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Abstract. As previous studies on anabelian geometry over p-adic local fields suggest, "ramifications of fields" play a key role in this area. In the present paper, more generally, we consider anabelian geometry of complete discrete valuation fields with perfect residue fields from the viewpoint of "ramifications of fields". Concretely, we establish mono-anabelian reconstruction algorithms of various invariants of these fields from their absolute Galois groups with ramification filtrations. By using these results, we reconstruct group-theoretically the isomorphism classes of mixed-characteristic complete discrete valuation fields with perfect residue fields under certain conditions. This result shows that these types of complete discrete valuation fields themselves have some "anabelianness". Moreover, we also investigate properties of homomorphisms between the absolute Galois groups of complete discrete valuation fields with perfect residue fields which preserve ramification filtrations.

Contents

Introduction

Grothendieck, who is the originator of anabelian geometry, considered that anabelian geometry should be developed over fields finitely generated over prime fields as seen in his conjecture given in 1980s. In 1990s, his conjecture for hyperbolic curves over fields finitely generated over $\mathbb{Q}$ was solved affirmatively by Nakamura (the case where $g = 0$, cf. [Nak1, Theorem C], [Nak2, (1.1)]), Tamagawa (the case where X is affine, cf. [T, Theorem 0.3]) and Mochizuki (the general case, cf. [Mo1, Theorem A]). Moreover, Mochizuki gave two important anabelian results over p-adic local fields. One is a certain analogue of the theorem of Neukirch-Uchida for the absolute Galois groups with ramification filtrations of p-adic local fields (cf. [Mo2, Theorem 4.2]), and the other is (the relative

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version of) the Grothendieck conjecture for hyperbolic curves over p-adic local fields (cf, [Mo3, Theorem A]). (Furthermore, in [Mo4, Theorem 4.12], Mochizuki also proved (the relative version of) the Grothendieck conjecture for hyperbolic curves over generalized sub-p-adic fields (i.e., fields isomorphic to subfields of fields finitely generated over the quotient field of the Witt ring with coefficients in an algebraic closure of $\mathbb{F}_p$).) Since then, anabelian phenomena over p-adic local fields have been one of main issues in anabelian geometry. However, in anabelian geometry over these fields, there are many difficulties which are not found in the finitely generated fields (especially, number fields) cases. For example, though number fields are reconstructed from their absolute Galois groups even in the sense of mono-anabelian reconstruction (i.e., a mono-anabelian version of the theorem of Neukirch-Uchida, cf. [Ho2, Theorem A]), the analogue of the theorem of Neukirch-Uchida for p-adic local fields fails to hold as it is. This failure of the analogue of the theorem of Neukirch-Uchida makes it difficult to study the absolute version of the Grothendieck conjecture for hyperbolic curves over p-adic local fields (which holds over number fields (cf. [Mo5, Corollary 1.3.5])). There are many studies trying to overcome these difficulties (see, e.g., [Mo6, §3], [Ho3] and [Mu1]). These studies and the above result of Mochizuki (an analogue of the theorem of Neukirch-Uchida) suggest that "ramifications of fields" play a key role in anabelian geometry over p-adic local fields.

On the other hand, Grothendieck's conjecture and developments of anabelian geometry over p-adic local fields raise the following question:

What kinds of fields are suitable for the base fields of anabelian geometry?

This question is a main theme of the present paper and [Mu2]. In the present paper, we consider this problem for complete discrete valuation fields with perfect residue fields from the viewpoint of "ramifications of fields". (In [Mu2], we consider this problem for higher local fields.) We mainly treat mixed-characteristic complete discrete valuation fields with perfect residue fields (which we shall abbreviate to GMLF's (cf. Definition 1.12 (i))). Concretely, we consider the following problems:

(A) Which invariants of GMLF's are reconstructed from the absolute Galois groups with ramification filtrations in the sense of mono-anabelian reconstruction?
(B) Does the analogue of the theorem of Neukirch-Uchida for GMLF's and the absolute Galois groups with ramification filtrations hold?

Moreover, we also investigate properties of homomorphisms between the absolute Galois groups of complete discrete valuation fields with perfect residue fields which preserve ramification filtrations.

For (A), we prove the following theorem:

Theorem A (cf. Propositions 2.9, 3.10)
Let K be a GMLF, $\mathfrak{G}_K$ the filtered absolute Galois group of K with the ramification filtration (cf. Definition 1.3 and Remark 1.4), and G_K the underlying profinite group of $\mathfrak{G}_K$ (i.e., the absolute Galois group of K). Then there exist mono-anabelian reconstruction algorithms of the following invariants from $\mathfrak{G}_K$:

- *the characteristic p of the residue field of K;*
- *the absolute ramification index e_K of K;*

- *the largest nonnegative integer a_K such that K contains a primitive p^{a_K}-th root of unity;*
- *the p-adic cyclotomic character $\chi_p : G_K \to \mathbb{Z}_p^\times$.*

By this theorem, in some special cases, the filtered absolute Galois group $\mathfrak{G}_K$ of a GMLF K and the isomorphism class of the residue field of K determine the isomorphism class of K (which gives an answer to (B)):

Theorem B (cf. Theorems 3.12, 3.13)
Let k be a perfect (resp. an algebraically closed) field of positive characteristic, K a GMLF with residue field k, $\mathfrak{G}_K$ the filtered absolute Galois group of K with the ramification filtration, G_K the underlying profinite group of $\mathfrak{G}_K$ (i.e., the absolute Galois group of K), and e_K and a_K as in Theorem A. Set $p := \operatorname{char} k$. Suppose that one of the following condition holds:

(i) $p \neq 2$.
(ii) $a_K \geq 2$.
(iii) $e_K = 1$ *(resp. e_K is prime to p).*

Suppose, moreover, that there exists a finite extension L of K satisfying the following conditions:

(a) *L is a totally ramified extension of K.*
(b) $e_L = p^{a_L-1}(p-1)$ *(resp. $e_L = p^{a_L-1}(p-1)n$, where n is a positive integer prime to p), where e_L and a_L are defined similarly to e_K and a_K.*

Then the isomorphism class of K is completely determined by $\mathfrak{G}_K$ and the isomorphism class of k.

Note that, in the situation of Theorem B, we may determine whether or not the conditions (i)$\sim$(iii) hold and whether or not a finite extension L of K satisfying the conditions (a) and (b) exists from the (filtered) group-theoretic data $\mathfrak{G}_K$ by Theorem A.

We shall review the contents of the present paper. In Section 1, we define R-filtered profinite groups, which are main objects of Sections 2 and 3. In Section 2, we discuss some generalities on complete discrete valuation fields with positive residue characteristic. We also obtain some injectivity results (cf. Propositions 2.14 and 2.16) on homomorphisms between the filtered absolute Galois groups of GMLF's (by using the theory of fields of norms and local class field theory), and prove a certain "Hom-version" of an analogue of the theorem of Neukirch-Uchida for complete discrete valuation fields with finite residue fields (and for the absolute Galois groups with ramification filtrations) (cf. Theorem 2.18), which is an improvement of Abrashkin's result. Moreover, by applying this result, we prove a certain "semi-absolute Hom-version" of the Grothendieck conjecture for hyperbolic curves over p-adic local fields (cf. Theorem 2.21), which is an improvement of Mocihzuki's result. In Section 3, by using theories in Section 2, we treat the problems (A) and (B), and prove Theorems A and B.

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0. Notations and conventions

Numbers:

We shall write

- $\mathbb{Z}$ for the set of integers;
- $\mathbb{Q}$ for the set of rational numbers;
- $\mathbb{R}$ for the set of real numbers;
- $\mathfrak{Primes}$ for the set of prime numbers.

For $a \in \mathbb{R}$ and $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, we shall write $\mathbb{X}_{\geq a}$ (resp. $\mathbb{X}_{>a}$, resp. $\mathbb{X}_{\leq a}$, resp. $\mathbb{X}_{<a}$) for

$$\{b \in \mathbb{X} \mid b \geq a \text{ (resp. } b > a\text{, resp. } b \leq a\text{, resp. } b < a)\}.$$

Fields:

For $p \in \mathfrak{Primes}$ and $n \in \mathbb{Z}_{>0}$, we shall write

- $\mathbb{Z}_p$ for the p-adic completion of $\mathbb{Z}$;
- $\mathbb{Q}_p$ for the quotient field of $\mathbb{Z}_p$;
- $\mathbb{F}_{p^n}$ for the finite field of cardinality p^n.

Profinite groups:

Let G be a profinite group and $p \in \mathfrak{Primes}$. Then we shall write G^{ab} for the abelianization of G (i.e., the quotient of G by the closure of the commutator subgroup of G), and $G(p)$ for the maximal pro-p quotient of G. For a subset X of G, we shall write $\overline{X}$ for the closure of X in G. For a closed subgroup H of G, we shall write

$$Z_G(H) := \{g \in G \mid g \cdot h = h \cdot g, \text{ for any } h \in H\}$$

for the *centralizer* of H in G.

We shall say that

- G is *slim* if for every open subgroup $H \subset G$, the centralizer $Z_G(H)$ is trivial;
- G is *elastic* if every topologically finitely generated closed normal subgroup $N \subset H$ of an open subgroup $H \subset G$ of G is either trivial or of finite index in G.

We denote the cohomological p-dimension of G by $\mathrm{cd}_p G$, and set:

$$\mathrm{cd} G := \sup_{p \in \mathfrak{Primes}} \mathrm{cd}_p G.$$

1. Preliminaries

In this section, suppose that K is a complete discrete valuation field. Moreover, we shall write

- p_K for the characteristic of K;
- K^{sep} for a separable closure of K;
- G_K for the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$;
- $\mathcal{O}_K$ for the ring of integers of K;
- $\mathfrak{M}_K$ for the maximal ideal of $\mathcal{O}_K$;
- v for the valuation of K such that $v(K^\times) = \mathbb{Z}$;
- $k = \mathcal{O}_K/\mathfrak{M}_K$ for the residue field of K;
- p_k for the characteristic of k;
- k^{sep} for the separable closure of k in the residue field of K^{sep};
- G_k for the Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.

For $a \in \mathcal{O}_K$, we denote the image of a in k by $\overline{a}$.

Let L be a finite Galois extension of K with Galois group G. For $\sigma \in G$, set:

$$i_G(\sigma) := \inf_{a \in \mathcal{O}_L} v_L(\sigma(a) - a);$$
$$s_G(\sigma) := \inf_{a \in L^\times} v_L(\sigma(a)a^{-1} - 1),$$

where $\mathcal{O}_L$ is the ring of integers of L and v_L is the valuation of L such that $v_L(L^\times) = \mathbb{Z}$. Then, for $u \in \mathbb{R}_{\geq -1}$, the *lower ramification subgroups of* G are defined as

$$G_u := \{\sigma \in G \mid i_G(\sigma) \geq u + 1\}.$$

For generalities on lower ramification subgroups, see [Se, IV] and [XZ, §1.1].

Lemma 1.1 (cf. [Hy, Lemma (2-16)], [XZ, §2.1])
Suppose that $p_k > 0$ and set $p := p_k$. Let L be a cyclic extension of K of degree p with Galois group G and $\sigma \in G$ a generator of G. Note that $i_G(\sigma)$ and $s_G(\sigma)$ are independent of the choice of σ.

(i) *Suppose that $p_K = 0$ and K contains a primitive p-th root of unity. Put $e_K := v(p)$. By Kummer theory, there exists an element $a \in K$ such that $L = K(a^{\frac{1}{p}})$. We can choose a with $v(a) = 1$ or $v(a) = 0$. In the latter case, we require that $l = v(a-1)$ is maximal. Then one (and only one) of the following occurs:*

(I) *If $v(a) = 1$, then L/K is a wild extension and*
$$s_G(\sigma) = \frac{pe_K}{p-1}.$$

(II) *If $v(a) = 0$ and $\overline{a} \notin k^p$, then L/K is a ferocious extension and*
$$s_G(\sigma) = \frac{e_K}{p-1}.$$

(III) *If $v(a) = 0$, $\overline{a} = 1$, $l < \dfrac{pe_K}{p-1}$ and p does not divide l, then L/K is a wild extension and*
$$s_G(\sigma) = \frac{pe_K}{p-1} - l.$$

(IV) *If* $v(a) = 0$, $\overline{a} = 1$, $l < \dfrac{pe_K}{p-1}$ *and* p *divides* l, *then* L/K *is a ferocious extension and*

$$s_G(\sigma) = \frac{1}{p}\left(\frac{pe_K}{p-1} - l\right).$$

(V) *If* $v(a) = 0$, $\overline{a} = 1$ *and* $l \geq \dfrac{pe_K}{p-1}$, *then* L/K *is an unramified extension and hence*

$$s_G(\sigma) = 0.$$

(In this case, in fact, we have $l = \dfrac{pe_K}{p-1}$*.)*

(ii) *Suppose that* $p_K \neq 0$ *(hence* $p_K = p$*). For* $x \in K$, *set* $\wp(x) := x^p - x$. *By Artin-Schreier theory, there exists an element* $a \in K$ *such that* $L = K(x)$, *where* $\wp(x) = a$. *Since* $\mathfrak{M}_K \subset \wp(K)$, *we have* $v(a) \leq 0$. *We require that* $v(a)$ *is maximal. Then one (and only one) of the following occurs:*

(I) *If* $v(a) = 0$, *then* L/K *is an unramified extension and hence*

$$s_G(\sigma) = 0.$$

(II) *If* $v(a) < 0$ *and* p *does not divide* $v(a)$, *then* L/K *is a wild extension and*

$$s_G(\sigma) = -v(a).$$

(III) *If* $v(a) < 0$ *and* p *divides* $v(a)$, *then* L/K *is a ferocious extension and*

$$s_G(\sigma) = -\frac{v(a)}{p}.$$

In the remainder of this section, we assume that k is a perfect field.

For $v \in \mathbb{R}_{\geq -1}$ and a finite Galois extension L of K with Galois group G, the *upper ramification subgroups of* G are defined as

$$G^v := G_{\psi_{L/K}(v)},$$

where the function $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is the inverse function of the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ given by

$$\varphi_{L/K}(u) := \int_0^u \frac{dt}{(G_0 : G_t)}.$$

Note that the function $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is given by

$$\psi_{L/K}(v) = \int_0^v (G^0 : G^w)dw.$$

Let L' be a finite separable extension of K (not necessarily Galois) and L'' a finite Galois extension of K containing L. Then we define functions $\varphi_{L'/K}, \psi_{L'/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ as follows:

$$\varphi_{L'/K} := \varphi_{L''/K} \circ \psi_{L''/L'},$$
$$\psi_{L'/K} := \varphi_{L''/L'} \circ \psi_{L''/K}.$$

Note that these functions coincide with $\varphi_{L'/K}$, $\psi_{L'/K}$ defined above in the case where L' is Galois over K and do not depend on the choice of L'' (cf. [XZ, §1.1]). For $v \in \mathbb{R}_{\geq -1}$ and an infinite Galois extension L of K with Galois group G, the *upper ramification subgroups of* G are defined as

$$G^v := \varprojlim_{K'} \mathrm{Gal}(K'/K)^v,$$

where K' runs through the set of finite Galois subextensions of L/K. For generalities on upper ramification subgroups, see [Se, IV] and [XZ, §3].

Let $\{G_K^v\}_{v\in\mathbb{R}_{\geq -1}}$ be the absolute Galois group of K with the upper ramification filtration. We shall write $I_K := \overline{\bigcup_{\varepsilon\in\mathbb{R}_{>0}} G_K^{-1+\varepsilon}}$ (resp. $P_K := \overline{\bigcup_{\varepsilon\in\mathbb{R}_{>0}} G_K^{0+\varepsilon}}$) for the inertia subgroup (resp. the wild inertia subgroup) of G_K.

Remark 1.2
We have the following two natural splitting short exact sequences:

$$1 \longrightarrow P_K \longrightarrow I_K \longrightarrow \hat{\mathbb{Z}}^{p_k'}(1) \longrightarrow 1,$$

$$1 \longrightarrow I_K \longrightarrow G_K \longrightarrow G_k \longrightarrow 1,$$

where $\hat{\mathbb{Z}}^{p_k'}$ is the maximal prime-to-p_k quotient of $\hat{\mathbb{Z}}$ (if $p_k = 0$, we set $\hat{\mathbb{Z}}^{p_k'} = \hat{\mathbb{Z}}$). Note that, if $p_k > 0$, then P_K is a non-trivial pro-p_k group. On the other hand, if $p_k = 0$, we have $P_K = \{1\}$ and hence $I_K \simeq \hat{\mathbb{Z}}(1)$.

Now we introduce a notion which gives a generalization of ramification filtrations:

Definition 1.3
For a profinite group G, a *filtration of R-type on* G (where “R” is understood as an abbreviation for “ramification”) consists of the following data:

- (i) A collection of closed normal subgroups $\mathfrak{G} = \{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ of G satisfying the following conditions:
  - (a) $G^{-1} = G$.
  - (b) If v_1, $v_2 \in \mathbb{R}_{\geq -1}$ satisfy $v_1 \geq v_2$, then $G^{v_1} \subset G^{v_2}$.
  - (c) $\bigcap_{v\in\mathbb{R}_{\geq -1}} G^v = \{1\}$.
- (ii) For any closed subgroup H of G such that HG^v is an open subgroup of G for any $v \in \mathbb{R}_{\geq -1}$, a collection of closed normal subgroups $\mathfrak{H} = \{H^v\}_{v\in\mathbb{R}_{\geq -1}}$ of H satisfying the following conditions:
  - (a) $H^0 = G^0 \cap H$.
  - (b) For $v \in \mathbb{R}_{\geq -1}$, we have

$$H^{\psi_{G/H}(v)} = G^v \cap H,$$

where $\psi_{G/H} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a function given by the following formula:

$$\psi_{G/H}(v) = \begin{cases} \int_0^v (G^0 : H^0 G^w) dw, & (v \geq 0); \\ v, & (-1 \leq v < 0). \end{cases}$$

Moreover, denote the inverse map of $\psi_{G/H}$ by $\varphi_{G/H}$. (Note that clearly we have $\lim\limits_{v\to\infty} \psi_{G/H}(v) = \infty$.)

We shall say that such H is a *subgroup of APF-type* (where "APF" is understood as an abbreviation for "arithmetically profinite").

(iii) For any closed normal subgroup H of G, a collection of closed normal subgroups $\{(G/H)^v\}_{v\in\mathbb{R}_{\geq -1}}$ of G/H such that, for any $v \in \mathbb{R}_{\geq -1}$,

$$(G/H)^v = G^v H/H.$$

(iv) For any open normal subgroup H of G (in particular, a subgroup of APF-type), a collection of (closed) normal subgroups $\{(G/H)_u\}_{u\in\mathbb{R}_{\geq -1}}$ of G/H such that, for any $u \in \mathbb{R}_{\geq -1}$,

$$(G/H)_u = (G/H)^{\varphi_{G/H}(u)}.$$

Note that the data in (ii), (iii) and (iv) are completely determined by the filtration $\mathfrak{G} = \{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ defined in (i). We shall say that $\mathfrak{G} = \{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ is an *R-filtered profinite group* and G is the *underlying profinite group of* $\mathfrak{G}$.

Remark 1.4
Suppose that G is the absolute Galois group of a complete discrete valuation field K with perfect residue field. Then the upper ramification filtration on G clearly determines a filtration of R-type on G. Let H be an open subgroup of G and L the finite separable extension of K corresponding to H. Then $\psi_{G/H}$ and $\varphi_{G/H}$ defined in Definition 1.3 (ii) coincide with $\psi_{L/K}$ and $\varphi_{L/K}$ defined in the argument following Lemma 1.1.

Definition 1.5
Let $\mathfrak{G}$ be an R-filtered profinite group and G its underlying profinite group.

(i) Let H be a closed subgroup of G of APF-type. Then $\mathfrak{G}$ determines a filtration of R-type on H. Denote the resulting R-filtered profinite group by $\mathfrak{H}$. In this case, we shall say that $\mathfrak{H}$ is an *R-filtered closed subgroup (of APF-type) of* $\mathfrak{G}$, and use the notation $\mathfrak{H} \subset \mathfrak{G}$. Moreover, if H is a(n) open (resp. normal) subgroup of G, we shall say that $\mathfrak{H}$ is an *R-filtered open* (resp. *normal*) *subgroup of* $\mathfrak{G}$.

(ii) Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of closed subgroups of G. Suppose that $\bigcap\limits_{\lambda\in\Lambda} H_\lambda$ is a subgroup of APF-type of G. (Note that, in this case, for any $\lambda \in \Lambda$, H_λ is a subgroup of APF-type of G, and $\mathfrak{G}$ determines a filtration of R-type on H_λ. Denote the resulting R-filtered profinite group by $\mathfrak{H}_\lambda$.) Then $\mathfrak{G}$ determines a filtration of R-type on $\bigcap\limits_{\lambda\in\Lambda} H_\lambda$. We denote the resulting R-filtered profinite group by $\bigcap\limits_{\lambda\in\Lambda} \mathfrak{H}_\lambda$.

Definition 1.6
Let $\mathfrak{G}$ be an R-filtered profinite group and G its underlying profinite group. For a finite quotient H of G and an element $\sigma \in H$, set:

$$s_H(\sigma) := \begin{cases} 0, & (\sigma \notin H_u \text{ for any } u \in \mathbb{R}_{>-1}); \\ \sup\{u \in \mathbb{R}_{\geq -1} \mid \sigma \in H_u\}, & (\text{otherwise}). \end{cases}$$

Remark 1.7
Let $\mathfrak{G}_K = \{G_K^v\}_{v \in \mathbb{R}_{\geq -1}}$ be the absolute Galois group of K with the upper ramification filtration (which is clearly an R-filtered profinite group). In the case where $\mathfrak{G} = \mathfrak{G}_K$, for any finite quotient H of $G = G_K$ and any $\sigma \in H$, $s_H(\sigma)$ defined the paragraph preceding Lemma 1.1 coincides with $s_H(\sigma)$ defined in Definition 1.6.

Definition 1.8
Let $\mathfrak{G}_1 = \{G_1^v\}_{v \in \mathbb{R}_{\geq -1}}$ and $\mathfrak{G}_2 = \{G_2^v\}_{v \in \mathbb{R}_{\geq -1}}$ be R-filtered profinite groups, G_1 and G_2 their underlying profinite groups, and $\alpha : G_1 \to G_2$ a homomorphism of profinite groups. We shall say that α is a *homomorphism of R-filtered profinite groups* if $\alpha(G_1)$ is a subgroup of APF-type of G_2 and for any $v \in \mathbb{R}_{\geq -1}$, the following condition holds:

$$\alpha(G_1^{\psi_{G_2/\alpha(G_1)}(v)}) = G_2^v \cap \alpha(G_1),$$

or, equivalently, for any $v \in \mathbb{R}_{\geq -1}$,

$$\alpha(G_1^v) = \alpha(G_1)^v.$$

In this case, we use the notation $\alpha : \mathfrak{G}_1 \to \mathfrak{G}_2$.

Remark 1.9
Let $\alpha : \mathfrak{G}_1 \to \mathfrak{G}_2$ and $\beta : \mathfrak{G}_2 \to \mathfrak{G}_3$ be homomorphisms of R-filtered profinite groups. Then $\beta \circ \alpha$ (the composite as homomorphisms of profinite groups) is not necessarily a homomorphism of R-filtered profinite groups.

Definition 1.10
Let $\alpha : \mathfrak{G}_1 \to \mathfrak{G}_2$ be a homomorphism of R-filtered profinite groups and $\mathfrak{G}$ an R-filtered profinite group.

(i) We shall say that α is a(n) *isomorphism* (resp. *open homomorphism*, resp. *injection*, resp. *surjection*) *of R-filtered profinite groups* if α is a(n) isomorphism (resp. open homomorphism, resp. injection, resp. surjection) as a homomorphism of profinite groups.

(ii) We shall say that α is *quasi-injective* (resp. *quasi-surjective*) if, for any homomorphism of R-filtered profinite groups $\beta : \mathfrak{H} \to \mathfrak{G}_1$ (resp. $\beta : \mathfrak{G}_2 \to \mathfrak{H}$), $\alpha \circ \beta$ (resp. $\beta \circ \alpha$) is also a homomorphism of R-filtered profinite groups.

(iii) We shall say that $\mathfrak{G}$ is *R-filtered hopfian* if every surjective homomorphism $\mathfrak{G} \to \mathfrak{G}$ of R-filtered profinite groups is an isomorphism.

Remark 1.11
Let α be a homomorphism of R-filtered profinite groups. If α is injective (resp. surjective), it is clear that α is quasi-injective (resp. quasi-surjective).

Definition 1.12 (cf. [Ho1, Definition 3.1])
Let K be a field, G a profinite group and $\mathfrak{G}$ an R-filtered profinite group.

(i) We shall say that K is a(n) *MLF* (resp. *GMLF*, resp. *PLF*, resp. *GPLF*) if K is isomorphic to a finite extension of $\mathbb{Q}_p$ for some prime number p (resp. a complete discrete valuation field of characteristic zero whose residue field is perfect and of positive characteristic, resp. a complete discrete valuation field of positive characteristic whose residue field is finite, resp. a complete discrete valuation field of positive characteristic whose residue field is perfect) (where "MLF" (resp. "GMLF", resp. "PLF", resp. "GPLF") is understood as an abbreviation for "Mixed-characteristic Local Field" (resp. "Generalized Mixed-characteristic Local Field", resp. "Positive-characteristic Local Field", resp. "Generalized Positive-characteristic Local Field")).

(ii) We shall say that G is of *MLF-type* (resp. *GMLF-type*, resp. *PLF-type*, resp. *GPLF-type*) if G is isomorphic, as a profinite group, to the absolute Galois group of a(n) MLF (resp. GMLF, resp. PLF, resp. GPLF).

(iii) We shall say that $\mathfrak{G}$ is of *R-MLF-type* (resp. *R-GMLF-type*, resp. *R-PLF-type*, resp. *R-GPLF-type*) if $\mathfrak{G}$ is isomorphic, as an R-filtered profinite group, to the absolute Galois group of a(n) MLF (resp. GMLF, resp. PLF, resp. GPLF) with the upper ramification filtration.

We give a group-theoretic characterization of profinite groups of MLF-type:

Proposition 1.13
Let G be a profinite group of GMLF-type. Then G is of MLF-type if and only if G is topologically finitely generated.

Proof.
We will prove this proposition in a similar way to the proof of [MT, Lemma 3.5].

If G is of MLF-type, then it is well-known that G is topologically finitely generated. Suppose that G is topologically finitely generated. Let K be a GMLF whose absolute Galois group is isomorphic to G, and $\mathfrak{M}_K$, k, p_k as in the beginning of this section. Set $p := p_k (> 0)$. It suffices to show that k is a finite field. Note that we have an isomorphism:

$$K^\times \simeq \mathbb{Z} \times k^\times \times (1 + \mathfrak{M}_K).$$

By taking an open normal subgroup of G if necessary, we may assume that K contains a primitive p-th root of unity. Then, by Kummer theory, we have:

$$H^1(G, \mathbb{Z}/p\mathbb{Z}) \simeq K^\times/(K^\times)^p \simeq \mathbb{Z}/p\mathbb{Z} \times (1 + \mathfrak{M}_K)/(1 + \mathfrak{M}_K)^p.$$

(Note that k is perfect.) Since $1 + \mathfrak{M}_K^2 \supset (1 + \mathfrak{M}_K)^p$, we have a surjection:

$$(1 + \mathfrak{M}_K)/(1 + \mathfrak{M}_K)^p \twoheadrightarrow (1 + \mathfrak{M}_K)/(1 + \mathfrak{M}_K^2) \simeq k.$$

If k is an infinite field, then $H^1(G, \mathbb{Z}/p\mathbb{Z})$ is an infinite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space. However, since (we have assumed that) G is topologically finitely generated, we obtain a contradiction. This completes the proof of Proposition 1.13. $\square$

Remark 1.14
A similar statement to Proposition 1.13 for GPLF does not hold. It is not clear to the author at the time of writing whether or not there exists a group-theoretic characterization of profinite groups of PLF-type.

Proposition 1.15
Let $\mathfrak{G}$ be an R-filtered profinite group of R-GMLF or R-GPLF-type and $\mathfrak{H} \subset \mathfrak{G}$ an R-filtered closed subgroup of APF-type of $\mathfrak{G}$. Then $\mathfrak{H}$ is an R-filtered profinite group of R-GPLF-type.

Proof.
Immediate from the theory of fields of norms (cf. [FW1, §2], [FW2, §4] and [W, Corollaire 3.3.6]). □

2. Generalities on GMLF's and GPLF's

In this section, we discuss some generalities on GMLF's and GPLF's.

For a GMLF or GPLF K, we shall write

- p_K for the characteristic of K;
- K^{sep} for a separable closure of K;
- G_K for the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$;
- $\mathfrak{G}_K = \{G_K^v\}_{v \in \mathbb{R}_{\geq -1}}$ for the R-filtered profinite group with underlying profinite group G_K determined by the ramification filtration on G_K;
- $I_K \subset G_K$ for the inertia subgroup of G_K;
- $P_K \subset I_K \subset G_K$ for the wild inertia subgroup of G_K;
- $\mathcal{O}_K$ for the ring of integers of K;
- $\mathfrak{M}_K$ for the maximal ideal of $\mathcal{O}_K$;
- U_K^i for the multiplicative group $1 + \mathfrak{M}_K^i$ ($i \in \mathbb{Z}_{>0}$);
- v_K for the valuation of K such that $v_K(K^\times) = \mathbb{Z}$;
- $k_K = \mathcal{O}_K/\mathfrak{M}_K$ for the residue field of K (by the definitions of GMLF and GPLF, k_K is perfect);
- $p_{k_K}(> 0)$ for the characteristic of k_K;
- $\overline{k_K}$ for the residue field of K^{sep}, which is an algebraic closure of k_K;
- G_{k_K} for the Galois group $\mathrm{Gal}(\overline{k_K}/k_K)$.

Moreover, if K is a GMLF, for a prime number p and an integer $n \in \mathbb{Z}_{>0}$, let $\zeta_{p^n} \in K^{\mathrm{sep}}$ be a primitive p^n-th root of unity.

Proposition 2.1
Let K be a GMLF and set $p := p_{k_K} > 0$.

(i) *We have $1 \leq \mathrm{cd}_p G_K \leq 2$.*
(ii) *Suppose that $k_K = \overline{k_K}$. Then we have $\mathrm{cd}_p G_K = 1$. Moreover, the maximal pro-p quotient $G_K(p)$ of G_K is a free pro-p group of infinite rank.*
(iii) *G_K and I_K are slim and elastic. Moreover, I_K is a projective group.*
(iv) *P_K is a free pro-p group of infinite rank. In particular, P_K is slim and elastic.*
(v) *Suppose that k_K is p-closed (i.e., k_K has no Galois extensions of degree p) and K contains a primitive p-th root of unity. Then the maximal pro-p quotient $G_K(p)$ of G_K is a free pro-p group of infinite rank.*

Proof.

First, let us consider (ii). The portion of (ii) concerning $\mathrm{cd}_p G_K$ is immediate from [NSW, Theorem 6.5.15]. In particular, $G_K(p)$ is a free pro-p group. Let k_0 be the algebraic closure of $\mathbb{F}_p$ in k_K, and K_0 (resp. K_{00}) the quotient field of the Witt ring with coefficients in k_K (resp. k_0) (therefore, $k_{K_0} = k_K$ and $k_{K_{00}} = k_0$). Then G_K is isomorphic to an open subgroup of G_{K_0}. Note that the natural inclusion $k_0 \hookrightarrow k_K$ induces an inclusion $K_{00} \hookrightarrow K_0$ (by the functorial property of Witt rings). By considering ramification indices, this inclusion induces a surjective homomorphism $G_{K_0} \twoheadrightarrow G_{K_{00}}$. Therefore, we obtain a surjection $G_{K_0}(p) \twoheadrightarrow G_{K_{00}}(p)$. On the other hand, we have an open homomorphism $G_K(p) \to G_{K_0}(p)$. So, to show the infiniteness of the rank of $G_K(p)$, it suffices to prove that $G_{K_{00}}(p)$ is a free pro-p group of infinite rank (note that $G_{K_{00}}(p)$ is a free pro-p group since k_K (hence also k_0) is algebraically closed). Let F be a subfield of k_0 which is a finite extension of $\mathbb{F}_p$, and K_F the quotient field of the Witt ring with coefficients in F. Then the inclusion $F \hookrightarrow k_0$ induces a homomorphism $G_{K_{00}}(p) \to G_{K_F}(p)$. Let J_{K_F} be the image of I_{K_F} in $G_{K_F}(p)$. Then the above homomorphism $G_{K_{00}}(p) \to G_{K_F}(p)$ induces a surjection $G_{K_{00}}(p) \to J_{K_F}$ (cf. the discussion concerning the surjectivity of $G_{K_0} \to G_{K_{00}}$). By local class field theory, we have the following homomorphism:

$$G_{K_F}(p)^{\mathrm{ab}} \simeq (1 + \mathfrak{M}_{K_F}) \times \mathbb{Z}_p.$$

Moreover, the natural morphism $G_{K_F}(p) \twoheadrightarrow G_{K_F}(p)^{\mathrm{ab}}$ induces a surjection $J_{K_F} \twoheadrightarrow 1 + \mathfrak{M}_{K_F}$. In particular, the image of the composite of $G_{K_{00}}(p) \to G_{K_F}(p)$ and $G_{K_F}(p) \twoheadrightarrow G_{K_F}(p)^{\mathrm{ab}}$ contains $1+\mathfrak{M}_{K_F}$ (note that $G_{K_{00}}(p) \to J_{K_F}$ is a surjection). Since $1+\mathfrak{M}_{K_F} \simeq \mathbb{Z}_p^{\oplus [F:\mathbb{F}_p]} \oplus$ (torsion elements), there exists a surjection $G_{K_{00}}(p) \twoheadrightarrow \mathbb{Z}_p^{\oplus [F:\mathbb{F}_p]}$. Since k_0 is algebraically closed (hence, in particular, an infinite extension of $\mathbb{F}_p$), for any $N \in \mathbb{Z}_{>0}$, there exists an intermediate field F of $k_0/\mathbb{F}_p$ such that $[F : \mathbb{F}_p] > N$. This shows the infiniteness of the rank of $G_{K_{00}}(p)$, as desired.

(iv) follows immediately from (ii) (note that we may regard I_K as the absolute Galois group of the completion of the maximal unramified extension of K, and that P_K surjects onto $I_K(p)$).

For (i), consider the following exact sequence (cf. Remark 1.2):

$$1 \longrightarrow I_K \longrightarrow G_K \longrightarrow G_{k_K} \longrightarrow 1.$$

By (ii), we have $\mathrm{cd}_p I_K = 1$ (hence $\mathrm{cd}_p G_K \geq 1$). Moreover, since k_K is of characteristic p, $\mathrm{cd}_p G_{k_K} \leq 1$ (cf. [NSW, Proposition 6.5.10]). Therefore, we obtain $\mathrm{cd}_p G_K \leq 2$. So (i) follows.

The portion of (iii) concerning the slimness and elasticity of G_K and I_K follows from [MT, Theorem C]. By considering the exact sequence in Remark 1.2 and the cohomological p-dimension of I_K (cf. (ii)), we have $\mathrm{cd} I_K = 1$. Therefore, I_K is projective.

(v) follows from [MT, Proposition 3.7] and its proof. □

Proposition 2.2

Let K be a GPLF and set $p := p_K = p_{k_K}$ (> 0).

(i) *We have* $\mathrm{cd}_p G_K = 1$.

(ii) *G_K and I_K are slim and elastic. Moreover, I_K is a projective group.*

(iii) *P_K is a free pro-p group of infinite rank. In particular, P_K is slim and elastic.*

(iv) *The maximal pro-p quotient $G_K(p)$ of G_K is a free pro-p group of infinite rank.*

Proof.

First, (iv) is immediate from [NSW, Proposition 6.1.7].

(i) is immediate from [NSW, Proposition 6.5.10] and (iv).

(iii) is immediate from (i) and (iv) (note that we may regard I_K as the absolute Galois group of the completion of the maximal unramified extension of K, and that P_K surjects onto $I_K(p)$).

The portion of (ii) concerning the slimness and elasticity of G_K and I_K follows from [MT, Theorem C]. The projectivity of I_K follows from a similar argument to the proof of (iii) of Proposition 2.1. $\square$

Proposition 2.3 (cf. [MW, Theorems 1, 2, 3])
Let K be a GMLF, L a cyclic extension of K of degree $p := p_{k_K}$ and σ a generator of $\mathrm{Gal}(L/K)$*. Suppose that L/K is a wild extension. Then we have*

$$s_{\mathrm{Gal}(L/K)}(\sigma) \leq \left\lfloor \frac{pv_K(p)}{p-1} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. The equality holds if and only if K contains a primitive p-th root of unity and $L = K(\alpha)$ where α is a root of $X^p - \beta \in K[X]$ ($\beta \in K^\times, v_K(\beta) \notin p\mathbb{Z}$).

Moreover, if K does not contain a primitive p-th root of unity, then $s_{\mathrm{Gal}(L/K)}(\sigma) \notin p\mathbb{Z}$.

Now we can determine whether or not a GMLF K contains a primitive p_{k_K}-th root of unity from the group-theoretic data $\mathfrak{G}_K$:

Proposition 2.4
Let K be a GMLF and set $p := p_{k_K}$. Then K contains a primitive p-th root of unity if and only if there exists a cyclic wild extension L/K of degree p such that $s_{\mathrm{Gal}(L/K)}(\sigma) \in p\mathbb{Z}$, where σ is a generator of $\mathrm{Gal}(L/K)$*. In particular, the (necessarily open normal) subgroup $G_{K(\zeta_p)}$ of G_K (possibly equal to G_K) is recovered from the R-filtered profinite group $\mathfrak{G}_K$.*

Moreover, the absolute ramification index $v_K(p)$ is also recovered from $\mathfrak{G}_K$.

Proof.

Note that, if K contains a primitive p-th root of unity, then $p-1$ divides $v_K(p)$. So, the equivalence follows immediately from Proposition 2.3. $G_{K(\zeta_p)}$ is characterized as the maximal open normal subgroup of G_K satisfying the latter condition of the equivalence.

Next, let us consider the absolute ramification index. Note that the subgroup $G_{K(\zeta_p)} \subset G_K$ is already recovered and hence the R-filtered profinite group $\mathfrak{G}_{K(\zeta_p)} = \{G^v_{K(\zeta_p)}\}_{v \in \mathbb{R}_{\geq -1}}$ is also recovered. Set:

$$s := \max\{s_{G_{K(\zeta_p)}/H}(\sigma_H) \mid H \text{ is an open normal subgroup of } G_{K(\zeta_p)} \text{ of index } p\},$$

where σ_H is a generator of $G_{K(\zeta_p)}/H$. By Proposition 2.3, we have:

$$v_{K(\zeta_p)}(p) = \frac{s(p-1)}{p}.$$

Since $v_{K(\zeta_p)}(p) = (I_K : I_{K(\zeta_p)})v_K(p)$, this completes the proof of Proposition 2.4. $\square$

Contrary to Proposition 2.3, we have the following for GPLF:

Proposition 2.5
Let K be a GPLF. Set $p := p_K = p_{k_K}$. Then, for any $N \in \mathbb{Z}_{>0}$, there exists a cyclic extension L of K of degree p such that $s_{\mathrm{Gal}(L/K)}(\sigma) > N$, where σ is a generator of $\mathrm{Gal}(L/K)$.

Proof.
Let us take an integer $M \in \mathbb{Z} \setminus p\mathbb{Z}$ such that $M > N$ and an element $a \in K$ such that $v_K(a) = -M$. Note that $a \notin \wp(K)$ (cf. Lemma 1.1). Clearly, we have the following:

$$-M = \max\{v_K(a+b) \mid b \in \wp(K)\}.$$

Let L be an extension of K generated by x satisfying $\wp(x) = a$. By Lemma 1.1 (ii), it follows that $s_{\mathrm{Gal}(L/K)}(\sigma) = M\ (> N)$, where σ is a generator of $\mathrm{Gal}(L/K)$. $\square$

Proposition 2.6
Let K_1 be a GMLF and K_2 a GPLF. For $i = 1, 2$, set $p_i := p_{k_{K_i}}\ (> 0)$, $G_i := G_{K_i}$ and $\mathfrak{G}_i := \mathfrak{G}_{K_i}$. Then there exist no open homomorphisms of R-filtered profinite groups from $\mathfrak{G}_1$ (resp. $\mathfrak{G}_2$) to $\mathfrak{G}_2$ (resp. $\mathfrak{G}_1$).

Proof.
Suppose that there exists an open homomorphism of R-filtered profinite groups $\alpha : \mathfrak{G}_1 \to \mathfrak{G}_2$ (resp. $\beta : \mathfrak{G}_2 \to \mathfrak{G}_1$). Since P_{K_i} is a (non-trivial) pro-p_i group, we may assume that $p := p_1 = p_2$. By replacing G_2 by $\alpha(G_1)$ (resp. G_1 by $\beta(G_2)$), we may assume that α (resp. β) is a surjection.

First, we consider α. By Proposition 2.5, there exists an open normal subgroup $H \subset G_2$ of index p such that $s_{G_2/H}(\sigma) > \dfrac{pv_{K_1}(p)}{p-1}$, where σ is a generator of G_2/H. Then $\alpha^{-1}(H)$ determines a cyclic extension of K_1 of degree p which violates Proposition 2.3. So, such α cannot exist.

Next, we consider β. There exists an open normal subgroup H_1 of G_1 of index n (n divides $p-1$) such that the extension L of K_1 corresponding to H_1 contains a primitive p-th root of unity. So, by replacing G_1 and G_2 by H_1 and $\beta^{-1}(H_1)$ if necessary, we may assume that K_1 contains a primitive p-th root of unity. Then, by Proposition 2.4, there exists an open normal subgroup $H \subset G_1$ of index p such that $s_{G_1/H}(\sigma) \in p\mathbb{Z}_{>0}$, where σ is a generator of G_1/H. $\beta^{-1}(H)$ determines a cyclic extension of K_2 of degree p, and we have $s_{G_2/\beta^{-1}(H)}(\tau) \in p\mathbb{Z}_{>0}$, where τ is a generator of $G_2/\beta^{-1}(H)$. However, since K_2 is of characteristic p, in light of Lemma 1.1 (ii), we obtain a contradiction (note that the residue field of K_2 is perfect). $\square$

Proposition 2.7
Let K be a GMLF and set $p := p_{k_K} > 0$. Suppose that K contains a primitive p^n-th root of unity (for $n \in \mathbb{Z}_{>0}$). Then K contains a primitive p^{n+1}-th root of unity if and only if there exists a surjection $\varphi : G_K \twoheadrightarrow H$ satisfying the following conditions:

(i) $H \simeq \mathbb{Z}/p^{n+1}\mathbb{Z}$;

(ii) *For $0 \leq i \leq n+1$, denote the (uniquely determined) subgroup of H of index p^i by H_i. Then, for $0 \leq i \leq n$,*

$$s_{H_i/H_{i+1}}(\sigma_i) = \frac{p^{i+1}v_K(p)}{p-1},$$

where σ_i is a generator of H_i/H_{i+1}.

(Note that H_i/H_{i+1} has a filtration determined by $\mathfrak{G}_K$.)

Proof.

First, suppose that K contains a primitive p^{n+1}-th root of unity. Let π be a uniformizer of K and set $L := K[X]/(X^{p^{n+1}} - \pi)$. Then L/K is a finite Galois extension, and clearly, the surjection φ determined by L/K satisfies the above conditions (i) and (ii) (cf. Lemma 1.1 (i)).

Next, suppose that K does not contain a primitive p^{n+1}-th root of unity, and that we have a surjection $\varphi : G_K \twoheadrightarrow H$ satisfying the above conditions (i) and (ii). Let L be the finite Galois extension of K corresponding to φ, and K_i the intermediate field of L/K corresponding to H_i for $0 \leq i \leq n+1$. Since $K\,(= K_0)$ (hence also K_1) contains a primitive p^n-th root of unity, by Kummer theory, there exists an element $\alpha \in K_1^\times \setminus (K_1^\times)^p$ such that $L = K_1[X]/(X^{p^n} - \alpha)$. Then $K_2 = K_1[X]/(X^p - \alpha)$. By the condition (ii), we have

$$s_{H_1/H_2}(\sigma_1) = \frac{p^2 v_K(p)}{p-1} = \frac{p v_{K_1}(p)}{p-1}.$$

By Lemma 1.1 (i), we may assume that α is a uniformizer of K_1. Then, by [Se, I, §6, Proposition 18], α is a root of an Eisenstein polynomial $f(X) \in \mathcal{O}_K[X]$ of degree p. Let $\pi_1 := \alpha$, $\pi_2, \cdots, \pi_p \in K^{\mathrm{sep}}$ be the (distinct) roots of $f(X)$. We have $K_1 = K[X]/(f(X))$, and $\pi_1, \cdots, \pi_p \in K_1$ since K_1/K is a Galois extension. Note that $L = K[X]/(f(X^{p^n}))$ and that the roots of $f(X^{p^n})$ consist of

$$\{\zeta_{p^n}^i \pi_j^{\frac{1}{p^n}} \mid 0 \leq i \leq p^n - 1,\ 1 \leq j \leq p\} (\subset L),$$

where $\pi_j^{\frac{1}{p^n}}$ is a p^n-th root of π_j for $1 \leq j \leq p$. Let σ be a generator of H whose image in $H_0/H_1\,(= H/H_1)$ is σ_1. Set:

$$u := \frac{\sigma(\pi_1)}{\pi_1} = \frac{\sigma_1(\pi_1)}{\pi_1}.$$

Since π_1 generates $\mathcal{O}_{K_1}$ as an $\mathcal{O}_K$-algebra, by [Se, IV, §1, Lemma 1] and the condition (ii),

$$u \in U_{K_1}^{\frac{pv_K(p)}{p-1}} \setminus U_{K_1}^{\frac{pv_K(p)}{p-1}+1}.$$

Clearly, we have $u \in (L^\times)^{p^n}$. We claim that $u \in (K_1^\times)^{p^n}$. Indeed, suppose that $u \in (K_1^\times)^{p^m} \setminus (K_1^\times)^{p^{m+1}}$ for some $0 \leq m \leq n-1$. Let $u' \in K_1^\times \setminus (K_1^\times)^p$ be an element such that $(u')^{p^m} = u$. Then $K_1((u')^{\frac{1}{p}})$ is a finite Galois extension of K_1 of degree p and contained in L. Therefore, $K_2 = K_1((u')^{\frac{1}{p}})$. However, since $v_{K_1}(u') = 0$, in light of Lemma 1.1 (i), this contradicts the condition (ii) (for $i = 1$). So, we obtain $u \in (K_1^\times)^{p^n}$.

On the other hand, for each $n \in \mathbb{Z}_{>0}$, the norm $N_{K_1/K}$ induces a homomorphism $N_n : U^n_{K_1}/U^{n+1}_{K_1} \to U^n_K/U^{n+1}_K$. By [Se, V, §3, Corollary 1], N_n is injective for $n \neq \dfrac{pv_K(p)}{p-1}$, and the kernel of $N_{\frac{pv_K(p)}{p-1}}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Since $N_{K_1/K}(\zeta_p) = 1$, the kernel of $N_{\frac{pv_K(p)}{p-1}}$ coincides with the subgroup of $U_{K_1}^{\frac{pv_K(p)}{p-1}}/U_{K_1}^{\frac{pv_K(p)}{p-1}+1}$ generated by ζ_p. (Note that $\zeta_p \in U_{K_1}^{\frac{v_{K_1}(p)}{p-1}} \setminus U_{K_1}^{\frac{v_{K_1}(p)}{p-1}+1}$ and $v_{K_1}(p) = pv_K(p)$.) Since it is immediate that $N_{K_1/K}(u) = 1$, there exists an element $w \in U_{K_1}^{\frac{pv_K(p)}{p-1}+1}$ such that $u = \zeta_p w$. Therefore, $N_{K_1/K}(w) = 1$. Since N_n is injective for $n > \dfrac{pv_K(p)}{p-1}$, by induction, we have that $w \in U^n_{K_1}$ for any $n > \dfrac{pv_K(p)}{p-1}$, hence $w = 1$. So, we obtain $u = \zeta_p \in (K_1^\times)^{p^n}$, and hence $K_1 = K(\zeta_{p^{n+1}})$. However, since $v_K(\zeta_{p^n}) = 0$, in light of Lemma 1.1 (i), this contradicts the condition (ii) (for $i = 0$). This completes the proof of Proposition 2.7. $\square$

Corollary 2.8
Let K be a GMLF and set $p := p_{k_K} > 0$. Then, for any $n \in \mathbb{Z}_{>0}$, we may determine whether or not K contains a primitive p^n-th root of unity from $\mathfrak{G}_K$. In particular, the (necessarily open normal) subgroup $G_{K(\zeta_{p^n})}$ of G_K (possibly equal to G_K) and the closed normal subgroup H_0 of G_K corresponding to $K(\zeta_{p^\infty})/K$ (where $K(\zeta_{p^\infty}) = \bigcup_{n\in\mathbb{Z}_{>0}} K(\zeta_{p^n})$) are recovered from $\mathfrak{G}_K$.

Proof.
Immediate from Propositions 2.4 and 2.7. $\square$

By the following proposition, we may recover the p_{k_K}-adic cyclotomic characters of GMLF's group-theoretically:

Proposition 2.9
Let K be a GMLF and $n \in \mathbb{Z}_{>0}$ a positive integer. Set $p := p_{k_K} > 0$. Then the modulo p^n cyclotomic character $\chi_{p,n} : G_K \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ is recovered from $\mathfrak{G}_K$. In particular, the p-adic cyclotomic character $\chi_p : G_K \to \mathbb{Z}_p^\times$ is recovered from $\mathfrak{G}_K$.

Proof.
To show this proposition, it suffices to recover the isomorphism class of μ_{p^n} as a G_K-module, where μ_{p^n} is the group of p^n-th roots of unity. Let $\mathfrak{G}_L$ be an R-filtered open normal subgroup of $\mathfrak{G}_K$ such that corresponding finite Galois extension L of K contains a primitive p^n-th root of unity. (Note that this is clearly a group-theoretic condition by Corollary 2.8.) Then we have a canonical isomorphism of G_K-modules

$$\varphi : L^\times/(L^\times)^{p^n} \xrightarrow{\sim} H^1(G_L, \mu_{p^n}).$$

Let us take an isomorphism of G_L-modules $\theta : \mu_{p^n} \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ and write

$$\psi : H^1(G_L, \mu_{p^n}) \xrightarrow{\sim} H^1(G_L, \mathbb{Z}/p^n\mathbb{Z}) = \mathrm{Hom}(G_L^{\mathrm{ab}}/p^nG_L^{\mathrm{ab}}, \mathbb{Z}/p^n\mathbb{Z})$$

for the isomorphism of G_L-modules induced by θ. On the other hand, we have the following exact sequence of G_K-modules:

$$1 \longrightarrow \mathcal{O}_L^\times/(\mathcal{O}_L^\times)^{p^n} \longrightarrow L^\times/(L^\times)^{p^n} \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0, \tag{2.1}$$

where the surjection is induced by v_L. Set $W_L := \mathcal{O}_L^\times/(\mathcal{O}_L^\times)^{p^n} \subset L^\times/(L^\times)^{p^n}$, $W_{L,\mu_{p^n}} := \varphi(W_L)$, and $W := \psi(W_{L,\mu_{p^n}})$. Note that the subgroup $W \subset \mathrm{Hom}(G_L^{\mathrm{ab}}/p^n G_L^{\mathrm{ab}}, \mathbb{Z}/p^n\mathbb{Z})$ does not depend on the choice of θ.

We claim that the subgroup $W \subset \mathrm{Hom}(G_L^{\mathrm{ab}}/p^n G_L^{\mathrm{ab}}, \mathbb{Z}/p^n\mathbb{Z})$ is recovered from $\mathfrak{G}_L$ (hence from $\mathfrak{G}_K$). Let f be an element of $\mathrm{Hom}(G_L^{\mathrm{ab}}/p^n G_L^{\mathrm{ab}}, \mathbb{Z}/p^n\mathbb{Z})$, and $0 \le m \le n$ the integer such that the image of $f : G_L^{\mathrm{ab}}/p^n G_L^{\mathrm{ab}} \to \mathbb{Z}/p^n\mathbb{Z}$ is $p^m\mathbb{Z}/p^n\mathbb{Z}$. Note that $\mathrm{Im}\, f \simeq p^m\mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Z}/p^{n-m}\mathbb{Z}$. Set $H := \mathrm{Im}\, f$, and for $0 \le i \le n-m$, denote the uniquely determined subgroup of H of index p^i by H_i. Then f belongs to W if and only if, for any $0 \le i < n-m$, the following condition holds:

$$s_{H_i/H_{i+1}}(\sigma_i) < \frac{p^{i+1} v_L(p)}{p-1},$$

where σ_i is a generator of H_i/H_{i+1} ($\simeq \mathbb{Z}/p\mathbb{Z}$). Indeed, the necessity is immediate from Lemma 1.1 (i). Let us show the sufficiency. Let $\tilde{L}$ be the finite extension of L of degree p^{n-m} associated to f, and L_i the intermediate field of $\tilde{L}/L$ associated to H_i (for $0 \le i \le n-m$). By Kummer theory, there exists an element $a \in L^\times$ such that $\tilde{L} = L(a^{\frac{1}{p^{n-m}}})$. We choose a such that $v_L(a)$ is minimal under the condition $v_L(a) \ge 0$. Then clearly we have $v_L(a) \in \{0, 1, p, \cdots, p^{n-m-1}\}$. However, if $v_L(a) \ne 0$, the condition on $s_{H_i/H_{i+1}}(\sigma_i)$ does not hold for i satisfying $\log_p v_L(a) \le i < n-m$. Therefore, we have that $v_L(a) = 0$ and hence that $f \in W$.

On the other hand, since the exact sequence (2.1) splits as an exact sequence of abelian groups, we have the following exact sequence of G_K-modules:

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}) \longrightarrow \mathrm{Hom}(L^\times/(L^\times)^{p^n}, \mu_{p^n}) \longrightarrow \mathrm{Hom}(W_L, \mu_{p^n}) \longrightarrow 0.$$

Moreover, by the Pontryagin duality, we have a canonical isomorphism of G_K-modules $\nu : \mathrm{Hom}(L^\times/(L^\times)^{p^n}, \mu_{p^n}) \xrightarrow{\sim} G_L^{\mathrm{ab}}/p^n G_L^{\mathrm{ab}}$. Set:

$$M_n := \{\sigma \in G_L^{\mathrm{ab}}/p^n G_L^{\mathrm{ab}} \mid f(\sigma) = 0 \text{ for all } f \in W\} \subset G_L^{\mathrm{ab}}/p^n G_L^{\mathrm{ab}}.$$

(Note that, since W is recovered from $\mathfrak{G}_K$, M_n is also recovered from $\mathfrak{G}_K$.) Then we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}) \arrow[r] \arrow[d] & \mathrm{Hom}(L^\times/(L^\times)^{p^n}, \mu_{p^n}) \arrow[r] \arrow[d, "\nu"] & \mathrm{Hom}(W_L, \mu_{p^n}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_n \arrow[r] & G_L^{\mathrm{ab}}/p^n G_L^{\mathrm{ab}} \arrow[r] & \mathrm{Hom}(W_L, \mu_{p^n}) \arrow[r] & 0,
\end{tikzcd}$$

where the horizontal sequences are exact, the vertical arrows are the isomorphisms of G_K-modules induced by ν. Note that since the upper sequence is an exact sequence of G_K-modules, the lower sequence is also an exact sequence of G_K-modules. In particular, we obtain an isomorphism of G_K-modules $\mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}) (= \mu_{p^n}) \xrightarrow{\sim} M_n$. Since M_n is group-theoretic, this completes the proof of Proposition 2.9. $\square$

Remark 2.10
In the case where K is an MLF, by using the Tate duality, Mochizuki recovered the p_{k_K}-adic cyclotomic character group-theoretically ([Mo2, Proposition 1.1]). However, we cannot apply this method to the case where K is a GMLF.

By Remark 1.2, for a GMLF or GPLF K, we have the following exact sequence:

$$1 \longrightarrow I_K \longrightarrow G_K \longrightarrow G_{k_K} \longrightarrow 1.$$

This exact sequence determines a homomorphism $\rho : G_{k_K} \to \mathrm{Out} I_K$.

Proposition 2.11
The above homomorphism $\rho : G_{k_K} \to \mathrm{Out} I_K$ is injective.

Proof.
Set $p := p_{k_K}$ (> 0). Denote the maximal unramified extension of K by K^{ur} $(\subset K^{\mathrm{sep}})$.

Consider the case where K is a GMLF. First, suppose that K contains a primitive p-th root of unity. In this case, by Kummer theory, we have an isomorphism of G_K-module $H^1(I_K, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} (K^{\mathrm{ur}})^\times/((K^{\mathrm{ur}})^\times)^p$. ρ determines an action of G_{k_K} on $H^1(I_K, \mathbb{Z}/p\mathbb{Z})$ and the above isomorphism is G_{k_K}-equivariant (note that $G_{k_K} \simeq \mathrm{Gal}(K^{\mathrm{ur}}/K)$). On the other hand, in a similar way to the proof of Proposition 1.13, we obtain a G_{k_K}-equivariant surjection

$$H^1(I_K, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} (K^{\mathrm{ur}})^\times/((K^{\mathrm{ur}})^\times)^p \twoheadrightarrow \overline{k_K}.$$

Since it is clear that G_{k_K} acts faithfully on $\overline{k_K}$, this shows the injectivity of ρ in the case where K contains a primitive p-th root of unity.

Next, suppose that K does not necessarily contain a primitive p-th root of unity. Set $L := K(\zeta_p)$. Since k_K is of positive characteristic, G_{k_K} is torsion-free (see, e.g., [Nag, VI, Exercise, §2, 1]). Therefore, to prove the injectivity of ρ, it suffices to show the injectivity of the restriction of ρ to an open subgroup of G_{k_K}. So, we may assume that $k_K = k_L =: k$. Then we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & I_L \arrow[r] \arrow[d, hook] & G_L \arrow[r] \arrow[d, hook] & G_k \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & I_K \arrow[r] & G_K \arrow[r] & G_k \arrow[r] & 1,
\end{tikzcd}$$

where the horizontal sequences are exact and the vertical arrows are the natural inclusions. This exact sequence determines a homomorphism $\rho_L : G_k \to \mathrm{Out} I_L$. By the first case, ρ_L is injective. Moreover, by Proposition 2.1 (iii), we can naturally identify I_K, I_L with $\mathrm{Inn} I_K$, $\mathrm{Inn} I_L$, respectively. Set:

$$\mathrm{Aut}_{I_L} I_K := \{\alpha \in \mathrm{Aut} I_K \mid \alpha \text{ preserves } I_L\}.$$

(In fact, by the choice of L, I_L is a characteristic subgroup of I_K, hence $\mathrm{Aut}_{I_L} I_K = \mathrm{Aut} I_K$.) By considering a section $s : G_k \hookrightarrow G_L$ (cf. Remark 1.2), we obtain a homomorphism $\varphi : G_k \to \mathrm{Aut}_{I_L} I_K / \mathrm{Inn} I_L$ (note that φ does not depend on the choice of s). So, we obtain the following commutative diagram:

$$\begin{tikzcd} G_k \arrow[d, "\varphi"] \arrow[dr, "\rho"] & \\ \mathrm{Aut}_{I_L} I_K/\mathrm{Inn} I_L \arrow[r] & \mathrm{Out} I_K, \end{tikzcd}$$

where the horizontal arrow is the natural homomorphism induced by the natural injection $\mathrm{Aut}_{I_L} I_K \hookrightarrow \mathrm{Aut} I_K$.

On the other hand, we have the following commutative diagram:

$$\begin{tikzcd} G_k \arrow[d, "\varphi"] \arrow[dr, "\rho_L"] & \\ \mathrm{Aut}_{I_L} I_K/\mathrm{Inn} I_L \arrow[r] & \mathrm{Out} I_L, \end{tikzcd}$$

where the horizontal arrow is the natural homomorphism induced by the natural homomorphism $\mathrm{Aut}_{I_L} I_K \to \mathrm{Aut} I_L$. Since ρ_L is injective, φ is also injective. Let σ be an element of $\mathrm{Ker}\,\rho$. Then $\varphi(\sigma) \in \mathrm{Inn} I_K/\mathrm{Inn} I_L \simeq I_K/I_L$. Since I_L is an open normal subgroup of I_K, σ is a torsion element of G_k (note that φ is injective). Since k is of positive characteristic (hence G_k is torsion-free), we obtain $\sigma = 1$. This completes the proof of Proposition 2.11 in the case where K is a GMLF.

Finally, suppose that K is a GPLF. By Artin-Schreier theory, we have an isomorphism of G_K-module $H^1(I_K, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} K^{\mathrm{ur}}/\wp(K^{\mathrm{ur}})$. ρ determines an action of G_{k_K} on $H^1(I_K, \mathbb{Z}/p\mathbb{Z})$ and the above isomorphism is G_{k_K}-equivariant. Let $\mathcal{O}_{K^{\mathrm{ur}}}$ be the ring of integers of K^{ur}, $\mathfrak{M}_{K^{\mathrm{ur}}}$ the maximal ideal of $\mathcal{O}_{K^{\mathrm{ur}}}$ and $v_{K^{\mathrm{ur}}}$ the valuation of K^{ur} such that $v_{K^{\mathrm{ur}}}((K^{\mathrm{ur}})^\times) = \mathbb{Z}$ (note that K^{ur} is a henselian discrete valuation field with residue field $\overline{k_K}$). Since the residue field of K^{ur} is algebraically closed, by Hensel's lemma, we have $\mathcal{O}_{K^{\mathrm{ur}}} \subset \wp(K^{\mathrm{ur}})$. So, we obtain a natural surjection $K^{\mathrm{ur}}/\mathcal{O}_{K^{\mathrm{ur}}} \twoheadrightarrow K^{\mathrm{ur}}/\wp(K^{\mathrm{ur}})$ (which is clearly G_{k_K}-equivariant). This surjection induces an injection $\mathfrak{M}_{K^{\mathrm{ur}}}^{-1}/\mathcal{O}_{K^{\mathrm{ur}}} \hookrightarrow K^{\mathrm{ur}}/\wp(K^{\mathrm{ur}})$. Indeed, let us take an element $a \in \mathfrak{M}_{K^{\mathrm{ur}}}^{-1} \setminus \mathcal{O}_{K^{\mathrm{ur}}}$, and suppose that there exists an element $b \in K^{\mathrm{ur}}$ such that $\wp(b) = a$. Clearly, we have $v_{K^{\mathrm{ur}}}(b) < 0$ and hence $\wp(b) \in p\mathbb{Z}_{<0}$. This is a contradiction. On the other hand, since $\mathfrak{M}_{K^{\mathrm{ur}}}^{-1}/\mathcal{O}_{K^{\mathrm{ur}}} \simeq \overline{k_K}$ and G_{k_K} acts faithfully on $\overline{k_K}$, it follows that ρ is injective. This completes the proof of Proposition 2.11 in the case where K is a GPLF, hence of Proposition 2.11. □

Corollary 2.12
Let K be a GMLF or GPLF. Then, for any open subgroup J of I_K,

$$Z_{G_K}(J) = \{1\}.$$

Proof.

Immediate from Propositions 2.1 (iii), 2.2 (ii), and 2.11, and the fact that the absolute Galois group of a field of positive characteristic is torsion-free (cf. the proof of Proposition 2.11). □

Lemma 2.13
Let G be a profinite group and N a closed subgroup of G which is a non-trivial pro-p group (for some $p \in \mathfrak{Primes}$). Then there exists an open subgroup H of G such that $N \subset H$ and the image of N in $H(p)^{\mathrm{ab}}$ is non-trivial.

Proof.

Since N is a non-trivial pro-p group, $N(p)^{\mathrm{ab}}$ $(= N^{\mathrm{ab}})$ is also non-trivial. On the other hand, since $N = \varprojlim_{N \subset H} H = \bigcap_{N \subset H} H$ (where H runs through the set of open subgroup of G containing N), we obtain

$$N(p)^{\mathrm{ab}} = \varprojlim_{N \subset H} H(p)^{\mathrm{ab}}.$$

□

Proposition 2.14
For $i = 1, 2$, let K_i be a GMLF, and set $\mathfrak{G}_i := \mathfrak{G}_{K_i}$. Let $\alpha : \mathfrak{G}_1 \to \mathfrak{G}_2$ be an open quasi-injective homomorphism of R-filtered profinite groups. Then α is an injection.

Proof.

Set $G_i := G_{K_i}$, $I_i := I_{K_i}$ and $P_i := P_{K_i}$ for $i = 1, 2$. Since α is open and P_i is a non-trivial pro-$p_{k_{K_i}}$ group for $i = 1, 2$ (cf. Proposition 2.1 (iv)), it follows that $p_{k_{K_1}} = p_{k_{K_2}}$. Set $p := p_{k_{K_1}} = p_{k_{K_2}}$. By replacing G_2 by $\alpha(G_1)$, we may assume that α is surjective. Let $N \subset G_1$ be the kernel of α. To prove this theorem, it suffices to show that $I_1 \cap N = \{1\}$. Indeed, if $I_1 \cap N = \{1\}$, α induces an isomorphism $I_1 \xrightarrow{\sim} I_2$, and we obtain the following commutative diagram:

$$\begin{tikzcd} G_{k_{K_1}} \arrow[r] \arrow[d, "\rho_1"] & G_{k_{K_2}} \arrow[d, "\rho_2"] \\ \mathrm{Out} I_1 \arrow[r, "\sim"] & \mathrm{Out} I_2, \end{tikzcd}$$

where the horizontal arrows are induced by α and the vertical arrows are the homomorphisms discussed in the argument preceding Proposition 2.11. Since ρ_1 is injective by Proposition 2.11, the image of N in $G_{k_{K_1}}$ is trivial.

Therefore, for $i = 1, 2$, by replacing K_i by the completion of the maximal unramified extension of K_i if necessary, we may assume that $G_i = I_i$. Then α induces the following commutative diagram (cf. Remark 1.2):

$$\begin{tikzcd} G_1 \arrow[r, two heads] \arrow[d, "\alpha"] & G_1/P_1(\simeq \hat{\mathbb{Z}}^{p'}(1)) \arrow[d] \\ G_2 \arrow[r, two heads] & G_2/P_2(\simeq \hat{\mathbb{Z}}^{p'}(1)). \end{tikzcd}$$

Since G_1/P_1 and G_2/P_2 are isomorphic as abstract profinite groups and topologically finitely generated (hence hopfian), it holds that $G_1/P_1 \twoheadrightarrow G_2/P_2$ is an isomorphism. Therefore, we have $N \subset P_1$. Suppose that N is non-trivial. By Lemma 2.13, there exists an open subgroup H of G_1 such that $N \subset H$ and the image of N in $H(p)^{\mathrm{ab}}$ is non-trivial. By replacing G_1 and G_2 by H and $\alpha(H)$, we may assume that the image of N in $G_1(p)^{\mathrm{ab}}$ is non-trivial. Let $\overline{N}$ be the (non-trivial) image of N in $G_1(p)^{\mathrm{ab}}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, two heads] & G_1 \arrow[r, "\alpha"] \arrow[d, two heads] & G_2 \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \overline{N} \arrow[r] & G_1(p)^{\mathrm{ab}} \arrow[r] & G_2(p)^{\mathrm{ab}} \arrow[r] & 1,
\end{tikzcd}$$

where the horizontal sequences are exact and the vertical arrows are the natural surjections. By Proposition 2.1 (iii), G_2 is a projective group. Therefore, the upper exact sequence splits, and hence also the lower exact sequence splits. In particular, there exists a retract $s : G_1(p)^{\mathrm{ab}} \to \overline{N}$. On the other hand, since $G_1(p)^{\mathrm{ab}}$ is a (non-trivial) torsion-free pro-p abelian group by Proposition 2.1 (ii), $\overline{N}$ is a non-trivial torsion-free pro-p abelian group. Therefore, $\overline{N}$ is isomorphic to $\prod_{\lambda\in\Lambda} \mathbb{Z}_p$ for some non-empty set Λ. So, there exists a surjection $\pi : \overline{N} \twoheadrightarrow \mathbb{Z}_p$. By composing π with s, we obtain a surjection $\pi \circ s : G_1(p)^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}_p$ whose restriction to $\overline{N}$ is also a surjection. Let $H_1 \subset G_1$ be the kernel of the composite of the natural surjection $G_1 \twoheadrightarrow G_1(p)^{\mathrm{ab}}$ and $\pi \circ s$. Clearly, H_1 is a closed normal subgroup of G_1 of APF-type, and hence determines an arithmetically profinite extension of K_1 (cf. [FW1, 1.3 (b)]). Let $\mathfrak{H}_1 \subset \mathfrak{G}_1$ be the resulting R-filtered profinite group and $\iota : \mathfrak{H}_1 \to \mathfrak{G}_1$ the natural inclusion. By Proposition 1.15, $\mathfrak{H}_1$ is of R-GPLF-type. Moreover, the composite $\alpha \circ \iota : \mathfrak{H}_1 \to \mathfrak{G}_2$ is a surjective homomorphism of R-filtered profinite groups (note that α is quasi-injective). Since $\mathfrak{G}_2$ is of R-GMLF-type, this contradicts Proposition 2.6. This completes the proof of Proposition 2.14. □

Lemma 2.15
Let G_i be a profinite group for $i = 1, 2$, and $\alpha : G_1 \to G_2$ a homomorphism of profinite groups. Suppose that G_1 is a prosolvable group. Moreover, suppose that α satisfies the following condition:

> *For any open subgroup $U_2 \subset G_2$, set $U_1 := \alpha^{-1}(U_2)$. Let $\alpha_U : U_1 \to U_2$ be the homomorphism induced by α. Then α_U induces an injection $\alpha_U^{\mathrm{ab}} : U_1^{\mathrm{ab}} \hookrightarrow U_2^{\mathrm{ab}}$.*

Then α is an injection.

Proof.
Let $N \subset G_1$ be the kernel of α. Then we have $N = \varprojlim_{U_2 \subset G_2} \alpha^{-1}(U_2)$, where U_2 runs through the set of open subgroups of G_2. By assumption, α induces an injection $\alpha^{-1}(U_2)^{\mathrm{ab}} \hookrightarrow U_2^{\mathrm{ab}}$ for any open subgroup U_2 of G_2. Therefore,

$$N^{\mathrm{ab}} = \varprojlim_{U_2 \subset G_2} (\alpha^{-1}(U_2)^{\mathrm{ab}}) \hookrightarrow \varprojlim_{U_2 \subset G_2} U_2^{\mathrm{ab}} = \{1\}.$$

(Note that, for any inverse system $\{H_\lambda\}_{\lambda\in\Lambda}$ of profinite groups, the natural surjection $\left(\varprojlim_{\lambda\in\Lambda} H_\lambda\right)^{\mathrm{ab}} \twoheadrightarrow \varprojlim_{\lambda\in\Lambda} H_\lambda^{\mathrm{ab}}$ is an isomorphism.) Therefore, $N^{\mathrm{ab}} = \{1\}$. On the other hand, since G_1 is prosolvable, N is also prosolvable. This implies that $N = \{1\}$, as desired. □

Let K_i be a GMLF or GPLF for $i = 1, 2$, and let us consider an open homomorphism of R-filtered profinite groups $\mathfrak{G}_{K_1} \to \mathfrak{G}_{K_2}$. If we suppose that K_1 is an MLF or PLF, we may show the injectivity of open homomorphisms of R-filtered profinite groups without assuming the quasi-injectivity (cf. Proposition 2.14):

Proposition 2.16
For $i = 1, 2$, let K_i be a GMLF or GPLF, and set $\mathfrak{G}_i := \mathfrak{G}_{K_i}$. Suppose that K_1 is an MLF (resp. a PLF). Suppose, moreover, that there exists an open homomorphism $\alpha : \mathfrak{G}_1 \to \mathfrak{G}_2$ of R-filtered profinite groups. Then K_2 is an MLF (resp. a PLF) and α is an injection.

Proof.
Set $G_i := G_{K_i}$, $I_i := I_{K_i}$ and $P_i := P_{K_i}$ for $i = 1, 2$. Since P_i is a non-trivial pro-$p_{k_{K_i}}$ group, it follows that $p_{k_{K_1}} = p_{k_{K_2}} =: p$. By replacing G_2 by $\alpha(G_1)$, we may assume that α is surjective.

Note that, by Proposition 2.6, if K_1 is an MLF (resp. a PLF), then K_2 is automatically a GMLF (resp. a GPLF). First, suppose that K_1 is an MLF. Then G_1 is topologically finitely generated (cf. Proposition 1.13). Since α is surjective, G_2 is also topologically finitely generated. By Proposition 1.13, this shows that K_2 is an MLF. Next, suppose that K_1 is a PLF. For $i = 1, 2$, set $Q_i := G_i^{\mathrm{ab}}/pG_i^{\mathrm{ab}}$, and let $\mathfrak{Q}_i = \{Q_i^v\}_{v\in\mathbb{R}_{\geq -1}}$ be the R-filtered profinite group with underlying profinite group Q_i determined by the natural surjection $G_i \twoheadrightarrow Q_i$. By Artin-Schreier theory, we have an isomorphism $K_i/\wp(K_i) \simeq H^1(Q_i, \mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z})$ for $i = 1, 2$. Moreover, by Lemma 1.1 and the Hasse-Arf theorem, we have the following isomorphisms for $i = 1, 2$:

$$\mathcal{O}_{K_i}/(\mathcal{O}_{K_i} \cap \wp(K_i)) \simeq \{f \in \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z}) \mid Q_i^1 = Q_i^0 \subset \mathrm{Ker}\, f\} =: R_i^1 \subset \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z}),$$
$$\mathfrak{M}_{K_i}^{-1}/(\mathfrak{M}_{K_i}^{-1} \cap \wp(K_i)) \simeq \{f \in \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z}) \mid Q_i^2 \subset \mathrm{Ker}\, f\} =: R_i^2 \subset \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z}).$$

By definition, we have the following commutative diagram for $i = 1, 2$:

$$\begin{tikzcd}
0 \arrow[r] & R_i^1 \arrow[r] \arrow[d, hook] & \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z})/R_i^1 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & R_i^2 \arrow[r] & \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z}) \arrow[r] & \mathrm{Hom}(Q_i, \mathbb{Z}/p\mathbb{Z})/R_i^2 \arrow[r] & 0,
\end{tikzcd}$$

where the horizontal sequences are exact, and the vertical arrows are the natural homomorphisms. By the Pontryagin duality, we have the following commutative diagram for $i = 1, 2$:

$$\begin{tikzcd}
0 \arrow[r] & Q_i^1 \arrow[r] & Q_i \arrow[r] & \mathrm{Hom}(R_i^1, \mathbb{Z}/p\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & Q_i^2 \arrow[r] \arrow[u, hook] & Q_i \arrow[r] \arrow[u, equal] & \mathrm{Hom}(R_i^2, \mathbb{Z}/p\mathbb{Z}) \arrow[r] \arrow[u, two heads] & 0,
\end{tikzcd}$$

where the horizontal sequences are exact. Therefore, for $i = 1, 2$, we obtain the following isomorphism:

$$Q_i^1/Q_i^2 \simeq \mathrm{Hom}(R_i^2/R_i^1, \mathbb{Z}/p\mathbb{Z}).$$

Moreover, we have $R_i^2/R_i^1 \simeq \mathfrak{M}_{K_i}^{-1}/\mathcal{O}_{K_i} \simeq k_{K_i}$. On the other hand, α induces a surjection $Q_1^1/Q_1^2 \twoheadrightarrow Q_2^1/Q_2^2$. Since K_1 is a PLF, k_{K_1} (hence also Q_1^1/Q_1^2) is finite. Therefore, Q_2^1/Q_2^2 (hence also k_{K_2}) is finite. This shows that K_2 is also a PLF. This completes the proof of the portion of Proposition 2.16 concerning the type of K_2.

We shall show the portion of Proposition 2.16 concerning the injectivity of α. Let N be the kernel of α. Then N is contained in P_1. Indeed, α induces a surjection $G_{k_{K_1}} \twoheadrightarrow G_{k_{K_2}}$. Since k_{K_i} is finite, $G_{k_{K_i}}$ is isomorphic to $\hat{\mathbb{Z}}$, and in particular, hopfian for $i = 1, 2$. Therefore, the above surjection is an injection and hence N is contained in I_1. Moreover, a similar argument to the proof of Proposition 2.14 shows that $N \subset P_1$.

Set $H_i := G_{K_i}^{\mathrm{ab}}$ for $i = 1, 2$. Let $\mathfrak{H}_i = \{H_i^v\}_{v\in\mathbb{R}_{\geq -1}}$ be the R-filtered profinite group with underlying profinite group H_i determined by the natural surjection $G_i \twoheadrightarrow H_i$ for $i = 1, 2$. Let $f_i \in \mathbb{Z}_{>0}$ be the integer such that the cardinality of k_{K_i} is p^{f_i}. α induces a surjection $\alpha^{\mathrm{ab}} : \mathfrak{H}_1 \twoheadrightarrow \mathfrak{H}_2$. By Lemma 2.15, to prove this proposition, it suffices to show that α^{ab} is an injection (note that G_1 is a prosolvable group). For any $v \in \mathbb{R}_{\geq -1}$, α^{ab} induces a surjection $(\alpha^{\mathrm{ab}})^v : H_1^v \twoheadrightarrow H_2^v$.

First, we claim that $f_1 = f_2$. Indeed, $(\alpha^{\mathrm{ab}})^0 : H_1^0 \twoheadrightarrow H_2^0$ induces a surjection $H_1^0/H_1^1 \twoheadrightarrow H_2^0/H_2^1$. Since the image of P_1 in H_1 is H_1^1 by local class field theory, this surjection is an injection (note that N is contained in P_1). On the other hand, again by local class field theory, we have $H_i^0/H_i^1 \simeq \mathcal{O}_{K_i}^\times/U_{K_i}^1 \simeq k_{K_i}^\times$ for $i = 1, 2$. This shows that $f_1 = f_2$.

Next, we claim that $(\alpha^{\mathrm{ab}})^1 : H_1^1 \twoheadrightarrow H_2^1$ is an injection. Indeed, let us take any non-trivial element $g \in H_1^1$. Let $n \in \mathbb{Z}_{>0}$ be an integer such that $g \in H_1^n \setminus H_1^{n+1}$. α induces the following commutative diagram:

$$\begin{tikzcd}
H_1^n \arrow[r, two heads, "(\alpha^{\mathrm{ab}})^n=(\alpha^{\mathrm{ab}})^1|_{H_1^n}"] \arrow[d] & H_2^n \arrow[d] \\
H_1^n/H_1^{n+1} \arrow[r, two heads] & H_2^n/H_2^{n+1}.
\end{tikzcd}$$

By local class field theory, we have $H_i^n/H_i^{n+1} \simeq U_{K_i}^n/U_{K_i}^{n+1} \simeq k_{K_i}$ for $i = 1, 2$. Since $f_1 = f_2$, the lower horizontal arrow is an injective. Therefore, g cannot belong to the kernel of $(\alpha^{\mathrm{ab}})^1$.

Now, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_1^1 \arrow[r] \arrow[d, two heads, "(\alpha^{\mathrm{ab}})^1"', "\wr"] & H_1 \arrow[r] \arrow[d, two heads, "\alpha^{\mathrm{ab}}"'] & H_1/H_1^1 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & H_2^1 \arrow[r] & H_2 \arrow[r] & H_2/H_2^1 \arrow[r] & 0,
\end{tikzcd}$$

where the horizontal sequences are exact and the vertical arrows are induced by α and surjective. By local class field theory, we have $H_i/H_i^1 \simeq \hat{\mathbb{Z}} \oplus k_{K_i}^\times$. Since $f_1 = f_2$, the right vertical arrow is injective. This shows that α^{ab} is injective, as desired. □

Remark 2.17

Let K be an MLF or PLF. If K is an MLF, it is well-known that G_K is hopfian (hence $\mathfrak{G}_K$ is R-filtered hopfian). (Note that G_K is topologically finitely generated.) Proposition 2.16 shows that $\mathfrak{G}_K$ is R-filtered hopfian also in the case where K is a PLF.

Theorem 2.18
For $i = 1, 2$, let K_i be a GMLF or GPLF. Suppose that K_1 is an MLF or PLF. Write $\mathrm{Hom}(K_2, K_1)$ *for the set of homomorphisms from K_2 to K_1,* $\mathrm{Hom}^{\text{R-op}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$ *for the set of open homomorphisms of R-filtered profinite groups from $\mathfrak{G}_{K_1}$ to $\mathfrak{G}_{K_2}$,* $\mathrm{Inn}(\mathfrak{G}_{K_2})$ *for the group of inner automorphisms of $\mathfrak{G}_{K_2}$. Then the natural map*

$$\mathrm{Hom}(K_2, K_1) \to \mathrm{Hom}^{\text{R-op}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})/\mathrm{Inn}(\mathfrak{G}_{K_2})$$

is bijective.

Proof.
Let $\mathrm{Hom}^{\text{R-op-inj}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$ be the set of open injective homomorphisms of R-filtered profinite groups from $\mathfrak{G}_{K_1}$ to $\mathfrak{G}_{K_2}$. Then Proposition 2.16 shows that the natural inclusion

$$\mathrm{Hom}^{\text{R-op-inj}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2}) \hookrightarrow \mathrm{Hom}^{\text{R-op}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$$

is bijective. Moreover, in the case where $\mathrm{Hom}^{\text{R-op-inj}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$ is not empty, K_2 is automatically an MLF (resp. a PLF) if K_1 is an MLF (resp. a PLF). So, by [A1, Theorem A] and [A2, Theorem A], the natural map

$$\mathrm{Hom}(K_2, K_1) \to \mathrm{Hom}^{\text{R-op-inj}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})/\mathrm{Inn}(\mathfrak{G}_{K_2})$$

is bijective. This completes the proof of Theorem 2.18. $\square$

Remark 2.19
As we see in the proof of Theorem 2.18, Theorem 2.18 is an improvement of [A1, Theorem A] and [A2, Theorem A].

Remark 2.20
In the situation of Theorem 2.18, suppose that K_1 is an MLF. Write $\mathrm{Hom}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$ for the set of homomorphisms of R-filtered profinite groups from $\mathfrak{G}_{K_1}$ to $\mathfrak{G}_{K_2}$. Then the natural map

$$\mathrm{Hom}(K_2, K_1) \to \mathrm{Hom}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})/\mathrm{Inn}(\mathfrak{G}_{K_2})$$

is bijective. Indeed, it suffices to show that the natural inclusion

$$\mathrm{Hom}^{\text{R-op}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2}) \hookrightarrow \mathrm{Hom}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$$

is bijective. We may assume that $\mathrm{Hom}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$ is not empty. Suppose that an element $f : \mathfrak{G}_{K_1} \to \mathfrak{G}_{K_2}$ of $\mathrm{Hom}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$ is not open. Then, by Proposition 1.15, f gives a surjective homomorphism of R-filtered profinite groups from $\mathfrak{G}_{K_1}$ to an R-filtered profinite group of R-GPLF-type. This contradicts Proposition 2.6.

In the remainder of this section, for $i = 1, 2$, we shall write

- K_i for a GMLF;
- $\overline{K_i}$ for an algebraic closure of K_i;
- G_{K_i} for the Galois group $\mathrm{Gal}(\overline{K_i}/K_i)$;
- X_i for a hyperbolic curve over K_i;
- $\pi_1(X_i)$ for the étale fundamental group of X_i (for some choice of basepoint);
- $\Delta_{X_i} = \pi_1(X_i \times_{\mathrm{Spec}\, K_i} \mathrm{Spec}\, \overline{K_i})$ for the geometric fundamental group of X_i (for some choice of basepoint).

By Theorem 2.18, we obtain a slight generalization of a part of [Mo6, Corollary 3.8]:

Theorem 2.21
Suppose that K_1 is an MLF. Write $\mathrm{Hom}^{\mathrm{dom}}(X_1/K_1, X_2/K_2)$ *for the set of dominant morphisms of schemes from X_1 to X_2 lying over a morphism* $\mathrm{Spec}\, K_1 \to \mathrm{Spec}\, K_2$*, and* $\mathrm{Hom}^{\text{SA-RF}}(\pi_1(X_1), \pi_1(X_2))$ *for the set of semi-absolute homomorphisms ϕ of profinite groups from $\pi_1(X_1)$ to $\pi_1(X_2)$ satisfying the following condition:*

> *The open homomorphism $\psi : G_{K_1} \to G_{K_2}$ induced by ϕ (cf. the condition that ϕ is semi-absolute) preserves ramification filtrations (i.e., determines a homomorphism of R-filtered profinite groups with respect to ramification filtrations).*

(For the definition of semi-absolute homomorphisms, see [Mo6, Definition 2.4 (ii)]*.) Moreover, write* $\mathrm{Inn}(\pi_1(X_2))$ *for the group of inner automorphisms of $\pi_1(X_2)$. Then the natural map*

$$\mathrm{Hom}^{\mathrm{dom}}(X_1/K_1, X_2/K_2) \to \mathrm{Hom}^{\text{SA-RF}}(\pi_1(X_1), \pi_1(X_2))/\mathrm{Inn}(\pi_1(X_2))$$

is a bijection.

Proof.
If $\mathrm{Hom}^{\text{SA-RF}}(\pi_1(X_1), \pi_1(X_2))$ is empty, the statement follows immediately. So, suppose that $\mathrm{Hom}^{\text{SA-RF}}(\pi_1(X_1), \pi_1(X_2))$ is non-empty. Then we have that K_2 is also an MLF (cf. Proposition 2.16). Therefore, by Theorem 2.18, for any $\phi \in \mathrm{Hom}^{\text{SA-RF}}(\pi_1(X_1), \pi_1(X_2))$, the induced homomorphism $\psi : G_{K_1} \to G_{K_2}$ is geometric, i.e., arises from a unique morphism of schemes $\mathrm{Spec}\, K_1 \to \mathrm{Spec}\, K_2$. By applying [Mo3, Theorem A] to the homomorphism $\pi_1(X_1) \to \pi_1(X_2 \times_{\mathrm{Spec}\, K_2} \mathrm{Spec}\, K_1) = \pi_1(X_2) \times_{G_{K_2}} G_{K_1}$ induced by ϕ, we obtain the desired bijection. $\square$

Remark 2.22
In fact, $\mathrm{Hom}^{\mathrm{dom}}(X_1/K_1, X_2/K_2)$ in Theorem 2.21 is equal to the set $\mathrm{Hom}^{\mathrm{dom}}(X_1, X_2)$ of dominant morphisms of schemes from X_1 to X_2.

Remark 2.23
By [Mo6, Example 2.13], for any X_2, there exist an X_1 and an absolute homomorphism $\phi : \pi_1(X_1) \to \pi_1(X_2)$ which is not semi-absolute. So we need to consider homomorphisms which are semi-absolute in Theorem 2.21.

3. Profinite groups of R-GMLF and R-GPLF-type

In this section, we establish mono-anabelian reconstruction algorithms of various invariants of GMLF's and GPLF's from their absolute Galois groups with ramification filtrations. Moreover, by using these results, we reconstruct the isomorphism classes of GMLF's under certain conditions.

Let $\mathfrak{G} = \{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ be an R-filtered profinite group of R-GMLF or R-GPLF-type, and G the underlying profinite group of $\mathfrak{G}$. In this section, we shall write

- $I(\mathfrak{G})$ for the closure of $\bigcup_{\varepsilon\in\mathbb{R}_{>0}} G^{-1+\varepsilon}$;
- $P(\mathfrak{G})$ for the closure of $\bigcup_{\varepsilon\in\mathbb{R}_{>0}} G^{0+\varepsilon}$;
- $\underline{G}(\mathfrak{G})$ for $G/I(\mathfrak{G})$.

For a GMLF or GPLF K, we shall write

- p_K for the characteristic of K;
- K^{sep} for an separable closure of K;
- G_K for the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$;
- $\mathfrak{G}_K = \{G_K^v\}_{v\in\mathbb{R}_{\geq -1}}$ for the R-filtered profinite group with underlying profinite group G_K determined by the ramification filtration on G_K;
- $I_K \subset G_K$ for the inertia subgroup of G_K;
- $P_K \subset I_K \subset G_K$ for the wild inertia subgroup of G_K;
- v_K for the valuation of K such that $v_K(K^\times) = \mathbb{Z}$;
- k_K for the residue field of K (by the definitions of GMLF and GPLF, k_K is perfect);
- p_{k_K} (> 0) for the characteristic of k_K;
- $\overline{k_K}$ for the residue field of K^{sep}, which is an algebraic closure of k_K;
- G_{k_K} for the Galois group $\mathrm{Gal}(\overline{k_K}/k_K)$.

Moreover, for a GMLF K, we shall write

- $\zeta_{p_{k_K}^n} \in K^{\mathrm{sep}}$ for a primitive $p_{k_K}^n$-th root of unity (for $n \in \mathbb{Z}_{>0}$);
- e_K for the absolute ramification index $v_K(p_{k_K})$;
- a_K for the largest nonnegative integer such that K contains a primitive $p_{k_K}^{a_K}$-th root of unity.

In the case where K is a GMLF, let $G_{K(\zeta_{p_{k_K}^\infty})}$ be the closed (normal) subgroup of G_K corresponding to $K(\zeta_{p_{k_K}^\infty}) := \bigcup_{n\in\mathbb{Z}_{>0}} K(\zeta_{p_{k_K}^n})$. Note that $G_{K(\zeta_{p_{k_K}^\infty})}$ is a subgroup of APF-type of G_K. Denote the R-filtered closed normal subgroup $\bigcap_{n\in\mathbb{Z}_{>0}} \mathfrak{G}_{K(\zeta_{p_{k_K}^n})}$ of $\mathfrak{G}_K$ by $\mathfrak{G}_{K(\zeta_{p_{k_K}^\infty})}$.

Remark 3.1
Note that, by Proposition 2.6, an R-filtered profinite group of R-GMLF-type (resp. R-GPLF-type) is not of R-GPLF-type (resp. R-GMLF-type).

Lemma 3.2
There exists a uniquely determined prime number p such that $P(\mathfrak{G})$ is a pro-p group.

Proof.
Immediate from Remark 1.2. □

Definition 3.3
We shall write $\underline{p}(\mathfrak{G})$ for the uniquely determined (cf. Lemma 3.2) prime number such that $P(\mathfrak{G})$ is a pro-$\underline{p}(\mathfrak{G})$ group.

Lemma 3.4
$\mathfrak{G}$ is of R-GMLF-type (resp. R-GPLF-type) if and only if there exists (resp. does not exist) an open subgroup $H \subset G\,(= G^{-1})$ such that, for some open normal subgroup H' of H of index $\underline{p}(\mathfrak{G})$, it holds that $s_{H/H'}(\sigma) \in \underline{p}(\mathfrak{G})\mathbb{Z}_{>0}$, where σ is a generator of H/H'. Moreover, if $\mathfrak{G}$ is of R-GMLF-type, then there exists a unique maximal subgroup H_0 of G satisfying the above condition for H.

Proof.
Immediate from Lemma 1.1 (ii) and Proposition 2.3. □

Definition 3.5

(i) We shall write

$$\overline{p}(\mathfrak{G}) := \begin{cases} 0, & (\mathfrak{G} \text{ is of R-GMLF-type}); \\ \underline{p}(\mathfrak{G}), & (\mathfrak{G} \text{ is of R-GPLF-type}). \end{cases}$$

(cf. Lemma 3.4.) Note that, by Remark 3.1, $\overline{p}(\mathfrak{G})$ is well-defined.

(ii) If $\overline{p}(\mathfrak{G}) = 0$, we shall denote the uniquely determined maximal subgroup H_0 of G in Lemma 3.4 by $G_{[1]}(\mathfrak{G})$.

(iii) If $\overline{p}(\mathfrak{G}) = 0$, $\mathfrak{G}$ determines a filtration of R-type on $G_{[1]}(\mathfrak{G})$. We shall denote the resulting R-filtered profinite group of R-GMLF-type by $\mathfrak{G}_{[1]}$.

(iv) If $\overline{p}(\mathfrak{G}) = 0$, set:

$$s(\mathfrak{G}) := \max\{s_{G_{[1]}(\mathfrak{G})/H}(\sigma_H) \mid H \text{ is an open normal subgroup of } G_{[1]}(\mathfrak{G}) \text{ of index } \underline{p}(\mathfrak{G})\},$$

where σ_H is a generator of $G_{[1]}(\mathfrak{G})/H$. Then we define:

$$e(\mathfrak{G}) := \frac{s(\mathfrak{G}) \cdot (\underline{p}(\mathfrak{G}) - 1)}{\underline{p}(\mathfrak{G}) \cdot [I(\mathfrak{G}) : I(\mathfrak{G}_{[1]})]}.$$

Definition 3.6
Suppose that $\mathfrak{G}$ is of R-GMLF-type. Set $p := \underline{p}(\mathfrak{G})$. For an integer $n \in \mathbb{Z}_{\geq 0}$, we shall say that $\mathfrak{G}$ satisfies $(*)_n$ if there exists a finite quotient H of G satisfying the following conditions:

(i) $H \simeq \mathbb{Z}/p^{n+1}\mathbb{Z}$;

(ii) For $0 \leq i \leq n+1$, denote the (uniquely determined) subgroup of H of index p^i by H_i. Then, for $0 \leq i \leq n$,

$$s_{H_i/H_{i+1}}(\sigma_i) = \frac{p^{i+1}e(\mathfrak{G})}{p-1},$$

where σ_i is a generator of H_i/H_{i+1}.

Lemma 3.7
Suppose that $\mathfrak{G}$ is of R-GMLF-type. Then one (and only one) of the following occurs:

(i) *$\mathfrak{G}$ does not satisfy $(*)_n$ for any $n \in \mathbb{Z}_{\geq 0}$;*

(ii) *There exists a unique integer $N \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{G}$ satisfies $(*)_N$ and does not satisfy $(*)_{N+1}$.*

The case (i) occurs if and only if $\mathfrak{G} \neq \mathfrak{G}_{[1]}$*. Moreover, for any* $n \in \mathbb{Z}_{\geq 0}$*, there exists a unique maximal open subgroup* H_n *of* G *whose resulting R-filtered profinite group of R-GMLF-type* $\mathfrak{H}_n$ *satisfies* $(*)_n$*.*

Proof.
Immediate from Propositions 2.3, 2.4 and 2.7 and their proofs. □

Definition 3.8
Suppose that $\mathfrak{G}$ is of R-GMLF-type.

(i) Define $a(\mathfrak{G})$ as follows:

$$a(\mathfrak{G}) := \begin{cases} 0, & \text{(if (i) of Lemma 3.7 occurs);} \\ N+1, & \text{(if (ii) of Lemma 3.7 occurs. } N \text{ is as in (ii) of Lemma 3.7).} \end{cases}$$

(ii) By Lemma 3.7, for $n \in \mathbb{Z}_{\geq 0}$, there exists a uniquely determined maximal open subgroup of G whose resulting R-filtered profinite group of R-GMLF-type satisfies $(*)_n$. We denote the subgroup by $G_{[n+1]}(\mathfrak{G})$ and the resulting R-filtered profinite group of R-GMLF-type by $\mathfrak{G}_{[n+1]}$.

Remark 3.9
For an R-filtered profinite group of R-GMLF-type $\mathfrak{G}$, $\mathfrak{G}_{[1]}$ defined in Definition 3.4 coincides with $\mathfrak{G}_{[1]}$ defined in Definition 3.8.

Proposition 3.10
Let K *be a GMLF or GPLF. Then it holds that*

- $I_K = I(\mathfrak{G}_K)$*;*
- $P_K = P(\mathfrak{G}_K)$*;*
- $G_{k_K} = \underline{G}(\mathfrak{G}_K)$*;*
- $p_{k_K} = \underline{p}(\mathfrak{G}_K)$*;*
- $p_K = \overline{p}(\mathfrak{G}_K)$*.*

Moreover, if K *is a GMLF, the following hold:*

- $e_K = e(\mathfrak{G}_K)$*;*
- $a_K = a(\mathfrak{G}_K)$*;*
- $\mathfrak{G}_{K(\zeta_{p^n})} = (\mathfrak{G}_K)_{[n]}$ *(for* $n \in \mathbb{Z}_{>0}$*);*
- $\bigcap_{n\in\mathbb{Z}_{>0}} (G_K)_{[n]}(\mathfrak{G}_K)$ *is a subgroup of APF-type of* G_K*. Moreover,* $\mathfrak{G}_{K(\zeta_{p^\infty})} = \bigcap_{n\in\mathbb{Z}_{>0}} (\mathfrak{G}_K)_{[n]}$*;*

where $p := p_{k_K} = \underline{p}(\mathfrak{G}_K)$*.*

Proof.
The assertions for I_K, P_K and G_{k_K} are immediate from definitions. The assertion for p_{k_K} follows from Remark 1.2 and Lemma 3.2. The assertion for p_K follows from the definition of $\overline{p}(\mathfrak{G})$. The assertion for e_K follows from Propositions 2.3, 2.4 and the definition of $e(\mathfrak{G})$. The assertion for a_K follows from Propositions 2.3, 2.4, 2.7, Lemma 3.7 and the definition of $a(\mathfrak{G})$. The last two assertions follow from Propositions 2.3, 2.4, 2.7 and Lemmas 3.4, 3.7. □

Proposition 3.11
For $i = 1, 2$, let $\mathfrak{G}_i = \{G_i^v\}_{v \in \mathbb{R}_{\geq -1}}$ be an R-filtered profinite group of R-GMLF or R-GPLF-type, and G_i the underlying profinite group of $\mathfrak{G}_i$. Suppose that there exists an open homomorphism $\alpha : \mathfrak{G}_1 \to \mathfrak{G}_2$ of R-filtered profinite groups.

(i) *It holds that $\underline{p}(\mathfrak{G}_1) = \underline{p}(\mathfrak{G}_2)$ and $\overline{p}(\mathfrak{G}_1) = \overline{p}(\mathfrak{G}_2)$. In particular, $\mathfrak{G}_1$ is of R-GMLF (resp. R-GPLF)-type if and only if $\mathfrak{G}_2$ is.*
(ii) *Suppose that at least one of the $\mathfrak{G}_i$ is (hence both of the $\mathfrak{G}_i$ are) of R-GMLF-type. Then it holds that*
- *$a(\mathfrak{G}_1) \geq a(\mathfrak{G}_2)$;*
- *$e(\mathfrak{G}_1) \geq e(\mathfrak{G}_2)$.*

Proof.
Since α is open and $P(\mathfrak{G}_i)$ is a non-trivial pro-$\underline{p}(\mathfrak{G}_i)$-group, the assertion (i) for $\underline{p}(\mathfrak{G}_i)$ follows immediately. Set $p := \underline{p}(\mathfrak{G}_1) = \underline{p}(\mathfrak{G}_2)$. Then $\overline{p}(\mathfrak{G}_i)$ ($i = 1, 2$) is either 0 or p. So, the assertion (i) for $\overline{p}(\mathfrak{G}_i)$ follows from Proposition 2.6.

Next, we consider the assertion (ii). Set $H := \alpha(G_1)$. Then H is an open subgroup of G_2, and $\mathfrak{G}_2$ determines a filtration of R-type on H. We shall denote the resulting R-filtered profinite group of R-GMLF-type by $\mathfrak{H}$. Then α decomposes into a surjection $\mathfrak{G}_1 \to \mathfrak{H}$ and an injection $\iota : \mathfrak{H} \to \mathfrak{G}_2$ of R-filtered profinite groups. By definition, it is clear that $a(\mathfrak{G}_1) \geq a(\mathfrak{H})$ and $e(\mathfrak{G}_1) \geq e(\mathfrak{H})$. Let K be a GMLF such that there exists an isomorphism of R-filtered profinite groups $\varphi : \mathfrak{G}_K \to \mathfrak{G}_2$, and L the finite extension of K corresponding to $\varphi^{-1}(H)$. Then we have the following commutative diagram:

$$\begin{tikzcd} \mathfrak{G}_L \arrow[r, "\sim", "\varphi|_{\mathfrak{G}_L}"'] \arrow[d, hook] & \mathfrak{H} \arrow[d, hook, "\iota"] \\ \mathfrak{G}_K \arrow[r, "\sim", "\varphi"'] & \mathfrak{G}_2. \end{tikzcd}$$

Clearly, we have that $a_L \geq a_K$ and $e_L \geq e_K$. Therefore, by Proposition 3.10, we obtain the following:

$$a(\mathfrak{H}) = a(\mathfrak{G}_L) = a_L \geq a_K = a(\mathfrak{G}_K) = a(\mathfrak{G}_2).$$

Moreover, we also have a similar inequality for $e(\mathfrak{H})$ and $e(\mathfrak{G}_2)$. This completes the proof of the assertion (ii), hence of Proposition 3.11. □

The following two theorems give mono-anabelian reconstruction algorithms of the isomorphism classes of GMLF's under certain conditions:

Theorem 3.12
Let k be a perfect field of positive characteristic, K a GMLF with residue field k, and K_0 the quotient field of the Witt ring with coefficients in k. Set $p := \operatorname{char} k = \underline{p}(\mathfrak{G}_K)$ (cf. Proposition 3.10). Suppose that one of the following condition holds:

(i) *$p \neq 2$.*
(ii) *$a(\mathfrak{G}_K) \geq 2$.*
(iii) *$e(\mathfrak{G}_K) = 1$.*

Suppose, moreover, that there exists an R-filtered open subgroup $\mathfrak{H}$ (with underlying profinite group H) of $\mathfrak{G}_K$ satisfying the following conditions:

(a) $[G_K : H] = [I(\mathfrak{G}_K) : I(\mathfrak{H})]$.
(b) $e(\mathfrak{H}) = p^{a(\mathfrak{H})-1}(p-1)$.

Set $L_0 := K_0(\zeta_{p^{a(\mathfrak{H})}})$. *Define a field* $K(\mathfrak{G}_K)$ *as follows:*

(I) *If* $p \neq 2$, *let* $K(\mathfrak{G}_K)$ *be the unique intermediate field of* L_0/K_0 *such that* $[K(\mathfrak{G}_K) : K_0] = e(\mathfrak{G}_K)$.
(II) *If* $p = 2$ *and* $a(\mathfrak{G}_K) \geq 2$, *set* $M_0 := K_0(\zeta_4)$ *(note that* $a(\mathfrak{H}) \geq a(\mathfrak{G}_K)\,(\geq 2)$ *(by Proposition 3.11 (ii)) and* L_0/M_0 *is a cyclic extension). Let* $K(\mathfrak{G}_K)$ *be the unique intermediate field of* L_0/M_0 *such that* $[K(\mathfrak{G}_K) : K_0] = e(\mathfrak{G}_K)$.
(III) *If* $p = 2$, $a(\mathfrak{G}_K) = 1$ *and* $e(\mathfrak{G}_K) = 1$, *set* $K(\mathfrak{G}_K) := K_0$ *(note that, since* $p = 2$, *in light of Proposition 3.10,* $a(\mathfrak{G}_K)$ *is necessarily positive).*

Then we have $K \simeq K(\mathfrak{G}_K)$.

Proof.
Let L be the finite extension of K corresponding to $\mathfrak{H}$. By the assumption on $\mathfrak{H}$, in light of Proposition 3.10, it is immediate that $L \simeq L_0$. Hence K is isomorphic to an intermediate field of L_0/K_0. On the other hand, since L_0/K_0 (resp. L_0/M_0) is a totally ramified cyclic extension, $K(\mathfrak{G}_K)$ as in the statement is the unique intermediate field of L_0/K_0 (resp. L_0/M_0) such that $e_{K(\mathfrak{G}_K)} = e(\mathfrak{G}_K) = e_K$ (cf. Proposition 3.10). This completes the proof of Theorem 3.12. $\square$

Theorem 3.13
Let k *be an algebraically closed field of positive characteristic,* K *a GMLF with residue field* k, *and* K_0 *the quotient field of the Witt ring with coefficients in* k. *Set* $p := \operatorname{char} k = \underline{p}(\mathfrak{G}_K)$ *(cf. Proposition 3.10). Suppose that one of the following condition holds:*

(i) $p \neq 2$.
(ii) $a(\mathfrak{G}_K) \geq 2$.
(iii) $e(\mathfrak{G}_K)$ *is prime to* p.

Suppose, moreover, that there exist an R-filtered open subgroup $\mathfrak{H}$ *of* $\mathfrak{G}_K$ *and an integer* $n \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}_{>0}$ *satisfying the following equality:*

$$e(\mathfrak{H}) = p^{a(\mathfrak{H})-1}(p-1)n.$$

Set $L_0 := K_0(\zeta_{p^{a(\mathfrak{H})}}, p^{\frac{1}{(p-1)n}})$. *Define a field* $K(\mathfrak{G}_K)$ *as follows:*

(I) *If* $p \neq 2$, *let* $K(\mathfrak{G}_K)$ *be the unique intermediate field of* L_0/K_0 *such that* $[K(\mathfrak{G}_K) : K_0] = e(\mathfrak{G}_K)$.
(II) *If* $p = 2$ *and* $a(\mathfrak{G}_K) \geq 2$, *set* $M_0 := K_0(\zeta_4)$ *(note that* $a(\mathfrak{H}) \geq a(\mathfrak{G}_K)\,(\geq 2)$ *(by Proposition 3.11 (ii)) and* L_0/M_0 *is a cyclic extension). Let* $K(\mathfrak{G}_K)$ *be the unique intermediate field of* L_0/M_0 *such that* $[K(\mathfrak{G}_K) : K_0] = e(\mathfrak{G}_K)$.
(III) *If* $p = 2$, $a(\mathfrak{G}_K) = 1$ *and* $e(\mathfrak{G}_K)$ *is prime to* $p\,(= 2)$, *set* $K(\mathfrak{G}_K) := K_0(p^{\frac{1}{e(\mathfrak{G}_K)}})$ *(note that, since* $p = 2$, *in light of Proposition 3.10,* $a(\mathfrak{G}_K)$ *is necessarily positive).*

Then we have $K \simeq K(\mathfrak{G}_K)$.

Proof.
This theorem is proved in a similar way to the proof of Theorem 3.12. (Note that, since k is algebraically closed, for $n \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}_{>0}$, K_0 has only one tamely ramified extension $K_0(p^{\frac{1}{(p-1)n}})$ of degree $(p-1)n$.) $\square$

The above two theorems give the following two bi-anabelian results:

Theorem 3.14
For $i = 1, 2$, let k_i be a perfect field of positive characteristic, and K_i a GMLF with residue field k_i. Suppose that the field k_1 is isomorphic to the field k_2. Set $p := \mathrm{char}\, k_1 = \mathrm{char}\, k_2 = \underline{p}(\mathfrak{G}_{K_1}) = \underline{p}(\mathfrak{G}_{K_2})$ (cf. Proposition 3.10). Suppose that one of the following condition holds:

(i) $p \neq 2$.
(ii) *Either* $a(\mathfrak{G}_{K_1}) \geq 2$ *or* $a(\mathfrak{G}_{K_2}) \geq 2$ *holds.*
(iii) *Either* $e(\mathfrak{G}_{K_1}) = 1$ *or* $e(\mathfrak{G}_{K_2}) = 1$ *holds.*

Suppose, moreover, that, for $i = 1, 2$, there exists an R-filtered open subgroup $\mathfrak{H}_i$ (with underlying profinite group H_i) of $\mathfrak{G}_{K_i}$ satisfying the following conditions:

(a) $[G_{K_i} : H_i] = [I(\mathfrak{G}_{K_i}) : I(\mathfrak{H}_i)]$.
(b) $e(\mathfrak{H}_i) = p^{a(\mathfrak{H}_i)-1}(p-1)$.

Then the following conditions are equivalent:

(I) *The field K_1 is isomorphic to the field K_2.*
(II) *There exists a surjective and quasi-injective homomorphism $\mathfrak{G}_{K_1} \twoheadrightarrow \mathfrak{G}_{K_2}$ of R-filtered profinite groups.*
(III) *There exists an isomorphism $\mathfrak{G}_{K_1} \xrightarrow{\sim} \mathfrak{G}_{K_2}$ of R-filtered profinite groups.*

Proof.
The implication (I)$\Longrightarrow$(II) is immediate. The implication (II)$\Longrightarrow$(III) follows from Proposition 2.14. The implication (III)$\Longrightarrow$(I) follows immediately from Proposition 3.10 and Theorem 3.12. □

Theorem 3.15
For $i = 1, 2$, let k_i be an algebraically closed field of positive characteristic, and K_i a GMLF with residue field k_i. Suppose that the field k_1 is isomorphic to the field k_2. Set $p := \mathrm{char}\, k_1 = \mathrm{char}\, k_2 = \underline{p}(\mathfrak{G}_{K_1}) = \underline{p}(\mathfrak{G}_{K_2})$ (cf. Proposition 3.10). Suppose that one of the following condition holds:

(i) $p \neq 2$.
(ii) *Either* $a(\mathfrak{G}_{K_1}) \geq 2$ *or* $a(\mathfrak{G}_{K_2}) \geq 2$ *holds.*
(iii) *Either* $e(\mathfrak{G}_{K_1})$ *or* $e(\mathfrak{G}_{K_2})$ *is prime to p.*

Suppose, moreover, that, for $i = 1, 2$, there exist an R-filtered open subgroup $\mathfrak{H}_i$ of $\mathfrak{G}_{K_i}$ and an integer $n_i \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}_{>0}$ satisfying the following equality:

$$e(\mathfrak{H}_i) = p^{a(\mathfrak{H}_i)-1}(p-1)n_i.$$

Then the following conditions are equivalent:

(I) *The field K_1 is isomorphic to the field K_2.*
(II) *There exists a surjective and quasi-injective homomorphism $\mathfrak{G}_{K_1} \twoheadrightarrow \mathfrak{G}_{K_2}$ of R-filtered profinite groups.*
(III) *There exists an isomorphism $\mathfrak{G}_{K_1} \xrightarrow{\sim} \mathfrak{G}_{K_2}$ of R-filtered profinite groups.*

Proof.

The implication (I)$\Longrightarrow$(II) is immediate. The implication (II)$\Longrightarrow$(III) follows from Proposition 2.14. The implication (III)$\Longrightarrow$(I) follows immediately from Proposition 3.10 and Theorem 3.13. $\square$

Finally, for future work, we formulate Isom and Hom-versions of Grothendieck conjectures for GMLF's and the absolute Galois groups with ramification filtrations:

Question 3.16
For $i = 1, 2$, let K_i be a GMLF. Then do the following statements hold?

(i) *Write* $\mathrm{Isom}(K_2, K_1)$ *for the set of isomorphisms from* K_2 *to* K_1, $\mathrm{Isom}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$ *for the set of isomorphisms of R-filtered profinite groups from* $\mathfrak{G}_{K_1}$ *to* $\mathfrak{G}_{K_2}$, $\mathrm{Inn}(\mathfrak{G}_{K_2})$ *for the set of inner automorphisms of* $\mathfrak{G}_{K_2}$. *Set* $\mathrm{Out}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2}) := \mathrm{Isom}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})/\mathrm{Inn}(\mathfrak{G}_{K_2})$. *Then the natural map*
$$\mathrm{Isom}(K_2, K_1) \to \mathrm{Out}^{\mathrm{R}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$$
is bijective.

(ii) *Write* $\mathrm{Hom}(K_2, K_1)$ *for the set of homomorphisms from* K_2 *to* K_1, $\mathrm{Hom}^{\mathrm{R\text{-}op}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})$ *for the set of open homomorphisms of R-filtered profinite groups from* $\mathfrak{G}_{K_1}$ *to* $\mathfrak{G}_{K_2}$, $\mathrm{Inn}(\mathfrak{G}_{K_2})$ *for the set of inner automorphisms of* $\mathfrak{G}_{K_2}$. *Then the natural map*
$$\mathrm{Hom}(K_2, K_1) \to \mathrm{Hom}^{\mathrm{R\text{-}op}}(\mathfrak{G}_{K_1}, \mathfrak{G}_{K_2})/\mathrm{Inn}(\mathfrak{G}_{K_2})$$
is bijective.

Remark 3.17
If (ii) of Question 3.16 holds, then clearly (i) of Question 3.16 holds.

Remark 3.18
(i) and (ii) of Question 3.16 for MLF's and PLF's are known affirmatively (cf. [Mo2, Theorem 4.2], [A1, Theorem A], [A2, Theorem A], and Theorem 2.18).

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