]. \tag{5}$$

Thus, the Langevin equation with Colomb friction law obeys the exponential VDF.

To check the validity of the new Langevin equation for the motion of particles, we evaluate both the diffusion coefficient D and the friction coefficient γ. At first, we have confirmed that the motion of particles in the horizontal plane is diffusive. Then, the diffusion coefficient is evaluated from the relation $< (\mathbf{r}(t) - \mathbf{r}(t_0))^2 >= 4D(t - t_0)$, where we choose t_0 as 25 cycles of the oscillation and simulate the motion of particles until 75 cycles. For each parameter of the oscillation we determine T from the second moment of VDF. Thus, we obtain the relation between D and T as in Fig.4. Honestly speaking, the best fitted relation is $\tilde{D} = 0.0925\tilde{T}^{0.470} - 0.011$, where $\tilde{D} = D/(d^{1/2}g^{3/2})$ and $\tilde{T} = T/(mgd)$, but this can be replaced by $\tilde{D} = 0.0945\sqrt{\tilde{T}} - 0.0099$ for 1,000 particles. If we include the data of 3,000 particles the fitted values are a little different from those presented here, but the tendency is the same. We also note that the collective vertical motion suppresses for larger T, and no diffusion takes place because of the insufficient kinetic energy for smaller T. Using the relation between $\tilde{D}$ and $\tilde{T}$ we can evaluate γ as $\gamma = 0.164g$ which is independent of T. This is because the dominant dissipative process is collisions between particles and the horizontal walls caused by the collective motion of particles in the vertical direction. Thus, we believe that the Langevin equation (2) can be used to describe the motion of particles.

IV. MATHEMATICAL ANALYSIS OF LANGEVIN EQUATION WITH COULOMB FRICTION

In the previous section we suggest that Langevin equation associated with Coulomb friction can be an effective equation in a vibrating system of granular particles through their simulation of granular particles. In this section, we investigate mathematical properties of Langevin equation with Coulomb friction following ref.[?]. We may add the external force $\mathbf{F}_{ex}$ in eq.(2) to discuss the linear response of $\mathbf{F}_{ex}$ to our system.

It is convenient to use dimensionless distribution function. For this purpose, we may introduce

$$P(\mathbf{v},t) = v_0(t)^{-d}\tilde{P}(\mathbf{c},t), \quad \mathbf{c} = \mathbf{v}/v_0(t) \tag{6}$$

with the normalization

$$\int d\mathbf{c}c^2\tilde{P}_0(\mathbf{c},t) = 1 \tag{7}$$

where d is the dimension, and $\tilde{P}_0$ is the scaled VDF without $\boldsymbol{F}_{ex}$. We adopt $\tilde{P}_0$ instead of $\tilde{P}$ in eq.(7), because the determination of the scaled factor is difficult if we use $\tilde{P}$. In fact, T is the function of $\boldsymbol{F}_{ex}$ and cannot be determined without the complete form of VDF.

Before we proceed to the next section, we give some comments on the model. First, Coulomb friction is singular at $\mathbf{v} = 0$ as in eqs.(2) and (4). Indeed the start and the stop of frictional motions of macroscopic materials are the singular processes such as the break up and the formation of force networks. In addition, the static friction coefficient is almost always larger than the dynamical friction coefficient. Therefore, we can guess that particles are condensed at $\mathbf{v} = 0$ in simple setups. Indeed, when we simulate the motion of frictional disks on an inclined surface with dense random scatters, most of particles are trapped by arrays of scatters but VDF for mobile particles seems to obey an exponential function in a steady state. Even when there are no condensed particles at $\mathbf{v} = 0$, the differentiation of VDF with respect to $\mathbf{v}$ in (??) is discontinuous at $\mathbf{v} = 0$. Thus, VDF can be singular at $\mathbf{v} = 0$. Here we are only interested in the case of $\mathbf{v} \neq 0$ in this paper.

Let us consider the situation that a particle is moving under the influence of $\boldsymbol{F}_{ex} = mg\hat{z}$ with the unit vector $\hat{z}$ parallel to the direction of the external force and Coulomb friction force associated with the random force. We assume that the system is a two-dimensional one, because the particle is typically located on a substrate when Coulomb friction is important. As mentioned in the previous section $\hat{z}$ is not direction of gravity but the direction of the slope on the horizontal plane in the setup of Kawarada and Hayakawa[26]. We are interested in the statistical steady state in the balance between the friction and the external force.

Let θ be the angle between the direction we consider and $\hat{z}$, the steady equation for $\mathbf{v} \neq 0$ becomes

$$g[\cos\theta\frac{\partial P}{\partial v} + \frac{\sin^2\theta}{v}\frac{\partial P}{\partial\cos\theta}] = \zeta\{\frac{P}{v} + \frac{\partial P}{\partial v}\} + D[\frac{\partial^2}{\partial v^2} + \frac{1}{v}\frac{\partial}{\partial v} + \frac{1}{v^2}\frac{\partial^2}{\partial\theta^2}]P, \tag{8}$$

where $v = |\mathbf{v}|$. It should be noted that the singularity at $v = 0$ still appears in the first term of the right hand side of (8).

Now we assume the expansion

$$P(v,\theta) = \sum_{n=0}^{\infty} P_n(v)\cos n\theta = P_0(v) + P_1(v)\cos\theta. \tag{9}$$

Here the terms with $n \geq 2$ are irrelevant, because the contribution of $n \geq 2$ is orthogonal to those of $n = 0, 1$, and the effect of gravity appears in the term of $n = 1$. Thus, the normalization (??) is reduced to

$$2\pi\int_0^\infty dvvP_0(v) = 1, \quad \text{and} \quad \int_0^{2\pi} d\theta\cos^2\theta\int_0^\infty dvvP_1(v) = 0. \tag{10}$$

It is easy to verify P_0 and P_1 can be represented by a function $f(v)$

$$P_0(v) = f(v)e^{-\eta v}, \quad P_1(v) = \frac{2D}{g}f'(v)e^{-\eta v}. \tag{11}$$

With the aid of eq.(11) eq.(8) can be rewritten as

$$f''' + \left(\frac{1}{v} - \eta\right)f'' - \left(\frac{1}{v^2} + \epsilon\eta^2\right)f' + \epsilon\eta^3 f = 0, \tag{12}$$

where $\epsilon = (g/\zeta)^2/2$.[27]

Since we do not know the general procedure to obtain the solution of the third order ordinary differential equation (12) and it may not be easy to get the numerical solution for eq.(12) around the singular point $v = 0$, we rewrite it as a set of the second order differential equations under the assumption of small ϵ. For this purpose, we expand

$$f(v) = \frac{\eta^2}{2\pi}[1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots]. \tag{13}$$

Skipping the detailed derivation we write the result of analytic calculation of $f^{(1)}$ as[27]

$$\begin{aligned} f^{(1)}(v) &= c_0\{G^{31}_{23}\left(\eta v\middle|\begin{matrix} 0 & 1 & \\ 0 & 0 & 0\end{matrix}\right) - \eta v(\gamma - 1) + (\gamma + \eta v)\ln\eta v - \frac{1}{2}(\ln\eta v)^2\} \\ &\quad + \frac{\eta^2}{2}v^2 + c_1 \end{aligned} \tag{14}$$

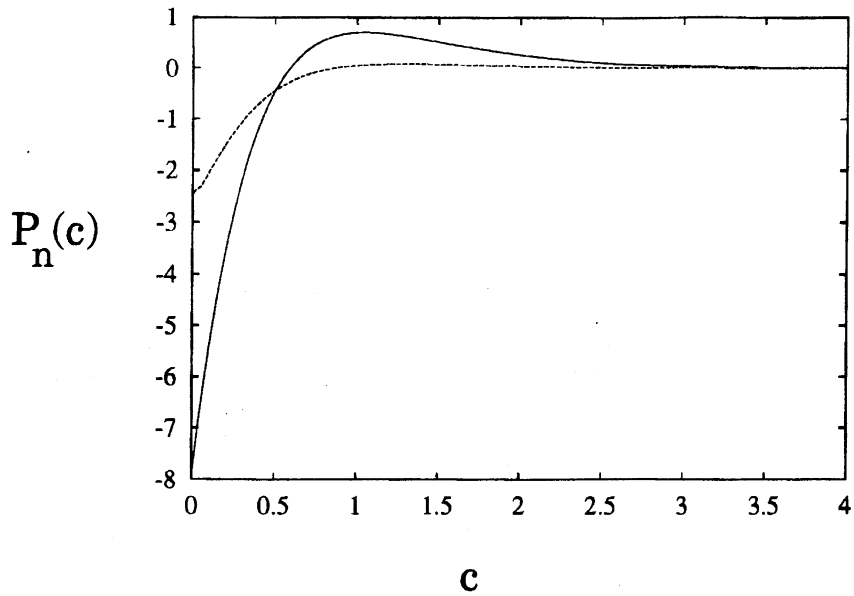

FIG. 5: The scaled $P_0(c)$ (solid line) and $P_1(c)$ (dashed line) as functions of $c = v/v_0$, where $P_n(c)$ denotes $P_0(c)$ or $P_1(c)$ in the figure. For $P_1(c)$ we plot $f^{(1)'}(c)e^{-\sqrt{6}c}$ to remove the effect of the expansion parameter ϵ.

where c_1 is a constant and $G^{31}_{23}\left(\eta v\middle|\begin{matrix}0 & 1 & \\ 0 & 0 & 0\end{matrix}\right)$ is Meijer's G function[29]. From the normalization condition constants c_0 and c_1 in eq.(14) become

$$c_0 = 1, \quad \text{and} \quad c_1 = 4 - \delta_1 + \frac{\pi^2}{12} - \frac{\gamma^2}{2} \simeq -2.98961, \tag{15}$$

respectively.

V. DISCUSSION

Let us discuss our result. First, we can obtain higher order terms such as $f^{(2)}(v)$ and $f^{(3)}(v)$, because the homogeneous solutions in eqs.(??) and (??) can be used in any order. However, these solutions may have complicated forms and we had better use numerical integrations to represent inhomogeneous terms. Because of the limitation of the length of this paper, we have omitted to give the explicit forms of higher order terms here.

Second, Kawarada and Hayakawa[26] assume that the diffusion coefficient in the real space is proportional to the diffusion constant in the velocity space, but this assumption may not be true for our case.

We also note that the behavior of VDF under the influence of g in granular particles in an inclined container with the vibration does not coincide with the theoretical prediction presented here quantitatively in our preliminary simulation. As shown in Fig.5, P_0 and P_1 have peaks around $c = 1$ and become negative near $c = 0$. The behavior of the simulation for granular particles is qualitatively similar, but the peak position is located near $c = 0.5$ and the negativity in the vicinity of $c = 0$ is not large in the simulation. It is not surprised in the difference between our theoretical result and our preliminary simulation mentioned here, because the theoretical model is oversimplified with ignoring vertical motion of particles and the collisions among mobile particles. In addition, our search for adequate parameters may not be enough, and higher order terms $f^{(n)}(v)$ with $n \geq 2$ may be important in the actual situation. In the simulation, we also find the existence of some peaks around the central peak at $v = 0$ of VDF. These extra peaks may come from the steady oscillation of particles among scatters. From the existence of these small peaks the peak positions of VDF in $P_0(v)$ and $P_1(v)$ may be entrained into small c region. In any case, we will have to check whether the connection between Langevin equation with Coulomb friction and the vibrated granular particles confined in a quasi-two-dimensional box is superficial. The detailed quantitative comparisons will be reported elsewhere.

Here, we comment on other possible situations to reproduce Langevin equation with Coulomb friction. One of simplest one is that the motion of frictional disks on an inclined slope with dense random scatters as in section

2. This system can reproduce an exponential VDF with macroscopic condensed particles in the immobile (zero temperature) state. However, VDF is highly anisotropic in this situation and most of particles' motions occur in the direction parallel to the external field, *i.e.* along the slope. Thus, we cannot use the expansion in eq.(9). We also need to check whether we can use our theory in the simpler situation of dense particles in a vibrating box. It is interesting that we look for physical situations that we can use Langevin equation with Coulomb friction.

VI. CONCLUSION

In conclusion, we propose an experimental accessible system for granular gases. The VDF in a 'steady state' obeys an exponential-like function but changes Gaussian-like distribution function when free-cooling starts. This exponential VDF is caused by Coulomb's friction force. Thus, we propose the Langevin equation with Coulomb's friction to reproduce the results of our simulation. we have also developed the theory of Langevin equation with Coulomb friction and obtained the steady solution of VDF under the influence of a steady external field.

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