On some subclasses of univalent functions
Mugur Acu¹, Shigeyoshi Owa²

ABSTRACT. In 1999, S. Kanas and F. Ronning introduced the classes of functions starlike and convex, which are normalized with $f(w) = f'(w) - 1 = 0$ and $w$ is a fixed point in $U$. In this paper we continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where $w$ is a fixed point in $U$.

2000 Mathematics Subject Classification: 30C45

Key words and phrases: Close to convex functions, $\alpha$ - convex functions, Briot-Bouquet differential subordination

1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$, $A = \{f \in \mathcal{H}(U): f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A: f$ is univalent in $U\}$.

We recall here the definitions of the well-known classes of starlike, convex, close to convex and $\alpha$ - convex functions:

$S^* = \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 , \ z \in U \right\}$,

$S^c = \left\{ f \in A : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 , \ z \in U \right\}$

$CC = \left\{ f \in A : \exists g \in S^*, \text{Re} \left( \frac{zf'(z)}{g(z)} \right) > 0 , \ z \in U \right\}$,

$M_\alpha = \left\{ f \in A : \frac{f(z) \cdot f'(z)}{z} \neq 0 , \ \text{Re} \ J(\alpha, f; z) > 0 , \ z \in U \right\}$

where $J(\alpha, f; z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)$.

Let $w$ be a fixed point in $U$ and $A(w) = \{ f \in \mathcal{H}(U): f(w) = f'(w) - 1 = 0\}$.

In [3] S. Kanas and F. Ronning introduced the following classes:

$S(w) = \{ f \in A(w): f$ is univalent in $U\}$

$ST(w) = S^*(w) = \left\{ f \in S(w): \text{Re} \left( \frac{(z-w)f'(z)}{f(z)} \right) > 0 , \ z \in U \right\}$

$CV(w) = S^c(w) = \left\{ f \in S(w): 1 + \text{Re} \left( \frac{(z-w)f''(z)}{f'(z)} \right) > 0 , \ z \in U \right\}$.
The class $S^*(w)$ is defined by the geometric property that the image of any circular arc centered at $w$ is starlike with respect to $f(w)$ and the corresponding class $S^c(w)$ is defined by the property that the image of any circular arc centered at $w$ is convex. We observe that the definitions are somewhat similar to the ones for uniformly starlike and convex functions introduced by A. W. Goodman in [1] and [2], except that in this case the point $w$ is fixed.

It is obvious that exists a natural "Alexander relation" between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \text{ if and only if } f(z) = (z-w)g'(z) \in S^*(w).$$

Let denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$ that are regular in $U$ and satisfy $p(w) = 1$ and $Re\, p(z) > 0$ for $z \in U$.

## 2 Preliminary results

If is easy to see that a function $f(z) \in \mathcal{A}(w)$ have the series expansions:

$$f(z) = (z-w) + a_2 (z-w)^2 + ...$$

In [7] J. K. Wald gives the sharp bounds for the coefficients $B_n$ of the function $p \in \mathcal{P}(w)$:

**Teorema 2.1** If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$, then

$$|B_n| \leq \frac{2}{(1+d)(1-d)^n}, \text{ where } d = |w| \text{ and } n \geq 1.$$

Using the above result, S. Kanas and F. Ronning obtain in [3]:

**Teorema 2.2** Let $f \in S^*(w)$ and $f(z) = (z-w) + a_2 (z-w)^2 + ...$ Then

$$|a_2| \leq \frac{2}{1-d^2}, \quad |a_3| \leq \frac{3+d}{(1-d^2)^2},$$

$$|a_4| \leq \frac{2}{3} \cdot \frac{(2+d)(3+d)}{(1-d^2)^3}, \quad |a_5| \leq \frac{1}{6} \cdot \frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4}$$

where $d = |w|$.

**Remark 2.1** It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.
The next theorem is the result of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [4], [5], [6]).

**Teorema 2.3** Let $h$ convex in $U$ and $\text{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ with $p(0) = h(0)$ and $p$ satisfied the Briot - Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z), \quad \text{then } p(z) < h(z).$$

### 3 Main results

Let consider the integral operator $L_a : A(w) \rightarrow A(w)$ defined by

$$f(z) = L_a F(z) = \left. \frac{1 + a}{(z-w)^a} \cdot \int_{w}^{z} F(t) \cdot (t-w)^{a-1} dt \right|_{a \in \mathbb{R}, \ a \geq 0}.$$  \hspace{1cm} (3)

We denote by $D(w) = \{ z \in U : \text{Re} \left[ \frac{w}{z} \right] < 1 \text{ and } \text{Re} \left[ \frac{z(1+z)}{(z-w)(1-z)} \right] > 0 \}. \text{ with } D(0) = U$, and $s(w) = \{ f : D(w) \rightarrow \mathbb{C} \} \cap S(w)$, where $w$ is a fixed point in $U$.

Denoting $s^*(w) = S^*(w) \cap s(w)$, where $w$ is a fixed point in $U$, we obtain:

**Teorema 3.1** Let $w$ be a fixed point in $U$ and $F(z) \in s^*(w)$. Then $f(z) = L_a F(z) \in S^*(w)$, where the integral operator $L_a$ is defined by (3).

**Proof.** By differentiating (3) we obtain:

$$f(z) = L_a F(z) = a \cdot f(z) + (z-w) \cdot f'(z). \hspace{1cm} (4)$$

From (4) we have:

$$f'(z) = (1+a) f'(z) + (z-w) f''(z). \hspace{1cm} (5)$$

Using (4) and (5) we obtain:

$$\frac{(z-w) F'(z)}{F(z)} = \frac{(1+a) \cdot (z-w) \cdot \frac{f'(z)}{f(z)} + (z-w)^2 \frac{f''(z)}{f(z)}}{a + (z-w) \frac{f'(z)}{f(z)}}. \hspace{1cm} (6)$$

With notation $p(z) = \frac{(z-w)f'(z)}{f(z)}$, where $p(z) \in \mathcal{H}(U)$ and $p(0) = 1$, we have:

$$(z-w)p'(z) = p(z) + (z-w)^2 \cdot \frac{f''(z)}{f(z)} - [p(z)]^2$$

...
and thus:

\[(7) \quad (z - w)^2 \frac{f''(z)}{f(z)} = (z - w)p'(z) - p(z)[1 - p(z)].\]

Using (6) and (7) we obtain:

\[(8) \quad \frac{(z - w)F'(z)}{F(z)} = p(z) + \frac{(z - w) \cdot p'(z)}{a + p(z)}.\]

Using \(F(z) \in s^*(w)\) from (8) we have:

\[p(z) + \frac{z - w}{a + p(z)} \cdot p'(z) < \frac{1 + z}{1 - z} \equiv h(z)\]

or

\[p(z) + \frac{1 - w}{a + p(z)} \cdot zp'(z) < \frac{1 + z}{1 - z}.\]

From hypothesis we have \(Re \left[ \frac{1 - w}{1 - z} \cdot h(z) + \frac{a}{1 - z} \right] > 0\) and thus from Theorem 2.3 we obtain \(p(z) < \frac{1 + z}{1 - z}\) or \(Re \left( \frac{(z - w)f'(z)}{g(z)} \right) > 0, z \in U\). This means \(f(z) \in S^*(w)\).

**Definition 3.1** Let \(f \in S(w)\) where \(w\) is a fixed point in \(U\). We say that \(f\) is \(w\)-close to convex if exists a function \(g \in S^*(w)\) such that \(Re \left( \frac{(z - w)f'(z)}{g(z)} \right) > 0, z \in U\). We denote this class with \(CC(z)\).

**Remark 3.1** If we consider \(f = g, g \in S^*(w)\), we have \(S^*(w) \subset CC(w)\).

If we take \(w = 0\) we obtain the well known close to convex functions.

**Theorem 3.2** Let \(w\) be a fixed point in \(U\) and \(f \in CC(w), f(z) = (z - w) + \sum_{n=2}^{\infty} b_n \cdot (z - w)^n\), with respect to the function \(g \in S^*(w), g(z) = (z - w) + \sum_{n=2}^{\infty} a_n \cdot (z - w)^n\). Then

\[|b_n| \leq \frac{1}{n} \left| a_n \right| + \sum_{k=1}^{n-1} \left| a_k \right| \cdot \frac{2}{(1 + d)(1 - d)^{n-k}}\]

where \(d = |w|, n \geq 2\) and \(a_1 = 1\).

**Proof.** Let \(f \in CC(w)\) with respect to the function \(g \in S^*(w)\). Then there exists a function \(p \in \mathcal{P}(w)\) such that

\[\frac{(z - w)f'(z)}{g(z)} = p(z)\]
where \( p(z) = 1 + \sum_{n=1}^{\infty} B_n(z - w)^n \).

Using the hypothesis through identification of \((z - w)^n\) coefficients we obtain:

\[
(9) \quad n \cdot b_n = a_n + \sum_{k=1}^{n-1} a_k \cdot B_{n-k}
\]

where \( a_1 = 1 \) and \( n \geq 2 \).

From (9) we have

\[
|b_n| \leq \frac{1}{n} \left[ |a_n| + \sum_{k=1}^{n-1} |a_k| \cdot |B_{n-k}| \right], \quad a_1 = 1, \quad n \geq 2.
\]

Applying the above and the estimates (1) we obtain the result.

**Remark 3.2** If we use the estimates (2) we obtain the same estimates for the coefficients \( b_n, n = 2, 3, 4, 5 \).

**Definition 3.2** Let \( \alpha \in \mathbb{R} \) and \( w \) be a fixed point in \( U \). For \( f \in S(w) \) we denote by

\[
J(\alpha, f, w; z) = (1 - \alpha) \left( \frac{(z - w)f'(z)}{f(z)} \right) + \alpha \left[ 1 + \frac{(z - w)f''(z)}{f'(z)} \right].
\]

We say that \( f \) is \( w - \alpha \)-convex function if \( \frac{f(z) \cdot f'(z)}{z - w} \neq 0 \) and \( \text{Re} J(\alpha, f, w; z) > 0, z \in U \). We denote this class with \( M_\alpha(w) \).

**Remark 3.3** It is easy to observe that \( M_\alpha(0) \) is the well known class of \( \alpha \)-convex functions.

**Teorema 3.3** Let \( w \) be a fixed point in \( U, \alpha \in \mathbb{R}, \alpha \geq 0 \) and \( m_\alpha(w) = M_\alpha(w) \cap S(w) \).

1. If \( f \in m_\alpha(w) \) then \( f \in S^*(w) \). This means \( m_\alpha(w) \subset S^*(w) \).

2. If \( \alpha, \beta \in \mathbb{R}, \) with \( 0 \leq \beta < \alpha < 1 \), then \( m_\alpha(w) \subset m_\beta(w) \).

**Proof.** From \( f \in m_\alpha(w) \) we have \( \text{Re} J(\alpha, f, w; z) > 0, z \in U \). Using the notation

\[
p(z) = \frac{(z - w)f'(z)}{f(z)}, \quad \text{with} \quad p \in \mathcal{H}(U) \quad \text{and} \quad p(0) = 1,
\]

we obtain:

\[
\text{Re} J(\alpha, f, w; z) = \text{Re} \left[ p(z) + \alpha \cdot \frac{(z - w)p'(z)}{p(z)} \right] > 0, \quad z \in U \quad \text{or}
\]

\[
p(z) + \frac{\alpha (1 - \frac{w}{z})}{p(z)} \cdot zp'(z) \prec \frac{1 + z}{1 - z} \equiv h(z).
\]

For \( \alpha = 0 \) we have \( p(z) \prec \frac{1 + z}{1 - z} \).
Using the hypothesis we have for $\alpha > 0$, $Re \left[ \frac{1}{\alpha \left( 1 - \frac{w}{z} \right)} \cdot h(z) \right] > 0$ and from Theorem 2.3 we obtain $p(z) < \frac{1 + z}{1 - z}$.

This means that $Re \left( \frac{(z - w)f'(z)}{f(z)} \right) > 0$, $z \in U$ and $\alpha \geq 0$ or $f \in S^*(w)$.

If we denote by $A = Re(p(z) - \frac{1 + z}{1 - z})$ and by $B = Re \left( \frac{(z - w)p'(z)}{p(z)} \right)$ we have $A > 0$ and $A + B \cdot \alpha > 0$, where $\alpha \geq 0$.

Using the geometric interpretation of the equation $y(x) = A + B \cdot x, x \in [0, \alpha]$ we obtain

$$y(\beta) = A + B \cdot \beta > 0 \quad \text{for every } \beta \in [0, \alpha].$$

This means $Re \left[ p(z) + \beta \cdot \frac{(z - w)p'(z)}{p(z)} \right] > 0$, $z \in U$ or $f \in m_\beta(w)$.

**Remark 3.4** From the above theorem we have:

$$m_1(w) \subseteq s^c(w) \subseteq m_\alpha(w) \subseteq s^*(w)$$

where $0 \leq \alpha \leq 1$ and $s^c(w) = S^c(w) \cap s(w)$. 
References


---

1 University "Lucian Blaga" of Sibiu
Department of Mathematics
Str. Dr. I. Rațiu, No. 5-7
550012 - Sibiu, Romania

2 Department of Mathematics
School of Science and Engineering
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan