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Some Coefficient Inequalities and Distortion Bounds Associated with Certain New Subclasses of Analytic Functions

Shigeyoshi Owa, Kyohei Ochiai and H. M. Srivastava

Abstract

The authors introduce and investigate two new subclasses $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ of normalized analytic functions satisfying certain coefficient inequalities in the open unit disk $\mathbb{U}$. The main results of the present paper provide various interesting properties of functions belonging to the classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$. Some of these properties include (for example) several coefficient inequalities, distortion bounds and inclusion relationships for the function classes which are considered here.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}. \tag{1.2}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are also *univalent* in $\mathbb{U}$.

Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.3}$$

A function $f \in \mathcal{S}^*(\alpha)$ is said to be *starlike of order* α in $\mathbb{U}$. Furthermore, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.4}$$

A function $f \in \mathcal{K}(\alpha)$ is said to be *convex of order* α in $\mathbb{U}$. We note that

$$f(z) \in \mathcal{K}(\alpha) \quad \Longleftrightarrow \quad zf'(z) \in \mathcal{S}^*(\alpha).$$

(See, for details, [1] and [2]; see also [3] and [6], and the references cited therein.)

About three decades ago, Silverman [5] gave the following coefficient inequalities for the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

Theorem A (Silverman [5]). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}(n - \alpha)|a_n| \leqq 1 - \alpha \quad (0 \leqq \alpha < 1), \tag{1.5}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1), \tag{1.6}$$

that is, that $f(z) \in \mathcal{S}^*(\alpha)$.

Theorem B (Silverman [5]). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}n(n - \alpha)|a_n| \leqq 1 - \alpha \quad (0 \leqq \alpha < 1), \tag{1.7}$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1), \tag{1.8}$$

that is, that $f(z) \in \mathcal{K}(\alpha)$.

More recently, Sekine and Owa [4] considered the subclass of functions $f \in \mathcal{A}$ which satisfy the following inequality:

$$\left|\frac{zf'(z)}{f(z)} - a\right| < a - \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ a > \alpha). \tag{1.9}$$

In this paper, we consider a new subclass $\mathcal{M}(\alpha)$ of the class $\mathcal{A}$ consisting of functions $f(z)$ such that

$$\left|\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U};\ 0 < \alpha < 1). \tag{1.10}$$

We also introduce and investigate here the subclass $\mathcal{N}(\alpha)$ of the class $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inclusion relationship:

$$zf'(z) \in \mathcal{M}(\alpha).$$

Let us now define the function $F(z)$ by

$$F(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{M}(\alpha)).$$

Then $f(z)$ satisfies the inequality:

$$F(z) + \bar{F}(z) > 2\alpha \quad (z \in \mathbb{U};\ 0 < \alpha < 1), \tag{1.11}$$

so that

$$\Re(F(z)) = \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 < \alpha < 1). \tag{1.12}$$

It follows from (1.12) that

$$\mathcal{M}(\alpha) \subset \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{N}(\alpha) \subset \mathcal{K}(\alpha).$$

Example. Let us consider the function given by

$$f(z) = z + \frac{1}{k}z^2 \quad (k \geqq 2). \tag{1.13}$$

Then we have

$$\frac{zf'(z)}{f(z)} - 1 = \frac{k+2z}{k+z} - 1 = \frac{z}{k+z}. \tag{1.14}$$

Since

$$\left|\frac{z}{k+z} + \frac{1}{k^2-1}\right| < \frac{k}{k^2-1} \quad (z \in \mathbb{U}), \tag{1.15}$$

we see that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{k-1} = 1 - \frac{k-2}{k-1} \tag{1.16}$$

which readily implies that

$$f(z) \in \mathcal{S}^*\left(\frac{k-2}{k-1}\right). \tag{1.17}$$

On the other hand, we observe that

$$\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} = \frac{k+z}{k+2z} - \frac{1}{2\alpha} = \frac{1}{2}\left(1 - \frac{1}{\alpha} + \frac{k}{k+2z}\right). \tag{1.18}$$

Noting also that

$$\left| \frac{k}{k+2z} - \frac{k^2}{k^2-4} \right| < \frac{2k}{k^2-4} \quad (z \in \mathbb{U}), \tag{1.19}$$

we have

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2}A(k,\alpha), \tag{1.20}$$

where

$$A(k,\alpha) := \max\left\{ \left|1 - \frac{1}{\alpha} + \frac{k}{k+2}\right|, \left|1 - \frac{1}{\alpha} + \frac{k}{k-2}\right| \right\}. \tag{1.21}$$

Thus we obtain

$$\frac{1}{\alpha} = \frac{1}{2}A(k,\alpha) \tag{1.22}$$

for $f(z) \in \mathcal{M}(\alpha)$. Let us put $\alpha = \alpha_0$. If α is given by

$$\alpha = \frac{1}{A(k,\alpha)}, \tag{1.23}$$

then $f(z) \in \mathcal{M}(\alpha_0)$. By the fact that $\mathcal{M}(\alpha) \subset \mathcal{S}^*(\alpha)$, we have

$$\alpha_0 \geqq \frac{k-2}{k-1}. \tag{1.24}$$

If we set

$$\alpha = \frac{k-2}{k-1}, \tag{1.25}$$

then we have

$$1 - \frac{1}{\alpha} + \frac{k}{k+2} = 1 - \frac{k-1}{k-2} + \frac{k}{k+2} = \frac{(k+1)(k-4)}{(k+2)(k-2)} \tag{1.26}$$

and

$$1 - \frac{1}{\alpha} + \frac{k}{k-2} = 1 - \frac{k-1}{k-2} + \frac{k}{k-2} = \frac{k-1}{k-2}. \tag{1.27}$$

Therefore, in the case when $k \geqq 4$, we have

$$\frac{k-1}{k-2} - \frac{(k+1)(k-4)}{(k+2)(k-2)} = \frac{2(2k+1)}{(k+2)(k-2)} \geqq 0. \tag{1.28}$$

Moreover, in the case when $2 \leqq k < 4$, we have

$$\frac{k-1}{k-2} - \frac{(k+1)(4-k)}{(k+2)(k-2)} = \frac{2(k^2-k-3)}{(k+2)(k-2)}. \tag{1.29}$$

Thus, if

$$2 \leqq k \leqq \frac{1+\sqrt{13}}{2} = 2.3027\cdots,$$

then we have

$$\frac{k-1}{k-2} \leqq \frac{(k+1)(4-k)}{(k+2)(k-2)}. \tag{1.30}$$

Therefore

$$A(k,\alpha) = \begin{cases} \dfrac{k-1}{k-2} & \left(k \geqq \dfrac{1+\sqrt{13}}{2}\right) \\[2ex] \dfrac{(k+1)(4-k)}{(k+2)(k-2)} & \left(2 \leqq k < \dfrac{1+\sqrt{13}}{2}\right). \end{cases} \tag{1.31}$$

By the condition (1.23), we have

$$\alpha_0 = \frac{k-2}{k-1} \qquad \left(k \geqq \frac{1+\sqrt{13}}{2}\right) \tag{1.32}$$

such that

$$f(z) \in \mathcal{M}\left(\frac{k-2}{k-1}\right) \qquad \left(k \geqq \frac{1+\sqrt{13}}{2} = 2.3027\cdots\right). \tag{1.33}$$

Thus we have

$$\frac{5-\sqrt{13}}{6} \leqq \frac{k-2}{k-1} < 1 \qquad \left(\frac{5-\sqrt{13}}{6} = 0.23241\cdots\right). \tag{1.34}$$

When $0 < \alpha \leqq \beta < 1$, we have the following inclusion relationship:

$$\mathcal{M}(\alpha) \supset \mathcal{M}(\beta), \tag{1.35}$$

which results from the definition of the class $\mathcal{M}(\alpha)$. Thus we conclude that

$$f(z) \in \mathcal{M}\left(\frac{k-2}{k-1}\right) \subset \mathcal{M}\left(\frac{5-\sqrt{13}}{6}\right) \subset \mathcal{S}^*\left(\frac{5-\sqrt{13}}{6}\right). \tag{1.36}$$

We now consider the following function:

$$f(z) = z + \frac{1}{2k}z^2 \quad (k \geqq 2), \tag{1.37}$$

which immediately yields

$$zf'(z) = z + \frac{1}{k}z^2 \quad (k \geqq 2). \tag{1.38}$$

Since, by definition,

$$f(z) \in \mathcal{N}(\alpha) \iff zf'(z) \in \mathcal{M}(\alpha),$$

we finally obtain the following inclusion relationship:

$$f(z) \in \mathcal{N}\left(\frac{k-2}{k-1}\right) \subset \mathcal{N}\left(\frac{5-\sqrt{13}}{6}\right) \subset \mathcal{K}\left(\frac{5-\sqrt{13}}{6}\right) \tag{1.39}$$

$$\left(k \geqq \frac{1+\sqrt{13}}{2} = 2.3027\cdots\right).$$

2 A Set of Coefficient Inequalities

Our first coefficient inequality is contained in Theorem 1 below.

Theorem 1. *Let* $0 < \alpha < 1$. *If* $f(z) \in \mathcal{A}$ *satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq \frac{1}{2}(1-|1-2\alpha|) = \begin{cases} \alpha & \left(0 < \alpha \leqq \frac{1}{2}\right) \\ 1-\alpha & \left(\frac{1}{2} \leqq \alpha < 1\right), \end{cases} \tag{2.1}$$

then $f(z) \in \mathcal{M}(\alpha)$.

Proof. By virtue of the condition (1.10), we have to show that

$$\left|\frac{2\alpha f(z)}{zf'(z)} - 1\right| < 1. \tag{2.2}$$

We first observe that

$$\left|\frac{2\alpha f(z) - zf'(z)}{zf'(z)}\right| = \left|\frac{1-2\alpha+\sum_{n=2}^{\infty}(n-2\alpha)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}na_n z^{n-1}}\right| \tag{2.3}$$

$$\leqq \frac{|1-2\alpha| + \sum_{n=2}^{\infty}(n-2\alpha)|a_n| \cdot |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n|a_n| \cdot |z|^{n-1}}$$

$$< \frac{|1-2\alpha| + \sum_{n=2}^{\infty}(n-2\alpha)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}.$$

Now, by using the coefficient inequality (2.1), we have

$$\frac{|1-2\alpha| + \sum_{n=2}^{\infty}(n-2\alpha)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|} \leqq 1, \tag{2.4}$$

which, in conjunction with (2.2) and (2.3), completes the proof of Theorem 1. □

By means of Theorem 1, we introduce the subclass $\mathcal{M}^*(\alpha)$ of the class $\mathcal{M}(\alpha)$ consisting of all functions $f(z)$ which satisfy the coefficient inequality (2.1) for some α $(0 < \alpha < 1)$.

Theorem 2. *Suppose that* $0 < \alpha < 1$. *If* $f(z) \in \mathcal{A}$ *satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq \frac{1}{2}(1 - |1-2\alpha|) = \begin{cases} \alpha & \left(0 < \alpha \leqq \frac{1}{2}\right) \\ 1-\alpha & \left(\frac{1}{2} \leqq \alpha < 1\right), \end{cases} \tag{2.5}$$

then $f(z) \in \mathcal{N}(\alpha)$.

Proof. The proof of Theorem 2 follows from Theorem 1 and the aforementioned fact that

$$f(z) \in \mathcal{N}(\alpha) \iff zf'(z) \in \mathcal{M}(\alpha).$$

□

By means of Theorem 2, we also introduce the subclass $\mathcal{N}^*(\alpha)$ of the class $\mathcal{N}(\alpha)$ consisting of all functions $f(z)$ which satisfy the coefficient inequality (2.5) for some α $(0 < \alpha < 1)$.

3 Distortion Bounds

For $f \in \mathcal{A}$, we define the integro-differential operators $I_k f(z)$ given by

$$I_{-1}f(z) = f'(z), \quad I_0 f(z) = f(z),$$

and

$$I_k f(z) = \int_0^z I_{k-1}f(t)dt \qquad (k \in \mathbb{N} := \{1, 2, 3, \cdots).$$

Then we find from (1.1) that

$$I_k f(z) = \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k}. \tag{3.1}$$

Theorem 3. *If $f(z) \in \mathcal{M}^*(\alpha)$, then*

$$\frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \leqq |I_k f(z)| \tag{3.2}$$

$$\leqq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2}$$

$$(z \in \mathbb{U};\ k \in \mathbb{N}_0 \cup \{-1\};\ \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Proof. We begin by noting that

$$|I_k f(z)| = \left| \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \right| \tag{3.3}$$

$$\leqq \frac{1}{(k+1)!}|z|^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \cdot |z|^{n+k}$$

$$< \frac{1}{(k+1)!}|z|^{k+1} + |z|^{k+2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n|.$$

Now it is easy to see that

$$\frac{(k+2)!(2-\alpha)}{2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \leqq \sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq \frac{1}{2}(1-|1-2\alpha|), \tag{3.4}$$

which implies that

$$\sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \leqq \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}. \tag{3.5}$$

Therefore, we have

$$|I_k f(z)| \leqq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \quad (z \in \mathbb{U}). \tag{3.6}$$

Also we can easily observe that

$$|I_k f(z)| \geqq \frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \quad (z \in \mathbb{U}). \tag{3.7}$$

□

By setting $k = -1, 0, 1$ in Theorem 3, we deduce Corollary 1 below.

Corollary 1. *If $f(z) \in \mathcal{M}^*(\alpha)$, then*

$$(3.8) \qquad 1 - \frac{1-|1-2\alpha|}{2-\alpha}|z| \leqq |f'(z)| \leqq 1 + \frac{1-|1-2\alpha|}{2-\alpha}|z| \qquad (k=-1),$$

$$(3.9) \qquad |z| - \frac{1-|1-2\alpha|}{2(2-\alpha)}|z|^2 \leqq |f(z)| \leqq |z| + \frac{1-|1-2\alpha|}{2(2-\alpha)}|z|^2 \qquad (k=0),$$

and

$$(3.10) \qquad \frac{1}{2}|z|^2 - \frac{1-|1-2\alpha|}{6(2-\alpha)}|z|^3 \leqq |I_1 f(z)| \leqq \frac{1}{2}|z|^2 + \frac{1-|1-2\alpha|}{6(2-\alpha)}|z|^3 \qquad (k=1).$$

For $f \in \mathcal{A}$, we consider again the following integro-differential operators:

$$I_{-2}f(z) = f''(z) = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}, \quad I_{-1}f(z) = f'(z), \quad I_0 f(z) = f(z),$$

and

$$I_k f(z) = \int_0^z I_{k-1} f(t)dt \qquad (k \in \mathbb{N}).$$

Next we state and prove the following result.

Theorem 4. *If $f(z) \in \mathcal{N}^*(\alpha)$, then*

$$(3.11) \qquad 2|a_2| - \frac{1-|1-2\alpha|}{2}|z| \leqq |I_{-2}f(z)| \leqq 2|a_2| + \frac{1-|1-2\alpha|}{2}|z|$$

and

$$(3.12) \qquad \frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \leqq |I_k f(z)|$$

$$\leqq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \qquad (z \in \mathbb{U};\ k \in \mathbb{N}_0 \cup \{-1\}).$$

Proof. We note that, for $k \in \mathbb{N}_0 \cup \{-1\}$,

$$(3.13) \qquad |I_k f(z)| = \left| \frac{1}{(k+1)!}z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \right|$$

$$\leqq \frac{1}{(k+1)!}|z|^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \cdot |z|^{n+k}$$

$$< \frac{1}{(k+1)!}|z|^{k+1} + |z|^{k+2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n|.$$

Since, for $f(z) \in \mathcal{N}^*(\alpha)$,

$$(3.14) \qquad (k+2)!(2-\alpha)\sum_{n=2}^{\infty}\frac{n!}{(n+k)!}|a_n| \leqq \sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq \frac{1}{2}(1-|1-2\alpha|),$$

we find that

$$(3.15) \qquad \sum_{n=2}^{\infty}\frac{n!}{(n+k)!}|a_n| \leqq \frac{1-|1-2\alpha|}{2(k+2)!(2-\alpha)}.$$

Therefore, we have

$$(3.16) \qquad \frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{2(k+2)!(2-\alpha)}|z|^{k+2} \leqq |I_k f(z)|$$

$$\leqq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{2(k+2)!(2-\alpha)}|z|^{k+2}$$

$$(z \in \mathbb{U};\ k \in \mathbb{N}_0 \cup \{-1\}).$$

In the exceptional case of (3.16) when $k = -2$, we have

$$(3.17) \qquad 2|a_2| - \frac{1-|1-2\alpha|}{2}|z| \leqq |I_{-2}f(z)| \leqq 2|a_2| + \frac{1-|1-2\alpha|}{2}|z|$$

$$(z \in \mathbb{U}).$$

□

By setting $k = -1, 0, 1$ in Theoem 4, we deduce the following corollary.

Corollary 2. *If $f(z) \in \mathcal{N}^*(\alpha)$, then*

$$(3.18) \qquad 1 - \frac{1-|1-2\alpha|}{2(2-\alpha)}|z| \leqq |f'(z)| \leqq 1 + \frac{1-|1-2\alpha|}{2(2-\alpha)}|z| \qquad (k=-1),$$

$$(3.19) \qquad |z| - \frac{1-|1-2\alpha|}{4(2-\alpha)}|z|^2 \leqq |f(z)| \leqq |z| + \frac{1-|1-2\alpha|}{4(2-\alpha)}|z|^2 \qquad (k=0),$$

and

$$(3.20) \qquad \frac{1}{2}|z|^2 - \frac{1-|1-2\alpha|}{12(2-\alpha)}|z|^3 \leqq |I_1 f(z)| \leqq \frac{1}{2}|z|^2 + \frac{1-|1-2\alpha|}{12(2-\alpha)}|z|^3 \qquad (k=1).$$

4 Inclusion Relationships Between the Function Classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$

Using the coefficient inequalities for the classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$, we now derive Theorem 5 below.

Theorem 5. *The following inclusion relationships hold true for the class* $\mathcal{M}^*(\alpha)$:

(A) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(1-\alpha) \qquad \left(0 < \alpha \leqq \frac{1}{2}\right)$.

(B) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*\left(1 - \frac{1}{3-2\alpha}\right) \qquad \left(\frac{1}{2} \leqq \alpha < 1\right)$.

(C) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(\beta) \qquad \left(0 < \alpha \leqq \beta \leqq \frac{1}{2}\right)$.

(D) $\mathcal{M}^*(\beta) \subset \mathcal{M}^*(\alpha) \qquad \left(\frac{1}{2} \leqq \alpha \leqq \beta < 1\right)$.

Proof. (A) For

$$0 < \alpha \leqq \frac{1}{2} \qquad \text{and} \qquad \frac{1}{2} \leqq \beta < 1,$$

we consider the maximum value of β such that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} |a_n| \leqq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha} |a_n| \leqq 1. \tag{4.1}$$

Thus we need to find the maximum value of β such that

$$\beta \leqq \frac{n(1-\alpha)-\alpha}{n-2\alpha} \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{4.2}$$

By taking the derivative of the right-hand side of (4.2) with respect to n, it is easily seen that the right-hand side of (4.2) is monotonically decreasing for n. Thus, upon letting $n \to \infty$, we have $\beta = 1 - \alpha$. Noting also that

$$\frac{1}{2} \leqq \beta < 1 \qquad \text{for} \qquad 0 < \alpha \leqq \frac{1}{2},$$

we have

$$\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(1-\alpha) \qquad \left(0 < \alpha \leqq \frac{1}{2}\right),$$

which evidently completes the proof of (A).

The proofs of (B), (C), and (D) are much akin to the proof of (A).

□

Finally, we consider some relationships between the function classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$.

Theorem 6. *Each of the following assertions holds true:*

(A) *If* $f(z) \in \mathcal{N}^*(\alpha)$ *for* $0 < \alpha \leqq \frac{1}{2}$, *then* $f(z) \in \mathcal{M}^*\left(\frac{4-4\alpha}{4-3\alpha}\right)$.

(B) *If* $f(z) \in \mathcal{N}^*(\alpha)$ *for* $\frac{1}{2} \leqq \alpha < 1$, *then* $f(z) \in \mathcal{M}^*\left(\frac{2-2\alpha}{5-3\alpha}\right)$.

(C) *If* $f(z) \in \mathcal{N}^*(\alpha)$ *for* $0 < \alpha \leqq \frac{1}{2}$, *then* $f(z) \in \mathcal{M}^*\left(\frac{2\alpha}{4-\alpha}\right)$.

(D) *If* $f(z) \in \mathcal{N}^*(\alpha)$ *for* $\frac{1}{2} \leqq \alpha < 1$, *then* $f(z) \in \mathcal{M}^*\left(\frac{2}{3-\alpha}\right)$.

Proof. (A) Let

$$0 < \alpha \leqq \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leqq \beta < 1.$$

We consider the maximum value of β such that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} |a_n| \leqq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{\alpha} |a_n| \leqq 1. \tag{4.3}$$

This means that

$$\beta \leqq \frac{n^2 - 2n\alpha}{n^2 - n\alpha - \alpha} \quad (n \in \mathbb{N} \setminus \{1\}). \tag{4.4}$$

If we take the derivative of the right-hand side of (4.4) with respect to n, then the numerator becomes

$$n^2\alpha - 2n\alpha + 2\alpha^2 \geqq 0 \qquad \left(0 < \alpha \leqq \frac{1}{2};\, n \in \mathbb{N} \setminus \{1\}\right). \tag{4.5}$$

Therefore, the right-hand side of (4.4) is monotonically increasing for n.
Thus, by setting $n = 2$, we have

$$\beta = \frac{4-4\alpha}{4-3\alpha}.$$

It is easy to see that

$$\frac{1}{2} \leqq \beta < 1 \quad \text{for} \quad 0 < \alpha \leqq \frac{1}{2},$$

which obviously completes the proof of (A).

The proofs of (B), (C), and (D) would run parallel to the proof of (A).

□

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Shigeyoshi Owa and Kyohei Ochiai
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-Mail: owa@math.kindai.ac.jp and ochiai@math.kindai.ac.jp

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
E-Mail: harimsri@math.uvic.ca