KIER DISCUSSION PAPER SERIES

KYOTO INSTITUTE
OF
ECONOMIC RESEARCH

Discussion Paper No.1036

"Short- and Long-run Impacts of Bursting Bubbles"

Takeo Hori and Ryonghun Im

August 2020

KYOTO UNIVERSITY

KYOTO, JAPAN

Short- and Long-run Impacts of Bursting Bubbles*

Takeo Hori† and Ryonghun Im‡

August 7, 2020

Abstract

Uninsured investment risks are introduced into a textbook AK model. There are no financial frictions. Depending on insurance market development, asset bubbles emerge in an infinitely-lived agent economy. A collapse of bubbles has short-run impacts. At the moment of the collapse of bubbles, aggregate demand decreases immediately. This instantly triggers sharp declines in all of GDP, consumption, investment, capital utilization, and wealth-to-GDP, although capital remains constant in the short run. Consistently with data, investment decreases more than consumption. The bubbles also has long-run impacts. The decreased investment depresses long-run growth. The economy falls into a prolonged recession.

Keywords: asset bubbles, uninsured idiosyncratic investment risks, instant contraction, aggregate demand, prolonged recession.

JEL classification numbers: E32, E44, G1

*The authors are grateful to Takashi Kamihigashi, Kazuo Mino, and Akihisa Shibata. Hori's research was financially supported by JSPS KAKENHI Grant No.18K01502. Im's research was financially supported by JSPS KAKENHI Grant No.16J04950. All remaining errors are ours.

†Department of Industrial Engineering and Economics, School of Engineering, Tokyo Institute of Technology, 2-12-1, Ookayama, Meguro-ku, Tokyo 152-8552, Japan. E-mail: hori.t.ag@m.titech.ac.jp.

‡JSPS Research Fellow at Kyoto University, Institute of Economic Research, Kyoto University, Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501, Japan. E-mail: ryonghunim@gmail.com

1 Introduction

Economic history has repeatedly experienced boom-bust in asset prices, which has significant impacts on real economies (Aliber and Kindleberger (2015)). A famous example is the Great Recession of 2007-2009 in the US economy. Some economists and policy makers believe that asset price busts may trigger the Great Recession in US.[1] In fact, Martin and Ventura (2012) document a drastic decrease in the wealth-to-GDP ratio during 2007-2009. The 2007-2009 crisis has the following two features (see Panels (a) and (b) in Figure 1).

[Figure 1]

Fact 1. Sharp and instant contraction: During the 2007-2009 crisis, economic growth slowed substantially and became even negative. In this period, per capita GDP, consumption and investment in the US decreased by 4.41%, 3.27%, and 23.1%, respectively. Notably, investment showed the largest decline.

Fact 2. Slow recovery and prolonged recession: After the crisis of 2007-2009, growth recovered. However, recovery was slow. The average growth rate of GDP per capita in the US during 2009-2013 is 1.27%, that is much lower than 2.09% average growth during 2003-2007.

These two features are also observed in Japanese stock and real estate markets boom around 1990 and U.S dotcom bubble around 2000.[2]

In traditional macroeconomic models of rational bubbles, bubbles suppress capital accumulation. Conversely, a collapse of bubbles accelerates capital accumulation in the long run (see Tirole (1985)). This prediction of traditional rational bubble models is inconsistent with the above facts. Recently, authors including Martin and Ventura (2012), Kunieda and Shibata (2016), Hirano and Yanagawa (2017), Miao and Wang (2018) attempt to overcome this shortcoming. These authors successfully construct models where a bust of bubbles suppresses capital accumulation. Since capital accumulation affects mainly long-run growth rather than short-run fluctuations, these models provide theoretical explanation for slow recovery and prolong recessions (Fact 2).

However, Panels (b) and (c) in Figure 1 show that during 2007-2009, growth of capital remained positive (although it slowed) even though GDP decreased sharply.[3] Thus, the above mentioned models may not explain sharp contractions in a short period (Fact 1). Indeed, in these models, GDP is determined (mainly) by capital and hence a collapse of bubbles has no impacts on GDP in the short run. Accordingly, asset bubbles affect only the division of output between aggregate consumption and investment. After a collapse of bubbles, these two aggregates move in the opposite direction in the short run. Significantly, these existing studies do not examine which decreases more, consumption or investment.

[1] A bubbles on asset is defined as the difference between the fundamental and market values of an asset.

[2] In Japan during 1986-1990, 1991-1993, and 1994-1996, the average growth rates of GDP per capita are 4.88%, -0.14%, and 2.68%, respectively. The average growth rates of investment (consumption) per capita are 8.82% (4.41%), -3.77% (1.34%), and 5.48% (2.05%), respectively. For the case of U.S dotcom bubble during 1995-1999, 2000-2002, and 2003-2006, the average growth rates of GDP per capita are 3.1%, 0.39%, and 2.47%, respectively. The average growth rates of investment (consumption) per capita are 7.11% (3.19%), -2.73% (1.51%), and 4.55% (2.54%), respectively. The source of data is the same as that in Figure 1.

[3] It is well known that on usual business cycles (not including the 2007-2009 crisis), capital stock is much less volatile than output. See Cooley and Prescott (1995) and King and Rebelo (1999).

The following question arises naturally; Does a collapse of bubbles explain Facts 1 and 2 simultaneously? We construct a single model where a burst of bubbles triggers instant and sharp declines in GDP and other aggregate variables that are followed by a prolonged recession, as shown in Panel (a) of Figure 2. We also address a larger decline in investment than consumption.

[Figure 2]

For our purpose, we construct an infinitely-lived agent model. Our model is quite simple and is composed of only four parameters. Importantly, our model is fairy close to a textbook macroeconomic model. Indeed, If we eliminate one parameter, our model reduces to a textbook AK model. Thus, we can provide main results and their mechanisms in an analytically clear manner. Moreover, the simplicity of our model allows us to easily show how a bust of bubbles triggers an instant contraction (Fact 1).

We conduct our analysis in two steps. In the first step, we introduce investment risks to a textbook AK model. We use this benchmark model to examine how bubbles affect capital accumulation in the long run. Each entrepreneur produces new capital, which is subject to idiosyncratic risks. The investment risks are not insured, which is an assumption common to recent literature on rational bubbles. In contrast to recent studies on rational bubbles, entrepreneurs in our model face no borrowing constraints. Entrepreneurs have an identical ex ante productivity of investment and investment risks realize only after investment takes place. Thus, there are no lending and borrowing among entrepreneurs. Borrowing constraints do not matter. We do not claim that the absence of borrowing constraints is realistic. However, it makes our analysis simpler.

Even without borrowing constraints, bubbly assets are valued in an infinitely-lived agent economy. Faced with investment risks, risk averse entrepreneurs reduce investment in capital production. Through saving-investment balance, the rate of return on *holding* capital is reduced. Since bubbly assets yield a high return, entrepreneurs hold bubbly assets for a speculative purpose. Only in economies with advanced technology and medium degrees of investment risks, asset bubbles arise.

Interestingly, even without borrowing constraints, asset bubbles accelerate capital accumulation and long-run growth. Asset bubbles make entrepreneurs wealthy. Wealthy entrepreneurs take more risks and produce more capital, which has a positive growth effect. Depending on production technology and insurance market development, the positive effect dominates an usual crowding-out effect of asset bubbles. Then, bubbles promote economic growth. Conversely, asset bubble bust decreases long-run growth, leading to a long-term depression.

However, the benchmark model fails to capture sharp and instant contractions (Fact 1). In fact, at the moment of bubble bust, the level of real GDP remains unchanged and thus consumption and investment move oppositely in the short run.

In the second step, we endogenize capital utilization to address Fact 1. If investment risk is removed, the extended model also returns to a textbook AK model. Panel (a) of Figure 2 shows how a collapse of bubbles at time t_1 affects GDP. At time t_1, the economy experiences instant and drastic falls in all of capital utilization, GDP, consumption, investment, the wealth-to-GDP ratio, and growth rate. Now, both consumption and investment decrease. Entrepreneurs' welfare also declines. After the initial contraction, because of depressed growth, the economy falls into a prolong recession.

Importantly, a sunspot shock triggers a collapse of bubbles in our model. This suggests that even without any changes in fundamentals, an instant contraction and a subsequent low growth may hit an economy.

How bubbles affect aggregate demand is a key to the initial decline at time t_1. A collapse of bubbles decreases entrepreneurs' wealth, which reduces aggregate consumption and investment. Faced with reduced aggregate demand, some production facilities (capital) seize operation, which decreases capital utilization. Although capital remains unchanged at time t_1, depressed capital utilization leads to instant declines in macroeconomic activities. The sharp decline in capital utilization is consistent with Panel (d) of Figure 1.[4]

The following two points should be emphasized. (i) At the moment of a collapse of bubbles (time t_1), capital stock remains constant (see Panel (b) of Figure 2). Even in a short period during which capital does not change, a collapse of bubbles triggers an instant contraction. (ii) Consistently with Fact 1, our model predicts that at time t_1, investment decreases more than consumption under a condition. This is because bubble bust depresses investment even if capital utilization is exogenously fixed.

Supply side is important for the prolonged recession after time t_1. Because of permanently reduced growth, capital grows slowly (see Panel (b) of Figure 2). In the long run, output is depressed and the recession prolongs.

We also consider the effect of a temporal negative technology shock. It induces bursting bubbles, which amplifies the impact of a negative technology shock. Even a temporal negative technology shock depresses economic activities permanently.

Our model provides some new insights that are not fully addressed by the existing studies on rational bubbles. We do not claim that our model is superior to the existing ones. Rather, our study complements the existing studies by highlighting how a collapse of bubbles induces an instant contraction as well as a long-term recession.

1.1 Related literature

This study constructs an infinitely-lived agents model of rational bubbles. In overlapping-generations (OLG) models, Tirole (1985) shows that bubbles may exist if the economy is dynamically inefficient. Abel *et al.* (1989) find empirically that developed economies are dynamically efficient. Farhi and Tirole (2011) and Martina and Ventura (2012) show that with borrowing constraints, bubbles arise even in dynamically efficient OLG economies.

Recently, Kocherlakota (2009), Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018) show that in the presence of financial frictions, bubbles exist in infinite-horizon models of production economies.[5] To ensure that borrowing constraints are binding occasionally, these studies assume that the insurance market is incomplete and productivity of agents changes frequently due to idiosyncratic shocks. As Aiyagari and McGrattan (1998) and Farhi and Tirole (2011) point out, occasionally binding borrowing constraints

[4]King and Rebelo (1999) and Stock and Watson (1999) show that on business cycles in the post-war period (not including the 2007-2009 recession), capacity utilization is much more volatile than output.

[5]Kocherlakota (1992) and Santos and Woodford (1997) provide examples of equilibrium with bubbles in infinite-horizon models of endowment economies with borrowing constraints.

shorten agents' planing horizon and make infinitely-lived agents' behavior similar to that of OLG models. This is crucial for the existence of asset bubbles in these models.

In our model, there are no borrowing constraints and entrepreneurs' productivity remains unchanged overtime.[6] We show that even without occasionally binding borrowing constraints, bubbles exist in an infinitely-lived agent model. Since there are no credit constraints, our model can not address the relation between credit booms and asset bubbles. However, the absence of borrowing constraints makes aggregation easy and simplifies our analysis.

Aoki *et al.* (2014) also show the existence of bubbles in an infinitely-lived agents model without borrowing constraints. In their model, agents hold bubbly assets to diversify idiosyncratic risks because bubbles are safety assets.[7] In our model, bubbly assets are risky and entrepreneurs hold bubbly assets only for speculative purpose. Moreover, in Aoki *et al* (2014), bubbles always lower long-run growth, which contrasts our results.

How is our long-run growth effect of bubbles related to the literature? Tirole (1985) shows that bubbles crowd investment out and lower capital accumulation and output in the long run.[8] Recent studies show that asset bubbles relax borrowing constraints and improve efficiency of resource allocation, which promotes capital accumulation and long-run growth (*e.g.,* Kocherlakota (2009), Farhi and Tirole (2011), Martin and Ventura (2012), Aoki and Nikolov (2015), Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018).[9]

Borrowing constraints are absent in our model. The key is incomplete insurance. Our result suggests that even without financial frictions, bubbles may enhance growth if insurance market is incomplete. However, there is a similarity between the existing and our models. In both models, bubbles positively affect agents' wealth, which boosts long-run growth.

Does a bursting of bubbles trigger instant and sharp contractions (Fact 1) in the existing studies we have just mentioned? Figure 2 in Aoki and Nikolov (2015) illustrates the answer well. The figure shows that after a collapse of bubbles, output decreases only gradually (the upper-left panel) and investment increases in the short run (the lower-right panel).[10] These results are obtained because output is mainly determined by capital and hence aggregate consumption and investment move in the opposite direction in the short run. The similar short-run effects are found also in Marin and Ventura (2012), Kunieda and Shibata (2016), Hirano and Yanagawa (2017).[11] In Kocherlakota (2009), when bubbles collapse, investment

---

[6]Ample empirical studies show that firm productivity is highly persistent. For example, Baily *et al* (1992) document that in the US economy, 58% of most productive firms remained most productive ten years later. See Bartelsman and Domes (2000), Foster *et al* (2001), Fukao and Kwon (2006), and Foster *et al* (2008). In a model with uninsured idiosyncratic shocks but without bubbles, Moll (2014) shows that persistency of productivity shocks affects results dramatically, arguing that persistent shocks are empirically relevant.

[7]Kitagawa (1994) show that agents demand bubbles as safety assets in an OLG model.

[8]Grossman and Yanagawa (1993), King and Ferguson (1993) and Futagami and Shibata (2000) find that asset bubbles retards long-run economic growth.

[9]Mitsui and Watanabe (1989) and Woodford (1990) are early studies showing that bubbles promote capital investment. These studies do not examine the impacts of bursting bubbles. Olivier (2000) and Tanaka (2011) investigate how stock bubbles stimulate R&D activities.

[10]In Aoki and Nikolov (2015), increased investment doe not necessarily accelerate capital accumulation because a collapse of bubbles reallocates resources from high productive firms to low productive ones. The same mechanism applies to Marin and Ventura (2012), Kunieda and Shibata (2016), and Hirano and Yanagawa (2017).

[11]In Martin and Ventura (2012), equation (11) shows that bubbles at period t affect period $t+1$ capital

instantly increases while consumption decreases (see footnote 5 in Kocherlakota (2009)) although output decreases sharply one period after the bubble bust. In Miao and Wang (2018), a collapse of bubbles induces an instant increase in aggregate consumption and output decreases only gradually (see Figure 3 in Miao and Wang (2018)). Besides, these studies do not address why investment decreases more than consumption.

These authors have advanced the rational bubble theory considerably, providing important insights on how bubbles promote long-run growth. However, the above discussion shows that more work still remains to be done in terms of the short-run effects of bubbles. We show that endogenizing capital utilization may solve some problems.[12] Remarkably, we show that investment decreases more than consumption under a condition.

Guerron-Quintana *et al.* (2019) also introduce capital utilization to a model of rational bubbles. Our study differs from theirs in several aspects. First, their focus is on the impacts of recurrently occurring bubbles. We focus on the short-run impacts of bubbles. Second, in their model, how bubbles affect economy depends on borrowing constraints. In our model, the degree of investment risks plays key roles. Finally, their results are based on numerical analysis. All of our results are theoretical. Thus, economic intuition and mechanism for results are analytically clear.

The rest of the paper is organized as follows: Section 2 presents our benchmark model. Section 3 examines how bubble emerges and how bubbles affect long-run growth in the benchmark model. Section 4 endogenizes capital utilization and shows that a collapse of bubbles triggers an instant contraction as well as a long-term recession. Concluding remarks are in Section 5.

2 A simple AK model

This section presents our benchmark model where capital utilization rate is exogenously fixed. Using this model, Section 3 examines the existence condition of bubbles and how bubbles affect long-run growth.

Time is continuous and runs from $t = 0$ to ∞. A single general good is produced by using an AK production function. The only input in general good production is called *capital*. Entrepreneurs own capital and bubbly assets. They can produce new capital using general good. They face idiosyncratic shocks when producing new capital. We can interpret capital broadly. Appendix R presents a model where production of new capital includes setting up new businesses or developing new technologies, which is subject to idiosyncratic risks.

and output. They assume heterogeneous productivity among agents. If $\delta = 1$, the right-hand side of equation (11) corresponds to aggregate investment. It shows that a collapse of bubbles increases aggregate investment. In Hirano and Yanagawa (2017), equation (16) shows that asset bubble busts increase aggregate investment and equation (8) shows that aggregate consumption and investment move in the opposite direction. Figure 3 in Kunieda and Shibata (2016) shows the same results. In Hirano and Yanagawa (2017) and Kunieda and Shibata (2016), a collapse of bubbles affects long-rung growth whereas it does not affect output level in the short run.

[12] Endogenous capital utilization in our model is motivated by King and Rebelo (1999) who show that capital utilization is important when considering short-run fluctuation in an RBC model.

2.1 General Good Sector

A single general good is used for both consumption and input of capital production. The general good is competitively produced by the following production function:

$$Y_t = AK_t, \quad A > 0, \tag{1}$$

where Y_t and K_t denote output and capital input, respectively. The general good is taken as a numeraire. Denote the rental rate of capital by q_t. Profit maximization yields

$$q_t = A. \tag{2}$$

2.2 Entrepreneurs

Preferences and Investment Risks: There is a continuum of infinitely-lived entrepreneurs whose measure is one. Entrepreneurs are risk averse. Entrepreneur $i \in [0, 1]$ has the following expected lifetime utility:

$$U_{i,t} = E_t \int_t^{\infty} (\log c_{i,t})\, e^{-\rho(s-t)} ds, \tag{3}$$

where $c_{i,t}$ is entrepreneur i's consumption, $\rho > 0$ is the subjective discount rate, and E_t is an expectation operator conditional on time t information. We assume that

$$A > \rho. \tag{4}$$

Capital production is irreversible and subject to risks. We assume that the risks are not fully insurable.[13] If entrepreneur i uses $I_{i,t} (\geq 0)$ units of general good for a time period of length dt, $dx_{i,t}$ units of new capital are produced as follows:

$$dx_{i,t} = \phi I_{i,t} dt + \sigma I_{i,t} dW_{i,t}, \qquad \phi = 1, \quad \sigma > 0, \tag{5}$$

where $W_{i,t}$ is a standard Brownian motion. Its increment, $dW_{i,t}$, represents idiosyncratic investment risks. We assume that $dW_{i,t}$ is independent and identically distributed across entrepreneurs. Parameters ϕ and σ are common to all entrepreneurs. As in a standard AK model, we assume $\phi = 1$. A large σ means a low insurance coverage and high risks. As insurance market develops, more risks are insurable and σ decreases. If $\sigma = 0$, our model reduces to a standard AK model. As mentioned earlier, $dx_{i,t}$ includes starting new businesses, developing new technologies, and so on (see Appendix R).

Asset holdings and budget constraint: Entrepreneurs sell capital that they newly produce. Denote the price of capital as v_t. Since general good price is one and $\phi = 1$, entrepreneur i earns the following profits:

$$(v_t - 1) I_{i,t} dt + \sigma v_t I_{i,t} dW_{i,t}. \tag{6}$$

The term $(v_t - 1) I_{i,t} dt$ represents the deterministic profits. The term $\sigma v_t I_{i,t} dW_{i,t}$ represents the stochastic profits that reflect investment risks. All entrepreneurs have the same (average)

[13] Asymmetric information is one of the sources of incomplete insurance. In Townsend (1979), the costly state verification causes asymmetric information and hence incomplete insurance. We do not model insurance contracts for simplicity.

productivity $\phi(=1)$ and learn shocks *after* output realizes. Thus, there are no borrowing and lending among entrepreneurs. Hence, borrowing constraints do not matter.

As in Tirole (1985), a bubbly asset is an intrinsically useless asset with zero fundamental value. Let p_t be the bubbly asset price at time t. The free disposability of bubbly assets ensures $p_t \geq 0$. In the *bubbleless economy*, p_t is zero (p_t=0). In the *bubble economy*, p_t is strictly positive ($p_t > 0$). Entrepreneur i holds $k_{i,t}$ units of capital and $b_{i,t}^n$ units of bubbly assets. His or her total assets holdings are given by

$$\omega_{i,t} = v_t k_{i,t} + p_t b_{i,t}^n = a_{i,t} + b_{i,t}, \tag{7}$$

where $a_{i,t} \equiv v_t k_{i,t}$ and $b_{i,t} \equiv p_t b_{i,t}^n$. We assume that $\omega_{i,0} > 0$ for all entrepreneurs.

We derive the evolution of $\omega_{i,t}$. Suppose that the bubble economy prevails between t and $t+dt$. Between t and $t+dt$, entrepreneur i earns capital rental income $q_t k_{i,t} dt$ and profits given by (6). He or she consumes $c_{i,t}dt$ units of general good, incurs capital depreciation $\delta \cdot v_t k_{i,t} dt$ ($\delta > 0$), and purchases $dk_{i,t}$ units of capital and $db_{i,t}^n$ units of bubbly assets. If he or she sells capital (bubbly assets), $dk_{i,t}$ ($db_{i,t}^n$) is negative. Thus, we have

$$c_{i,t}dt + \delta v_t k_{i,t} dt + v_t dk_{i,t} + p_t db_{i,t}^n = q_t k_{i,t} dt + (v_t - 1) I_{i,t} dt + \sigma v_t I_{i,t} dW_{i,t}. \tag{8}$$

From (7), we have $d\omega_{i,t} = (dv_t)k_{i,t} + v_t dk_{i,t} + (dp_t) b_{i,t}^n + p_t db_{i,t}^n$. By using (7) and (8), we derive

$$d\omega_{i,t} = [r_t a_{i,t} + \psi_t b_{i,t} + (v_t - 1) I_{i,t} - c_{i,t}]\, dt + \sigma v_t I_{i,t} dW_{i,t}. \tag{9}$$

The rates of return on holding capital and bubbly assets are, respectively, given by

$$r_t dt \equiv \frac{qdt + dv_t - \delta v_t dt}{v_t} \quad \text{and} \quad \psi_t dt \equiv \frac{dp_t}{p_t}.$$

Note that r_t is deterministic. In the bubbleless economy, we have $p_t = b_{i,t} = \psi_t = 0$ in (9).

Given $\omega_{i,t} \equiv v_t k_{i,t} + p_t b_{i,t}^n$, there is a trade-off between holding capital and bubbly assets. However, there is no trade-off between bubbly assets $b_{i,t}^n$ and capital production $I_{i,t}$. Thus, entrepreneurs can not diversify investment risks by holding bubbly assets. This contrasts with Aoki *et al.* (2014) in which the rate of return on holding capital is stochastic and individuals hold (safe) bubbly assets to diversify capital holding risks.[14]

Following Weil (1987), we consider the stochastic bubbles which may burst in the future. The literature often assumes that once bubbles burst, they will never be valued in the subsequent future. Consider a sunspot shock that follows a Poisson process with a constant arrival rate $\mu > 0$. The sunspot shock triggers a asset bubble bust. Given that $p_t > 0$, p_{t+dt} remains strictly positive with probability $1 - \mu dt$. Otherwise, we have $p_{t+dt} = 0$.[15] Asset bubble bust is an aggregate shock that is independent of idiosyncratic shocks, $\sigma dW_{i,t}$. A larger μ means riskier bubbles. All entrepreneurs know the value of μ.

Utility maximization: Given $\omega_{i,0} > 0$, entrepreneur i maximizes (3) subject to (7) and (9). We do not impose the non-negativity constraints, $k_{i,t} \geq 0$ and $b_{i,t}^n \geq 0$. Since all

[14] They consider a budget constraint like $d\omega_{i,t} = [r_t a_{i,t} + \psi_t b_{i,t} - c_{i,t}]\, dt + \sigma_k a_{i,t} dW_{i,t}^k$, where $a_{i,t}$ is capital holdings and $W_{i,t}^k$ is a standard Brownian motion. If $\sigma_k > 0$, the rate of return on capital, $r_t dt + \sigma_k dW_{i,t}^k$, is stochastic. Holding safe bubbly asset, $b_{i,t}$, diversifies this capital holding risk.

[15] All the propositions in this study hold even if bubbles never burst $\mu = 0$.

entrepreneurs have the same ex-ante productivity, they do not have incentives to lend and borrow and hence the short sales constraint $b^n_{i,t} \geq 0$ never binds.[16] Appendix A shows that the behavior of entrepreneur i is summarized as follows:

$$c_{i,t} = \rho\omega_{i,t}, \tag{10a}$$

$$a_{i,t} = (1 - s_t)\omega_{i,t}, \tag{10b}$$

$$b_{i,t} = s_t\ \omega_{i,t}, \tag{10c}$$

$$s_t = \begin{cases} 1 - \frac{\mu}{\psi_t - r_t} & \text{in the bubble economy } (p_t > 0), \\ 0 & \text{in the bubbleless economy } (p_t = 0), \end{cases} \tag{10d}$$

$$I_{i,t} = \frac{v_t - 1}{(\sigma v_t)^2}\omega_{i,t}, \tag{10e}$$

$$d\omega_{i,t} = \left[r_t(1 - s_t) + \psi_t s_t + \left(\frac{v_t - 1}{\sigma v_t}\right)^2 - \rho \right] \omega_{i,t}dt + \left(\frac{v_t - 1}{\sigma v_t}\right)\omega_{i,t}dW_{i,t}. \tag{10f}$$

Here, we assume an inner solution for $I_{i,t} \geq 0$, which is satisfied in equilibrium we consider. The transversality condition is satisfied as follows:

$$\lim_{t\to\infty} E_t\left[\frac{\omega_{i,t}}{c_{i,t}}e^{-\rho t}\right] = \lim_{t\to\infty}\frac{1}{\rho}e^{-\rho t} = 0. \tag{11}$$

We focus only on equilibria where all of r_t, ψ_t, v_t and s_t are constant. Thus, (10f) shows that $\omega_{i,t}$ follows a geometric Brownian motion, which ensures that $\omega_{i,t} > 0$ since $\omega_{i,0} > 0$. (10a) is an usual consumption function under a logarithmic utility function.

(10b)–(10d) summarize entrepreneur i's portfolio choice between capital and bubbly assets. Particularly, s_t represents an incentive for holding bubbly assets. Since s_t is independent of i, all entrepreneurs hold the same fraction of their wealth as bubbly assets. Later, we observe that $s_t \in (0, 1)$ holds in the bubble economy, which ensures $k_{i,t} > 0$ and $b^n_{i,t} > 0$.

We mention the following three points. First, investment risk σ does not directly affect s_t, which means that entrepreneurs do not hold bubbly assets to diversify investment risk $\sigma I_{i,t}dW_{i,t}$. This contrasts with Aoki *et al.* (2014). Second, one may guess that when a positive shock hits entrepreneur i, he or she may accumulate bubbly assets (increase s_t) as a self-insurance and then resell bubbly assets (decrease s_t) when hit by a negative shock. This guess is not the case. Since s_t is independent of i, realization of idiosyncratic shock does not affect entrepreneurs' portfolio. Entrepreneurs do not use bubbly assets as a self-insurance.

Finally, entrepreneurs hold bubbly assets for purely speculative motive. In the bubble economy, the term $\psi_t - r_t$ in (10d) is the risk premium on bubbles that is positive in equilibrium if $\mu > 0$ (see (20d)).[17] Only if the risk premium is high enough to compensate for risks of bubble bust $\psi_t - r_t > \mu$, entrepreneurs hold bubbly assets $s_t > 0$.

(10e) shows decisions on capital production. Only if capital price is high enough to compensate capital production risks ($v_t > 1$), risk averse entrepreneurs choose positive capital production. As σ increases, entrepreneurs decrease capital production. If $\sigma = 0$, we have $v_t = 1$ and hence our model reduces to a textbook AK model.

[16] Kocherlakota (1992) shows that if individuals borrow and lend, a short sales constraint $b^n_{i,t} \geq 0$ is needed for the existence of bubbles.

[17] If $\mu = 0$, $\psi_t = r_t$ holds in the bubble economy. In this case, s_t is indeterminate at the individual entrepreneurs' level. However, this does not affect our main results.

2.3 Aggregation and competitive equilibrium

Let us define the following aggregate valuables, $C_t = \int_0^1 c_{i,t}di$, $I_t = \int_0^1 I_{i,t}di$, $K_t = \int_0^1 k_{i,t}di$, $b_t^n = \int_0^1 b_{i,t}^n di$, and $\omega_t = \int_0^1 \omega_{i,t}di$. Then, we have

$$\omega_t = v_t K_t + p_t b_t^n, \tag{12a}$$
$$C_t = \rho\omega_t, \tag{12b}$$
$$I_t = \frac{v_t - 1}{(\sigma v_t)^2}\omega_t. \tag{12c}$$

Since $I_{i,t}$ and $dW_{i,t}$ are independent and $dW_{i,t}$ follows a normal distribution with zero mean, we aggregate (5) as $dK_t \equiv \int_0^1(dx_{i,t})di - \delta K_t dt = [I_t + \sigma\int_0^1 I_{i,t}di\int_0^1(dW_{i,t})di - \delta K_t]dt = [I_t - \delta K_t]dt$. The long-run growth rate of economy is given by

$$g_t = \frac{\dot{K}_t}{K_t} = \frac{I_t}{K_t} - \delta. \tag{13}$$

Since total nominal supply of bubbly assets is constant at $M > 0$, the market for bubbly assets clears as $b_t^n = M$. The general good market clears as

$$Y_t = C_t + I_t, \tag{14}$$

For later use, let us define V_t and B_t as follows:

$$V_t \equiv \frac{1}{v_t} \quad \text{and} \quad B_t \equiv \frac{p_t M}{v_t K_t}. \tag{15}$$

V_t is price of general good in terms of capital and B_t is the value of bubbles relative to value of capital. We have $B_t > 0$ in the bubble economy, whereas we have $B_t = 0$ in the bubbleless economy. Since $p_t M = s_t\omega_t$ holds from (10c) and $b_t^n = M$, we have $s_t = B_t/(1 + B_t)$. Thus, $s_t \in (0, 1)$ holds in the bubble economy ($B_t > 0$). Both V_t and B_t are jump variables. A steady state equilibrium is an equilibrium where V_t and B_t are constant. At a steady state equilibrium g_t becomes constant and K_t, C_t, Y_t, and p_t grow at the same rate.

2.4 Economy without investment risks: $\sigma = 0$

If $\sigma = 0$ holds, our model reduces to a standard AK model and asset bubbles can not exist, as shown in the following proposition.

Proposition 1 *Suppose that $\sigma = 0$ and (4) hold. (i) There exist an unique bubbleless equilibrium where V_t, r_t, and g_t satisfy*

$$V_t = 1 \equiv V_{NR}, \quad r_t = A - \delta \equiv r_{NR}, \quad \text{and} \quad g_t = A - \rho - \delta \equiv g_{NR} \ \ (< r_{NR}). \tag{16}$$

Inequality (4) ensures $I_t > 0$. (ii) There exists no bubble economy.

(Proof) See Appendix B.

3 Investment risks and the long-run effects of bubbles

This section shows that with investment risks $\sigma > 0$, asset bubbles emerge in the benchmark economy. We also examine how bubbles affect long-run growth. We first provide a set of equations that characterize equilibrium dynamics.

Proposition 2 *Suppose $\sigma > 0$. In an equilibrium where $I_t > 0$ holds, V_t and B_t satisfy*

$$A = \left[\frac{\rho}{V_t} + \frac{1 - V_t}{\sigma^2}\right](1 + B_t), \tag{17a}$$

$$\dot{B}_t = \left\{\mu(1 + B_t) + AV_t - \frac{1 - V_t}{\sigma^2}(1 + B_t)\right\} B_t. \tag{17b}$$

(Proof) See Appendix C.

(17a) comes from general good market equilibrium condition (14). The left-hand side (LHS) shows general good supply (Y_t/K_t) while the right-hand side (RHS) shows general good demand ($(C_t + I_t)/K_t$). The dynamics of B_t follow (17b).

3.1 Bubbleless economy

In the bubbleless economy where $B_t = \dot{B}_t = 0$ holds, (17a) alone determines equilibrium V_t. We prove the following proposition.

Proposition 3 *Suppose that $\sigma > 0$. If and only if (4) holds, there exists a unique bubbleless steady-state equilibrium such that $I_t > 0$ holds and V_t, r_t, and g_t satisfy*

$$V_t = V_L \quad (< V_{NR} \equiv 1), \tag{18a}$$

$$r_t = AV_L - \delta \equiv r_L \quad (< r_{NR}), \tag{18b}$$

$$g_t = \frac{1 - V_L}{\sigma^2} - \delta \equiv g_L \quad (< g_{NR}), \tag{18c}$$

where $V_L \in (\rho/A,\ 1)$ is a positive solution of (17a) under $B_t = 0$.

(Proof) See Appendix D.

Faced with investment risk ($\sigma > 0$), entrepreneurs invest less in capital production, compared to no risk case $\sigma = 0$. Hence, growth rate is reduced ($g_L < g_{NR}$). The reduced capital production increases capital price $v_t = V_t^{-1}$ ($V_L < V_{NR}$) and hence decreases the return on capital holding ($r_L < r_{NR}$). In other words, risks depress investment and then lower the return on capital through saving-investment balance. This creates a basis for bubbles.

3.2 Bubble economy

By using (17a) and (17b), we show the existence of a bubble steady state where a low rate of return on holding capital induces entrepreneurs to hold bubbly assets with a high return.

Proposition 4 *Suppose that $\sigma > 0$.*
(i) If $A \le \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}$, the bubble steady-state equilibrium does not exist.
(ii) If

$$A > \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}, \tag{19}$$

there exist σ_1 and σ_2, where $0 < \sigma_1 < \sigma_2 < 1/(\rho+\mu)^{\frac{1}{2}}$, such that

(a) if $\sigma \notin (\sigma_1, \sigma_2)$, the bubble steady-state equilibrium does not exist;

(b) if $\sigma \in (\sigma_1, \sigma_2)$, there exists a unique bubble stead-state equilibrium where $I_t > 0$ holds and V_t, B_t, r_t, ψ_t and g_t satisfy

$$V_t = 1 - \sigma(\rho+\mu)^{\frac{1}{2}} \equiv V^* \quad (\in (0,\ V_{NR})), \tag{20a}$$

$$B_t = \frac{A\left[1 - \sigma(\rho+\mu)^{\frac{1}{2}}\right]}{\frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}} - \mu} - 1 \equiv B^* \quad (> 0), \tag{20b}$$

$$r_t = AV^* - \delta \equiv r^*, \tag{20c}$$

$$\psi_t - r_t = \mu(1+B^*) > 0, \tag{20d}$$

$$g_t = \frac{1-V^*}{\sigma^2}(1+B^*) - \delta \equiv g^*. \tag{20e}$$

(Proof) See Appendix E.

Proposition 4 (i) implies that asset bubbles do not exist if technology level is extremely low (small A). The intuition is simple. If A is small, capital accumulates at a considerably low rate, which cannot sustain expansion of asset bubbles.[18]

Only in economies with advanced technology (large A), asset bubbles may exist on the condition that insurance market is moderately developed, $\sigma \in (\sigma_1, \sigma_2)$ (Proposition 4 (ii)).[19] With a large risk ($\sigma > \sigma_1$), entrepreneurs reduce capital investment considerably. Through saving-investment balance, the rate of return on holding capital r_t decreases, which leads to a positive risk premium on bubbly assets $\psi_t - r_t > 0$. Thus, entrepreneurs have an incentive to hold bubbly assets. Only if investment risk is not too large ($\sigma < \sigma_2$), capital accumulates at a sufficiently high rate and can sustain expansion of asset bubbles. Thus, only for medium investment risks ($\sigma \in (\sigma_1, \sigma_2)$), the bubble steady state exists.[20]

Our mechanism behind asset bubbles is different from those of the existing models. Let us focus on infinitely-lived agent models. In Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018) who consider financial frictions, borrowing constraints play an important role, which is absent from our model. In Aoki *et al.* (2014) where return on holding capital bears risks (see footnote 14), entrepreneurs hold bubbly assets to diversify

[18]We have $Y_t = AK_t \ge C_t \ge \rho p_t M$ because of (1), (12a), (12b), (14), $I_t \ge 0$, and $b_t^n = M$. Thus, $p_t M$ can not grow faster than K_t ($\psi(\equiv \dot{p}_t/p_t) < g^*$). In the steady-state equilibrium, $\psi \le g^*$ must hold.

[19]Hirano and Yanagawa (2017) show that asset bubbles are likely to arise in an economy with large inequality in productivity among firms.

[20]Idiosyncratic nature of investment risk is essential for the existence of bubbles. If a positive (negative) shock $dW_{i,t} > 0$ ($dW_{i,t} < 0$) hits an entrepreneur, he or she accumulates more (less) wealth than the average entrepreneurs (see (10f)). (Note that he or she does not change the share of bubbly assets holdings, s.) This heterogeneity triggers trade of assets, including bubbly assets, among entrepreneurs.

the risks and bubbly assets provide a lower rate of return than capital. In our model, the rate of return on bubbly assets is higher than capital because investment risks depresses the rate of return on holding capital. This stimulates entrepreneurs to hold bubbly assets for purely speculative motive.

Remark: We can show that a bubble steady state exists if and only if $r_L < g_L - \mu$ holds in the bubbless steady-state equilibrium (see Appendix G). Previous studies provide similar existence conditions. Our mechanism behind a low rate of return on capital is different from previous studies again. In overlapping-generations models, overaccumulation of capital results in a low interest rate. In Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018), borrowing constraints depresses demand for borrowing, which results in a low interest rate. In Aoki *et al.* (2014), risk premium on holding capital generates a low risk-free rate. In our model, the uninsured risks depress investment and hence lower the rate of return on capital r_L through saving-investment balance.

3.3 Coexistence of the bubble and bubbleless steady states

Since (19) implies (4), we immediately obtain the following corollary.

Corollary 1 *Suppose that $\sigma > 0$ and that (19) holds. If $\sigma \in (\sigma_1, \sigma_2)$, there exist two steady-state equilibria; the bubble and bubbleless steady-state equilibria.*

Corollary 1 states that the bubble and bubbleless steady states coexist under the same parameter set. Figure 3 shows the phase diagram (see Appendix F). The bubble steady state is unstable while the bubbleless one is totally stable.

[Figure 3]

A sunspot shock triggers a collapse of bubbles. Assume that an economy is in the bubble steady state at time 0. At time $t_1 (> 0)$, a sunspot shock hits the economy. (The shock follows a Poisson process with an arrival rate μ.) Then, asset bubbles burst. Since both V_t and B_t are jump variables, the economy immediately jumps to the bubbleless steady state. The remaining of this section examines how a collapse of bubbles affects long-run growth.

3.4 Growth effects of bubbles

The following proposition shows that even though there is no borrowing constraint in our model, asset bubbles enhance long-run growth under some conditions.

Proposition 5 *Suppose that both the bubble and bubbleless steady-state equilibrium exist.*
(i) If $\mu + 2\rho + 2\{\rho(\mu + \rho)\}^{1/2} < A \leq 2(\mu + 2\rho)$, we have $g^* > g_L$.
(ii) If $A > 2(\mu + 2\rho)$, then there exists $\overline{\sigma} \in (\sigma_1, \sigma_2)$ such that
(a) if $\sigma = \overline{\sigma}$, we have $g^* = g_L$;
(b) if $\sigma \in (\sigma_1, \overline{\sigma})$, we have $g^* < g_L$;
(c) if $\sigma \in (\overline{\sigma}, \sigma_2)$, we have $g^* > g_L$.

(Proof) See Appendix H.

Proposition 5 (i) shows that in the economy with relatively low technology (small A), bubbles always enhance long-run growth. Proposition 5 (ii) (c) shows that in the economy with advanced technology (large A), Bubbles boost long-run growth if the insurance market is less developed.

To understand Proposition 5 intuitively, we rewrite the first-order condition for $I_{i,t}$, (A.14), as

$$(v_t - 1)I_{i,t} = \frac{(\sigma \cdot v_t I_{i,t})^2}{v_t k_{i,t} + b_{i,t}}. \tag{21}$$

The LHS shows the (average) investment return $(v_t - 1)I_{i,t}$. This represents entrepreneurs' incentive of increasing $I_{i,t}$. The RHS shows investment risk $(\sigma v_t I_{i,t})^2$ relative to entrepreneur i's wealth, which represents entrepreneurs' incentive of reducing $I_{i,t}$ to avoid investment risks.

Asset bubbles $b_{i,t}$ have a direct effect on the RHS. If v_t remains constant, asset bubbles increase entrepreneur i's wealth from $v_t k_{i,t}$ to $v_t k_{i,t} + b_{i,t}$, which negatively affects the RHS. In sum, asset bubbles make entrepreneurs wealthy, which raises entrepreneurs' tolerance to investment risks and encourage them to take more investment risks.

Besides, asset bubbles indirectly affect the RHS through v_t. General good market equilibrium $Y_t = C_t + I_t$, or equivalently (17a), shows this effect. The bubbleless steady state is characterized by (17a) alone. Figure 4 shows the graphs of both sides of (17a). Asset bubbles make entrepreneurs wealthy, which has positive effects on demand for general good $C_t + I_t$ (see (12b) and (12c)). Since supply of general good is fixed at $Y_t = AK_t$ in the short run, price of general good relative to capital, $1/v_t$, increases. Thus, we have $v_t^* < v_{L,t}$ (see Appendix G for a formal proof). A decrease in v_t reduces the variance of investment shock $(\sigma \cdot v_t I_{i,t})^2$, which reduces the RHS and crowds in capital production. A decrease in v_t have a negative effect on the LHS, which crowds out capital production.

[Figure 4]

If A is small, the average growth rate of capital is low. Thus, entrepreneurs care more about investment risks. If σ is large, the RHS of (21) becomes relatively important. Thus, the crowding-in effect dominates the crowding-out effect. Proposition 5 (i) and (ii)(c) hold.

We emphasize the importance of the direct wealth effect of bubbles. If there is no direct effect, (21) reduces to

$$(v_t - 1)I_{i,t} = \frac{(\sigma \cdot v_t I_{i,t})^2}{v_t k_{i,t}} \quad \Rightarrow \quad I_{i,t} = \frac{1}{\sigma^2}\left(1 - \frac{1}{v_t}\right) k_{i,t}.$$

As discussed above, asset bubbles decrease v_t. The above equation shows that without the direct wealth effect, asset bubbles discourage capital production of each entrepreneur. We conclude that asset bubbles stimulate growth mainly because asset bubbles make entrepreneurs wealthy and then encourage them to take more investment risks.

In infinitely-lived agent models without borrowing constraints, Aoki *et al.* (2014) show that asset bubbles always decrease growth. Recent studies show that in the presence of borrowing constraints, bubbles may increase growth. See Kocherlakota (2009), Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and and Miao and Wang (2018). Note that those authors assume incomplete insurance to ensure binding borrowing constraints.

In our model, there are no borrowing constraints. Nevertheless, asset bubbles stimulate growth. The key to this result is incomplete insurance. Our result suggests that even without financial frictions, bubbles have a growth enhancing effect if insurance is incomplete. We emphasize the similarity between the existing models and our models. In both models, bubbles positively affect entrepreneurs' wealth, which in turn boosts long-run growth.

3.5 Collapse of bubbles and the long-run effect

From Corollary 1 and Proposition 5, we obtain the following corollary.

Corollary 2 *Suppose that either Proposition 5 (i) or (ii)(c) holds. In addition, suppose that the economy is initially in the bubble steady state and that at time $t_1(>0)$, a sunspot shock triggers a collapse of asset bubbles. Then, at time t_1, the economy jumps to the bubbleless steady state and long-run growth rate of the economy declines suddenly and permanently.*

Even without any fundamental changes, the collapse of bubbles causes a permanent decline in growth, leading to a prolonged recession.

The benchmark model fails to capture instant and sharp contractions (Fact 1). At the moment of asset bubble bust, general good production remains constant since it is determined by capital, $Y_t = AK_t$. A collapse of bubbles affects only the division between C_t and I_t in the short run, because $Y_t = C_t + I_t$ holds. Thus, C_t and I_t moves in the opposite direction in the short run.

4 Capital Utilization and the short-run effects of bubbles

We now endogenize capital utilization and then show that under this extension, a collapse of bubbles causes instant declines in all of capital utilization rate, consumption, capital production, general good production, GDP, wealth-to-GDP ratio, and long-run growth.

Our formulation of capital utilization follows Miao (2014).[21] Denote capital utilization rate of entrepreneur i as $\zeta_{i,t} \geq 0$. We do not impose any upper bounds on $\zeta_{i,t}$ for simplicity. If entrepreneur i lends one unit of capital with utilization rate $\zeta_{i,t}$, he or she earns capital rental income of $q_t\zeta_{i,t}$. As capital utilization increases, capital depreciates more. We assume that the depreciation rate is given by $\bar{\delta}(\zeta_{i,t}) = \delta\zeta_{i,t}^2/2$, where $\delta > 0$. The rate of return on holding one unit of capital is given by

$$r_{i,t}dt = \frac{q_t\zeta_{i,t} \cdot dt - \bar{\delta}(\zeta_{i,t})v_t \cdot dt + dv_t}{v_t}. \tag{22}$$

Given q_t and v_t, entrepreneur i chooses $\zeta_{i,t}$ to maximize $r_{i,t}$, which yields

$$\zeta_{i,t} = \frac{q_t V_t}{\delta} \equiv \zeta_t, \tag{23}$$

where $V_t = 1/v_t$ again. A high capital price means a large value of capital depreciation. Thus, ζ_t decreases with capital price $v_t = V_t^{-1}$. Since all entrepreneurs choose the same

[21] Guerron-Quintana *et al.* (2019) also consider endogenous capital utilization in a model with bubbles, although they focus on the effect of recurrent bubbles.

level of capital utilization, we have $r_{i,t} = r_t$. In other respects, the optimization problem of entrepreneurs is the same as the benchmark model. Thus, (10a)–(10f) and (11) hold.

The production function of the general good is given by $Y_t = AK_t^D$, where K_t^D is capital input. The capital rental rate is $q_t = A$. Since $K_t^D = \zeta_t K_t$ holds in equilibrium, the aggregate production function is given by $Y_t = A\zeta_t K_t$. Since the capital depreciation rate is $\bar{\delta}(\zeta_t) = (AV_t)^2/(2\delta)$, growth rate of the economy is given by

$$g_t = \frac{1 - V_t}{\sigma^2}(1 + B_t) - \frac{(AV_t)^2}{2\delta}. \tag{24}$$

We focus on economies with investment risks, $\sigma > 0$. Analogously to Proposition 2, in an equilibrium, V_t and B_t satisfy

$$A\zeta_t = \left(\frac{\rho}{V_t} + \frac{1 - V_t}{\sigma^2}\right)(1 + B_t), \tag{17a$'$}$$

$$\dot{B}_t = \left[\mu(1 + B_t) + A\zeta_t V_t - \frac{1 - V_t}{\sigma^2}(1 + B_t)\right] B_t, \tag{17b$'$}$$

were ζ_t is given by

$$\zeta_t = \frac{AV_t}{\delta}. \tag{25}$$

(17a$'$) comes from general good market equilibrium (14) that relates general good supply, Y_t/K_t, of the LHS with general good demand, $(C_t + I_t)/K_t$, on the RHS.

Remark: If there are no investment risks $\sigma = 0$, we have $V_t = 1$. Capital utilization rate is determined by exogenous parameters, $\zeta_t = A/\delta$. Thus, the model with endogenous capital utilization works the same way as a textbook AK model.

4.1 Existence of steady states

In the following discussion, a hat above each variable indicates that capital utilization is endogenized. We first show the existence of bubbleless steady-state equilibrium.

Proposition 6 *Suppose that $\sigma > 0$. If and only if $\delta > 0$ is small enough to satisfy*

$$\delta\rho < A^2, \tag{26}$$

there exists a unique bubbleless steady-state equilibrium such that $I_t > 0$ holds and $V_t = \hat{V}_L$, $\zeta_t = \hat{\zeta}_L$, and $g_t = \hat{g}_L$, where $\hat{V}_L \in (0, 1)$, $\hat{\zeta}_L$, and $\hat{g}_L$ are defined in Appendix I.

(Proof) See Appendix I.

Next, we show the existence of bubble steady-state equilibrium.

Proposition 7 *Suppose that the following two inequalities hold;*

$$\sigma < (\rho + \mu)^{-\frac{1}{2}}, \tag{27}$$

$$\delta < AV^*(1 + B^*), \tag{28}$$

where V^* and B^* are given by (20a) and (20b), respectively. Then, there exists a unique bubble steady-state equilibrium where $I_t > 0$ holds and $V_t = \hat{V}^*$, $B_t = \hat{B}^*$, $\zeta_t = \hat{\zeta}^*$, and $g_t = \hat{g}^*$, where $\hat{V}^* \in (0, 1)$, $\hat{B}^* > 0$, $\hat{\zeta}^*$, and $\hat{g}^*$ are defined in Appendix J.

(Proof) See Appendix J.

From (27) and (28), we can derive lower and upper bounds of σ that ensure the existence of the bubble steady state. Thus, the intuition behind Proposition 7 is similar to that of Proposition 4.[22]

Appendix K shows that conditions (27) and (28) implies (26). The following corollary holds.

Corollary 3 *Suppose that conditions* (27) *and* (28) *hold. Then, there exist two steady-state equilibria; the bubble and bubbleless steady-state equilibria.*

As in the benchmark AK model, a sunspot shock triggers a collapse of bubbles. Since both of V_t and B_t are jump variables, after a collapse of the bubbles, the economy immediately moves from the bubble steady state to bubbleless steady state.

4.2 Comparison between the bubble an bubbless economies

We examine how asset bubbles affect aggregate variables. Denote capital price, capital utilization rate, aggregate wealth, wealth-to-GDP ratio, aggregate consumption, capital investment, general good production, and real and nominal GDP in the bubble (bubbleless) steady state as $\hat{v}^*$, $\hat{\zeta}^*$, $\hat{\omega}_t^*$, $\hat{\Upsilon}^*$, $\hat{C}_t^*$, $\hat{I}_t^*$, $\hat{Y}_t^*$, $\widehat{GDP}_t^*$, and $\widehat{nGDP}_t^*$ ($\hat{v}_L$, $\hat{\zeta}_L$, $\hat{\omega}_{L,t}$, $\hat{\Upsilon}_L$, $\hat{C}_{L,t}$, $\hat{I}_{L,t}$, $\hat{Y}_{L,t}$, $\widehat{GDP}_{L,t}$, and $\widehat{nGDP}_{L,t}$), respectively. We omit time index t from $\hat{v}^*$, $\hat{\zeta}^*$, $\hat{\Upsilon}^*$, $\hat{v}_L$, $\hat{\zeta}_L$, and $\hat{\Upsilon}_L$ because they are constant at a steady state.

Nominal GDP and the wealth-to-GDP ratio are given by

$$\widehat{nGDP}_t = \hat{C}_t + \hat{v}\hat{I}_t, \quad \text{and} \quad \hat{\Upsilon} = \hat{\omega}_t / \widehat{nGDP}_t,$$

respectively. In the real GDP, we set capital price at the bubble steady state price, v^*.

$$\widehat{GDP}_t = \hat{C}_t + \hat{v}^* \hat{I}_t.$$

In the above equations, we omit asterisks and subscript L, except for v^* in $\widehat{GDP}_t$.

Now, we prove the following proposition.

Proposition 8 *Suppose that both the bubble and bubbleless steady state equilibria exist. Then, the following statements are true.*
(i) We have

$$\hat{\zeta}^* > \hat{\zeta}_L \quad \textit{and} \quad \hat{\Upsilon}^* > \hat{\Upsilon}_L.$$

(ii) Suppose that both steady states have the same level of capital stock at time t. Then, we have

$$\hat{C}_t^* > \hat{C}_{L,t} \quad \textit{and} \quad \hat{Y}_t^* > \hat{Y}_{L,t}.$$

[22] Condition (28) can be written as $A^2(\rho+\mu)\sigma^3 - 2A^2(\rho+\sigma)^{1/2}\sigma^2 + (A^2+\mu\delta)\sigma - \delta(\rho+\mu)^{1/2} > 0$. This inequality and condition (27) give lower and upper bounds of σ.

If $\rho > 0$ is sufficiently small, or if $\sigma \in (\underline{\sigma},\ (\rho+\mu)^{-1/2})$, we have

$$\hat{I}_t^* > \hat{I}_{L,t} \quad \text{and} \quad \widehat{GDP}_t^* > \widehat{GDP}_{L,t}. \tag{29}$$

(iii) Finally, if $\rho > 0$ is sufficiently small, we have

$$\hat{g}^* > \hat{g}_L. \tag{30}$$

$\underline{\sigma}$ is defined in Appendix L.

(Proof) See Appendix L.

Asset bubbles promote various macroeconomic performance. How bubbles affect aggregate demand is a key to this result. Let us use general good market equilibrium condition $\hat{Y}_t = \hat{C}_t + \hat{I}_t$, or equivalently (17a′). In the bubbleless economy where $B_t = 0$ holds, (17a′) and (25) determine equilibrium values of $\hat{V}_t$ and $\hat{\zeta}_t$. The LHS of (17a′) is general good supply relative to capital, $\hat{Y}_t/\hat{K}_t = A\hat{\zeta}_t$. Since a high $\hat{v}_t$ discourages capital utilization, $\hat{Y}_t/\hat{K}_t$ decreases with $\hat{v}_t(\equiv \hat{V}_t^{-1})$ (see Figure 5). The RHS of (17a′) is demand for general good relative to capital, $(\hat{C}_t + \hat{I}_t)/\hat{K}_t$ (see (12b) and (12c) too). This increases with $\hat{v}_t$ partly because $\hat{v}_t$ increases entrepreneurs' wealth $\hat{\omega}_t = \hat{v}_t\hat{K}_t + \hat{b}_t$.

[Figure 5]

Asset bubbles make entrepreneurs wealthy and thus stimulate aggregate consumption (see (12b)). Besides, as Proposition 5 in Section 3.4 shows, asset bubbles encourage entrepreneurs to take more risk and hence increase capital investment. Asset bubbles increase aggregate demand for general good. The graph of $(\hat{C}_t + \hat{I}_t)/\hat{K}_t$ moves upward and $\hat{v}_t$ decreases. A reduction in $\hat{v}_t$ encourages capital utilization (ζ increases from $\hat{\zeta}_L$ and $\hat{\zeta}^*$). Hence, general good production increases. In sum, asset bubbles stimulate aggregate demand for general good, leading to an increase in capital utilization and production.

Since asset bubbles increase both aggregate consumption and investment, the bubble steady state has a larger real GDP than the bubbleless one. Since asset bubbles increase wealth more than nominal GDP, the wealth-to-GDP ratio is higher in the bubble economy than in the bubbleless economy.[23]

4.3 Collapse of bubbles: the short- and long-run effects

Corollary 3 and Proposition 8 immediately yield the following important result.

Corollary 4 *Suppose that there exist both the bubble and bubbleless steady state equilibria and that $\rho > 0$ is small enough to ensure* (29) *and* (30)*. In addition, suppose that the economy is initially in the bubble steady state and that at time $t_1(> 0)$, a sunspot shock causes a collapse of asset bubbles. Then, at time t_1, the economy jumps to the bubbleless steady state and hence experiences instant declines in all of capital utilization rate, aggregate consumption, aggregate investment, general good production, economic growth rate, real GDP, and the wealth-to-GDP ratio.*

[23] We can show that even if utilization rate is exogenously fixed, asset bubbles increase the wealth-to-GDP ratio because (L.2) in Appendix L holds even if utilization rate is fixed.

A collapse of bubbles initially depresses various macroeconomic performances at time t_1. Since growth is depressed permanently, the economy experiences a long-term depression. See Figure 2. This prediction of our model is consistent with recent economic crisis.

Demand side is relatively important for initial declines, whereas supply side is relatively important for the long-term depression. When bubbles burst, entrepreneurs become less wealthy. The aggregate demand for general good decreases. Then, some of production facilities (capital) become inactive ($\hat{\zeta}_t$ decreases) although the number of production facilities (capital) remains constant. This causes instant and simultaneous declines in GDP and other aggregate variables . Because of decreased growth, the number of production facilities (capital) gradually decreases in the long run. Accordingly, the supply of general good decreases in the long run, which results in a prolonged recession.

The following four points are worth mentioning. First, even without any changes in fundamentals, a collapse of bubbles causes simultaneous declines in all of capital utilization, aggregate consumption, capital and general goods production, GDP, and growth.

Second, a collapse of bubbles has not only long-run but also short-run negative impacts. For example, at the moment of a collapse of bubbles (time t_1), real GDP decreases from $\widehat{GDP}_t^*$ to $\widehat{GDP}_{L,t}$. In the long run, depressed growth further depresses macroeconomic performance. Remember that as discussed in Introduction, the existing models of rational pay less attention to the short-run effects of a collapse of bubbles because they focus primary on how bubbles affect long-run capital accumulation.[24]

Third, at the moment of a collapse of bubbles (time t_1), capital remains unchanged in our model. Even in a short period during which capital does not change, a collapse of bubbles induces an instant contraction at time t_1. This is because a collapse of bubbles induces an instant decline in capital utilization, which allows general good production $\hat{Y}_t(= A\hat{\zeta}_t\hat{K}_t)$ to decline instantly. This is consistent with Panels (b), (c), and (d) in Figure 1.

Finally, we examine if aggregate investment decreases more or less than consumption in real terms. We show the following proposition.

Proposition 9 *Suppose that both the bubble and bubbleless steady state equilibria exist. Then, for $\sigma \in (\underline{\sigma},\ (\rho+\mu)^{-1/2})$, we have*

$$0 > \frac{\hat{C}_{L,t} - \hat{C}_t^*}{\hat{C}_t^*} > \frac{\hat{v}^*\hat{I}_{L,t} - \hat{v}^*\hat{I}_t^*}{\hat{v}^*\hat{I}_t^*}.$$

(Proof) See Appendix M.

Since the bubble steady state equilibrium exists, inequality $\sigma < (\rho+\mu)^{-1/2}$, (27), is satisfied. Condition $\sigma \in (\underline{\sigma}, (\rho+\mu)^{-1/2})$ means a relatively large σ. Proposition 9 says that with relatively large σ, at the moment of a collapse of bubbles, aggregate investment decreases more than aggregate consumption. As Proposition 5 (ii) shows, if σ is large enough, asset bubbles promote investment even if capital utilization is exogenous. Thus, a collapse of bubbles induces a large decline in aggregate investment, which is consistent with data.

[24] As discussed in Introduction, the existing models of rational bubbles do not capture an initial decline in final output (real GDP) because final output is determined mainly by capital. Consequently, in the short run, a collapse of bubbles affects only the division of output between aggregate consumption and investment, and hence the two aggregate variables move in the opposite direction.

4.4 Temporal technology shock and collapse of bubbles

This section argues that in the presence of asset bubbles, even a temporal technology shock can have a permanent effect on macroeconomic performance.

We first point out that a technology shock may trigger a collapse of bubbles. Appendix K shows that the right-hand side of (28) increases with A (see (K.1)). Thus, if a negative shock on A hits the economy, condition (28) is violated and then the bubble steady state no longer exists. Note that (26) can be satisfied even if (28) is not satisfied (see Appendix K and Corollary 3). Thus, the bubbleless steady state still exists.

Let us consider the following scenario. Suppose that at time 0, technology level of the economy, A, satisfies condition (28) and then the economy is in the bubble steady state. At time $t_1(> 0)$, an unexpected negative technology shock hits the economy (a decrease in A) and then condition (28) breaks. Asset bubbles collapse and the economy jumps to the bubbleless steady state. At time $t_2(> t_1)$, technology level unexpectedly returns to the original level. After time $t_2(> t_1)$, the bubbleless economy continues to prevail.

Under this scenario, at time t_1, the economy experiences instant declines in capital utilization, aggregate consumption and investment, general goods production, GDP, and growth. These instant declines are caused by two factors. Of course, the negative shock on A itself has a negative impact on macroeconomy. In addition, the collapse of bubbles causes the economy to jump to the bubbleless steady state. This means that asset bubbles amplify the negative impact of the negative technology shock.

When A returns to the original level at time t_2, macroeconomic performance recovers slightly. However, since bubbles no longer exist, the economy remains to show a poorer performance than the initial economy. Since growth remains lower than the initial growth rate, economic activities are depressed further in the long run. Thus, even a temporally negative technology shock has permanent negative impacts on macroeconomic performance.

4.5 Fundamental shocks without bubbles

Without bubbles, changes in fundamentals can not explain some of comovements among aggregate variables. Appendix N shows that in the bubbleless steady state, (i) each of a decrease in A, an increase in δ, and an increase in σ reduces growth rate $\hat{g}_L$ but increases the wealth-to-GDP ratio $\hat{\Upsilon}_L$ and (ii) an increase in ρ reduces $\hat{g}_L$ but increase capital utilization $\hat{\zeta}_L$. These are not surprising results since our model is quite simple. Nevertheless, once asset bubbles are allowed, our simple model provides an empirically relevant prediction.

4.6 Bubbles and welfare

The aggregate capital at time t is $\hat{K}_t$. Entrepreneur i holds $\hat{k}_{i,t}$ units of capital at time t, where $\int \hat{k}_{i,t} di = \hat{K}_t$. Appendix O shows that in the bubbleless steady state, utility of entrepreneur i at time t, $W_L(\hat{k}_{i,t})$, is given by

$$\rho W_L(\hat{k}_{i,t}) = \log \frac{\hat{C}_{L,t}}{\hat{K}_t}\hat{k}_{i,t} + \frac{1}{\rho}\left[\hat{g}_L - \frac{1}{2}\sigma^2(\hat{g}_L + \bar{\delta}(\hat{\zeta}_L))^2\right]. \quad (31)$$

The term $-\frac{\sigma^2}{2\rho}(\hat{g}_L + \bar{\delta}(\hat{\zeta}_L))^2$ captures utility loss from investment risks. Naturally, a large σ implies a large utility loss. Similarly, in the bubble steady state, utility of entrepreneur i at time t, $W^*(\hat{k}_{i,t})$, is given by

$$\rho W^*(\hat{k}_{i,t}) = \log \frac{\hat{C}_t^*}{\hat{K}_t}\hat{k}_{i,t} + \frac{1}{\rho}\left[\hat{g}^* - \frac{1}{2}\left\{\frac{\sigma\left(\hat{g}^* + \bar{\delta}(\hat{\zeta}^*)\right)}{1+\hat{B}^*}\right\}^2\right] - \mu\left[W^*(\hat{k}_{i,t}) - W_L(\hat{k}_{i,t})\right]. \quad (32)$$

See Appendix O again. The last term represents utility loss of bubbles burst. The term $-\frac{\sigma^2}{2\rho}\left(\frac{\hat{g}^*+\bar{\delta}(\hat{\zeta}^*)}{1+\hat{B}^*}\right)^2$ captures utility loss from investment risks and shows that $\hat{B}^*$ mitigate the utility loss. Intuition is simple. As we discuss just after Proposition 5, given v_t, asset bubbles make entrepreneurs wealthier and then increase entrepreneurs' tolerance to investment risks.

From (31) and (32), we obtain

$$(\rho+\mu)\left[W^*(\hat{k}_{i,t}) - W_L(\hat{k}_{i,t})\right] = \log\frac{\hat{C}_t^*}{\hat{C}_{L,t}} + \frac{\hat{g}^* - \hat{g}_L}{\rho} + \frac{\sigma^2}{2\rho}\left\{\left(\hat{g}_L + \bar{\delta}(\hat{\zeta}_L)\right)^2 - \left(\frac{\hat{g}^* + \bar{\delta}(\hat{\zeta}^*)}{1+\hat{B}^*}\right)^2\right\}.$$

From Proposition 8, the first and second terms are positive. Since asset bubbles mitigate utility loss from investment risks, the term $(\hat{g}_L + \bar{\delta}(\hat{\zeta}_L))^2 - \left(\frac{\hat{g}^*+\bar{\delta}(\hat{\zeta}^*)}{1+\hat{B}^*}\right)^2$ is always positive (see Appendix P for a proof). We obtain the following proposition.[25]

Proposition 10 *Suppose that both the bubble and bubbleless steady-state equilibria exist and that $\rho > 0$ is small enough to ensure that $\hat{g}^* > \hat{g}_L$. At the moment of a collapse of bubbles caused by a sunspot shock, welfare of all entrepreneurs decreases instantly.*

5 Discusssion and Conclusion

We construct an infinitely-lived agent model of rational bubbles. Even without borrowing constraints, asset bubbles emerge and accelerate long-run growth. If capital utilization is endogenized, a burst of bubbles causes an instant contraction of various economic activities as well as a prolonged low growth. Since the existing studies on rational bubbles do not fully address how the instant contraction, we believe that our model makes some progress.

Still, our model has some shortcomings. We shortly discuss the limitations of the present study and the possible future works. First, the present study is purely qualitative. Thus, it is important to examine the impact of a burst of bubbles quantitatively. Second, as Mendoza and Terrones (2012) show, not all but many of credit booms end with an economic crisis. The present study does not address how asset bubbles are related to credit booms. Thus, incorporating credit frictions into our model would be an important extension. Third, no policy interventions are considered in this study. It is said that contractionary monetary policy might have triggered the asset bubble bust around 1990 in Japan. How policy interventions affect the existence of bubbles and the impacts of bubbles would be interesting. Fourth, asset

[25] Appendix Q shows that bubbles improve welfare even if capital utilization is exogenously fixed.

bubbles might be contagious internationally. Considering effects of bubbles in multi-country settings has to be done.

Many authors have already tackled these issues by using mainly OLG models.[26] However, since our model is quite simple and fairy close to standard macroeconomic models that are widely used in modern macroeconomic literature, we believe that our model would be an useful basis for these extensions.

References

[1] Abel, A. B., Mankiw, N. G., Summers, L. H., and Zeckhauser, R. J. (1989) Assessing dynamic efficiency: Theory and evidence. *Review of Economic Studies*, 56, 1-19.

[2] Aiyagari, S. R. and McGrattan, E. R. (1998) The optimum quantity of debt. *Journal of Monetary Economics*, 42, 447-469.

[3] Aliber, R. Z. and Kindleberger, C. P. (2015) Manias, Panics and Crashes: A History of Financial Crises, 7th edition, Palgrave Macmillan

[4] Aoki, K., Nakajima, T., and Nikolov, K. (2014) Safe asset shortages and asset price bubbles. *Journal of Mathematical Economics*, 53, 164-174.

[5] Aoki, K., and Nikolov, K. (2015) Bubbles, banks and financial stability. *Journal of Monetary Economics*, 74, 33-51

[6] Baily, M. N., Hulten, C., Campbell, D., Bresnahan, T., and Caves, R. E. (1992) Productivity dynamics in manufacturing plants. *Brookings papers on economic activity. Microeconomics*, 187-267.

[7] Bartelsman, E. J. and Doms, M. (2000) Understanding productivity: Lessons from longitudinal microdata. *Journal of Economic literature*, 38, 569-594.

[8] Caballero, R. J. and Krishnamurthy, A. (2006) Bubbles and capital flow volatility: Causes and risk management. *Journal of monetary Economics*, 53, 35-53.

[9] Cooley, T. F. and Prescott, E. C. (1995) Economic growth and business cycles. Frontiers of business cycle research, 1.

[10] Farhi, E. and Tirole, J. (2011) Bubbly liquidity. *Review of Economic Studies,* 79, 678-706.

[11] Feenstra, Robert C., Robert Inklaar, and Marcel P. Timmer. (2015) The next generation of the penn world table. *American Economic Review*, 105, 3150-3182.

[12] Foster, L., Haltiwanger, J. C., and Krizan, C. J. (2001) Aggregate productivity growth: Lessons from microeconomic evidence. In New developments in productivity analysis (pp. 303-372). University of Chicago Press.

[26] Gali (2014) consider the effect of monetary policy rules on bubbles in an OLG model. Caballero and Krishnamurthy (2006) and Martin and Vetura (2015) construct open economy models of rational bubbles, based on OLG models.

[13] Foster, L., Haltiwanger, J., and Syverson, C. (2008) Reallocation, firm turnover, and efficiency: selection on productivity or profitability? *American Economic Review,* 98, 394-425.

[14] Fukao, K. and Kwon, H. U. (2006) Why did Japan's TFP growth slow down in the lost decade? An empirical analysis based on firm-level data of manufacturing firms. *Japanese Economic Review*, 57, 195-228.

[15] Futagami, K. and Shibata, A (2000) Growth effects of bubbles in an endogenous growth model. *Japanese Economic Review*, 51, 221-235.

[16] Galí, J. (2014) Monetary policy and rational asset price bubbles. *American Economic Review*, 104, 721-52.

[17] Grossman, G. and Yanagawa, N. (1993) Asset bubbles and endogenous growth. *Journal of Monetary economics*, 31, 3-19.

[18] Guerron-Quintana, P. A., Hirano, T., and Jinnai, Ryo. (2019) Recurrent bubbles and economic growth. Center for Macroeconomic-DP2020-12.

[19] Hirano, T. and Yanagawa, N. (2017) Asset bubbles, endogenous growth, and financial frictions. *Review of Economic Studies*, 84, 406-433.

[20] King, M. and Ferguson, D (1993) Dynamic inefficiency, endogenous growth, and Ponzi games. *Journal of Monetary Economics*, 32, 79-104.

[21] King, R. G. and Rebelo, S. T. (1999) Resuscitating real business cycles. In Handbook of macroeconomics 1: 927-1007.

[22] Kitagawa, A. (1994) Risky storage and the value of money. *Economics Letters*, 45, 73-77.

[23] Kocherlakota, N. R. (1992) Bubbles and constraints on debt accumulation. *Journal of Economic Theory*, 57, 245-256.

[24] Kocherlakota, N.R. (2009) Bursting bubbles: Consequences and cures. Workin paper, Federal Reserve Bank of Minneapolis.

[25] Kunieda, T. and Shibata, A. (2016) Asset bubbles, economic growth, and a self-fulfilling financial crisis. *Journal of Monetary Economics*, 82, 70-84.

[26] Martin, A. and Ventura, J. (2012) Economic growth with bubbles. *American Economic Review*, 102, 3033-3058.

[27] Martin, A. and Ventura, J. (2015) The international transmission of credit bubbles: theory and policy. *Journal of Monetary Economics*, 76, S37-S56.

[28] Mendoza, E. G., and Terrones, M. E. (2012) An anatomy of credit booms and their demise. National Bureau of Economic Research working paper No. 18379.

[29] Miao, J. and Wang, P. (2018) Asset bubbles and credit constraints. *American Economic Review*, 108, 2590-2628.

[30] Mitsui, T., and Watanabe, S. (1989) Monetary growth in a turnpike environment. *Journal of Monetary Economics*, 24, 123-137.

[31] Moll, B. (2014) Productivity losses from financial frictions: Can self-financing undo capital misallocation? *American Economic Review*, 104, 3186-3221.

[32] Olivier, J. (2000) Growth-enhancing bubbles. *International Economic Review*, 41, 133-151.

[33] Tirole, J. (1985) Asset bubbles and overlapping generations. *Econometrica*, 53, 1499-1528.

[34] Townsend, R. M. (1979) Optimal contracts and competitive markets with costly state verification. *Journal of Economic Theory*, 21, 265-293.

[35] Santos, S, M. and Woodford, M. (1997) Rational asset pricing bubbles. *Econometrica*, 65, 19-58.

[36] Stock, J. H. and Watson, M. W. (1999) Business cycle fluctuations in us macroeconomic time series. Handbook of Macroeconomics, 1, 3-64.

[37] Tanaka, J. (2011) Re-examination of the growth-enhancing bubble. *Japanese Economic Review*, 62, 170-183.

[38] Weil, P. (1987) Confidence and the real value of money in an overlapping generations economy. *Quarterly Journal of Economics*, 102, 1-22.

[39] Woodford, M. (1990) Public debt as private liquidity. *American Economic Review*, 80, 382-388.

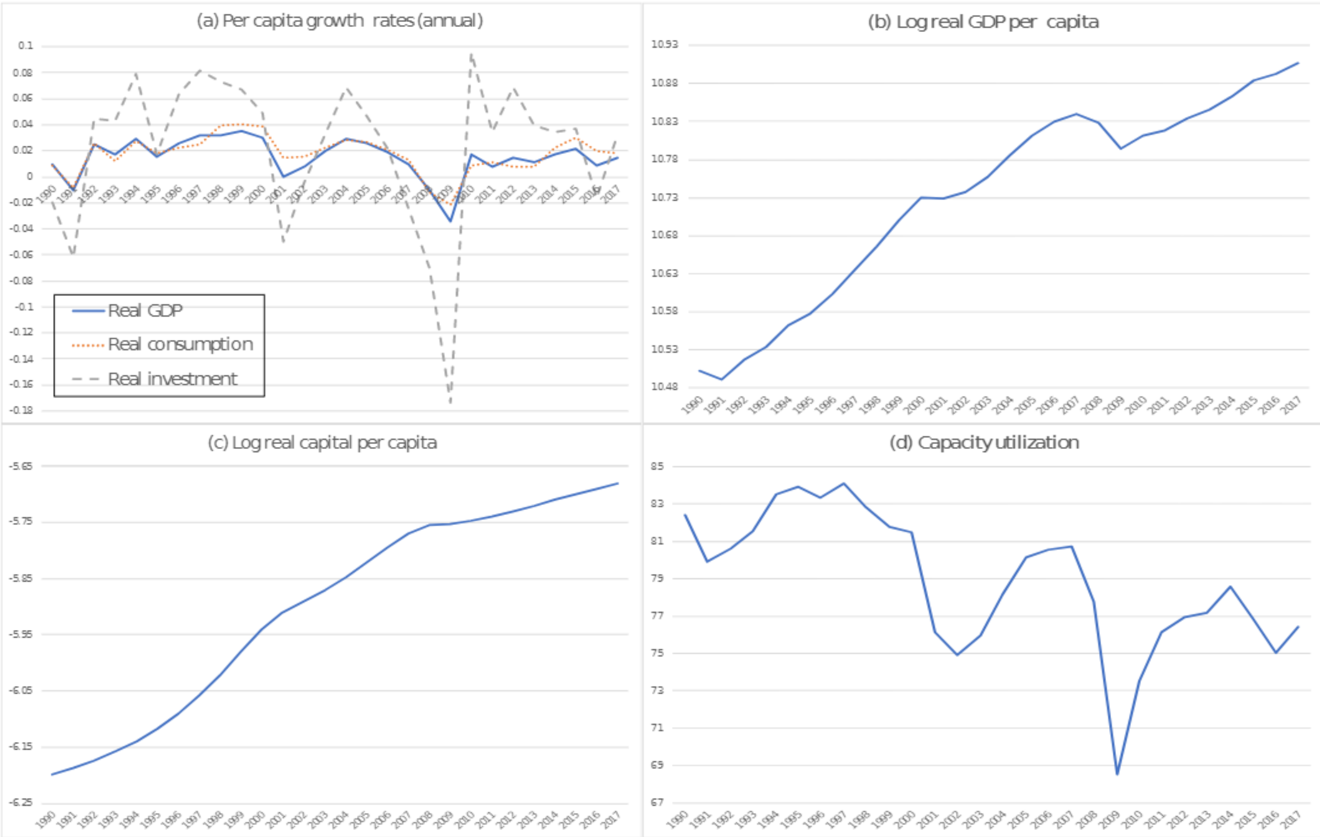

Figure 1

Source: Data on GDP and population come from the Penn World Table 9.1(Feenstra, Inklaar, and Timmer (2015)). Consumption and investment data are obtained from National Accounts data at Penn Word Table 9.1. Data on capital are obtained from "rkna" in Penn World Table 9.1. This measure of capital is adjusted for difference in marginal product among different types of capital and is a proper measure of capital input. Capacity utilization is obtained by Board of Governors of the Federal Reserves System (G.17 Industrial Production and Capacity utilization). All of these are the annual data.

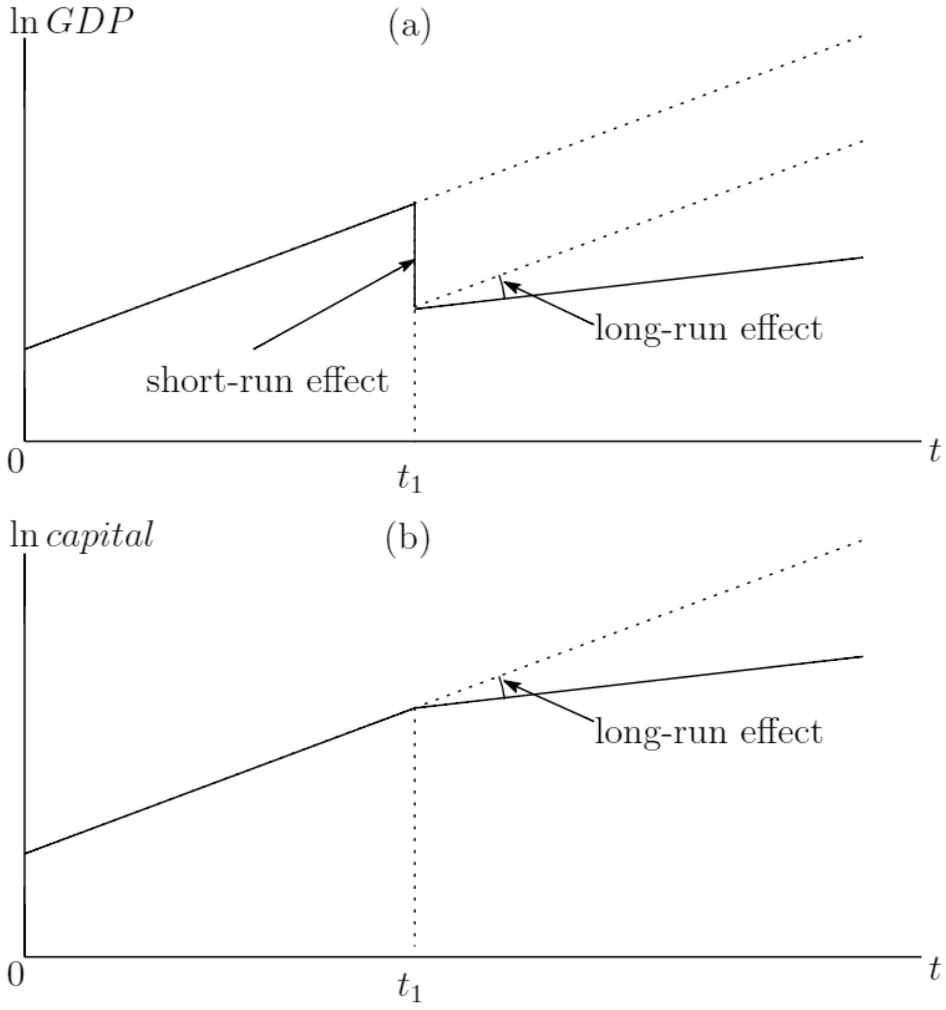

Figure 2 Collapse of Bubbles

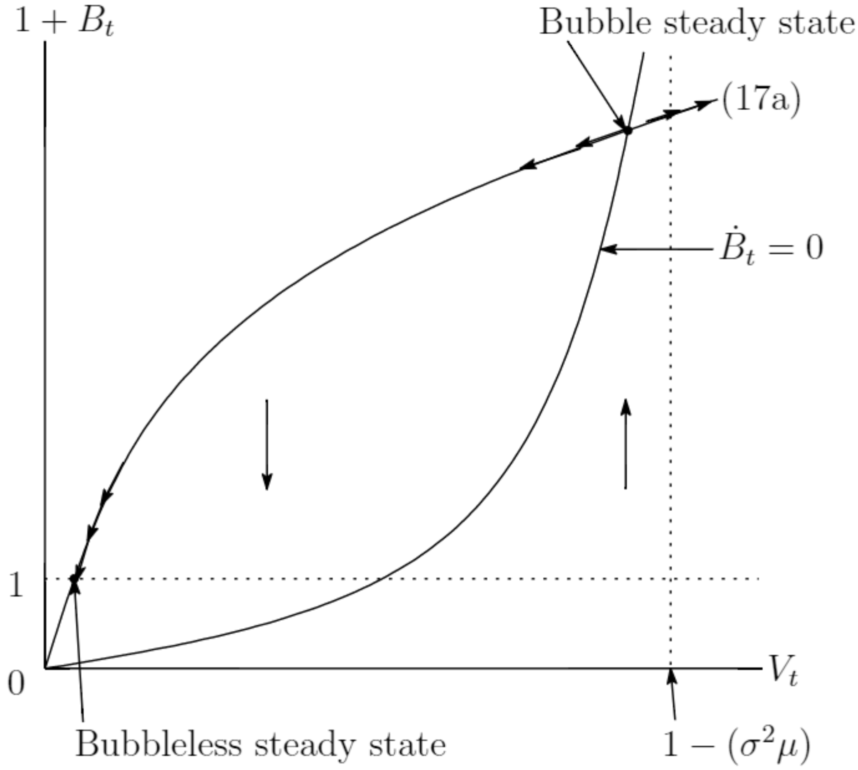

Figure 3 Phase Diagram

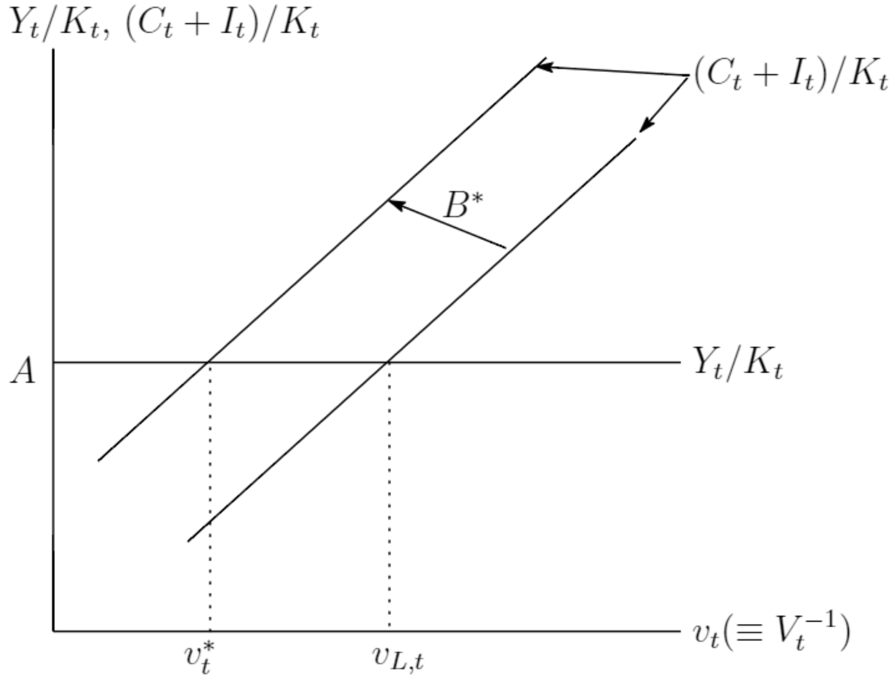

Figure 4 General Good Market and Bubbles

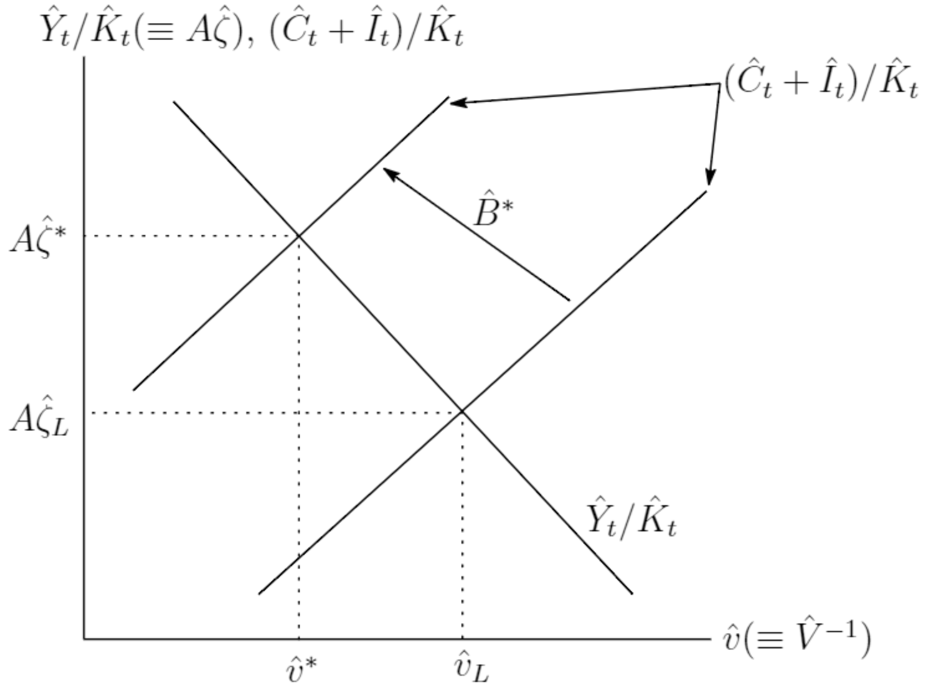

Figure 5 General Good Market and Bubbles with Endogenous Capital Utilization

Appendix

A Bellman equation and the optimal behavior of an entrepreneur

In the bubble economy, let us denote the value function of entrepreneur i with $\omega_{i,t}$ by $U^*(\omega_{i,t}, t)$. In the bubbleless econoomy, we have $\omega_{i,t} = a_{i,t}$. Then, $U(a_{i,t}, t)$ is the value function for the bubbleless economy.

Following Stokey (2009, chapter 3), we derive the Bellman equations for $U^*(\omega_{i,t}, t)$ and $U(a_{i,t}, t)$. Consider an infinitesimal short time interval of length dt. Since bubbles burst with probability μdt, the Bellman equation of an entrepreneur with asset $\omega_{i,t}$ satisfies

$$U^*(\omega_{i,t}, t) = \max_{c_{i,t}, I_{i,t}, k_{i,t}, b^n_{i,t}} \Bigg\{ (\log c_{i,t}) dt + \frac{1}{1+\rho dt} E_t \left[(1-\mu dt) \cdot U^*(\omega_{i,t+dt}, t+dt) + \mu dt \cdot U(a_{i,t+dt}, d+dt) \right] \Bigg\},$$

where the maximization is subject to Eqs. (7) and (9). We rearrange the above equation by using $U^*(\omega_{i,t+dt}, t+dt) = U^*(\omega_{i,t}, t) + dU^*(\omega_{i,t}, t)$ as follows:

$$\rho U^*(\omega_{i,t}, t) = \max_{c_{i,t}, I_{i,t}, k_{i,t}, b^n_{i,t}} \Bigg\{ (1+\rho dt) \log c_{i,t} + E_t \left[\frac{dU^*(\omega_{i,t}, t)}{dt} - \mu (U^*(\omega_{i,t+dt}, t+dt) - U(a_{i,t+dt}, d+dt)) \right] \Bigg\}.$$

Taking a limit of $dt \to 0$ in the above equation yields

$$\rho U^*(\omega_{i,t}, t) = \max_{c_{i,t}, I_{i,t}, k_{i,t}, b^n_{i,t}} \left\{ \ln c_{i,t} + \frac{EdU^*(\omega_{i,t}, t)}{dt} - \mu [U^*(\omega_{i,t}, t) - U(a_{i,t}, t)] \quad s.t \ \ (7) \text{ and } (9) \right\}. \tag{A.1}$$

Similarly, in the bubbleless economy, we have

$$\rho U(a_{i,t}, t) = \max_{c_{i,t}, I_{i,t}} \left\{ \ln c_{i,t} + \frac{EdU(a_{i,t}, t)}{dt} \quad s.t \ \ (7) \text{ and } (9) \right\}. \tag{A.2}$$

We guess that $U^*(\omega_{i,t}, t) = D^*(\ln \omega_{i,t} + u^*_t)$ and $U(a_{i,t}, t) = D(\ln a_{i,t} + u_t)$. The asset holdings $\omega_{i,t}$ follows a stochastic process with a Brownian motion $W_{i,t}$. If we use Ito's lemma, the functional form of $dU^*(\omega_{i,t}, t)$ is given by

$$dU^*(\omega_{i,t}, t) = D^* \frac{d\omega_{i,t}}{\omega_{i,t}} - \frac{D^*}{2} \left(\frac{d\omega_{i,t}}{\omega_{i,t}} \right)^2 + D^* du^*_t. \tag{A.3}$$

From (9), $(d\omega_{i,t})^2$ is computed as follows

$$\begin{aligned} (d\omega_{i,t})^2 &= \left[r_t a_{i,t} + \psi_t b_{i,t} + (v_t - 1) I_{i,t} - c_{i,t} \right]^2 (dt)^2 \\ &\quad + 2 \left[r_t a_{i,t} + \psi_t b_{i,t} + (v_t - 1) I_{i,t} - c_{i,t} \right] \sigma v_t I_{i,t} dt dW_{i,t} + (\sigma v_t I_{i,t} dW_{i,t})^2 \\ &= (\sigma v_t I_{i,t})^2 dt, \end{aligned} \tag{A.4}$$

where we use $(dt)^2 = 0$, $dtdW_{i,t} = 0$, and $(dW_{i,t})^2 = dt$. We substitute (9) and (A.4) into (A.3) and then take an expectation to obtain

$$\begin{aligned} E_t dU^*(\omega_{i,t}, t) &= E_t \left\{ D^* \frac{[r_t a_{i,t} + \psi_t b_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}]\, dt + \sigma v_t I_{i,t} dW_{i,t}}{\omega_{i,t}} \right. \\ &\quad \left. - \frac{D^*}{2} \left(\frac{\sigma v_t I_{i,t}}{\omega_{i,t}} \right)^2 dt + D^* du_t^* \right\} \\ &= D^* \frac{[r_t a_{i,t} + \psi_t b_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}]\, dt}{\omega_{i,t}} - \frac{D^*}{2} \left(\frac{\sigma v_t I_{i,t}}{\omega_{i,t}} \right)^2 dt + D^* du_t^*, \quad \text{(A.5)} \end{aligned}$$

where the second line uses $E_t dW_{i,t} = 0$. $E_t dU(a_{i,t}, t)$ is given in the same manner.

The Bellman equation in the bubbleless economy is given by

$$\rho U(a_{i,t}, t) = \max_{c_{i,t}, I_{i,t}} \left\{ \log c_{i,t} + D \frac{r_t a_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}}{a_{i,t}} - \frac{D}{2} \left(\frac{\sigma v_t I_{i,t}}{a_{i,t}} \right)^2 + D \dot{u}_t \right\}. \quad \text{(A.6)}$$

The first-order conditions are given by

$$c_{i,t} \ : \quad \frac{1}{c_{i,t}} = \frac{D}{a_{i,t}} \quad \text{(A.7)}$$

$$I_{i,t} \ : \quad \frac{v_t - 1}{a_{i,t}} = \left(\frac{\sigma v_t}{a_{i,t}} \right)^2 I_{i,t} \quad \text{(A.8)}$$

If we use (A.6), (A.7), and (A.8), we obtain

$$\rho D \log a_{i,t} + \rho D u_t = \log a_{i,t} - \log D + D \left[r_t + \left(\frac{v_t - 1}{\sigma v_t} \right)^2 \right] - 1 - \frac{D}{2} \left(\frac{v_t - 1}{\sigma v_t} \right)^2 + D \dot{u}_t. \quad \text{(A.9)}$$

Therefore, we obtain

$$D = \frac{1}{\rho} \quad \text{(A.10)}$$

$$\rho u_t = \rho \ln \rho + r_t + \left(\frac{v_t - 1}{\sigma v_t} \right)^2 - \rho - \frac{1}{2} \left(\frac{v_t - 1}{\sigma v_t} \right)^2 + \dot{u}_t. \quad \text{(A.11)}$$

Then, we have

$$c_{i,t} = \rho a_{i,t},$$

The transversality condition is satisfied:

$$\lim_{t \to \infty} E_t \left[\frac{a_{i,t}}{c_{i,t}} e^{-\rho t} \right] = \lim_{t \to \infty} \rho e^{-\rho t} = 0.$$

We next consider the Bellman equation in the bubbly economy. We distinguish capital price in the bubble economy v_t^* from that in the bubbleless economy v_t because the existence bubble may affect value of capital. If we use $U^*(\omega_{i,t},t) = D^*(\log\omega_{i,t} + u_t^*)$ and $U(a_{i,t},t) = D(\log a_{i,t} + u_t)$, the Bellman equation in the bubbly economy can be written as

$$\begin{aligned}\rho U^*(\omega_{i,t},t) = \max_{c_{i,t},I_{i,t},b^n_{i,t}} \Bigg\{ &\log c_{i,t} + D^* \frac{r_t(\omega_{i,t} - b_{i,t}) + \psi_t b_{i,t} + (v_t^* - 1)I_{i,t} - c_{i,t}}{\omega_{i,t}} \\ &- \frac{D^*}{2}\left(\frac{\sigma v_t^* I_{i,t}}{\omega_{i,t}}\right)^2 + D^* \dot{u}_t^* \\ &- \mu\left[D^*(\log\omega_{i,t} + u_t^*) - D\left(\log\frac{v_t}{v_t^*}(\omega_{i,t} - b_{i,t}) + u_t\right)\right]\Bigg\}. \qquad (A.12)\end{aligned}$$

The first line of the above equation uses $v_t^* k_{i,t} = \omega_{i,t} - b_{i,t}$. The third line uses $a_{i,t} = v_t k_{i,t} = v_t(\omega_{i,t} - b_{i,t})/v_t^*$.

In the bubble economy, the first-order conditions are given by

$$c_{i,t} \;:\quad \frac{1}{c_{i,t}} = \frac{D^*}{\omega_{i,t}} \qquad (A.13)$$

$$I_{i,t} \;:\quad \frac{v_t^* - 1}{\omega_{i,t}} = \left(\frac{\sigma v_t^*}{\omega_{i,t}}\right)^2 I_{i,t} \qquad (A.14)$$

$$b_{i,t} \;:\quad D^* \frac{\psi_t - r_t}{\omega_{i,t}} = D\frac{\mu}{\omega_{i,t} - b_{i,t}} \qquad (A.15)$$

Let us define $s_{i,t} \equiv b_{i,t}/\omega_{i,t}$. From (7) and (A.15), we obtain

$$s_{i,t} = 1 - \frac{D}{D^*}\frac{\mu}{\psi_t - r_t} = s_t. \qquad (A.16)$$

Thus, all entrepreneur holds the same fraction of their wealth as bubbly assets.

Using (A.13), (A.14), (A.15), and (A.16), we rewrite (A.12) as

$$\begin{aligned}\rho D^* \log\omega_{i,t} + \rho D^* u_t^* = &\log\omega_{i,t} - \log D^* + D^*\left[r_t(1 - s_{i,t}) + \psi_t s_{i,t} + \left(\frac{v_t^* - 1}{\sigma v_t^*}\right)^2\right] - 1 + D^* \dot{u}_t^* \\ &- \frac{D^*}{2}\left(\frac{v_t^* - 1}{\sigma v_t^*}\right)^2 - \mu\left[D^*(\ln\omega_{i,t} + u_t^*) - D\left(\ln\frac{v_t}{v_t^*}(1 - s_t)\omega_{i,t} + u_t\right)\right]. \qquad (A.17)\end{aligned}$$

Therefore, we obtain

$$D^* = \frac{1}{\rho}(= D) \qquad (A.18)$$

$$\begin{aligned}\rho u_t^* = \rho\ln\rho \;&+\; r_t(1 - s_t) + \psi_t s_t + \left(\frac{v_t^* - 1}{\sigma v_t^*}\right)^2 - \rho - \frac{1}{2}\left(\frac{v_t^* - 1}{\sigma v_t^*}\right)^2 \\ &+\; \mu\left\{\ln\left[(1 - s_t)\frac{v_t^*}{v_t}\right] - u_t^* + u_t\right\} + \dot{u}_t^*. \qquad (A.19)\end{aligned}$$

The behavior of entrepreneur i is summarized by (10a)–(10e) and the transversality condition holds as (11). Note that in these equations, we do not distinguish v_t^* from v_t for simplicity. Substituting (10a)–(10e) into (9) yields (10f).

B Proof of Proposition 1

Consider the case where there are no risks concerning capital production $\sigma = 0$. From the first-order condition for $I_{i,t}$, (A.8) or (A.14), we obtain the first equation of (16). Hence, the capital price v_t is constant at $1(=\phi^{-1})$ and $\dot{v}_t = 0$. The rate of return on capital is given by the second equation of(16).

We next show that $B_t = 0$. If we use $B_t = p_t M/(v_t K_t)$, (12b), and the first equation of (16), the good market clearing condition (14) can be written as

$$A = \rho(1 + B_t) + \frac{I_t}{K_t}. \tag{B.1}$$

Since $I_t \geq 0$, $B_t = p_t M/(v_t K_t) \geq 0$ must be bounded above. Suppose that price of bubbly assets is positive, $p_t > 0$. Then, we have

$$\begin{aligned}
\dot{B}_t &= \left(\psi_t - \frac{\dot{K}_t}{K_t}\right) B_t \\
&= \{\psi_t - A + \rho\,(1 + B_t) + \delta\}\, B_t \\
&= \{\psi_t - r_t + \rho\,(1 + B_t)\}\, B_t \\
&= (\mu + \rho)\,(1 + B_t)\, B_t.
\end{aligned} \tag{B.2}$$

The first line uses $v_t = 1$, $\dot{v}_t = 0$, and $\psi \equiv \dot{p}_t/p_t$. The second line uses (13) and (B.1). The third line uses $v_t = 1$, $dv_t = 0$, and $r_t \equiv \frac{q+\dot{v}_t-\delta v_t}{v_t}$. The last line uses $v_t = 1$, (10b), (10d), and

$$\frac{\mu}{\psi_t - r_t} = 1 - s_t = \frac{v_t K_t}{v_t K_t + p_t M} = \frac{1}{1 + B_t}. \tag{B.3}$$

Since $B_t \geq 0$ must be bounded, the solution of (B.2) is $B_t = 0$. Thus, there is no bubble equilibrium. From $B_t = 0$ and (B.1), we have $I_t/K_t = A - \rho > 0$. From (13) and (B.1), we obtain the last equation of (16).

C Proof of Proposition 2

If $I_t > 0$, then (12c) holds. We substitute (7), (12b) and (12c) into (14) and the after some rearrangement by using $B_t = p_t M/(v_t K_t)$, we obtain (17a).

In the bubble economy $B_t > 0$, we can derive the dynamics of B_t as follows

$$\frac{\dot{B}_t}{B_t} = \frac{\dot{p}_t}{p_t} - \frac{\dot{v}_t}{v_t} - \frac{\dot{K}_t}{K_t} = \mu(1 + B_t) + AV_t - \frac{1 - V_t}{\sigma^2}(1 + B_t).$$

In the second equality, we use (2), (12c), (13), and (B.3), $r_t \equiv \frac{q+\dot{v}_t-\delta v_t}{v_t}$ and $\psi_t \equiv \dot{p}_t/p_t$.

Note that in the bubbleless economy, we have $p_t = 0$, which implies that $B_t = \dot{B}_t = 0$. Then, (17b) holds in both the bubble and bubbleless economies.

D Proof of Proposition 3

In the bubbleless economy where $B_t = \dot{B}_t = 0$ holds, (17a) reduces to

$$A - \frac{\rho}{V_t} = \frac{1 - V_t}{\sigma^2}. \tag{D.1}$$

From (12c) and $V_t \equiv 1/v_t$, we know that in the bubbleless economy, we have $I_t/K_t = (1 - V_t)/\sigma^2$. Thus, if and only if the right-hand of (D.1) is positive, we have $I_t > 0$. In addition, we have $C_t = \rho K_t/V_t$. Thus, $C_t > 0$ if and only if $V_t > 0$. We examine a condition under which (D.1) has a positive solution V_t ensuring that the right-hand of (D.1) is positive.

The left-hand side of (D.1) increases from zero to $A - \rho$ as V_t increases from ρ/A to 1 (see Figure A1). The right-hand of (D.1) decreases from $1/\sigma^2$ to zero as V_t increases from zero to 1. Thus, if and only if $A - \rho > 0$, (D.1) has a unique solution $V_L \in \left(\frac{\rho}{A}, 1\right)$ ensuring the right-hand of (D.1) is positive and hence $I_t > 0$.

[Figure A1]

Substituting (2), $v_t = 1/V_L$, and $dv_t = 0$ into $r_t = (A - \dot{v}_t\delta)/v_t$ yields (18b). Since $V_L < 1 (\equiv V_{NR})$, we have $r_L < r_{NR}$. Substituting $\omega_t = v_t K_t = K_t/V_t$ and (12c) into (13) yields (18c). Because of (D.1), we can rewrite (18c) as

$$g_L = \left(A - \frac{\rho}{V_L}\right) - \delta. \tag{D.2}$$

Since $V_L < 1 (\equiv V_{NR})$, we have $g_L < g_{NR}$.

E Proof of Proposition 4

We first prove the following lemma.

Lemma A1 *Suppose that $\sigma > 0$. If and only if*

$$A\left[1 - \sigma(\rho + \mu)^{\frac{1}{2}}\right] > \frac{1}{\sigma}(\rho + \mu)^{\frac{1}{2}} - \mu > 0, \tag{E.1}$$

there exists a unique bubble steady-state equilibrium where $I_t > 0$ holds and V_t, B_t, r_t, ψ_t, and g_t satisfy (20a)*,* (20b)*,* (20c)*,* (20d)*, and* (20e)*, respectively.*

Proof: If we assume that $B_t > 0$, (17b) and $\dot{B}_t = 0$ imply

$$AV_t = \left(\frac{1 - V_t}{\sigma^2} - \mu\right)(1 + B_t). \tag{E.2}$$

Solving (17a) and (E.2) for V_t yields $V = 1 \pm \sigma(\rho + \mu)^{1/2}$. Note that if we use (15), (12c) can be written as $I_t = (1 - V_t)(1 + B_t)K_t/\sigma^2$. To ensure $I_t > 0$, we must have $V_t < 1$. Thus, (20a) holds. From $s = B/(1 + B)$ and (10d), we obtain (20d). Substituting (20a) into (E.2) yields (20b).

Condition (E.1) implies that $1 > \sigma(\rho + \mu)^{1/2}$, which ensures that $V^* > 0$. Condition (E.1) also ensures that $B^* > 0$. Then, (E.1) ensures that $V^* > 0$ and $B^* > 0$.

Conversely, suppose that $V^* > 0$ and $B^* > 0$. Then, $V^* > 0$ implies that $1 > \sigma(\rho+\mu)^{1/2}$. Thus, $B^* > 0$ implies that condition (E.1).

Since V_t is constant at V^*, we obtain (20c) from $r = (A - \dot{v}_t - \delta)/v_t$. Substituting $\omega_t = v_t K_t + p_t M = (1 + B_t)K_t/V_t$ and (12c) into (13) yields (20e). Lemma (A1) is proved. □

Note that (E.1) implies

$$\sigma < \min\left\{\frac{1}{(\rho+\mu)^{\frac{1}{2}}},\ \ \frac{(\rho+\mu)^{\frac{1}{2}}}{\mu}\right\}. \tag{E.3}$$

We also have

$$\frac{1}{(\rho+\mu)^{\frac{1}{2}}} < \frac{1}{(\rho+\mu)^{\frac{1}{2}}}\frac{\rho+\mu}{\mu} = \frac{(\rho+\mu)^{\frac{1}{2}}}{\mu}. \tag{E.4}$$

$\sigma < 1/(\rho+\mu)^{\frac{1}{2}}$ implies the second inequality of (E.1). Hence, (E.1) holds if and only if

$$\sigma < \frac{1}{(\rho+\mu)^{\frac{1}{2}}}, \tag{E.5}$$

$$A\left[1 - \sigma(\rho+\mu)^{\frac{1}{2}}\right] > \frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}} - \mu. \tag{E.6}$$

Since $\sigma > 0$, we can rewrite (E.6) as

$$\Gamma(\sigma) \equiv A(\rho+\mu)^{\frac{1}{2}}\sigma^2 - (A+\mu)\sigma + (\rho+\mu)^{\frac{1}{2}} < 0. \tag{E.7}$$

Thus, the following lemma holds.

Lemma A2 *The bubble steady-state equilibrium exists if and only if* (E.5) *and* (E.7) *hold.*

$\Gamma(\sigma)$ has following properties:

$$\begin{aligned}
\Gamma(0) &= (\rho+\mu)^{\frac{1}{2}} > 0,\\
\Gamma\left(\frac{1}{(\rho+\mu)^{\frac{1}{2}}}\right) &= \frac{\rho}{(\rho+\mu)^{\frac{1}{2}}} > 0,\\
\Gamma'(\sigma) &= 2A(\rho+\mu)^{\frac{1}{2}}\sigma - (A+\mu)\\
\Gamma'(0) &= -(A+\mu) < 0\\
\Gamma'\left(\frac{1}{(\rho+\mu)^{\frac{1}{2}}}\right) &= A - \mu,
\end{aligned} \tag{E.8}$$

Note that if $A - \mu \leq 0$, $\Gamma(\sigma)$ is a decreasing function for $\sigma \in (0,\ 1/(\rho+\mu)^{\frac{1}{2}})$. Because of (E.8), $\Gamma(\sigma) > 0$ holds for $\sigma \in (0,\ 1/(\rho+\mu)^{\frac{1}{2}})$ (see panel (a) of Figure A2). We obtain the following lemma.

Lemma A3 *Suppose that $A - \mu \leq 0$. Then the bubble steady-state equilibrium does not exist.*

[Figure A2]

Equation, $\Gamma(\sigma) = 0$, has real solutions if and only if

$$\begin{aligned} 0 &< (A+\mu)^2 - 4A(\rho+\mu)^{\frac{1}{2}}(\rho+\mu)^{\frac{1}{2}} \\ &= A^2 - 2(\mu+2\rho)A + \mu^2 \equiv H(A) \end{aligned} \tag{E.9}$$

Note the following points.

- If $H(A) \leq 0$, then $\Gamma(\sigma) \geq 0$ holds for all $\sigma > 0$ because of $\Gamma(0) > 0$. See panel (b) of Figure A2.
- If $H(A) > 0$ holds, $\Gamma(\sigma) = 0$ has two solution, σ_1 and σ_2. In addition, if $A > \mu$ holds, we have $\Gamma'(1/(\rho+\mu)^{\frac{1}{2}}) > 0$. Remember that $\Gamma(0) > 0$, $\Gamma(1/(\rho+\mu)^{\frac{1}{2}}) > 0$, and $\Gamma'(0) < 0$. Thus, we have $\sigma_1, \sigma_2 \in (0,\ 1/(\rho+\mu)^{\frac{1}{2}})$. Besides, $\Gamma(\sigma) < 0$ for $\sigma \in (\sigma_1, \sigma_2)$ and $\Gamma(\sigma) \geq 0$ for $\sigma \notin (\sigma_1, \sigma_2)$. See panel (c) of Figure A2.

From the discussion so far, we can prove the next lemma.

Lemma A4
(i) If $H(A) \leq 0$, there is no bubble steady-state equilibrium.
(ii) If $A > \mu$ and $H(A) > 0$ hold, there are σ_1 and $\sigma_2 \in (0,\ 1/(\rho+\mu)^{\frac{1}{2}})$. If $\sigma \in (\sigma_1,\ \sigma_2)$, there exists a bubble steady state. If $\sigma \notin (\sigma_1,\ \sigma_2)$, the bubble steady state does not exist.

We next examine the properties of $H(A)$. We evaluate $H(A)$ at $A = 0$ and $A = \mu$:

$$H(0) = \mu^2 > 0, \tag{E.10}$$

$$H(\mu) = -4\rho\mu < 0. \tag{E.11}$$

Besides, $H(A) = 0$ has the following solutions:

$$A = \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} > \mu. \tag{E.12}$$

Figure A3 shows that graph of $H(A)$.

[Figure A3]

We obtain the following lemma.

Lemma A5
(i) If $A \leq \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}$, then we have either $H(A) \leq 0$ or $A - \mu \leq 0$.
(ii) If $A > \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}$, then we have both $H(A) > 0$ and $A - \mu > 0$.

From Lemmas A1–A5, we obtain Proposition 4.

F Phase Diagram

The resource constraint (17a) can be written as

$$1 + B_t = \frac{A}{\frac{\rho}{V_t} + \frac{1-V_t}{\sigma^2}} = \frac{AV_t}{\rho + \frac{(1-V_t)V_t}{\sigma^2}}.$$

The right-hand side increases from zero to A/ρ as V_t increases from zero to 1. Since this equation represents the resource constraint, the economy is always on this line. We set $\dot{B}_t = 0$ in (17b) and then solve for $1 + B_t$ to obtain

$$1 + B_t = \frac{AV_t}{\frac{1-V_t}{\sigma^2} - \mu}.$$

The right-hand side increases from zero to $+\infty$ as V_t increases from zero to $1 - \mu\sigma^2$. In the region above (below) $\dot{B}_t = 0$ locus, we have $\dot{B}_t > 0$ ($\dot{B}_t < 0$). The phase diagram is shown in Figure 3. The phase diagram shows that the bubble steady state is unstable whereas the bubbleless one is stable.

G Existence of bubbles: g_L and r_L

This appendix proves the following proposition.

Proposition A1 *Suppose that $\sigma > 0$ and that the bubbless steady-state equilibrium exists. A bubble steady-state equilibrium exists if and only if $r_L < g_L - \mu$ holds in the bubbless steady-state equilibrium.*

(Proof) We first show that $V_L < V^* < 1 (\equiv V_{NR})$ holds. Suppose that both the bubble and bubbleless steady states exist. From (17a), we have

$$A = \frac{\rho}{V_L} + \frac{1 - V_L}{\sigma^2} = \left[\frac{\rho}{V^*} + \frac{1 - V^*}{\sigma^2}\right](1 + B^*). \tag{G.1}$$

Since $B^* > 0$, the above relation implies that

$$\frac{\rho}{V_L} + \frac{1 - V_L}{\sigma^2} > \frac{\rho}{V^*} + \frac{1 - V^*}{\sigma^2}. \tag{G.2}$$

Since the left-hand side decreases with V_L, we thus we have

$$V_L < V^* (\equiv 1 - \sigma(\rho + \mu)^{1/2}) < 1 (\equiv V_{NR}). \tag{G.3}$$

Suppose that the stochastic bubbly steady state exists. Then, (20a) holds. We have

$$\begin{aligned}
(20a) \quad &\iff \quad \left(\frac{1 - V^*}{\sigma}\right)^2 = \mu + \rho, \\
&\Rightarrow \quad \left(\frac{1 - V_L}{\sigma}\right)^2 > \mu + \rho, \\
&\iff \quad r_L + \left(\frac{1 - V_L}{\sigma}\right)^2 - \rho > r_L + \mu, \\
&\iff \quad g_L > r_L + \mu.
\end{aligned} \tag{G.4}$$

The second line uses (G.3). In the bubbleless steady state, $\omega_t(= v_t K_t)$ grows at $g_L = \dot{K}_t/K_t$. If we integrate (10f) using the fact that $\omega_{i,t}$ and $dW_{i,t}$ are independent, we obtain the last line because of $s_t = 0$.

Next, suppose that $g_L > r_L + \mu$ holds in the bubbleless steady state. From the second to the last lines of (G.4), we have that $\left(\frac{1-V_L}{\sigma}\right)^2 > \mu + \rho$. Since $V_L < 1$, there exists a $\hat{V}$ such that $\hat{V} > V_L$ and

$$\left(\frac{1-\hat{V}}{\sigma}\right)^2 = \mu + \rho \;\Rightarrow\; \hat{V} = 1 - \sigma(\mu+\rho)^{\frac{1}{2}} \equiv V^*. \tag{G.5}$$

Since $\hat{V} = V^* > V_L$, we obtain $B^* > 0$ from (G.1) and (G.2). Then, there exists the bubble steady state.

H Proof of Proposition 5

We first show the following lemma.

Lemma A6 *If both the bubble and bubbleless steady states exist, we have*

$$g^* < (=)(>) g_L \iff \sigma(\rho+\mu)^{1/2} < (=)(>) V_L.$$

(Proof) Irrespective of whether asset bubbles exist or not, (12c) and (17a) hold. We can rearrange (17a) as

$$\frac{1+B_t}{V_t} = \frac{A}{\frac{(1-V_t)V_t}{\sigma^2} + \rho},$$

where $(V_t, B_t) = (V_L, 0)$ and $(V_t, B_t) = (V^*, B^*)$ hold in the bubbleless and bubble economies, respectively. Then, (12c) can be written as

$$\frac{I_t}{K_t} = \frac{1-V_t}{\sigma^2}(1+B_t) = \frac{(1-V_t)V_t}{\sigma^2}\frac{A}{\frac{(1-V_t)V_t}{\sigma^2} + \rho},$$

where $(V_t, B_t) = (V_L, 0)$ or $(V_t, B_t) = (V^*, B^*)$. Since $V_L \in (0,\ 1)$ and $V^* \in (0,\ 1)$, the above equation and (13) show that growth rate increases with $(1-V_t)V_t$. Thus, we have

$$\text{sign}\{g^* - g_L\} = \text{sign}\left\{(1-V^*)V^* - (1-V_L)V_L\right\}.$$

We have the following relationship:

$$\begin{aligned}
\text{sign}\{g^* - g_L\} &= \text{sign}\left\{(1-V^*)V^* - (1-V_L)V_L\right\} \\
&= \text{sign}\left\{\sigma(\rho+\mu)^{1/2}V^* - (V^* + \sigma(\rho+\mu)^{1/2} - V_L)V_L\right\} \\
&= \text{sign}\left\{\sigma(\rho+\mu)^{1/2}(V^* - V_L) - V_L(V^* - V_L)\right\} \\
&= \text{sign}\left\{[\sigma(\rho+\mu)^{1/2} - V_L](V^* - V_L)\right\} \\
&= \text{sign}\left\{\sigma(\rho+\mu)^{1/2} - V_L\right\}
\end{aligned}$$

The second line uses $V^* \equiv 1 - \sigma(\rho+\mu)^{1/2}$. In the last line, we use $V^* > V_L$ (see (G.3)). Lemma A6 is proved. □

We next prove the following lemma.

Lemma A7

$$\sigma < (=)(>)\overline{\sigma} \iff V_L > (=)(<)\sigma(\rho+\mu)^{1/2},$$

where

$$\overline{\sigma} = \frac{-\mu + \sqrt{\mu^2 + 4A(\rho+\mu)}}{2A(\rho+\mu)^{1/2}} > 0. \tag{H.1}$$

(Proof) Note that V_L is a positive solution of (D.1). We evaluate both sides of (D.1) at $V_t = \sigma(\rho+\mu)^{1/2}$. As shown in Figure A4, we have the following relationship:

$$\begin{aligned} V_L < (=)(>)\sigma(\rho+\mu)^{1/2} &\iff A - \frac{\rho}{\sigma(\rho+\mu)^{1/2}} > (=)(<)\frac{1-\sigma(\rho+\mu)^{1/2}}{\sigma^2}, \\ &\iff G(\sigma) \equiv A(\rho+\mu)^{1/2}\sigma^2 + \mu\sigma - (\rho+\mu)^{1/2} > (=)(<)0. \end{aligned}$$

[Figure A4]

Since $G(0) < 0$ and $G(\infty) > 0$, $G(\sigma) = 0$ has a unique positive solution $\overline{\sigma}$ that is defined by (H.1). Then, we have $\sigma < (=)(>)\overline{\sigma} \iff G(\sigma) < (=)(>)0 \iff V_L > (=)(<)\sigma(\rho+\mu)^{1/2}$. Lemma A7 is proved. □

The following lemma examines whether $\overline{\sigma} \in (\sigma_1,\ \sigma_2)$ holds.

Lemma A8

$$\overline{\sigma} \begin{cases} \notin (\sigma_1,\ \sigma_2), & \textit{if}\ \ \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} < A \leq 2(\mu+2\rho) \\ \in (\sigma_1,\ \sigma_2), & \textit{if}\ \ A > 2(\mu+2\rho), \end{cases}$$

(Proof) Remember that σ_1 and σ_2 are solution of $\Gamma(\sigma) \equiv A(\rho+\mu)^{\frac{1}{2}}\sigma^2 - (A+\mu)\sigma + (\rho+\mu)^{\frac{1}{2}} = 0$ and that $\Gamma(\sigma) < 0$ holds if and only if $\sigma \in (\sigma_1,\ \sigma_2)$. Thus, the following relationships holds:

$$\overline{\sigma} \begin{cases} \in (\sigma_1,\ \sigma_2) & \text{if}\ \ \Gamma(\overline{\sigma}) < 0 \\ \notin (\sigma_1,\ \sigma_2) & \text{if}\ \ \Gamma(\overline{\sigma}) > 0 \end{cases} \tag{H.2}$$

We evaluate $\Gamma(\overline{\sigma})$ as follows:

$$\begin{aligned} \Gamma(\overline{\sigma}) &= A(\rho+\mu)^{\frac{1}{2}}\overline{\sigma}^2 - (A+\mu)\overline{\sigma} + (\rho+\mu)^{\frac{1}{2}} \\ &= A(\rho+\mu)^{\frac{1}{2}}\overline{\sigma}^2 - (A+\mu)\overline{\sigma} + A(\rho+\mu)^{1/2}\overline{\sigma}^2 + \mu\overline{\sigma} \\ &= \overline{\sigma}\{2A(\rho+\mu)^{\frac{1}{2}}\overline{\sigma} - A\} \\ &= \overline{\sigma}\left\{\sqrt{\mu^2 + 4A(\rho+\mu)} - (A+\mu)\right\}. \end{aligned}$$

The second line uses $G(\overline{\sigma}) \equiv A(\rho+\mu)^{1/2}\overline{\sigma}^2 + \mu\overline{\sigma} - (\rho+\mu)^{1/2} = 0$. The last line uses the definition of $\overline{\sigma}$, (H.1). Because of $\overline{\sigma} > 0$, we have

$$\begin{aligned} \Gamma(\overline{\sigma}) < (=)(>)0 &\iff \sqrt{\mu^2 + 4A(\rho+\mu)} < (=)(>)(A+\mu) \\ &\iff \mu^2 + 4A(\rho+\mu) < (=)(>)(A+\mu)^2 \\ &\iff A > (=)(<)2(\mu+2\rho) \end{aligned} \tag{H.3}$$

Remember that σ_1 and σ_2 exist if $A > \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}$. Note that

$$\mu + 2\rho + 2[\rho(\mu+\rho)]^{1/2} < 2(\mu + 2\rho) \tag{H.4}$$

holds because we have

$$\begin{aligned}&2(\mu+2\rho) - \{\mu + 2\rho + 2[\rho(\mu+\rho)]^{1/2}\} = \mu + 2\rho - 2[\rho(\mu+\rho)]^{1/2}\\ &(\mu+2\rho)^2 - \{2[\rho(\mu+\rho)]^{1/2}\}^2 = \mu^2 > 0.\end{aligned}$$

From (H.2) and (H.3), $\overline{\sigma} \in (\sigma_1,\ \sigma_2)$ holds if $A > 2(\mu+2\rho)$, while $\overline{\sigma} \notin (\sigma_1,\ \sigma_2)$ holds if $\mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} < A \leq 2(\mu+2\rho)$. Lemma A8 is proved. □

We finally prove the next lemma.

Lemma A9 *If $\mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} < A \leq 2(\mu+2\rho)$, we have $\overline{\sigma} \leq \sigma_1$.*

(Proof) Lemma A8 implies that $\overline{\sigma} \notin (\sigma_1, \sigma_2)$. σ_1 and σ_2 are solutions of a quadratic equation $\Gamma(\sigma) = 0$ such that $\sigma_1 < \sigma_2$. The quadratic term σ^2 of $\Gamma(\sigma)$ has a positive coefficient. Thus, if $\overline{\sigma} \notin (\sigma_1, \sigma_2)$ satisfies $\Gamma'(\overline{\sigma}) < (>) 0$, we have $\overline{\sigma} \leq \sigma_1$ ($\overline{\sigma} \geq \sigma_2$). We evaluate $\Gamma'(\sigma)$ at $\sigma = \overline{\sigma}$.

$$\Gamma'(\overline{\sigma}) = \sqrt{\mu^2 + 4A(\rho+\mu)} - (A + 2\mu).$$

We define

$$\Psi(A) \equiv \left(\sqrt{\mu^2 + 4A(\rho+\mu)}\right)^2 - (A+2\mu)^2 = -A^2 + 4\rho A - 3\mu^2.$$

$\Gamma'(\overline{\sigma})$ and $\Psi(A)$ have the same signs. $\Psi(A)$ has the following properties:

1. The coefficient of A^2 in $\Psi(A)$ is negative.
2. $\Psi(\ \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}\ ) = -4\{\mu^2 + \rho\mu + \mu[\rho(\rho+\mu)]^{1/2}\} < 0.$
3. $\Psi'(\ \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}\ ) = -2\mu - 4\{\rho(\mu+\rho)\}^{1/2} < 0.$

Thus, we have $\Psi(A) < 0$ for $A \in (\mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}, 2(\mu+2\rho)]$ (see Figure A5). Then, $\Gamma'(\overline{\sigma}) < 0$ holds, which means $\overline{\sigma} \leq \sigma_1$. Lemma A9 is proved. □

[Figure A5]

If $\mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} < A \leq 2(\mu+2\rho)$, Lemma A9 indicates that $\sigma > \overline{\sigma}$ holds for $\sigma \in (\sigma_1, \sigma_2)$. Lemmas A6 and A7 imply Proposition 5 (i).

If $A > 2(\mu+2\rho)$, Lemma A8 implies that $\overline{\sigma} \in (\sigma_1, \sigma_2)$. We obtain Proposition 5 (ii) from Lemmas A6 and A7.

I Proof of Proposition 6

In the bubbleless steady state where $B_t = 0$, the following equation holds from (17a′) and (25):

$$\frac{A^2V}{\delta} - \frac{\rho}{V} = \frac{1-V}{\sigma^2}. \tag{I.1}$$

The left-hand side of (I.1) increases from 0 to $+\infty$ as V increases from $\sqrt{\rho\delta}/A$ to ∞. The right-hand side of (I.1) decrease from $1/\sigma^2 > 0$ to 0 as V increases from 0 to 1. Thus, (26) ensures that (I.1) has a unique positive solution $\hat{V}_L \in (\sqrt{\rho\delta}/A,\ 1)$. Therefore, we have $V_t = \hat{V}_L$. Since $\hat{V}_L < 1$, $\hat{I}_t > 0$.

Since growth rate is given by (24), we have

$$g_t = \frac{1-\hat{V}_L}{\sigma^2} - \frac{(A\hat{V}_L)^2}{2\delta} \equiv \hat{g}_L. \tag{I.2}$$

Since (23) holds, we have

$$\zeta_t = \frac{A\hat{V}_L}{\delta} \equiv \hat{\zeta}_L. \tag{I.3}$$

J Proof of Proposition 7

If we set $\dot{B}_t = 0$ in (17b′), we obtain

$$\left(\frac{1-V_t}{\sigma^2} - \mu\right)(1+B_t) = A\zeta_t V_t. \tag{J.1}$$

If we eliminate B_t from (17a′) and (J.1), we obtain

$$V_t = 1 - \sigma(\rho+\mu)^{\frac{1}{2}} \equiv \hat{V}^*(= V^* \in (0,1)),$$

where V^* is given by (20a). Since $\hat{V}^* < 1$, we have $I_t > 0$. Since (23) holds, we have

$$\zeta_t = \frac{A\hat{V}^*}{\delta} \equiv \hat{\zeta}^*. \tag{J.2}$$

Condition (27) ensures that $\hat{V}^* > 0$. If we substitute $V_t = \hat{V}^*$ into (J.1), we obtain

$$B_t = \hat{\zeta}^*(1+B^*) - 1 \equiv \hat{B}^*(> 0).$$

Because of (E.4), condition (27) ensures that $1 + B^* > 0$. Then, condition (28) ensures that $\hat{B}^* > 0$. Since growth rate is given by (24), we have

$$g_t = \frac{1-\hat{V}^*}{\sigma^2}(1+\hat{B}^*) - \frac{(A\hat{V}^*)^2}{2\delta} \equiv \hat{g}^*.$$

K Proof of Corollary 3

We show that conditions (27) and (28) imply condition (26). Since $0 < V^* < 1$, condition (28) implies

$$\delta < A(1+B^*) = A^2 \frac{1-\sigma(\rho+\mu)^{1/2}}{\frac{1}{\sigma}(\rho+\mu)^{1/2}-\mu}. \tag{K.1}$$

We can show the following relationship:

$$\begin{aligned} \text{sign}\left\{\frac{1}{\rho} - \frac{1-\sigma(\rho+\mu)^{1/2}}{\frac{1}{\sigma}(\rho+\mu)^{1/2}-\mu}\right\} &= \text{sign}\left\{\frac{1}{\sigma}(\rho+\mu)^{1/2} - \mu - \rho\{1-\sigma(\rho+\mu)^{1/2}\}\right\} \\ &= \text{sign}\left\{\rho\sigma^2(\rho+\mu)^{1/2} + (\rho+\mu)^{1/2} - \sigma(\mu+\rho)\right\} \\ &= \text{sign}\left\{\rho\sigma^2(\rho+\mu)^{-1/2} + (\rho+\mu)^{-1/2} - \sigma\right\} \\ &> 0. \end{aligned} \tag{K.2}$$

The first equality holds because the term $\frac{1}{\sigma}(\rho+\mu)^{1/2} - \mu$ is positive because of (E.4) and condition (27). The last inequality holds because of condition (27). The inequalities (K.1) and (K.2), implies $\delta < A^2/\rho$, which is equivalent to (26). Thus, conditions (27) and (28) imply condition (26).

L Proof of Proposition 8

Proof of (i): From (17a′) and (25), we have

$$\frac{A^2}{\delta} = \left(\frac{\rho}{V^2} + \frac{1-V}{\sigma^2 V}\right)(1+B), \tag{L.1}$$

where $(V, B) = (\hat{V}^*, \hat{B}^*)$ or $(\hat{V}_L, \hat{B}_L)$. Since $\hat{B}^* > 0$, the above equation implies

$$\frac{\rho}{\hat{V}^{*2}} + \frac{1-\hat{V}^*}{\sigma^2\hat{V}^*} < \frac{\rho}{\hat{V}_L^2} + \frac{1-\hat{V}_L}{\sigma^2\hat{V}_L}.$$

The left-hand side decreases with $\hat{V}^*$. Thus, we have $\hat{V}^* > \hat{V}_L$. From (I.3) and (J.2), we have $\hat{\zeta}^* > \hat{\zeta}_L$. The wealth-to-GDP ratio can be rewritten as

$$\Upsilon = \frac{\omega_t}{\rho\omega_t + v_t I_t} = \frac{1+B}{\rho(1+B) + I_t/K_t} = \frac{1}{\rho + \frac{1-V}{\sigma^2}}, \tag{L.2}$$

where $(V, \Upsilon) = (\hat{V}_L, \hat{\Upsilon}_L)$ or $(\hat{V}^*, \hat{\Upsilon}^*)$. The last equality holds because of (12c). Then, Υ increases with V. Since $\hat{V}^* > \hat{V}_L$, we have $\hat{\Upsilon}^* > \hat{\Upsilon}_L$.

Proof of (ii): Since $Y_t = A\zeta_t K_t$, we have $\hat{Y}_t^* > \hat{Y}_{L,t}$ if both steady states have the same level of K_t.

We rearrange (L.1) as

$$\frac{1+B}{V} = \frac{A^2/\delta}{\frac{\rho}{V} + \frac{1-V}{\sigma^2}}. \tag{L.3}$$

where $(V,B)=(\hat{V}^*,\hat{B}^*)$ or $(\hat{V}_L,\hat{B}_L)$. Thus, we have

$$C_t=\rho\omega_t=\rho\frac{1+B}{V}K_t=\rho\frac{A^2/\delta}{\frac{\rho}{V}+\frac{1-V}{\sigma^2}}K_t. \tag{L.4}$$

The last term increases with V. Thus, we have $\hat{C}^*_t>\hat{C}_{L,t}$ if both steady states have the same level of capital stock.

The first inequality (29) is proved as follows. From $Y_t=C_t+I_t$, $Y_t=A\zeta_tK_t$, and (L.4), we have

$$I_t=\frac{A^2}{\delta}V\frac{(1-V)V}{\rho\sigma^2+(1-V)V}K_t. \tag{L.5}$$

Note that $\hat{V}_L$ is a positive solution of (I.1) that can be rewritten as

$$\Lambda(V)\equiv(\sigma^2A^2+\delta)V^2-\delta V-\rho\delta\sigma^2=0.$$

Thus, we have

$$\hat{V}_L=\frac{\delta+\sqrt{\delta^2+4\rho\delta\sigma^2(\sigma^2A^2+\delta)}}{2(\sigma^2A^2+\delta)}\rightarrow\frac{\delta}{\sigma^2A^2+\delta}, \tag{L.6}$$

as $\rho\rightarrow 0$. Besides, we have

$$\hat{V}^*=1-\sigma\sqrt{\rho+\mu}\rightarrow 1-\sigma\sqrt{\mu},\quad\text{as}\quad\rho\rightarrow 0. \tag{L.7}$$

Thus, both of $\hat{V}_L$ and $\hat{V}^*$ converge to a constant as $\rho\rightarrow 0$. From (L.5), we have

$$I_t\rightarrow\frac{A^2}{\delta}VK_t.$$

Since $\hat{V}^*>\hat{V}_L$, the first equality of (29) holds if $\rho>0$ is sufficiently small. Then, the second equality of (29) also holds.

We continue the proof of the first inequality (29). (L.5) shows that since $\hat{V}^*>\hat{V}_L$, we have

$$(1-\hat{V}^*)\hat{V}^*>(1-\hat{V}_L)\hat{V}_L\quad\Rightarrow\quad\hat{I}^*_t>\hat{I}_{L,t}.$$

As in proof of Lemma A6, because of $\hat{V}^*=1-\sigma(\rho+\mu)^{1/2}$, we have

$$(1-\hat{V}^*)\hat{V}^*>(1-\hat{V}_L)\hat{V}_L\iff\hat{V}_L<\sigma(\rho+\mu)^{1/2}.$$

Note that $\hat{V}_L$ is a positive solution of $\Lambda(V)\equiv(\sigma^2A^2+\delta)V^2-\delta V-\rho\delta\sigma^2=0$. Clearly, $\Lambda(0)=-\rho\delta\sigma^2<0$ holds. Thus, we have $\hat{V}_L<\sigma(\rho+\mu)^{1/2}$ if and only if $\Lambda\left(\sigma(\rho+\mu)^{1/2}\right)>0$, which can be rearranged as

$$\Lambda\left(\sigma(\rho+\mu)^{1/2}\right)>0\iff\Xi(\sigma)\equiv A^2(\rho+\mu)\sigma^3+\delta\mu\sigma-\delta\sqrt{\rho+\mu}>0.$$

Since $\Xi(0) < 0$ and $\Xi'(\sigma) > 0$ hold, $\Xi(\sigma) = 0$ has a unique and positive solution $\underline{\sigma}$. Since the bubble steady state exists, condition (27), $\sigma < (\rho+\mu)^{-\frac{1}{2}}$, must be satisfied. We evaluate $\Xi(\sigma)$ at $\sigma = (\rho+\mu)^{-\frac{1}{2}}$.

$$\Xi\left((\rho+\mu)^{-1/2}\right) = \frac{1}{\sqrt{\rho+\mu}}(A^2 - \delta\rho) > 0.$$

Condition (26) ensures the above inequality. Thus, we have $\Xi(\sigma) > 0$ for $\sigma \in (\underline{\sigma}, (\rho+\mu)^{-1/2})$. Therefore, for $\sigma \in (\underline{\sigma}, (\rho+\mu)^{-1/2})$, we have $\hat{V}_L < \sigma(\rho+\mu)^{1/2}$ and hence $(1-\hat{V}^*)\hat{V}^* > (1-\hat{V}_L)\hat{V}_L$. The two inequalities of (29) hold.

Proof of (iii): If we use (L.3), the growth rate is written as

$$g = \frac{1-V}{\sigma^2}(1+B) - \frac{(AV)^2}{2\delta} = \frac{1-V}{\sigma^2}\frac{A^2V/\delta}{\frac{\rho}{V}+\frac{1-V}{\sigma^2}} - \frac{(AV)^2}{2\delta}, \tag{L.8}$$

where $V = \hat{V}_L$ or $\hat{V}^*$. Since both of $\hat{V}_L$ and $\hat{V}^*$ converge to a constant as $\rho \to 0$, we have

$$g \to \frac{A^2}{\delta}\left(V - \frac{V^2}{2}\right), \tag{L.9}$$

where $V = \hat{V}_L$ or $\hat{V}^*$. We differentiate the last term of the above equation with respect to V:

$$\frac{\partial}{\partial V}\frac{A^2}{\delta}\left(V - \frac{V^2}{2}\right) = \frac{A^2}{\delta}(1-V) > 0 \quad \text{for} \quad V < 1. \tag{L.10}$$

Thus, since $0 < \hat{V}_L < \hat{V}^* < 1$, we have (30) for sufficiently small $\rho > 0$.

M Proof of Proposition 9

From (L.4) and (L.5), we have

$$\frac{\hat{I}_{L,t}}{\hat{I}^*_t} = \frac{\hat{V}_L(1-\hat{V}_L)}{\hat{V}^*(1-\hat{V}^*)}\frac{\hat{C}_{L,t}}{\hat{C}^*_t},$$

ans thus

$$(1-\hat{V}^*)\hat{V}^* > (1-\hat{V}_L)\hat{V}_L \iff \frac{\hat{I}_{L,t}}{\hat{I}^*_t} < \frac{\hat{C}_{L,t}}{\hat{C}^*_t}.$$

Appendix L shows that for $\sigma \in (\underline{\sigma}, (\rho+\mu)^{-1/2})$, we have $(1-\hat{V}^*)\hat{V}^* > (1-\hat{V}_L)\hat{V}_L$. Proposition 9 is proved.

N Comparative statics in the bubbless steady state

Since $\hat{V}_L$ is a positive solution of (I.1), we have $\partial\hat{V}_L/\partial A = -(2A\hat{V}_L)/(A^2+\rho\delta/\hat{V}_L^2+\delta/\sigma^2)^{-1} < 0$. The last equality of (L.2) shows that $\hat{\Upsilon}_L$ increases with $\hat{V}_L$. Thus, we have $\partial\hat{\Upsilon}_L/\partial A < 0$.

We rearrange $\hat{g}_L$, (I.2), as

$$\hat{g}_L = \frac{1-\hat{V}_L}{\sigma^2} - \frac{A\hat{V}_L}{2}\frac{A\hat{V}_L}{\delta} = \frac{1-\hat{V}_L}{\sigma^2} - \frac{A\hat{V}_L}{2}\frac{1}{A}\left(\frac{1-\hat{V}_L}{\sigma^2} + \frac{\rho}{\hat{V}_L}\right)$$
$$= \frac{1}{\sigma^2}(1-\hat{V}_L)\left(1-\frac{\hat{V}_L}{2}\right) - \frac{\rho}{2}. \tag{N.1}$$

The second equality uses (I.1). Since $0 < \hat{V}_L < 1$, the last line shows that $\hat{g}_L$ decreases with $\hat{V}_L$. Thus, we have $\partial\hat{g}_L/\partial A > 0$.

From (I.1), we have $\partial\hat{V}_L/\partial\delta = A^2\hat{V}_L(\delta A^2 + \rho\delta^2/\hat{V}_L^2 + \delta^2/\sigma^2)^{-1} > 0$. From the last equality of (L.2), we have $\partial\hat{\Upsilon}_L/\partial\delta > 0$. From the last equality of (N.1), we have $\partial\hat{g}_L/\partial\delta < 0$.

From (I.1), we have $\partial\hat{V}_L/\partial\sigma = -2[(1-\hat{V}_L)/\sigma^3](A^2/\delta + \rho/\hat{V}_L^2 + 1/\sigma^2)^{-1} < 0$. We use (I.1) to rearrange (L.2) as

$$\hat{\Upsilon}_L = \frac{1}{\rho + \frac{1-\hat{V}_L}{\sigma^2}} = \frac{1}{\rho + \frac{A^2\hat{V}_L}{\delta} - \frac{\rho}{\hat{V}_L}}.$$

$\hat{\Upsilon}_L$ decreases with $\hat{V}_L$. Thus, we have $\partial\hat{\Upsilon}_L/\partial\sigma > 0$. Using (I.1), we rearrange $\hat{g}_L$, (I.2), as

$$\hat{g}_L = \frac{A^2\hat{V}_L}{\delta} - \frac{\rho}{\hat{V}_L} - \frac{(A\hat{V}_L)^2}{2\delta}. \tag{N.2}$$

Then, we have

$$\frac{\partial\hat{g}_L}{\partial\sigma} = \left[\frac{A^2(1-\hat{V}_L)}{\delta} + \frac{\rho}{\hat{V}_L^2}\right]\frac{\partial\hat{V}_L}{\partial\sigma} < 0. \tag{N.3}$$

From (I.1), we have $\partial\hat{V}_L/\partial\rho = \hat{V}_L^{-1}(A^2/\delta + \rho/\hat{V}_L^2 + 1/\sigma^2)^{-1} > 0$. Thus, we have $\partial\hat{\zeta}_L/\partial\rho > 0$. From (I.2), we know that $\hat{g}_L$ deceases with $\hat{V}_L$. Thus, we have $\partial\hat{g}_L/\partial\rho < 0$.

O Derivation of (31) and (32)

We first derive (31) and (32). In both the bubble and bubbleless economies, we have $\omega_t = K_t/V_t + p_tM = (1+B_t)K_t/V_t$. Both V_t and B_t are constant in the steady state. Thus, in the steady state, we have

$$g_t = \frac{\dot{K}_t}{K_t} = \frac{\dot{\omega}_t}{\omega_t} = r_t(1-s_t) + \psi_t s_t + \left(\frac{1-V_t}{\sigma}\right)^2 - \rho. \tag{O.1}$$

To obtain the last equality, we aggregate (10f) over i using the facts that $\omega_{i,t}$ and $dW_{i,t}$ are independent and $dW_{i,t}$ follows a normal distribution with zero mean.

Since $\hat{g}_L = (1-\hat{V}_L)/\sigma^2 - \bar{\delta}(\hat{\zeta}_L)$ holds in the bubbleless economy, we have

$$\frac{1-\hat{V}_L}{\sigma} = \sigma\left(\hat{g}_L + \bar{\delta}(\hat{\zeta}_L)\right). \tag{O.2}$$

From $s_t = 0$, (A.10), (A.11), (O.2) and $U(a_{i,t}, t) = D(\log a_{i,t} + u_t)$, we have

$$\rho U(\hat{a}_{i,t}, t) = \log \hat{a}_{i,t} + \log \rho + \frac{1}{\rho}\left[\hat{r}_L + \left(\frac{1-\hat{V}_L}{\sigma}\right)^2 - \rho\right] - \frac{1}{2\rho}\left\{\sigma\left(\hat{g}_L + \bar{\delta}(\hat{\zeta}_L)\right)\right\}^2. \quad (O.3)$$

From (O.1) and (O.3), we obtain

$$\rho U(\hat{a}_{i,t}, t) = \log \rho \hat{a}_{i,t} + \frac{1}{\rho}\left[\hat{g}_L - \frac{1}{2}\left\{\sigma\left(\hat{g}_L + \bar{\delta}(\hat{\zeta}_L)\right)\right\}^2\right]. \quad (O.4)$$

We have $\rho \hat{a}_{i,t} = \rho \hat{v}_L \hat{k}_{i,t} = (\hat{C}_{L,t}/\hat{K}_t)\hat{k}_{i,t}$ in the bubbleless steady state. Thus, (O.4) is rewritten as (31).

Since $\hat{g}^* = (1-\hat{V}^*)(1+\hat{B}^*)/\sigma^2 - \bar{\delta}(\hat{\zeta}^*)$ holds in the bubble steady state, we have

$$\frac{1-\hat{V}^*}{\sigma} = \frac{\sigma\left(\hat{g}^* + \bar{\delta}(\hat{\zeta}^*)\right)}{1+\hat{B}^*}. \quad (O.5)$$

From (A.18), (A.19), (O.5) and $U^*(\omega_{i,t}, t) = D^*(\log \omega_{i,t} + u_t^*)$, we have

$$\begin{aligned}\rho U^*(\hat{\omega}_{i,t}, t) = \log \hat{\omega}_{i,t} + \log \rho + \frac{1}{\rho}\left[\hat{r}^*(1-\hat{s}^*) + \psi \hat{s}^* + \left(\frac{1-\hat{V}^*}{\sigma}\right)^2 - \rho\right] \\ -\frac{1}{2\rho}\left\{\frac{\sigma\left(\hat{g}^* + \bar{\delta}(\hat{\zeta}^*)\right)}{1+\hat{B}^*}\right\}^2 - \mu\left[U^*(\hat{\omega}_{i,t}, t) - U(\hat{a}_{i,t}, t)\right]. \quad (O.6)\end{aligned}$$

From (O.1) and (O.6), we obtain

$$\rho U^*(\hat{\omega}_{i,t}, t) = \log \rho \hat{\omega}_{i,t} + \frac{1}{\rho}\left[\hat{g}^* - \frac{1}{2}\left\{\frac{\sigma\left(\hat{g}^* + \bar{\delta}(\hat{\zeta}^*)\right)}{1+\hat{B}^*}\right\}^2\right] - \mu\left[U^*(\hat{\omega}_{i,t}, t) - U(\hat{a}_{i,t}, t)\right]. \quad (O.7)$$

In the bubble steady state, we have

$$\begin{aligned}\rho \hat{\omega}_{i,t} &= \rho \hat{\omega}_t \frac{\hat{\omega}_{i,t}}{\hat{\omega}_t} \\ &= \hat{C}_t^* \frac{\hat{v}^* \hat{k}_{i,t}}{\hat{v}^* \hat{K}_t} \\ &= \frac{\hat{C}_t^*}{\hat{K}_t}\hat{k}_{i,t}. \quad (O.8)\end{aligned}$$

The second equality uses $\hat{v}^* \hat{k}_{i,t} = (1-\hat{s}^*)\hat{\omega}_{i,t}$ (see (10b)), $\hat{K}_t = \int_0^1 \hat{k}_{i,t} di$ and $\hat{\omega}_t = \int_0^1 \hat{\omega}_{i,t} di$. From (O.7) and (O.8), we obtain (32).

P Proof of Proposition 10

From (O.2) and (O.5), we have

$$\text{sign}\left\{\left(\hat{g}_L+\bar{\delta}(\hat{\zeta}_L)\right)^2-\left(\frac{\hat{g}^*+\bar{\delta}(\hat{\zeta}^*)}{1+\hat{B}^*}\right)^2\right\}=\text{sign}\left\{(1-\hat{V}_L)^2-(1-\hat{V}^*)^2\right\}.$$

Since $0<\hat{V}_L<\hat{V}^*<1$ holds, we have $(1-\hat{V}_L)^2>(1-\hat{V}^*)^2$.

Q Bubbles and welfare

This section investigates how asset bubbles affect welfare of entrepreneurs. The initial aggregate capital is K_0. We assume that in both the bubble and bubbleless economies, entrepreneur i holds the same amounts of capital $k_{i,0}$ at time 0, where $\int k_{i,0}di=K_0$. Appendix Q.1 shows that in the bubbleless steady state, utility of entrepreneur i at time 0 is given by

$$\rho W_L(k_{i,0})=\log k_{i,0}+Z(g_L)-\frac{\sigma^2}{2\rho}(g_L+\delta)^2, \tag{Q.1}$$

where

$$Z(g)\equiv\log\{A-(g+\delta)\}+\frac{g}{\rho}.$$

$Z(g)$ represents utility from consumption and its growth. The term $-\frac{\sigma^2}{2\rho}(g_L+\delta)^2$ captures utility loss from investment risks. Naturally, a large σ implies a large utility loss. Similarly, in the bubble steady state, utility of entrepreneur i at time 0 is given by

$$\rho W^*(k_{i,0})=\log k_{i,0}+Z(g^*)-\frac{\sigma^2}{2\rho}\left(\frac{g^*+\delta}{1+B^*}\right)^2-\mu\left[W^*(k_{i,0})-W_L(k_{i,0})\right]. \tag{Q.2}$$

The last term represents utility loss of bubbles burst. The term $-\frac{\sigma^2}{2\rho}\left(\frac{g^*+\delta}{1+B^*}\right)^2$ captures utility loss from investment risks and shows that B^* mitigate the utility loss. Intuition is simple. As we discuss just after Proposition 5, given v_t, asset bubbles make entrepreneurs wealthier and then increase entrepreneurs' tolerance to investment risks.

From (Q.1) and (Q.2), we obtain

$$(\rho+\mu)\left[W^*(k_{i,0})-W_L(k_{i,0})\right]=Z(g^*)-Z(g_L)+\frac{\sigma^2}{2\rho}\left\{(g_L+\delta)^2-\left(\frac{g^*+\delta}{1+B^*}\right)^2\right\}. \tag{Q.3}$$

The term $(g_L+\delta)^2-\left(\frac{g^*+\delta}{1+B^*}\right)^2$ is always positive (see Appendix Q.2), because asset bubbles increase entrepreneurs' tolerance to investment risks and then have a positive welfare effect. The term $Z(g^*)-Z(g_L)$ can be positive or negative depending on parameters of the model. Thus, asset bubbles has a negative welfare effect. However, we can show that the overall welfare effect of bubbles is always positive. In sum, asset bubbles have a significantly large positive welfare effect because asset bubbles increase entrepreneurs' tolerance to investment risks. We obtain the following proposition.

Proposition A2 *Suppose that both the bubble and bubbleless steady-state equilibrium exist. Then, asset bubbles always improve welfare of all entrepreneurs.*

(Proof) See Appendix Q.2.

Q.1 Derivation of (Q.1) and (Q.2)

We first derive (Q.1) and (Q.2). In both the bubble and bubbleless economies, we have $\omega_t = K_t/V_t + p_t M = (1+B_t)K_t/V_t$. Both V_t and B_t are constant in the steady state. Thus, in the steady state, we have

$$g_t = \frac{\dot{K}_t}{K_t} = \frac{\dot{\omega}_t}{\omega_t} = r_t(1-s_t) + \psi_t s_t + \left(\frac{1-V_t}{\sigma}\right)^2 - \rho. \tag{Q.4}$$

To obtain the last equality, we aggregate (10f) over i using the facts that $\omega_{i,t}$ and $dW_{i,t}$ are independent and $dW_{i,t}$ follows a normal distribution with zero mean.

Since $g_L = (1-V_L)/\sigma^2 - \delta$ holds in the bubbleless economy, we have

$$\frac{1-V_L}{\sigma} = \sigma(g_L + \delta). \tag{Q.5}$$

From $s_t = 0$, (A.10), (A.11), (Q.5) and $U(a_{i,t}, t) = D(\log a_{i,t} + u_t)$, we have

$$\rho U(a_{i,t}, t) = \log a_{i,t} + \log \rho + \frac{1}{\rho}\left[r_L + \left(\frac{1-V_L}{\sigma}\right)^2 - \rho\right] - \frac{1}{2\rho}\left\{\sigma(g_L+\delta)\right\}^2. \tag{Q.6}$$

From (Q.4) and (Q.6), we obtain

$$\rho U(a_{i,0}, 0) = \log \rho a_{i,0} + \frac{1}{\rho}\left[g_L - \frac{1}{2}\left\{\sigma(g_L+\delta)\right\}^2\right]. \tag{Q.7}$$

We have $\rho a_{i,0} = \rho v_0 k_{i,0} = (C_0/K_0)k_{i,0}$ in the bubbleless steady state. From (1), (13), and (14), we have

$$\frac{C_0}{K_0} = A - (g_t + \delta), \tag{Q.8}$$

in both the bubble and bubbleless economies. Thus, (Q.7) is rewritten as (Q.1).

Since $g^* = (1-V^*)(1+B^*)/\sigma^2 - \delta$ holds in the bubble steady state, we have

$$\frac{1-V^*}{\sigma} = \frac{\sigma(g^*+\delta)}{1+B^*}. \tag{Q.9}$$

From (A.18), (A.19), (Q.9) and $U^*(\omega_{i,t}, t) = D^*(\log \omega_{i,t} + u_t^*)$, we have

$$\begin{aligned}\rho U^*(\omega_{i,t}, t) = \log \omega_{i,t} + \log \rho + \frac{1}{\rho}\left[r^*(1-s) + \psi s + \left(\frac{1-V^*}{\sigma}\right)^2 - \rho\right] \\ -\frac{1}{2\rho}\left\{\frac{\sigma(g^*+\delta)}{1+B^*}\right\}^2 - \mu\left[U^*(\omega_{i,t}, t) - U(a_{i,t}, t)\right].\end{aligned} \tag{Q.10}$$

From (Q.4) and (Q.10), we obtain

$$\rho U^*(\omega_{i,0}, 0) = \log \rho\omega_{i,0} + \frac{1}{\rho}\left[g^* - \frac{1}{2}\left\{\frac{\sigma(g^*+\delta)}{1+B^*}\right\}^2\right] - \mu\left[U^*(\omega_{i,0}, 0) - U(a_{i,0}, 0)\right]. \tag{Q.11}$$

In the bubble steady state, we have

$$\begin{aligned}\rho\omega_{i,0} &= \rho\omega_0\frac{\omega_{i,0}}{\omega_0} \\ &= C_0\frac{v_0 k_{i,0}}{v_0 K_0} \\ &= \frac{C_0}{K_0}k_{i,0}. \end{aligned} \tag{Q.12}$$

The second equality uses $v_0 k_{i,0} = (1-s_0)\omega_{i,0}$ (see (10b)), $K_0 = \int_0^1 k_{i,0}di$ and $\omega_0 = \int_0^1 \omega_{i,0}di$. From (Q.8), (Q.11), and (Q.12), we obtain

$$(\rho+\mu)U^*(\omega_{i,0},0) = \log k_{i,0} + Z(g^*) - \frac{\sigma^2}{2\rho}\left(\frac{g^*+\delta}{1+B^*}\right)^2 + \mu W_L(k_{i,0},0) \equiv (\rho+\mu)W^*(k_{i,0},0).$$

After rearranging the above equation, (Q.2) is derived.

Q.2 Proof of Proposition A2

From (Q.5) and (Q.9), we have

$$\text{sign}\ \left\{(g_L+\delta)^2 - \left(\frac{g^*+\delta}{1+B^*}\right)^2\right\} = \ \text{sign}\ \left\{(1-V_L)^2 - (1-V^*)^2\right\} \tag{Q.13}$$

From (G.3), we have $V_L < V^* < 1$. Thus, $(1-V_L)^2 > (1-V^*)^2$ holds.

We show that $\max\{g_L, g^*\} < g_{NR}$, where $g_{NR} \equiv A - \delta - \rho$ is the growth rate under $\sigma = 0$ (see the last equation of (16)). (18c) ensures that $g_L < g_{NR}$. Using (20e), we show $g^* < g_{NR}$ as follows:

$$\begin{aligned} g^* &= \left(A - \rho\frac{1+B^*}{V^*}\right) - \delta \\ &< \left(A - \rho\frac{1}{V^*}\right) - \delta \\ &< A - \rho - \delta \equiv g_{NR}, \end{aligned} \tag{Q.14}$$

The first line uses (17a) in (20e). The second line uses $B^* > 0$. The last line uses $V^* < 1$.

Function $Z(g)$ has the following properties:

$$Z'(g) = \frac{g_{NR} - g}{\rho(g_{NR} + \rho - g)} > 0 \ \text{ for } \ g < g_{NR}, \tag{Q.15}$$

$$Z''(g) = \frac{-1}{(g_{NR} + \rho - g)^2} < 0 \ \text{ for } \ g < g_{NR}. \tag{Q.16}$$

We consider the following two cases; (i) $g^* \geq g_L$ and (ii) $g^* < g_L$.

(i) If $g^* \geq g_L$ holds, we have $Z(g^*) \geq Z(g_L)$ because of (Q.14) and (Q.15). Since $(1-V_L)^2 > (1-V^*)^2$ holds, we have $U^*(\omega_{i,0},0) > U(a_{i,0},0)$.

(ii) If $g^* < g_L$ holds, we have $Z(g^*) < Z(g_L)$ because of $\max\{g_L, g^*\} < g_{NR}$ and (Q.15). Furthermore, since $Z(g)$ is an increase and concave function for $g < g_{NR}$, we have

$$0 < Z(g_L) - Z(g^*) < Z'(g^*)(g_L - g^*). \tag{Q.17}$$

In addition, because of $V_L < V^*$ (see (G.3)), we have

$$\begin{aligned}(1-V_L)^2 - (1-V^*)^2 &= (1-V_L+1-V^*)(V^*-V_L)\\ &> 2(1-V^*)(V^*-V_L).\end{aligned} \tag{Q.18}$$

From (Q.3), (Q.17), and (Q.18), we obtain

$$(\rho+\mu)\left[U^*(\omega_{i,0},0) - U(a_{i,0},0)\right] > Z'(g^*)(g^*-g_L) + \frac{1}{\rho\sigma^2}(1-V^*)(V^*-V_L). \tag{Q.19}$$

If we use $g_{NR} \equiv A - \delta - \rho$ and the first line of (Q.14), we have

$$Z'(g^*) = \frac{1}{\rho}\left(1 - \frac{\rho}{A-\delta-g^*}\right) = \frac{1}{\rho}\left(1 - \frac{V^*}{1+B^*}\right). \tag{Q.20}$$

Using (18c), (20e) and (Q.20), we examine the sign of the second line of (Q.19) as follows:

$$\begin{aligned}
&\text{sign}\left\{Z'(g^*)(g^*-g_L) + \frac{1}{\rho\sigma^2}(1-V^*)(V^*-V_L)\right\}\\
&= \text{sign}\left\{\left(1-\frac{V^*}{1+B^*}\right)\{(1-V^*)(1+B^*)-(1-V_L)\} + (1-V^*)(V^*-V_L)\right\}\\
&= \text{sign}\left\{(1-V^*) - V_L(1-V^*) + (1-V^*)B^* - (1-V_L) + \frac{V^*(1-V_L)}{1+B^*}\right\}\\
&= \text{sign}\left\{(1-V_L)(1-V^*) + (1-V^*)B^* + \frac{(V^*-1)(1-V_L)-(1-V_L)B^*}{1+B^*}\right\}\\
&= \text{sign}\left\{\frac{(1-V_L)(1-V^*)B^*}{1+B^*} + (1-V^*)B^* - \frac{(1-V_L)B^*}{1+B^*}\right\}\\
&= \text{sign}\left\{(1-V^*) - \frac{(1-V_L)V^*}{1+B^*}\right\}\\
&= \text{sign}\left\{\frac{(1-V^*)}{V^*}(1+B^*) - (1-V_L)\right\}\\
&= \text{sign}\left\{A\frac{\sigma(\rho+\mu)^{\frac{1}{2}}}{\frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}}-\mu} - (1-V_L)\right\}.
\end{aligned} \tag{Q.21}$$

The last line uses (20a) and (20b). Note that the first term in the last line of (Q.21) increases with μ. Since $\mu > 0$, we have

$$\begin{aligned}
A\frac{\sigma(\rho+\mu)^{\frac{1}{2}}}{\frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}}-\mu} - (1-V_L) &> \sigma^2\left(A - \frac{1-V_L}{\sigma^2}\right)\\
&= \frac{\rho\sigma^2}{V_L}\\
&> 0.
\end{aligned} \tag{Q.22}$$

The second line uses (D.1).

From (Q.19), (Q.21), and (Q.22), we obtain $U^*(\omega_{i,0},0) > U(a_{i,0},0)$.

R A variety expansion model

In our model, we can interpret capital more broadly. To see this fact, this section modifies the variety expansion model proposed by Barro and Sala-i-martin (2004). Entrepreneurs can set up new businesses, including establishments, which is subject to the idiosyncratic shocks. The number of firms in the economy accumulates through entrepreneurial activities. Main results obtained in our AK model hold in this variety-expansion model. Thus, in our model, capital, K, can include not just physical capital but also businesses and innovations.

A general good is produced by using intermediate goods and labor. Labor is supplied inelastically by workers. Entrepreneurs can create new firms and accumulate their own firms.

Production sector: A competitive general good firm has the following production technology:

$$Y_t = ZL_t^{\alpha} \int_0^{n_t} X_t^{1-\alpha}(j)dj, \quad Z > 0, \quad 0 < \alpha < 1 \tag{R.1}$$

where n_t is the number of varieties, L_t and $X_t(j)$ represent labor and intermediate good j inputs, respectively. Profit-maximizing yields $X_t(j) = [(1-\alpha)Z]^{\frac{1}{\alpha}} L_t p_t^X(j)^{-\frac{1}{\alpha}}$, where $p_t^X(j)$ denotes the price of intermediate good j. See Appendix R.1 for the derivation of the optimal behavior of the general and intermediate good firms.

Each intermediate good j is produced by a monopolistically competitive firm. The production of one unit of intermediate good requires $\eta > 0$ units of general goods. Profits of each intermediate good is given by $\pi_t(j) = \left[p_t^X(j) - \eta\right] X_t(j)$. Appendix R.1 shows that from the profit maximization problem by firm j, we obtain

$$\pi_t(j) = \alpha \left[\eta^{\alpha-1}(1-\alpha)^{2-\alpha} z\right]^{\frac{1}{\alpha}} L \equiv \pi \tag{R.2}$$

where we use the labor market condition $L_t = L$. Thus, π is constant over time.

Entrepreneurs: Entrepreneurs create new firms using their investment projects given by $dx_{i,t}^N = I_{i,t}dt + \sigma I_{i,t}dW_{i,t}$, where $dx_{i,t}^N$ denotes the number of newly created firms by entrepreneur i. Idiosyncratic shocks include risks of starting a new business. Entrepreneurs are the owners of intermediated goods firms. Entrepreneur i owns $n_{i,t}$ units of intermediate good firms. The market value of an intermediate good firm is v_t^N. Then, total assets holdings of entrepreneurs i are given by $\omega_{i,t}^N = v_t^N n_{i,t} + p_t b_{i,t}^n = a_{i,t}^N + b_{i,t}$, where $a_{i,t}^N \equiv v_t^N n_{i,t}$. Between $t + dt$, entrepreneur i receives operating profits from intermediates goods firms $\pi_t n_{i,t} dt$ and earns profit income from creating new intermediate goods firms $v_t^N dx_{i,t}^N - I_{i,t}dt = (v_{i,t}^N - 1)I_{i,t}dt + \sigma v_t^N I_{i,t}dW_{i,t}$. Thus, the budget constraint of entrepreneur i at $t + dt$ is as follows:

$$c_{i,t}dt + \delta v_t^N n_{i,t}dt + v_t^N dn_{i,t} + p_t db_{i,t}^n = \pi n_{i,t}dt + (v_t^N - 1)I_{i,t}dt + \sigma v_t^N I_{i,t}dW_{i,t}. \tag{R.3}$$

where $\delta \in [0, 1]$ denotes an exogenous destruction rate of an intermediate good firm and π corresponds to the rental price q of the AK model presented in chapter 2. The detailed derivation of the evolution of $\omega_{i,t}^N$ and optimal plans of entrepreneur i is given by Appendix R.2.

Workers: The population size of workers is L. Each worker inelastically supplies one unit of labor and earns wage income. For simplicity, we assume that the workers are hand-to-mouth consumers, which means they consume their current income entirely. Then, the aggregate consumption of workers, C_t^w, is given by

$$C_t^w = w_t L \tag{R.4}$$

where w_t denotes the wage rate. The labor market clears as $L_t = L$.

Equilibrium: Let us define $V_t^N \equiv 1/v_t^N$ and $B_t^N \equiv p_t M/(v_t^N n_t)$, where B_t^N denotes the value of bubbles relative to the market value of intermediate goods firms and $n_t \equiv \int_0^1 n_{i,t} di$ denotes the aggregate number of intermediated good firms. Then, the aggregate assets holdings is given by $\omega_t^N = v_t^N n_t + p_t M$, where we use $\int_0^1 b_{i,t}^n di = M$. We derive the low of motion of the number of firms as $dn_t \equiv \int_0^1 (dx_{i,t}^N) di - \delta n_t dt = (I_t - \delta n_t)dt$, where I_t is given by (R.21). The growth rate of economy is as follows

$$g_t^N = \frac{\dot{n}_t}{n_t} = \frac{I_t}{n_t} - \delta. \tag{R.5}$$

The following proposition gives a set of equations that characterizes the bubble and bubbleless equilibria in the variety expansion model.

Proposition A3 *Suppose that $\sigma > 0$. Then, the bubble and bubbleless equilibria with $I_t > 0$ are characterized by*

$$\pi = \left[\frac{\rho}{V_t^N} + \frac{1 - V_t^N}{\sigma^2} \right] (1 + B_t^N), \tag{R.6}$$

$$\dot{B}_t^N = \left[\mu(1 + B_t^N) + \pi V_t^N - \frac{1 - V_t^N}{\sigma^2}(1 + B_t^N) \right] B_t^N, \tag{R.7}$$

where π is given by (R.2).

Proof: See Appendix R.3.

Compare (R.6) and (R.7) with (17a) and (17b), respectively. If we replace A by π in (17a) and (17b), these two equations become equivalent to (R.6) and (R.7), respectively. This means that if we substitute π into A, Propositions 3-5 and Corollaries 1 and 2 hold even in this variety-expansion model. This fact shows that in our benchmark AK model, capital K include not just physical capital but also businesses and innovations.

R.1 The optimal behavior of production sector

The profits of the general good firm is given by $\pi_t = Y_t - \int_0^{n_t} p_t^X X_t(j) dj - w_t L_t$. The first-order conditions are given by

$$X_t(j) \quad : \quad (1-\alpha) Z L_t^{\alpha} X_t^{-\alpha}(j) = p_t^X(j), \quad j \in [0, n_t] \tag{R.8}$$

$$L_t \quad : \quad \alpha Z L_t^{\alpha-1} \int_0^{n_t} X_t^{1-\alpha}(j) dj = w_t. \tag{R.9}$$

From (R.8), we obtain

$$X_t(j) = [(1-\alpha)Z]^{\frac{1}{\alpha}} L_t p_t^X(j)^{-\frac{1}{\alpha}}, \quad j \in [0, n_t]. \tag{R.10}$$

The profits of intermediate good firm j is $\pi_t(j) = \left[p_t^X(j) - \eta\right] X_t(j)$. Intermediate good firm j maximizes the profits subject to (R.10). We can obtain the following equations

$$p_t^X(j) = \frac{\eta}{1-\alpha} \equiv p^X. \tag{R.11}$$

$$X_t(j) = \left[\frac{(1-\alpha)^2 Z}{\eta}\right]^{\frac{1}{\alpha}} L \equiv X, \tag{R.12}$$

where we use $L_t = L$. From (R.11), (R.12), and $\pi_t(j) = \left[p_t^X(j) - \eta\right] X_t(j)$ we heve (R.2).

R.2 The evolution of $\omega_{i,t}^N$ and optimal behavior of an entrepreneur in variety expansion model

We can derive the evolution of $\omega_{i,t}^N$ using the procedure presented in (9). From $\omega_{i,t}^N = v_t^N n_{i,t} + p_t b_{i,t}^n$, we obtain $d\omega_{i,t}^N = (dv_t^N)n_{i,t} + v_{i,t}^N(dn_{i,t}) + (dp_t)b_{i,t}^n + p_t(db_{i,t}^n)$. Then, (R.3) can be rearrange as:

$$d\omega_{i,t}^N = \left[r_t^N a_{i,t}^N + \psi_t b_{i,t} + (v_t^N - 1)I_{i,t} - c_{i,t}\right] dt + \sigma v_t^N I_{i,t} dW_{i,t}, \tag{R.13}$$

where $a_t^N \equiv v_t^N n_{i,t}$ and $r_t^N dt \equiv (\pi dt + dv_t^N - \delta v_t^N dt)/v_t^N$ and we use $\psi \equiv \dot{p}_t/p_t$.

Entrepreneur i maximizes (3) subject to $\omega_{i,t}^N = v_t^N n_{i,t} + p_t b_{i,t}^n$ and (R.13). If we replace $\omega_{i,t}$, r_t and $v_{i,t}$ by $\omega_{i,t}^N$, r_t^N, and v_t^N in (9), (R.13) are equivalent to (9). By using procedure presented in Appendix A, we can derive the optimal behavior as follows:

$$c_{i,t} = \rho\omega_{i,t}^N, \tag{R.14}$$

$$a_{i,t}^N = (1-s_t)\omega_{i,t}^N, \tag{R.15}$$

$$b_{i,t} = s_t\, \omega_{i,t}^N, \tag{R.16}$$

$$s_t = \begin{cases} 1 - \frac{\mu}{\psi_t - r_t^N} & \text{in the bubble economy } (p_t > 0), \\ 0 & \text{in the bubbleless economy } (p_t = 0), \end{cases} \tag{R.17}$$

$$I_{i,t} = \frac{v_t^N - 1}{(\sigma v_t^N)^2}\omega_{i,t}^N. \tag{R.18}$$

If $v_t^N > 1$, then $I_{i,t} > 0$ holds. The transversality condition holds:

$$\lim_{t\to\infty} E_t\left[\frac{\omega_{i,t}^N}{c_{i,t}} e^{-\rho t}\right] = \lim_{t\to\infty} \frac{1}{\rho} e^{-\rho t} = 0. \tag{R.19}$$

R.3 Proof of Proposition A3

From (R.14), (R.15), (R.18), and $\omega_t^N = v_t^N n_t + p_t M$, aggregate consumption and investment are given by

$$C_t = \rho\omega_t^N, \tag{R.20}$$

$$I_t = \frac{v_t^N - 1}{(\sigma v_t^N)^2}\omega_t^N. \tag{R.21}$$

The market clearing condition for general goods is given by

$$Y_t = C_t + C_t^w + I_t + \eta X n_t, \tag{R.22}$$

where we use $X_t(j) = X$ from (R.12). Under a competitive economy, $Y_t = p^X X n_t + w_t L$ holds. Then, (R.22) can be rewritten as

$$\begin{aligned} & p^X X n_t + w_t L = C_t + C_t^w + I_t + \eta X n_t \\ \iff & (p^X - \eta) X n_t = C_t + I_t \\ \iff & \pi n_t = \rho \omega_t^N + \frac{v_t^N - 1}{(\sigma v_t^N)^2} \omega_t^N \end{aligned} \tag{R.23}$$

The first line uses $Y_t = p^X X n_t + w_t L$. The second line uses (R.4). The third line uses $\pi = (p^X - \eta)X$, (R.20), and (R.21). Dividing both sides of (R.23) by n_t and the after some rearrangement by using $\omega_t^N = v_t^N n_t + p_t M$, $V_t^N = 1/v_t^N$ and $B_t^N = p_t M/(v_t^N n_t)$, (R.6) is derived.

The dynamics of asset bubbles is given by $\dot{B^N}_t/B_t^N = \dot{p}_t/p_t - \dot{v_t^N}/v_t^N - \dot{n}_t/n_t$, where $\dot{n}_t/n_t$ given by (R.5). Using the procedure presented in the derivation of (17b), (R.7) is obtained. □

References

[1] Barro, R. and Sala-i-Martin, X. (2004) Economic growth. MIT Press, Cambridge, MA.

[2] Stokey, N. L. (2009) The Economics of Inaction: Stochastic Control models with fixed costs. Princeton University Press.

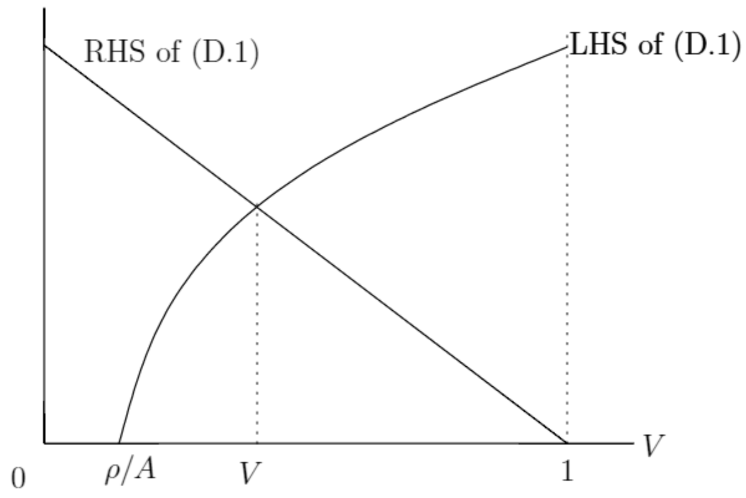

Figure A1 Bubbleless Steady State

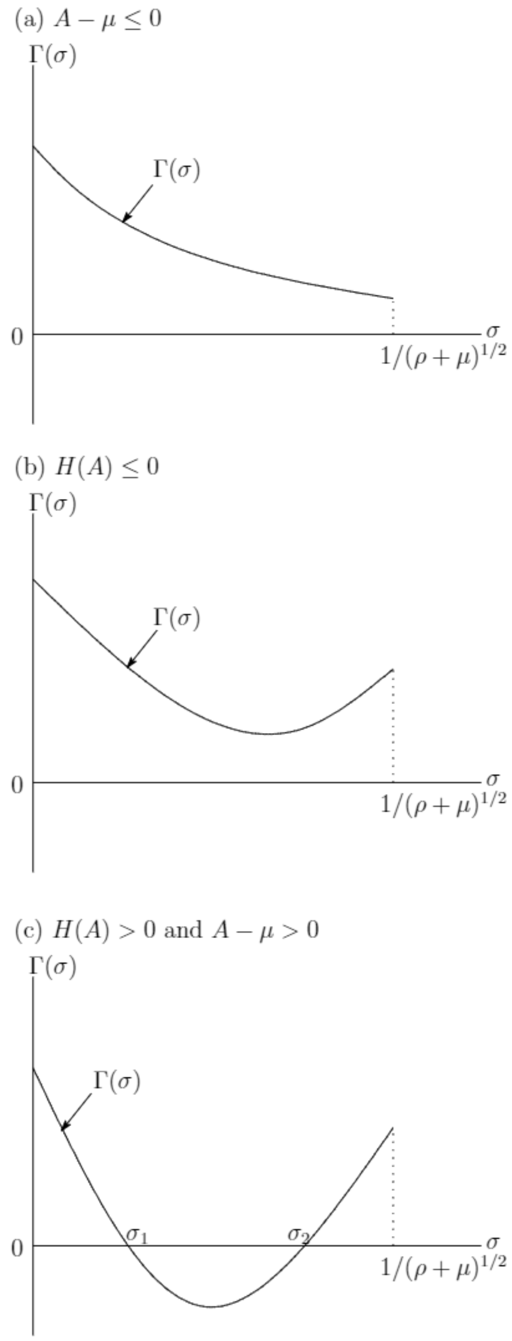

Figure A2 Function $\Gamma(\sigma)$

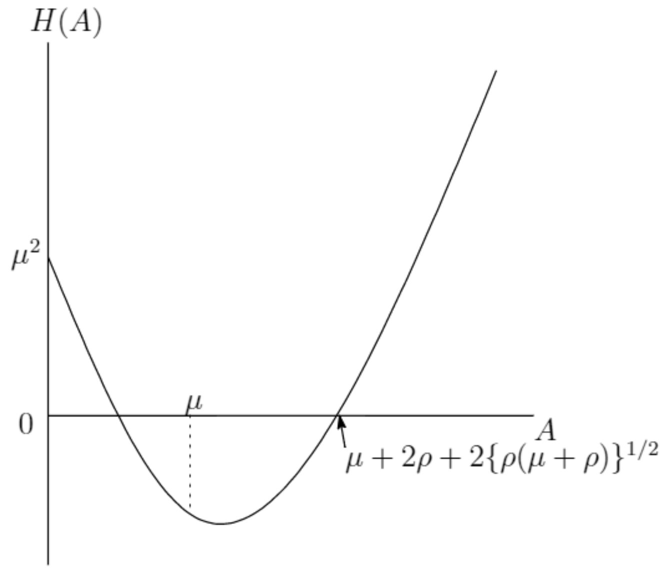

Figure A3 Function $H(A)$

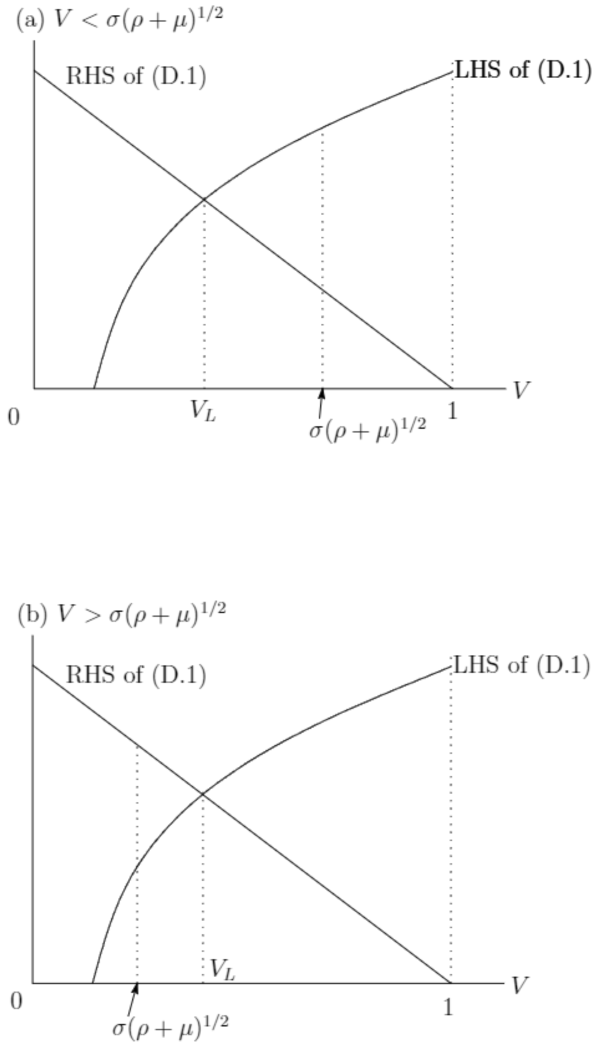

Figure A4 Relationship between V and $\sigma(\rho+\mu)^{1/2}$

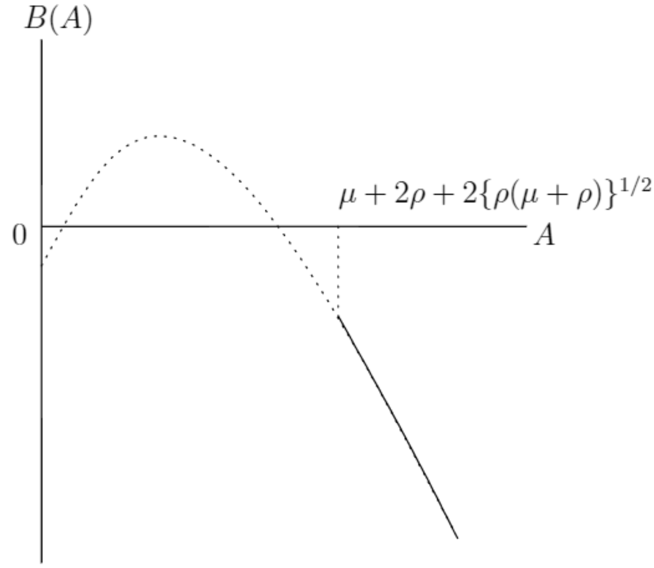

Figure A5 Function $\Psi(A)$