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ON UNIVALENCE CRITERIA FOR MEROMORPHIC FUNCTIONS

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ABSTRACT. We propose a way of deduction of various univalence criteria for meromorphic functions on the outside of the unit circle in terms of the range of their derivatives. This is a summary of the forthcoming joint paper [15] of S. Ponnusamy and the author.

1. INTRODUCTION

Let $\mathcal{A}$ denote the set of analytic functions $f$ in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. The set $S$ of univalent functions in $\mathcal{A}$ has been intensively studied by many authors. Let $\Sigma$ denote the set of univalent functions $F$ in the domain $\Delta = \{ \zeta : |\zeta| > 1 \}$ of the form

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}.$$ 

Note that the function $1/f(1/\zeta)$ belongs to $\Sigma$ for each $f \in S$. The converse is, however, not true in general. More precisely, for $F \in \Sigma$, the function $f(z) = 1/F(1/z)$ belongs to $S$ if and only if $F$ omits 0, namely, $F(\zeta) \neq 0$ for $\zeta \in \Delta$.

In parallel with the analytic case, we consider the set $\mathcal{M}$ of meromorphic functions in $\Delta$ with the expansion (1.1) around $\zeta = \infty$. For some technical reason, we also consider the sets $\mathcal{A}_n = \{ f \in \mathcal{A} : f^{(m)}(0) = 0 \text{ for } m = 2, \ldots, n \}$ and $\mathcal{M}_n = \{ F \in \mathcal{M} : b_0 = \cdots = b_n = 0 \}$. Note that $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{M}_{-1} = \mathcal{M}$.

Practically, it is an important problem to determine univalence of a given function in $\mathcal{A}_n$ or in $\mathcal{M}_n$. The best known conditions for univalence are probably those involving pre-Schwarzian or Schwarzian derivatives, which are defined by

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2.$$
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We define quantities for functions $f \in A$ and $F \in M$ by

$$B(f) = \sup_{|z|<1} (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right|,$$

$$B(F) = \sup_{|\zeta|>1} (|\zeta|^2 - 1) \left| \frac{\zeta F''(\zeta)}{F'(\zeta)} \right|,$$

$$N(f) = \sup_{|z|<1} (1 - |z|^2)^2 |S_f(z)|,$$

$$N(F) = \sup_{|\zeta|>1} (|\zeta|^2 - 1)^2 |S_F(\zeta)|.$$

Note that these quantities may take $\infty$ as their values. For example, if $F$ has a pole at a finite point, then $B(F) = \infty$.

If $f \in A$ and $F \in M$ have the relation $f(z) = 1/F(1/z)$, then we can easily see that

$$(1 - |z|^2)^2 S_f(z) = (|\zeta|^2 - 1)^2 S_F(\zeta)$$

holds for $z = 1/\zeta$. In particular, we have $N(f) = N(F)$.

**Theorem A** (Nehari [14]). Every $f \in S$ satisfies $N(f) \leq 6$. Conversely, if $f \in A$ satisfies $N(f) \leq 2$ then $f$ must be univalent. The constants 6 and 2 are best possible. The same is true for meromorphic $F$.

Though $zf'(z)/f(z) = \zeta F'((\zeta)/F(\zeta)$, there is no such a simple relation between $zf''(z)/f'(z)$ and $\zeta F''((\zeta)/F(\zeta)$, and thus, between $B(f)$ and $B(F)$ for $f(z) = 1/F(1/z)$, $\zeta = 1/z$. Nevertheless, it is rather surprising that the formally same conclusions can be deduced for $f$ and $F$. Compare Theorem B with Theorem C.

**Theorem B.** Every $f \in S$ satisfies $B(f) \leq 6$. Conversely, if $f \in A$ satisfies $B(f) \leq 1$ then $f \in S$. Moreover, if $B(f) \leq k < 1$, then $f$ extends to a $k$-quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.

Here and hereafter, a quasiconformal mapping $g$ is called $k$-quasiconformal if its Beltrami coefficient $\mu = g_2/g_z$ satisfies $\|\mu\|_{\infty} \leq k$.

The sufficiency of univalence and quasiconformal extendibility are due to Becker [6]. The sharpness of the constant 1 is due to Becker and Pommerenke [8]. The sharp inequality $B(f) \leq 6$ follows from a standard argument in the coefficient estimation (see, e.g., [9, Theorem 2.4]).

**Theorem C.** Every $F \in \Sigma$ satisfies $B(F) \leq 6$. Conversely, if $F \in M$ satisfies $B(F) \leq 1$ then $F \in \Sigma$. Moreover, if $B(F) \leq k < 1$, then $F$ extends to a $k$-quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.

The sufficiency of univalence and quasiconformal extendibility are due to Becker [7]. The sharpness of the constant 1 is also due to Becker and Pommerenke [8]. On the other hand, the estimate $B(F) \leq 6$ lies deeper. Avhadiev [3] first showed the sharp inequality $B(F) \leq 6$ by appealing to Goluzin's inequality (see [10, p. 139]).
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Note that many authors use a different norm for the pre-Schwarzian derivative of $f \in \mathcal{A}$, namely,

$$
||T_f|| = \sup_{|z|<1} (1 - |z|^2)|T_f(z)|.
$$

By definition, we observe $\mathcal{B}(f) \leq ||T_f||$.

Recall that a plane domain $\Omega \subset \mathbb{C}$ is called hyperbolic if $\partial \Omega$ contains at least two points. Let $\Omega$ be a hyperbolic plain domain such that $1 \in \Omega$ but $0 \notin \Omega$ and set

$$
\Pi(\Omega) = \{F \in \mathcal{M} : F'(\zeta) \in \Omega \text{ for all } \zeta \in \Delta\}.
$$

Set also $\Pi_n(\Omega) = \Pi(\Omega) \cap \mathcal{M}_n$ for $n = -1, 0, 2, \ldots$. One of our main results in the present paper is an estimate of $\mathcal{B}(F)$ for $F \in \Pi(\Omega)$. The proof is given in [15].

**Theorem 1.** Let $\Omega$ be a domain such that $1 \in \Omega$ but $0 \notin \Omega$. For every $F \in \Pi_n(\Omega)$, $n \geq 0$, the inequality

$$
\mathcal{B}(F) \leq C_n W(\Omega)
$$

holds, where $C_n$ is the constant given by

$$
(1.2) \quad C_n = \sup_{0<r<1} \frac{(n+2)(1-r^2)r^n}{1-r^{2n+4}}
$$

and $W(\Omega)$ is the circular width of $\Omega$ with respect to the origin, namely,

$$
W(\Omega) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{p'(z)}{p(z)} \right|
$$

for an analytic universal covering projection $p$ of $\mathbb{D}$ onto $\Omega$.

Note that $W(\Omega)$ does not depend on the particular choice of $p$. For more details on circular width, see [12]. As one sees easily, $C_0 = 2$ and $1 \leq C_n \leq (n+2)/(n+1)$. If we write $F \in \Pi(\Omega)$ in the form $F = F_0 + b_0$, where $F_0 \in \Pi_0(\Omega)$, the relation $\mathcal{B}(F) = \mathcal{B}(F_0)$ holds. Therefore, the above theorem can be applicable to the whole family $\Pi(\Omega)$. We note that the analytic counterpart of this theorem is known and much simpler to prove (see [11, Theorem 4.1]); $\mathcal{B}(f) \leq ||T_f|| \leq W(\Omega)$ holds for $f \in \mathcal{A}$ with $f'(\mathbb{D}) \subset \Omega$.

As is well known, if $f \in \mathcal{A}$ satisfies $\text{Re} f' > 0$ then $f$ is necessarily univalent (cf. [9, Theorem 2.16]). However, the meromorphic counterpart does not hold (see, for instance, the example given in Section 3). The following univalence criterion is due to Aksent’ev [1] (see also [5, Theorem 11]). Later, Krzyż [13] gave quasiconformal extensions for the functions.

**Theorem D** (Aksent’ev, Krzyž). Let $0 \leq k \leq 1$. If $F \in \mathcal{M}$ satisfies the inequality

$$
(1.3) \quad |F'(\zeta) - 1| \leq k, \quad |\zeta| > 1,
$$

then $F$ is univalent. Furthermore, if $k < 1$, then $F$ extends to a $k$-quasiconformal mapping of the extended plane. The radii $1$ and $k$ are best possible.

Note that the range of $F'$ cannot be enlarged to $\{w : |w - 1| < a\}$, $a > 1$, for univalence [2].
The following examples can be found in [12].

**Example 1** (sectors). For \( S(\beta) = \{ w : |\arg w| < \pi \beta/2 \}, \ 0 < \beta \leq 2 \), we have \( W(S(\beta)) = 2\beta \).

**Example 2** (annuli). For the annulus \( A(r, R) = \{ w : r < |w| < R \}, \ 0 < r < R < \infty \), we have \( W(A(r, R)) = (2/\pi) \log(R/r) \).

**Example 3** (disks). Let \( \mathbb{D}(a, r) = \{ w : |w - a| < r \} \) for \( 0 < r \leq a \). Then
\[
W(\mathbb{D}(a, r)) = \frac{2r/a}{1 + \sqrt{1 - (r/a)^2}}.
\]

**Example 4** (parallel strips). Let \( P(a, b) = \{ w : a < \mathrm{Re} w < b \} \) for \( 0 \leq a < b < \infty \). Then
\[
W(P(a, b)) = \left[ \frac{2t \cos \theta}{1 - t \theta} \right]_{0}^{\pi/2},
\]
where \( t \) is a number with \( 0 < t \leq 2/\pi \) determined by
\[
\frac{\pi t}{2} = \frac{b - a}{b + a}.
\]

**Example 5** (truncated wedges). Let \( S(\beta, r, R) = \{ w : |\arg w| < \pi \beta/2, r < |w| < R \}, \ 0 < \beta \leq 2, 0 < r < R < \infty \). Then
\[
W(S(\beta, r, R)) = \frac{\log(R/r)}{(1 + t)\mathcal{K}(t)}.
\]
where
\[
\mathcal{K}(t) = \int_{0}^{1} \frac{dx}{\sqrt{(1 - x^2)(1 - t^2x^2)}}
\]
is the complete elliptic integral of the first kind and \( 0 < t < 1 \) is a number such that
\[
\frac{\mathcal{K}(\sqrt{1 - t^2})}{\mathcal{K}(t)} = \frac{2\pi \beta}{\log(R/r)}.
\]

3. **Applications**

We apply Theorem 1 and Theorem C to the above examples to obtain several results on univalence of meromorphic functions. As samples, we state a few theorems. Note that the univalence criteria in Theorems 2 and 3 were first given by Avhadiev and Aksent’ev [4].

Let \( x_2 \approx 0.4198 \) denote the unique zero of the equation
\[
\sqrt{x} \log((1 + \sqrt{x})/(1 - \sqrt{x})) = 1
\]
in \( 0 < x < 1 \).

**Theorem 2.** Let \( 0 \leq k \leq 1 \). Suppose that a function \( F \in \mathcal{M} \) satisfies the condition
\[
|\arg F'(\zeta)| \leq \frac{k\pi}{8}, \quad |\zeta| > 1,
\]
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then \( F \) must be univalent. Furthermore, if \( k < 1 \), then \( F \) extends to a \( k \)-quasiconformal mapping of the extended plane. As for univalence, the constant \( \pi/8 \) cannot be replaced by any smaller number than \( (4/\pi) \arctan x_2 \).

Note that \( (4/\pi) \arctan x_2 \approx 0.506057 \approx 1.28866(\pi/8) \). The number \( x_2 \) appears in the following example.

We consider the function \( F_n \in \mathcal{M} \) given by
\[
F_n(\zeta) = \zeta - 2 \sum_{j=1}^{\infty} \frac{\zeta^{1-nj}}{nj-1}
\]
for each integer \( n \geq 2 \), where \( {}_2F_1(1, -\frac{1}{n}; 1 - \frac{1}{n}; \zeta^{-n}) - 1 \), \(|\zeta| > 1\),

from that \( F_n \) has the \( n \)-fold symmetry
\[
F_n(e^{2\pi i/n} \zeta) = e^{2\pi i/n} F_n(\zeta)
\]
and belongs to the class \( \mathcal{M}_{n-2} \). Since the function \( h_n \) defined by
\[
h_n(x) = 2 {}_2F_1(1, -\frac{1}{n}; 1 - \frac{1}{n}; x) - 1 \quad (x \in (0, 1))
\]
has the properties that \( h_n \) is monotone decreasing, \( h_n(0) = 1 \) and \( \lim_{x \to 1-} h_n(x) = -\infty \), there is the unique point \( x_n \) such that \( h(x_n) = 0 \) in the interval \( 0 < x < 1 \). Hence, the function \( F_n \) has the \( n \) zeros \( e^{2\pi ij/n} x_n^{-1/n} \), \( j = 0, 1, \ldots, n - 1 \), in \( \Delta \) and, in particular, is not univalent in \( \Delta \). On the other hand, we have
\[
F_n'(\zeta) = 1 + 2 \sum_{j=1}^{\infty} \zeta^{-nj} = p(\zeta^{-n}),
\]
where \( p(z) \) is the function given by \( p(z) = (1 + z)/(1 - z) \). It is a standard fact that \( p \) maps the unit disk onto the right half-plane \( \mathbb{H} = \{ w \in \mathbb{C} : \text{Re} w > 0 \} \). Therefore, \( F_n' \) maps \( \Delta \) onto \( \mathbb{H} \) in an \( n \)-to-1 way and thus \( \text{Re} F_n' > 0 \) holds.

In the next criterion, \( F' \) may take values with negative real part.

**Theorem 3.** Let \( 0 \leq k \leq 1 \). Suppose that a function \( F \in \mathcal{M} \) satisfies the condition
\[
|\log |F'(\zeta)|| \leq \frac{k\pi}{8}, \quad |\zeta| > 1,
\]
then \( F \) must be univalent. Furthermore, if \( k < 1 \), then \( F \) extends to a \( k \)-quasiconformal mapping of the extended plane. As for univalence, the constant \( \pi/8 \) cannot be replaced by any smaller number than \( \log((1 + x_2)/(1 - x_2)) \).

Note that \( \log((1 + x_2)/(1 - x_2)) \approx 0.894894 \approx 2.27883(\pi/8) \). In these results, if we assume \( F \) to be in \( \mathcal{M}_n \) for larger \( n \), then we can make the involved constants better.

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