<table>
<thead>
<tr>
<th>Title</th>
<th>Some Doubly Infinite, Finite and Mixed Infinite Sums derived from the N-Fractional Calculus of A Power Function: with Some Examinations (Coefficient Inequalities in Univalent Function Theory and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nishimoto, Katsuyuki; Romero, Susana S. de</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1414: 26-39</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/26228">http://hdl.handle.net/2433/26228</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Some Doubly Infinite, Finite and Mixed Infinite Sums derived from The N-Fractional Calculus of A Power Function (with Some Examinations)

Katsuyuki Nishimoto and Susana S. de Romero

Abstract

In this article theorems for some doubly infinite, finite and mixed infinite sums derived from the N-fractional calculus of a power function are reported. Moreover some numerical examinations for the theorems are reported too.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto)([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

$C_-$ be a curve along the cut joining two points $z$ and $-\infty + i \text{Im}(z)$,

$C_+$ be a curve along the cut joining two points $z$ and $\infty + i \text{Im}(z)$,

$D_-$ be a domain surrounded by $C_-$, $D_+$ be a domain surrounded by $C_+$.

(Here $D$ contains the points over the curve $C$).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu(z) = (f)_\nu = (f)_\nu = \frac{\Gamma(v+1)}{2\pi i} \int_{C} \frac{f(\zeta)}{(\zeta-z)^{v+1}} d\zeta \quad (v \notin \mathbb{Z})$$

(1)

$$\lim_{\nu \to m} (f)_\nu = (m \in \mathbb{Z}^+)$$

(2)

where $-\pi \leq \arg(\zeta-z) \leq \pi$ for $C_-$, $0 \leq \arg(\zeta-z) \leq 2\pi$ for $C_+$,

$\zeta = z$, $z \in C$, $\nu \in R$, $\Gamma$; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order $\nu$ (derivatives of order $\nu$ for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to $z$, of the function $f$, if $|f_\nu| < \infty$.

(II) On the fractional calculus operator $N^\nu$ [3]
Theorem A. Let fractional calculus operator (Nishimoto’s Operator) \( N^v \) be
\[
N^v = \left( \frac{\Gamma(v+1)}{2\pi i} \int_{C} \frac{d\zeta}{(\zeta-z)^{v+1}} \right) \quad (v \notin \mathbb{Z}^+) \tag{3}
\]
with
\[
N^{-m} = \lim_{v \to -m} N^v \quad (m \in \mathbb{Z}^+), \tag{4}
\]
and define the binary operation \( \circ \) as
\[
N^{\beta} \circ N^{\alpha} f = N^{\beta}N^{\alpha} f = N^{\beta}(N^{\alpha} f) \quad (\alpha, \beta \in \mathbb{R}), \tag{5}
\]
then the set
\[
\{ N^v \} = \{ N^v | v \in \mathbb{R} \} \tag{6}
\]
is an Abelian product group (having continuous index \( v \)) which has the inverse transform operator \( (N^v)^{-1} = N^{-v} \) to the fractional calculus operator \( N^v \), for the function \( f \) such that \( f \in F = \{ f; 0 \neq |f_v| < \infty, v \in \mathbb{R} \} \), where \( f = f(z) \) and \( z \in \mathbb{C} \).

(For our convenience, we call \( N^\beta \circ N^\alpha \) as product of \( N^\beta \) and \( N^\alpha \).)

Theorem B. "F.O.G. \( \{ N^v \} \)" is an "Action product group which has continuous index \( v \)" for the set of \( F \).(F.O.G.; Fractional calculus operator group)

Theorem C. Let
\[
S := \{ \pm N^v \} \cup \{ 0 \} = \{ N^v \} \cup \{ -N^v \} \cup \{ 0 \} \quad (v \in \mathbb{R}) \tag{7}
\]
Then the set \( S \) is a commutative ring for the function \( f \in F \), when the identity
\[
N^{\alpha} + N^{\beta} = N^{\gamma} \quad (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) \tag{8}
\]
holds. [5]

(III) Lemma. We have [1]

(i) \( ((z-c)^{\alpha})_{\beta} = e^{-i\pi \alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(\beta)} (z-c)^{\beta-\alpha} \quad \left( \left| \frac{\Gamma(\alpha-\beta)}{\Gamma(\beta)} \right| < \infty \right) \),

(ii) \( \left( \log(z-c) \right)_{\alpha} = e^{-i\pi \alpha} \frac{\Gamma(\alpha)}{\Gamma(-\alpha)} \log(z-c) \quad (\left| \Gamma(\alpha) \right| < \infty) \),

(iii) \( ((z-c)^{-\alpha})_{-\beta} = e^{i\pi \alpha} \log(z-c) \quad (\left| \Gamma(\alpha) \right| < \infty) \),

where \( z-c \neq 0 \) in (i), and \( z-c \neq 0,1 \) in (ii) and (iii). (\( \Gamma; \) Gamma function),

(iv) \( (u \cdot v)_{\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} \frac{u_{\alpha-k}}{u_k} v_k \quad \left( \begin{array}{c} u = u(z), \\ v = v(z) \end{array} \right) \).
§ 1. Doubly Infinite, Finite and Mixed Sums

In the following $\alpha, \beta, \gamma \in \mathbb{R}$.

**Theorem 1.** Let

$$G(\alpha, \beta, \gamma ; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(m-\beta)\Gamma(k-m-\alpha+\gamma)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}.$$  \hspace{1cm} (1)

(i) When $\alpha, \beta, \gamma \notin \mathbb{Z}$, we have the following doubly infinite sums;

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(\alpha, \beta, \gamma ; k, m) \left( \frac{z-c}{z} \right)^m \left( \frac{z-c}{z} \right)^k \subseteq \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left( \frac{z-c}{z} \right)^{\gamma-a},$$  \hspace{1cm} (2)

where

$$|\frac{z-c}{z}| < 1, \quad |c/(z-c)| < 1,$$

and

$$\left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|, \quad \left|\frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)}\right| < \infty.$$  \hspace{1cm} (3)

The identity (notation = ) holds for $(\gamma-\alpha) \in \mathbb{Z}$.

(ii) When $\alpha, \gamma \notin \mathbb{Z}^+$, we have the following mixed infinite sums;

$$\sum_{k=0}^{\infty} \sum_{m=0}^{s} G(\alpha, s, \gamma ; k, m) \left( \frac{z-c}{z} \right)^m \left( \frac{z-c}{z} \right)^k \subseteq \frac{\Gamma(\gamma-\alpha-s)}{\Gamma(-\alpha-s)} \left( \frac{z-c}{z} \right)^{\gamma-a},$$  \hspace{1cm} (3)

for $s \in \mathbb{Z}^+$ where

$$|c/(z-c)| < 1, \quad |z-c|/z < \infty,$$

and

$$\left|\frac{\Gamma(\gamma-\alpha-s)}{\Gamma(-\alpha-s)}\right|, \quad \left|\frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)}\right| < \infty.$$  \hspace{1cm} (3)

The identity (notation = ) holds for $(\gamma-\alpha) \in \mathbb{Z}$.

**Proof of (i).** We have

$$z^\alpha = (z-c)^\alpha \left( 1 - \frac{c}{c-z} \right)^\alpha$$  \hspace{1cm} (4)

$$= (z-c)^\alpha \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} (c-z)^k \left( |c-z| > |c| \right)$$  \hspace{1cm} (5)
\[
\sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} (z-c)^{\gamma-k}
\]

Next make \((6) \times z^\beta\), then operate \(N\)-fractional calculus operator \(N'\) to its both sides, we obtain

\[
(z^\alpha z^\beta)' = \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} ((z-c)^{\alpha-k} \cdot z^\beta)',
\]

by Lemma (i i).

Now we have

\[
(z^\alpha z^\beta)' = e^{-i\pi \gamma} \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} z^{\alpha+\beta - \gamma},
\]

by Lemma (i), respectively.

Therefore, substituting (9), (10) and (11) into (8) we obtain

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{G}(\alpha, \beta, \gamma; k, m) \left( \frac{z-c}{z} \right)^{n\iota} \left( \frac{c}{z-c} \right) = \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \frac{z^{\gamma-\alpha}}{\Gamma(k-\alpha)}.
\]

However the LHS (left hand side) of (12) is always one valued function, on the contrary the RHS (right hand side) of (12) is many valued function for \((\gamma - \alpha) \notin \mathbb{Z}\) and one valued one for \((\gamma - \alpha) \in \mathbb{Z}\).

Hence we must calculate as

\[
\left( \frac{z-c}{z} \right)^{\gamma-\alpha} = \left( e^{i2n\pi} \frac{z-c}{z} \right)^{\gamma-\alpha} \left( \frac{\gamma - \alpha \notin \mathbb{Z}}{n \in \mathbb{Z}} \right),
\]

because we are now being in the field of complex analysis.
Moreover when \((\gamma - \alpha) \in \mathbb{Z}\) both of the LHS and the RHS of (12) are one valued functions respectively. In this case we have (12) strictly.

Therefore, we have (2) from (12), considering (13) for \((\gamma - \alpha) \notin \mathbb{Z}\).

**Proof of (1 i i).** Set \(\beta = s \in \mathbb{Z}^*\) in (2), we have then (3) clearly, under the conditions.

**Corollary 1.** When \(r, p \in \mathbb{Z}^*\) we have the doubly finite sums;

\[
\sum_{k=0}^{r} \sum_{m=0}^{p} G(r, \beta, p; k, m) \left( \frac{z-c}{z} \right)^{m} \left( \frac{c}{z-c} \right)^{k} = \frac{\Gamma(p-r-\beta)}{\Gamma(-r-\beta)} \left( \frac{z-c}{z} \right)^{p-r}, \quad (14)
\]

for \((p-r) \in \mathbb{Z}\), where

\[|c/(z-c)|, \ |(z-c)/z| < \infty,\]

and

\[\left| \frac{\Gamma(p-r-\beta)}{\Gamma(-r-\beta)} \right|, \ \left| \frac{\Gamma(k-r+p-m)}{\Gamma(k-r)} \right| < \infty.\]

**Proof.** Set \(\alpha = r\) and \(\gamma = p\) in (2), we have then this corollary clearly.

Now both of the LHS and RHS of (14) are one valued functions respectively, hence the identity (notation \(=\)) holds in (2).

**Corollary 2.** When \(r, p \in \mathbb{Z}^*\) we have the doubly finite sums;

\[
\sum_{k=0}^{r} \sum_{m=0}^{s} G(r, s, p; k, m) \left( \frac{z-c}{z} \right)^{m} \left( \frac{c}{z-c} \right)^{k} = \frac{\Gamma(p-r-s)}{\Gamma(-r-s)} \left( \frac{z-c}{z} \right)^{p-r}, \quad (15)
\]

where

\[|c/(z-c)|, \ |(z-c)/z| < \infty,\]

and

\[\left| \frac{\Gamma(p-r-s)}{\Gamma(-r-s)} \right|, \ \left| \frac{\Gamma(k-r+p-m)}{\Gamma(k-r)} \right| < \infty.\]

**Proof.** Set \(\alpha = r\) and \(\gamma = p\) in (3), we have then this corollary under the conditions, clearly.

Now both of the LHS and RHS of (15) are one valued functions respectively, hence the identity (notation \(=\)) holds in (3).

**§ 2. Direct calculation of the doubly infinite sums**

The direct calculation (without the use of N-fractional calculas) of the doubly infinite sum in LHS of § 1. (2) is shown as follows.
Theorem 2. Let
\[ G = G(\alpha, \beta, \gamma; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(m-\beta)\Gamma(k-m-\alpha+\gamma)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)} \cdot \frac{(z-c)}{z-c}^k \] (1)
and
\[ P = P(\alpha, \beta, \gamma) := \frac{\sin \pi x \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \alpha)} \cdot \frac{(z-c)}{z-c}^k \] (2)

We have then
\[ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G \cdot \frac{(z-c)}{z-c}^k \approx P \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)} \] (3)
where \( |c/(z-c)| < 1 \),
and
\( (\alpha+\beta), (\gamma-\alpha), (\gamma-\alpha-\beta) \not\in \mathbb{Z} \).

Proof. Now we have
\[ G \cdot \frac{(z-c)}{z-c}^m \cdot \frac{c}{z-c}^k \]
\[ = \frac{\Gamma(\gamma - \alpha)\cdot[-\beta]_m\cdot[-\gamma]_m\cdot[y - \alpha - m]_k}{\Gamma(-\alpha)\cdot k! \cdot m! \cdot [1 + \alpha - \gamma]_m} \cdot \frac{(z-c)}{z-c}^m \cdot \frac{c}{z-c}^k \] (4)
using the identity
\[ \Gamma(\lambda+1-k) = (-1)^{-k} \Gamma(\lambda+1)\Gamma(-\lambda) \] (5)
and
\[ \Gamma(k + \gamma - \alpha - m) = (-1)^{-m} \Gamma(\gamma - \alpha)\frac{[\gamma - \alpha - m]_k}{[1 + \alpha - \gamma]_m} \] (6)
where
\[ [\lambda]_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) = \Gamma(\lambda + k) / \Gamma(\lambda) \], with \([\lambda]_0 = 1\) (notation of Pochhammer).

We have then
\[ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G \cdot \frac{(z-c)}{z-c}^m \cdot \frac{c}{z-c}^k = \frac{\Gamma(\gamma - \alpha)}{\Gamma(-\alpha)} \]
\[ \times \sum_{m=0}^{\infty} \frac{[-\beta]_m\cdot[-\gamma]_m\cdot[y - \alpha - m]_k}{m! \cdot [1 + \alpha - \gamma]_m} \cdot \frac{(z-c)}{z-c}^m \sum_{k=0}^{\infty} \frac{[\gamma - \alpha - m]_k}{k!} \cdot \frac{(-c)}{z-c}^k \] (7)
\[
= \frac{\Gamma(\gamma - \alpha)}{\Gamma(-\alpha)} \left(\frac{z}{z-c}\right)^{\alpha - \gamma} \sum_{m=0}^{\infty} \frac{[-\beta]_m [-\gamma]_m}{m! [1 + \alpha - \gamma]_m}
\]

\[(8)\]

\[
= \frac{\Gamma(\gamma - \alpha)}{\Gamma(-\alpha)} \left(\frac{z}{z-c}\right)^{\alpha - \gamma} \ {}_2F_1(-\beta, -\gamma ; 1 + \alpha - \gamma ; 1)
\]

\[(9)\]

\[
= \frac{\Gamma(\gamma - \alpha) \Gamma(1 + \alpha - \gamma) \Gamma(1 + \alpha + \beta)}{\Gamma(-\alpha) \Gamma(1 + \alpha) \Gamma(1 + \alpha + \beta - \gamma)} \left(\frac{z}{z-c}\right)^{\gamma - \alpha}
\]

\[(10)\]

where
\[
\left|\frac{c}{z-c}\right| < 1, \left|\frac{z-c}{z}\right| < 1, \ \text{Re}(\alpha + \beta) > -1.
\]

Because we have
\[
\sum_{k=0}^{\infty} \frac{[\gamma - \alpha - m]_k}{k!} \left(\frac{-c}{z-c}\right)^k = \left(\frac{z}{z-c}\right)^{\alpha - \gamma} \left(\frac{z}{z-c}\right)^m
\]

\[(11)\]

since
\[
\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1 - z)^{-\lambda}
\]

\[(12)\]

and
\[
\ {}_2F_1(a,b;c;1) = \sum_{m=0}^{\infty} \frac{[a]_m [b]_m}{m! [c]_m} = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \left(\text{Re}(c - a - b) > 0, \ c \notin \mathbb{Z}_0^\times \right).
\]

\[(13)\]

Moreover we have the identity
\[
\Gamma(\lambda) \Gamma(1 - \lambda) = \frac{\pi}{\sin \pi \lambda}, \ \ (\lambda \notin \mathbb{Z})
\]

\[(14)\]

then applying (14) to (10) we obtain (3) under the conditions.

\[\text{§ 3. Some Numerical Examinations for Theorem 1}\]

[1] Examination of Theorem 1. (2) (Doubly infinite sum)

Set
\[c = 1, \ z = 3, \ \alpha = 1/4 \ \text{and} \ \beta = \gamma = 1/2\]

in Theorem 1. (2) we obtain
\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(1/4,1/2,1/2;k,m) \left(\frac{1}{2}\right)^k \left(\frac{2}{3}\right)^m \leq \frac{\Gamma(-1/4)}{\Gamma(-3/4)} \left(\frac{2}{3}\right)^{1/4}
\]

\[(1)\]
\[3 \cdot \frac{\Gamma(3/4)}{\Gamma(1/4)} \left( e^{i2n \times \frac{2}{3}} \right)^{1/4} \quad (n \in \mathbb{Z}) \quad (2)\]

\[
\begin{cases}
0.91622\cdots & \text{(for } n = 0) \\
i \cdot 0.91622\cdots & \text{(for } n = 1) \\
-0.91622\cdots & \text{(for } n = 2) \\
-i \cdot 0.91622\cdots & \text{(for } n = 3)
\end{cases}
\quad (3-6)
\]

Now the LHS of (1) is real, then we must choose (3) and (5) from the set \{(3), (4), (5), (6)\}.

And now we have

\[G(\frac{1}{4}, \frac{1}{2}, \frac{1}{2} ; 0, 0) \left( \frac{1}{2} \right)^0 \left( \frac{2}{3} \right)^0 \left( \frac{3}{2} \right)^0 < 0. \quad (7)\]

Then choosing (5) from the set \{(3), (5)\}, since the sign of the double infinite sum of the LHS of (1) is decided by the sign of its first term (with \(k = m = 0\)), when

\[
\left| G_{k,m} \left( \frac{1}{3} \right)^{k} \left( \frac{2}{3} \right)^{m} \right| > \left| G_{k+1,m+1} \left( \frac{1}{3} \right)^{k+1} \left( \frac{2}{3} \right)^{m+1} \right| 
\]

we have then

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(\frac{1}{4}, \frac{1}{2}, \frac{1}{2} ; k, m) \left( \frac{1}{2} \right)^{k} \left( \frac{2}{3} \right)^{m} = -0.91622\cdots \quad (5)
\]

from (1), considering (7).

Indeed we have

\[
\text{LHS of (5)} = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{4})}{k! \Gamma(\frac{5}{4}-k)} \left( \frac{1}{2} \right)^{k} \left[ \frac{\Gamma(k+\frac{1}{4})}{\Gamma(k-\frac{1}{4})} + \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{4}) \Gamma(k-\frac{3}{4}) \left( \frac{2}{3} \right)}{2! \Gamma(-\frac{1}{2}) \Gamma(-\frac{1}{2}) \Gamma(k-\frac{3}{4}) \left( \frac{2}{3} \right)^2} \right]
\]
\[
\begin{align*}
&+ \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)\Gamma\left(k-\frac{1}{4}\right)}{3!\Gamma\left(-\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(k-\frac{3}{4}\right)} \left(\frac{2}{3}\right)^{3} + \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)\Gamma\left(k-\frac{15}{4}\right)}{4!\Gamma\left(-\frac{5}{2}\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(k-\frac{3}{4}\right)} \left(\frac{2}{3}\right)^{4} \\
&+ \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{9}{2}\right)\Gamma\left(k-\frac{19}{4}\right)}{5!\Gamma\left(-\frac{7}{2}\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(k-\frac{5}{4}\right)} \left(\frac{2}{3}\right)^{5} + \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{11}{2}\right)\Gamma\left(k-\frac{23}{4}\right)}{6!\Gamma\left(-\frac{9}{2}\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(k-\frac{7}{4}\right)} \left(\frac{2}{3}\right)^{6} + \ldots \\
= &\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \left\{ -\frac{1}{4} + \frac{1}{2 \cdot 4^{2}} - \frac{5}{2! 2^{2} \cdot 4^{2}} + \frac{5 \cdot 9}{3! 2^{3} \cdot 4^{4}} - \frac{5 \cdot 9 \cdot 13}{4! 2^{4} \cdot 4^{6}} + \frac{5 \cdot 9 \cdot 13 \cdot 17}{5! 2^{5} \cdot 4^{8}} - \ldots \right\} \\
+ &\frac{\Gamma\left(\frac{1}{4}\right)}{3 \Gamma\left(\frac{1}{2}\right)} \left\{ -\frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 2^{2} \cdot 4^{2}} - \frac{5}{3! 2^{3} \cdot 4^{2}} + \frac{5 \cdot 9}{4! 2^{4} \cdot 4^{2}} - \frac{5 \cdot 9 \cdot 13}{5! 2^{5} \cdot 4^{4}} - \ldots \right\} \\
+ &\frac{\Gamma\left(\frac{1}{4}\right)}{2! 2^{2} \cdot 3^{2} \Gamma\left(\frac{1}{2}\right)} \left\{ -\frac{4}{3 \cdot 7} + \frac{1}{2 \cdot 3} - \frac{1}{2 \cdot 2^{2} \cdot 4} + \frac{1}{3! 2^{3} \cdot 4^{2}} - \frac{5}{4! 2^{4} \cdot 4^{2}} + \frac{5 \cdot 9}{5! 2^{5} \cdot 4^{4}} - \ldots \right\} \\
- &\frac{\Gamma\left(\frac{1}{4}\right)}{3! 2^{3} \cdot 3 \Gamma\left(\frac{1}{2}\right)} \left\{ \frac{4^{2}}{3 \cdot 7 \cdot 11 + 1} + \frac{4}{2 \cdot 3 \cdot 7} + \frac{1}{2 \cdot 3} + \frac{1}{3! 2^{3}} \cdot 4 \\
&\quad - \frac{5}{4! 2^{4} \cdot 4^{2}} + \frac{1}{5! 2^{5} \cdot 4^{4}} - \frac{5 \cdot 9 \cdot 13 \cdot 17}{6! 2^{6} \cdot 4^{8}} - \ldots \right\} \\
+ &\frac{5^{2} \Gamma\left(\frac{1}{4}\right)}{4! 2^{4} \cdot 3^{2} \Gamma\left(\frac{1}{2}\right)} \left\{ -\frac{4^{3}}{3 \cdot 7 \cdot 11 \cdot 15} + \frac{4^{2}}{2 \cdot 3 \cdot 7 \cdot 11} - \frac{4}{2 \cdot 2^{2} \cdot 3 \cdot 7} - \frac{1}{3! 2^{3} \cdot 3} \\
&\quad - \frac{4}{4! 2^{4} \cdot 4} + \frac{1}{5! 2^{5} \cdot 4^{2}} - \ldots \right\} \\
- &\frac{5^{2} \cdot 7^{2} \Gamma\left(\frac{1}{4}\right)}{5! 2^{5} \cdot 3^{3} \Gamma\left(\frac{1}{2}\right)} \left\{ \frac{4^{4}}{3 \cdot 7 \cdot 11 \cdot 15 \cdot 19 + 1} + \frac{4^{3}}{2 \cdot 3 \cdot 7 \cdot 11 \cdot 15} + \frac{4^{2}}{2 ! 2^{2} \cdot 3 \cdot 7 \cdot 11} \\
&\quad + \frac{4}{3 ! 2^{3} \cdot 3 \cdot 7} + \frac{1}{4 ! 2^{4} \cdot 3} + \frac{1}{5! 2^{5} \cdot 4} + \ldots \right\} + \ldots \\
= &- (0.66861\ldots) - (0.22272\ldots) - (0.01591\ldots) - (0.00690\ldots) \\
- (0.00180\ldots) - (0.00093\ldots) - \ldots \\
= &- 0.9168\ldots \\
= &- 0.9168\ldots \\
\end{align*}
\]
Examination of Theorem 1. (3) (Mixed infinite sum)

Set
\[ c = 1, \ z = 3, \ \alpha = 1/4, \ \gamma = 1/2 \quad \text{and} \quad s = 1 \]
in Theorem 1. (3) we obtain
\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(1/4, 1, 1/2 ; k, m) \left( \frac{1}{2} \right)^k \left( \frac{2}{3} \right)^m \leq \frac{\Gamma(-3/4)}{\Gamma(-5/4)} \left( \frac{2}{3} \right)^{1/4} \tag{12}
\]
\[
= -\frac{5}{12} \frac{\Gamma(1/4)}{\Gamma(3/4)} \left( e^{i2\pi n \frac{2}{3}} \right)^{1/4} (n \in \mathbb{Z}) \tag{13}
\]
\[
= \begin{cases} 
-1.113943\ldots \quad \text{(for } n = 0) \\
-i \cdot 1.113943\ldots \quad \text{(for } n = 1) \\
1.113943\ldots \quad \text{(for } n = 2) \\
i \cdot 1.113943\ldots \quad \text{(for } n = 3).
\end{cases} \tag{14}
\]

Now the LHS of (1) is real, then we must choose (14) and (16) from the set \{(14), (15), (16), (17)\}.

And now we have
\[
G(1/4, 1, 1/2 ; 0, 0) \left( \frac{1}{2} \right)^0 \left( \frac{2}{3} \right)^0 \left( \text{first term of the LHS of (12)} \right)
\]
\[
= -\frac{\Gamma(1/4)}{\Gamma(-1/4)} = -\frac{1}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)} < 0. \tag{18}
\]

Then choosing (14) from the set \{(14), (16)\}, since the sign of the double infinite sum of the LHS of (12) is decided by the sign of its first term (with \( k = m = 0 \)), when
\[
\left| G_{k,m} \left( \frac{1}{2} \right)^k \left( \frac{2}{3} \right)^m \right| > \left| G_{k+1,m+1} \left( \frac{1}{2} \right)^{k+1} \left( \frac{2}{3} \right)^{m+1} \right|, \quad G_{k,m} = G(\alpha, s, \gamma ; k, m),
\]
we have then
\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(1/4, 1, 1/2 ; k, m) \left( \frac{1}{2} \right)^k \left( \frac{2}{3} \right)^m = -1.113943\ldots, \tag{19}
\]
from (13), considering (18).

Indeed we have
\[ \text{LHS of (12)} = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{k! \Gamma\left(\frac{3}{2} - k\right) \Gamma\left(k - \frac{1}{2}\right)} \left(\frac{1}{2}\right)^k \]
\[ \times \sum_{m=0}^{1} \frac{\Gamma\left(\frac{3}{2}\right) \Gamma(m - 1) \Gamma(k - m + \frac{1}{2})}{m! \Gamma\left(\frac{3}{2} - m\right) \Gamma(-1)} \left(\frac{2}{3}\right)^m \]  
\[ = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{5}{2}\right) \Gamma(k + \frac{1}{2})}{k! \Gamma\left(\frac{5}{2} - k\right) \Gamma\left(k - \frac{3}{4}\right)} \left(\frac{1}{2}\right)^k \]
\[ + \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}) \Gamma\left(\frac{5}{2}\right) \Gamma(k - \frac{3}{4})}{k! \Gamma\left(\frac{5}{4} - k\right) \Gamma\left(k - \frac{1}{4}\right)} \left(\frac{1}{2}\right)^k \]  
\[ = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \left\{ - \frac{1}{4} + \frac{1}{2 \cdot 4^2} - \frac{5}{2 ! 2^3 \cdot 4^3} + \frac{5 \cdot 9}{3 ! 2^3 \cdot 4^4} - \frac{5 \cdot 9 \cdot 13}{4 ! 2^4 \cdot 4^5} + \frac{5 \cdot 9 \cdot 13 \cdot 17}{5 ! 2^5 \cdot 4^6} - \ldots \right\} \]
\[ - \frac{\Gamma\left(\frac{1}{4}\right)}{3 \Gamma\left(\frac{3}{4}\right)} \left\{ \frac{1}{3} + \frac{1}{2 ! 8^2} + \frac{5}{3 ! 8^3} - \frac{5 \cdot 9}{4 ! 8^4} + \frac{5 \cdot 9 \cdot 13}{5 ! 8^5} - \ldots \right\} \]  
\[ = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \left\{ - 0.25 + 0.03125 - (0.009765\ldots) + (0.003662\ldots) \right. 
\[ - (0.00487\ldots) + (0.000632\ldots) - \ldots \} \]  
\[ - \frac{1}{3} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \left\{ (0.333333\ldots) + 0.125 - (0.007812\ldots) + (0.001627\ldots) \right. 
\[ - (0.000457\ldots) + (0.000148\ldots) - \ldots \} \]  
\[ = (- 0.667796\ldots) + (- 0.445583\ldots) \]  
\[ = - 1.113379\ldots \]  

[1111] Examination of Corollary 1 (Doubly finite sum)

Set \( c = 1, \ z = 3, \ r = 3, \ \beta = -1/2 \) and \( p = 2 \) in Corollary 1, we obtain
\[ \sum_{k=0}^{3} \sum_{m=0}^{2} G(3, -1/2, 2; k, m) \left(\frac{1}{2}\right)^k \left(\frac{2}{3}\right)^m = \frac{\Gamma(-1/2)}{\Gamma(-5/2)} \left(\frac{2}{3}\right)^{-1} = 45 \]
\[ = 8 \]  

Indeed we have
LHS of (26) = \sum_{k=0}^{3} \frac{3! \cdot \underline{?}!}{k! \Gamma(4-k)} \left( \frac{1}{2} \right)^k \sum_{m=0}^{2} \Gamma(m+\frac{1}{2}) \Gamma(k-m-1) \left( \frac{2}{3} \right)^m 

= \sum_{k=0}^{3} \frac{6 \Gamma(k-1)}{k! \Gamma(4-k) \Gamma(k-3)} \left( \frac{1}{2} \right)^k + \sum_{k=0}^{3} \frac{4 \Gamma(k-2)}{k! \Gamma(4-k) \Gamma(k-3)} \left( \frac{1}{2} \right)^k 

= \left\{ \frac{\Gamma(-1)}{\Gamma(-3)} + \frac{3}{2} \frac{\Gamma(0)}{\Gamma(-2)} + \frac{3}{4} \frac{1}{\Gamma(-1)} + \frac{1}{8} \frac{1}{\Gamma(0)} \right\} 

+ \left\{ \frac{2}{3} \frac{\Gamma(-2)}{\Gamma(-3)} + \frac{\Gamma(-1)}{\Gamma(-2)} + \frac{1}{2} \frac{\Gamma(0)}{\Gamma(-1)} + \frac{1}{12} \frac{1}{\Gamma(0)} \right\} 

+ \left\{ \frac{1}{3} + \frac{1}{2} + \frac{1}{4} + \frac{1}{24} \right\} 

= \{6+3+0+0\} + \{ -2 - 2 - (1/2) + 0\} + (9/8) = \frac{45}{8} .

(27) (28) (29) (30)

In this example both of the LHS and RHS of (26) are one valued functions respectively, then we have the identity (notation =), without the ad hoc shown in [11].

§ 4. Commentary

[11] Applying N-fractional calculus to some power functions we obtain the result shown by § 1. (2).

On the other hand we obtain the identity § 2. (3), by the direct calculation of its LHS. However

the LHS of § 2. (3) is always one valued function,

on the contrary

the RHS of § 2. (3) is many valued function for $(\gamma-\alpha) \notin \mathbb{Z}$,

and one valued one for $(\gamma-\alpha) \in \mathbb{Z}$.

We have then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G \left( \frac{\frac{\gamma-\alpha-\beta}{\gamma-\alpha}}{z} \right) c \left( \frac{\frac{\gamma-\alpha-\beta}{\gamma-\alpha}}{z} \right) \subseteq P \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left( \frac{\gamma-\alpha}{z} \right)^{\gamma-\alpha}$$

(1)

from § 2. (3), considering § 1. (13), in complex analysis.
Therefore, when
\[ |P| = |P(\alpha, \beta, \gamma)| = 1, \tag{2} \]
§ 1. (2) and (1) overlap each other, because
\[ \left\{ \text{Set of the elements of } \left( \frac{z-c}{z} \right)^{\gamma-\alpha} \right\} = \left\{ \text{Set of the elements of } -\left( \frac{z-c}{z} \right)^{\gamma-\alpha} \right\} \tag{3} \]
\[ ((\gamma-\alpha) \not\in \mathbb{Z}). \]

That is, the result § 1. (2) which is obtained by the use of N-fractional calculus is a special case of (1) which is the result obtained by the direct calculation of its LHS.

Namely the space \[ |P| = |P(\alpha, \beta, \gamma)| = 1 \] in which the result § 1. (2) holds is a subspace of the space such that \[ |P| = |P(\alpha, \beta, \gamma)| = M < \infty \] in which the result (1) holds good.

Note. When
\[ c = 1, \quad z = 3, \quad \gamma = 1/2, \quad \alpha = 1/4, \]
we have
\[ \text{LHS of (3)} = \left\{ (2/3)^{1/4}, \quad i(2/3)^{1/4}, \quad -(2/3)^{1/4}, \quad -i(2/3)^{1/4} \right\} \]
and
\[ \text{RHS of (3)} = \left\{ -(2/3)^{1/4}, \quad -i(2/3)^{1/4}, \quad (2/3)^{1/4}, \quad i(2/3)^{1/4} \right\}, \]
for example.

References


Katsuyuki Nishimoto
Institute of Applied Mathematics
Descartes Press Co.
2 - 13 - 10 Kaguie, Koriyama
963 - 8833 Japan