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Does Automation Technology increase Wage?*

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Abstract

This paper examines the relationship between automation technology diffusion and the wage. In this model, producers either choose automation or non-automation technology, whichever is more profitable. When they introduce the automation technology, they have to pay fixed costs, which are different between industries. The main results of this paper are that the productivity improvement of automation technology, which promotes automation diffusion, decreases labor share, and this improvement also decreases the wage when the level of automation technology diffusion is high enough.

Keywords: automation, the wage, labor share decline, technology choice

JEL codes: E24, J23, O3

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1 Introduction

In recent years, many studies have been showing the decline in labor income share for the last several decades. Technological change is considered to be one of the main factors influencing this phenomenon (Karabarbounis and Neiman, 2014; Grossman et al., 2017; Alvarez-Cuadrado et al., 2018).[*1] Acemoglu and Restrepo (2018b) and Autor et al. (2018) indicate that automation technology, such as robots and machines, which save labor force for production, is one of the most important latest technological changes. As labor force is replaced by capital in the production process through the diffusion of automation, labor income share decreases. Because a change in labor income share is associated with a change of the wage, this paper focuses on a relationship between the diffusion of automation technology and the average wage.

In empirical studies, there is currently no consensus regarding the impact of automation diffusion on the wage. Graetz and Michaels (2018) use data regarding robot adoption and conclude that, on average, an adoption of industrial robots by an industry positively affects worker wages. On the other hand, Acemoglu and Restrepo (2020a) focus on local labor markets divided by commuting zones and demonstrate that an introduction of industrial robots negatively affects the wage by using data between 1990 and 2007. Dauth et al. (2019) and Chiacchio et al. (2018) apply the approach of Acemoglu and Restrepo (2020a) to German and 6 countries in EU (Finland, France, Germany, Italy, Spain and Sweden) data, respectively. The former research indicates that the average wage does not increase as a result of robot adoption, and the latter indicates that the result of the decrease in the wage in Acemoglu and Restrepo (2020a) is not robust. Because these empirical researches propose different

[*1] Several types of drivers of the decline in labor income share have been presented. Elsby et al. (2013) and Boehm et al. (2019) show that globalization, which promoted offshoring and reduced the ratio of labor-intensive industry in OECD countries, is important for the decline in labor income share. Autor et al. (2017, 2020), and Barkai (2020) focus on rising market concentration of sales and markups. As market sales concentrate to a small fraction of productive firms and their markup rates increase, the income share for profits compresses the income share for workers.

results, making a theoretical research that shows conditions of the decrease in the wage is crucial.

A theoretical studies on the relationship between automation diffusion and the wage is conducted by Acemoglu and Restrepo (2018b). Using the task-based model presented by Zeira (1998),*2 they analyze the impact of advances in automation technology (automation technology becomes available for more tasks) on the wage. The adoption of automation technology by enterprises has the effect displacing workers, but it also increases aggregate production and the labor demand. As a result of the mixture of positive and negative effects on the labor demand, their static analysis proposes that the wage might decline. However, taking capital accumulation into account, the extended model concludes that the adoption of automation technology increases the wage because capital accumulation has an additional positive effect. This paper presents a general equilibrium model in which the wage decreases even if capital accumulation exists.

The model is constructed using the task-based model presented by Zeira (1998), and the over-lapping generations model by Diamond (1965). Households live for two periods; they inelastically supply labor as workers in the first period and manage production units in industries as managers in the second period. They choose either automation technology or non-automation technology for the management of the production units to maximize profits. To introduce automation technology, they must pay fixed costs, which are different across industries.*3 The fixed cost is considered as one of the main factors of technology choice in theoretical studies (e.g., Hall, 2004; Jovanovic and Lach, 1989). Sirkin et al. (2015) specify the fixed costs, estimate their magnitude, and state that their burden impedes the adoption of robots.

The main result of this paper is that an improvement of automation productivity, which promotes automaton diffusion, causes a decrease (an increase) in the wage when

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*2 Acemoglu and Autor (2011); Nakamura (2009); Nakamura and Nakamura (2008, 2019); Nakamura and Zeira (2018); Yuki (2016); Hémous and Olsen (2014); Aghion et al. (2019); Martinez et al. (2018) etc utilize the task-based model.

*3 Fixed costs order industries and the fixed cost function is assumed to increase with respect to industry.

the level of automation diffusion is high (low) enough. Whether the wage decreases or not is associated with the fixed cost function. As Acemoglu and Restrepo (2018b), the diffusion displaces workers while it increments the aggregate production. The former has a negative effect, and the latter has a positive effect on the wage. The technology choice of managers depends on net profits, including the fixed cost for automation technology and the return from the production. The expansion of the automation diffusion means that managers who face higher fixed costs and use labor before the expansion newly introduce automation technology. Thus, the return, which is proportional to the aggregate production, is sure to increase. When the fixed costs, and thus aggregate production, do not increase rapidly with respect to the diffusion level, the positive effect is weaker than the negative effect of displacement. Then, in contrast to Acemoglu and Restrepo (2018b), the wage decreases when the level of the automation diffusion is high enough, even if capital accumulation on the wage exists.

In this model, the labor income share decreases with the automation diffusion. While the income share for managers increases with the diffusion, the income share for workers decreases with it. Because the latter impact is larger than the former, the labor income share, the sum of them, decreases. This result is consistent with the trend for several decades.

This paper also considers two extensions. In the first extension, we introduce a technology frontier. Because of the technology frontier, some managers cannot utilize automation technology; otherwise, they use it.[*4] This analysis shows that the expansion of the frontier decreases the wage when the level of the frontier is high enough. The second extension considers a policy that subsidizes the fixed costs. Because the technology choice depends on the net return from production, which subtracts the fixed cost, such a policy ultimately affects the diffusion of automation technology. The benefits of subsidy policy vary across households. This paper shows that there exist subsidy rates, which improve the welfare of all households.

The rest of the paper is organized as follows. Section 2 constructs the model.

[*4] Acemoglu and Restrepo (2018b) adopt this assumption. This extension highlights the difference between their model and ours.

Section 3 solves the equilibrium and describes the conditions of the existence of a unique steady state. Section 4 examines the comparative statics and presents an extension considering the creation of new intermediate goods. Section 5 discusses the difference between this paper and Acemoglu and Restrepo (2018b), and modifies the model of this paper by introducing automation technology frontier to demonstrate clearly the difference. Section 6 introduces a subsidy policy into the model. Section 7 concludes.

2 The Model

The consumer side of the model is based upon the two-period OLG model presented by Diamond (1965). Each household supplies labor as a worker during the first period of her life and manages a production unit producing intermediate goods in an industry during the second period. There are many industries over which households are distributed during the first period. The producer side consists of managers producing intermediate goods and final goods producers which are used for consumption and investment.The factor of production for operating production unit in an industry of intermediate goods is either labor or capital and managers choose the more profitable input. When they choose capital, they can make use of automation technology,[*5] which enables them to produce without using labor, but instead they must pay fixed costs that are heterogeneous over industries.

2.1 Households

A unit measure of households live for two periods. During the first (young) period, they inelastically supply labor as workers and allocate their wages to consumption and savings. During the second period (old), they manage production units in industries of intermediate goods to receive profits and they consume the returns from their savings and profits. Let us consider a household born in period t and working in industry j.

[*5] This structure of the production side is similar to that of Zeira (1998), who is the pioneer in the literature which examines mechanization by using the task-based model.

Her utility maximization problem is expressed as follows:

$$U_t(j) = \max_{c_t^y, c_{t+1}^o} \log(c_t^y(j)) + \beta \log(c_{t+1}^o(j)), \quad \beta \in (0,1) \tag{1}$$

$$s.t.\ c_t^y(j) + s_t(j) = w_t, \tag{2}$$
$$c_{t+1}^o = R_{t+1} s_t(j) + \pi_{t+1}(j),$$

where $U_t(j)$, $c_t^y(j)$, $c_{t+1}^o(j)$, $s_t(j)$, and $\pi_{t+1}(j)$ represent lifetime utility, consumption during the first period, consumption during the second period, savings, and the profits which she receives during the second period, respectively; R_{t+1} is the gross interest rate, and β is the discount factor. She is assigned to industry $j \in [0,1]$ before she decide consumption and savings during the first period. Thus, the decision takes it into account which industry she will belong to in the second period.

From her optimal conditions, savings $s_t(j)$ is:

$$s_t(j) = \frac{\beta}{1+\beta} w_t - \frac{\pi_{t+1}(j)}{(1+\beta)R_{t+1}}. \tag{3}$$

2.2 Producers

There are two types of producers: final goods producers and managers producing intermediate goods. Final goods producers combine a unit measure of intermediate goods. Their technology is:

$$\log Y_t = \int_0^1 \log y_t(j) dj, \tag{4}$$

where Y is the amount of final goods and $y(j)$ is the amount of the intermediate good $j \in [0,1]$. [*6] From the profit maximization problem of final goods producers, the demand for each intermediate good $y(j)$ is:

$$y_t(j) = \frac{Y_t}{p_t(j)}, \tag{5}$$

[*6] In the model of Acemoglu and Restrepo (2018b), the range of intermediate goods is from $N-1$ and N, where $N \geq 1$ and N increases over time. They interprets the change of N as the creation of new tasks. We set $N = 1$ here, but section 4.3 develops the model in which the range is from $N-1$ to N and analyzes the effect of an increase in N on the wage.

where $p_t(j)$ is the price of intermediate good j.

A manager operating a production unit in industry j produces intermediate good j. In each industry, a unit measure of managers exists.*7 To produce intermediate goods, managers have two technology choices: non-automation technology or automation technology. When they choose non-automation technology, they employ only labor as their factor of production. On the other hand, when they choose automation technology, they use only capital as their factor of production. To introduce automation technology, they must pay fixed costs before commencing operations.

Acemoglu and Restrepo (2018b) assume that labor productivity varies across tasks, and the tasks with low labor productivity have the comparative advantage of labor for capital. In tasks with comparative advantage of capital, automation technology is introduced. In our model, the costs for using capital is the source of different comparative advantage because the fixed costs to introduce automation technology vary across industries. Sirkin et al. (2015) specify the costs and estimated their magnitude. They state that the price of the robotic hardware and software to introduce a spot welding robot as an example of automation technology is only one-quarter of that total cost.*8 Some of the other components of the costs are fixed costs, such as the cost of programming and integrating a robot into a factory, and the cost of safety structures, which protect workers and robots themselves. Those costs account for two-thirds of the total. The argument that fixed costs inhibit the adoption of new technologies has been the subject of many theoretical papers (e.g., Hall, 2004; Jovanovic and Lach, 1989).

Because managers gain profits from producing intermediate goods, they choose the more profitable technology:

$$\pi_{t+1}(j) = \max\{\pi^l_{t+1}(j), \pi^k_{t+1}(j)\}, \tag{6}$$

*7 This assumption ensures that the intermediate goods market is competitive, in spite of households' immovability between industries.

*8 Spot welding is a type of electric resistance welding used to weld various sheet metal products, and it achieves a labor-saving in a welding line (Takayama and Takahashi, 2014). It is applied to production processes, such as the production of automobile and aerospace vehicles (Li and Duarte, 2018).

where $\pi^l_{t+1}(j)$ and $\pi^k_{t+1}(j)$ are the profits from producing intermediate goods by using non-automation technology and automation technology in industry j, respectively.

When a manager utilizes non-automation technology, she can produce intermediate goods without incurring any fixed costs. The production function is given by:[*9]

$$y^l_t(j) = l_t(j), \tag{7}$$

where $l_t(j)$ is the amount of labor inputs. The profit maximization problem, then, is:

$$\pi^l_t(j) = \max_{l_t(j)} \; p^l_t(j) y^l_t(j) - w_t l_t(j), \tag{8}$$

where w_t represents the wage. Since the production function is linear, the price of the intermediate goods produced by using non-automation technology is

$$p^l_t(j) = w_t. \tag{9}$$

Thus, the profit from non-automation technology is equal to zero.

The production function for automation technology is:[*10]

$$y^k_t(j) = \gamma k_t(j)^\alpha, \quad \alpha \in (0,1), \; \gamma > 0 \tag{10}$$

where $k_t(j)$ is the capital used in industry j and γ is the productivity of automation technology. The profit maximization problem for managers choosing automation technology is:

$$\pi^k_t(j) = \max_{k_t(j)} \; p^k_t(j) y^k_t(j) - R_t k_t(j) - C(j), \tag{11}$$

$$where \;\; C(j) \equiv C \cdot j, \quad C > 0. \tag{12}$$

In the above equation, $C(j)$ represents the fixed cost function required to introduce automation technology into industry j and is paid by final goods.[*11] Since fixed costs are increasing in industry index j, the comparative advantage of labor to capital is

[*9] For simplicity, the production function for non-automation technology is linear. The case in which the function is non-linear is discussed in section 4.2.

[*10] Rosenfeld et al. (2004) studied the productivity of foraging robots. They indicate that the marginal productivity is decreasing in the number of robots.

[*11] Chen and Koebel (2017) estimates fixed costs by industry and show that they are different.

also increasing in j.[*12] The first order condition is:

$$R_t = p_t^k(j)\gamma\alpha k_t(j)^{\alpha-1}. \tag{13}$$

By substituting this equation into (11), profit becomes:

$$\pi_t^k(j) = \frac{1-\alpha}{\alpha}(p_t^k(j)\gamma\alpha)^{\frac{1}{1-\alpha}} R_t^{\frac{\alpha}{\alpha-1}} - C(j). \tag{14}$$

3 Equilibrium

At equilibrium, the optimal conditions for consumers and producers and each market clearing condition are satisfied. By using (5), (10) and (13), from the intermediate goods market clearing condition, the price of the intermediate good j produced by using automation technology $p_t^k(j)$ is given by

$$p_t^k(j) = \frac{Y_t^{1-\alpha}}{\gamma}(\frac{R_t}{\alpha})^{\alpha}. \tag{15}$$

Substituting this equation into (14), $\pi_t^k(j)$ can be rewritten as:

$$\pi_t^k(j) = (1-\alpha)Y_t - C(j). \tag{16}$$

The fixed costs are increasing in the industry index. This implies that a threshold $j^* \in (0,1)$ exists below which managers choose automation technology. This is because $j^* = 0$ and $j^* = 1$ do not satisfy the conditions of the equilibrium. What $j^* = 0$ means is that no manager introduces automation technology, but managers in industry $j = 0$, in which the fixed cost for automation are not needed, inevitably use automation technology. When automation is introduced in all industries, that is $j^* = 1$, the wage is zero since there are no industry using labor. Then, the amount of aggregate capital is zero since the fund for capital is the wage.

Since the profit from non-automation technology is equal to zero, the return from automation technology for industry j_t^* is zero. Thus, from using (16) and (12):

$$j_t^* = \frac{(1-\alpha)Y_t}{C}. \tag{17}$$

[*12] In the model of Acemoglu and Restrepo (2018b), the productivity of labor depends on the index. Our setting differs, but it is the same in terms of comparative advantage.

From (13) and (15),

$$k_t(j) = \frac{\alpha Y_t}{R_t}. \tag{18}$$

The capital market clearing condition is:

$$K_t = \int_0^{j_t^*} k_t(j)dj, \tag{19}$$

where K_t represents aggregate capital. Thus, the interest rate is equal to:

$$R_t = \frac{j_t^* \alpha Y_t}{K_t}. \tag{20}$$

From (5), (7), and (9),

$$l_t(j) = \frac{Y_t}{w_t}. \tag{21}$$

Since the labor supply is unity, the labor market clearing condition is:

$$1 = \int_{j^*}^1 l_t(j)dj. \tag{22}$$

Thus, the equation for the wage equals:

$$w_t = (1 - j_t^*)Y_t, . \tag{23}$$

By combining (18) and (21) with (4), (7) and (10), the amount of final goods Y_t is:

$$Y_t = \left[\gamma \left(\frac{K_t}{j_t^*}\right)^\alpha\right]^{j_t^*} \left(\frac{1}{1 - j_t^*}\right)^{1-j_t^*}. \tag{24}$$

By using (17), this equation can be rewritten as:

$$\frac{Cj_t^*}{1-\alpha} = \left[\gamma \left(\frac{K_t}{j_t^*}\right)^\alpha\right]^{j_t^*} \left(\frac{1}{1 - j_t^*}\right)^{1-j_t^*}. \tag{25}$$

Thus, K_t equals:

$$K_t = \left(\frac{Cj_t^*}{1-\alpha}\right)^{\frac{1}{\alpha j_t^*}} (1 - j_t^*)^{\frac{1-j_t^*}{\alpha j_t^*}} j_t^* \gamma^{-\frac{1}{\alpha}} \equiv \Phi(j_t^*). \tag{26}$$

From the final goods market clearing condition, savings are used to produce capital for the next period. Thus, the equation for capital accumulation is:

$$K_{t+1} = \int_0^1 s_t(j)dj. \tag{27}$$

By combining (3), (16), (17), (20), and (23), this equation can be rewritten as:

$$K_{t+1} = \frac{2\alpha\beta C}{(1-\alpha)\{2\alpha(1+\beta)+(1-\alpha)\}}(1-j_t^*)(j_t^*) \equiv \Psi(j_t^*). \tag{28}$$

Since $K_{t+1} = \Phi(j_{t+1}^*)$ from (26), the equation for capital accumulation is then converted to the dynamic equation for j^*:

$$\Phi(j_{t+1}^*) = \Psi(j_t^*). \tag{29}$$

From this equation, the conditions of the existence and the stability of an unique steady state are obtained as the following propositions.

Proposition 1. *When the parameter of the fixed cost function C is sufficiently small, there exists a unique steady state.*

Proposition 2. *Suppose that a unique steady state exists. When the automation productivity γ is small enough or C is large enough that the level of automation diffusion j^* is sufficiently low, this steady state is stable.*

In Appendix A, we provide more formal conditions and proofs for the propositions above and show the dynamics graphically. The analysis demonstrated in this paper focus only on the case in which a unique steady state exists, and it is stable.

4 Analysis

4.1 Comparative Statics

In this section, we focus on the unique steady state and analyze the long-run effects on the wage and income shares of an improvement in automation technology productivity, γ, and of the cost-saving technological changes which reduces the parameter of the fixed cost function, C. The following proposition examines the former effect. Proofs of the propositions presented in this section are given in Appendix.B.

Proposition 3. *Suppose that Proposition 1 holds. If the productivity of automation technology γ is large (small) enough or if the fixed cost parameter C is small (large)* *enough that the equilibrium threshold of industries j^* is higher (lower) than $1/2$, an improvement of automation productivity reduces (raises) the wage in the long-run.*

This proposition shows that the wage decreases (increases) due to the improvement of the automation productivity when the threshold of a unique steady state j^* is more (less) than $1/2$. From (17) and (23), the wage in the steady state is:

$$w = (1 - j^*)Y, \tag{30}$$

$$= \frac{C}{1-\alpha} j^*(1 - j^*). \tag{31}$$

Lemma B.1 presented in Appendix.B shows that j^* increases as γ rises or C falls. (31) demonstrates that the level of j^* determines the sign of the marginal effect of γ on the wage. This proposition shows that when γ is large (small) enough or C is small (large) enough that $j^* > 1/2$ ($j^* < 1/2$), an increase in γ reduces (raises) the wage.

From (30), the wage is the product of the mass of industries employing labor $1-j^*$ time the amount of final goods Y. When j^* increases, labor is substituted for capital in the newly automated industries. Thus, this channel decreases labor demand and the wage. On the other hand, an increase in j^* causes an increase in Y since from (17), Y is proportional to the fixed costs, which is increasing in j^*. For automation technology to be introduced in higher fixed-cost industries, Y needs to be large. An increase in Y pulls up the wage from (30). When $j^* > 1/2$ ($j^* < 1/2$), the negative (positive) effect is dominant, thus the wage decreases (increases) with an increase in γ. This result might appear to depend on the assumption that the fixed cost function is linear. We discuss how robust the result is when the fixed cost function is more general.

The following proposition shows the effect of a cost-saving technological change on the wage.

Proposition 4. *Suppose that Proposition 1 holds. If γ is large (small) enough or C is small (large) enough that j^* is higher (lower) than a value $j^c > 1/2$, a decrease in C reduces (raises) the wage in the long-run.*

This proposition shows that a cost-saving technological change that reduces the fixed cost of automation technology has a similar effect on the wage, as does an improvement in automation productivity. In contrast to the case of γ, from (31) ,the change in C influences the wage directly. Since this direct effect is positive, in order for the wage to decrease, it is necessary to achieve a higher level of automation diffusion.

The results that there exist both possibilities of an increase and decrease in the wage through automation diffusion are consistent with empirical researches. Some researches indicate that the diffusion of automation raises the wage (e.g., Autor and Salomons, 2017; Graetz and Michaels, 2018). Others, such as Acemoglu and Restrepo (2020a) and Dauth et al. (2019) and Chiacchio et al. (2018), indicate that the effect of automation on the wage is negative or ambiguous. These propositions show that whether automation diffusion raises or reduces the wage is determined by the level of automation diffusion in the economy. The effect of the diffusion level before technological changes has not empirically examined yet. These results suggest a new empirical question about the relationship between automation diffusion and the wage.

The following proposition analyzes the effect of an increase in γ or a decrease in C on the labor income share. In this model, there exist workers and managers. In the statistics, the labor income share includes the income share of managers, so in the next proposition, labor income share is the sum of the income shares for workers and managers.[*13]

Proposition 5. *When the conditions in Proposition 1 hold, the income share of managers is increasing (decreasing), and of workers is decreasing (increasing) with* γ *(*C*). The total income share of managers and workers is decreasing (increasing) with* γ *(*C*).*

The income share of managers increases (decreases), and of workers decreases (increases) with γ (C). Their total income share decreases (increases) with γ (C). This

[*13] Let $\mathcal{C}$ be the aggregate fixed cost, and Π be the aggregate profit. The income share of managers and workers are $\Pi/(Y-\mathcal{C})$ and $w/(Y-\mathcal{C})$, respectively.

result is consistent with many researches, such as Karabarbounis and Neiman (2014) and Elsby et al. (2013), which show the decline of labor income share in many countries since the 1980s.[*14] As Piketty and Saez (2003) and Jones and Kim (2018) suggest, the gap of income share between workers and entrepreneurs widens in the US since the 1970s.

(Figure 1 around here)

Figure 1 illustrates graphically the effect of a productivity improvement of automation technology on the extent of automation technology diffusion, the amount of final good production, wage, the income share of workers, the income share of managers, and the total labor income share at the steady state. The parameters are set to be $\alpha = 0.3$, $\beta = 0.82$, and $C = 4.4$. The time preference β is a standard value for OLG model. The value of the parameter of the production function of automation technology α does not change qualitative results. The value of C and the range of γ are chosen to ensure the unique stable steady state. As shown in Proposition 3, when j^* is over $1/2$ (γ is around 5), aggregate capital and the wage decrease with γ. As shown in Proposition 5, the income share of workers declines, the income share of managers increases, and the total income share decreases with γ.

4.2 Why Can Wage Decrease?

In the model of Acemoglu and Restrepo (2018b) and many other task-based models analyzing mechanization, the wage increases by technological change when the technology choice depends on producers' decision.[*15][*16] In contrast, our model proposes

[*14] Karabarbounis and Neiman (2014) divide labor income share and profit share, but our result is not in contradiction with them.

[*15] Acemoglu and Restrepo (2018b) suggest the possibility of a decrease in the wage when some producers cannot utilize automation technology because of a technology frontier. The model considering the technology frontier will be discussed in Section 5. Berg et al. (2018) also focus on the negative effect of automation diffusion on the wage, but the wage always increases in the long-run while decreases in the short-run in their model.

[*16] Acemoglu and Restrepo (2020b) analyze the effect of an automation diffusion on labor markets, including the wage by focusing on a skill difference. Then, they suggest theoretically and

that the wage decreases with an increase in the automation productivity γ when the level of automation diffusion j^* is large enough. This section explains the mechanism of the decrease in the wage presented in Proposition 3 and 4.

We assume that the fixed cost function, which was linear in the previous section, is a continuous, differentiable, and increasing function $C(j)$, $C' > 0$. Then, the threshold j^* determining the choice of technology in the steady state is transformed from (17) to:

$$C(j^*) = (1-\alpha)Y. \tag{32}$$

In industry j^*, the price of intermediate goods is equal to the average cost because the profit from automation technology is zero. The price depends on the demand for the intermediate goods, which is proportional to Y from (5). The average cost includes the fixed cost. Thus, in industry j^*, Y relates to the fixed cost. (32) implies that an increase in Y enhances the automation diffusion j^* since the fixed cost function is increasing in j^*. This is because, as the output of final goods increases, the difference between the returns from automation technology and non-automation technology becomes higher, allowing more industries to pay fixed costs and introduce automation.

The wage in the steady state is:

$$\begin{aligned} w &= (1-j^*)Y, \\ \Leftrightarrow \log w &= \log(1-j^*) + \log Y. \end{aligned} \tag{33}$$

Thus, the effect of an increase in j^* on the logarithm of the wage is:

$$\frac{d\log w}{dj^*} = -\frac{1}{1-j^*} + \frac{d\log Y}{dj^*}. \tag{34}$$

empirically that the automation diffusion enhances the gap of the wage between low-skill and high-skill labor and the decline of the wage of low-skill labor. In their model, low-skill labor works in tasks in which automation technology is able to replace labor. Still, high-skill labor does not because they are engaged in tasks that automation technology cannot perform. However, as pointed out by Autor (2015), the progress of automation tends to be encroaching upward in abstract tasks in that high-skill labor is apt to engage. Thus, the approach of Acemoglu and Restrepo (2020b) has a limitation. This paper proposes the mechanism of the decrease in the wage that does not depend on skill differences.

Following Acemoglu and Restrepo (2018b), we call the first term the displacement effect, and the second term the productivity effect. The diffusion of automation technology captured by an increase in j^* displaces labor because the industries in which automation technology is newly introduced cease employing labor. The displacement effect has a negative effect on the labor demand, and then on the wage. On the other hand, the diffusion has a positive effect on the wage through the increase in the output of final goods. When the negative effect dominates the positive effect, the wage decreases.

The scale of productivity effect depends on the fixed cost function from (32):

$$\frac{d \log Y}{dj^*} = \frac{d \log C(j^*)}{dj^*}. \tag{35}$$

Because the productivity effect depends on the fixed cost function, whether the wage increases or decreases with the automation diffusion depends on the fixed costs function. From (34) and (35):

$$\frac{d \log w}{dj^*} = -\frac{1}{1-j^*} + \frac{d \log C(j^*)}{dj^*}. \tag{36}$$

In our model, $d \log w/dj^* < 0$ when $j^* > 1/2$ because of the linear fixed cost function. The condition of linearity can be relaxed. For example, when the fixed cost function is an exponential function $C(j) = \exp(a \cdot j)$, where $a > 1$, $d \log w/dj^* < 0$ when $j^* > 1 - 1/a$. Even if the degree of increase in C is large, the same result holds, in other words, $\lim_{j \to 1} C(j) < \infty$ is not necessary. Let us consider the case of the following function.

$$C(j) = \frac{\exp(a_1 j + a_3)}{(1-j)^{a_2}}, \ where \ 0 < 1 - a_2 < a_1, \ a_2 > 0.$$

This function satisfies that $\lim_{j \to 1} C(j) = \infty$ and $\lim_{j \to 1} C'(j)/C(j) = \infty$. In this case, $d \log w/dj^* < 0$ when $j^* > (a_1 + a_2 - 1)/a_1$. Therefore, the results of Proposition 3 and 4 hold without the linearity of the fixed cost function. For simplicity, we assume the linear function as (12) below.

4.3 Effects of New Industries

Acemoglu and Restrepo (2018b) point out that the progress of automation technology creates new tasks and show theoretically that this creation, which increases the tasks performed by non-automation technology, stimulates labor demand, and increases wages. This subsection examines the effect of the creation of new industries on wages. In Acemoglu and Restrepo (2018b), the displacement of labor by capital due to the diffusion of automation technology is viewed as the factor that reduces the wage, and the task creation is viewed as the countervailing factor that increases the wage. In contrast, the creation works as a factor in reducing the wage in this paper.

In this subsection, to introduce new industries, the range of intermediate good industries is modified to $j \in [N-1, N]$. The technology of final goods is rewritten as:

$$\log Y_t = \int_{N-1}^{N} \log y_t(j) dj,$$

where N indicates the number of industries. An increase in N is interpreted as the creation of new industries. This setup is the same as Acemoglu and Restrepo (2018b). (23), (26) and (28) are rewritten as:

$$\begin{aligned}
w &= (N - j_t^*) Y_t, \\
\Phi^N(j_{t+1}^*) &= \left(\frac{C j_{t+1}^*}{1-\alpha}\right)^{\frac{1}{\alpha(j_{t+1}^* - (N-1))}} (1 - j_{t+1}^*)^{\frac{N - j_{t+1}^*}{\alpha(j_{t+1}^* - (N-1))}} (j_{t+1}^* - (N-1)) \gamma^{-\frac{1}{\alpha}} \\
\Psi^N(j_t^*, j_{t+1}^*) &= \left[1 + \frac{1-\alpha}{2\alpha(1+\beta)} \frac{j_{t+1}^* - N + 1}{j_{t+1}^*}\right]^{-1} \frac{\beta C}{(1+\beta)(1-\alpha)} (N - j_t^*) j_t^*
\end{aligned}$$

From these equations, let us demonstrate the effects of new industries on the steady state numerically. The values of α, β, and C are the same as before, and the value of γ is 4.65. The range of N is set to be $(1, 1.15)$ and the parameters here ensure the existence of the equilibrium. As N increases, the fixed cost for automation technology increases.[*17]

[*17] When N is sufficiently large, the fixed cost is too high to introduce the automation technology even for managers in industry $N-1$, and no one uses this technology. In this case, the equilibrium conditions do not hold.

(Figure 2 around here)

Figure 2 demonstrates how each variable in the steady state depends on N. This figure shows that an increase in N leads to a decrease in j^*, Y, w. As in the previous subsections, the effect on the wage is separated into two channels; the effect through $1 - j^*$ and the effect through Y. The former effect on the wage is positive because an increase in N and the resulting decrease in j^* stimulate labor demand. However, the latter effect on the wage is negative because the decrease in j^* leads to a decrease in Y, and is stronger than the former effect. Thus, the creation of new industries typically leads to a decrease in wages in this model.[*18]

5 Automation Technology Frontier

In the model in the previous section, the automation technology is potentially applied to all industries. However, some industries may not be automated because the technology to automate them does not exist. Thus, this section considers the case in which automated industries are limited to industry $j \leq \bar{j} < j^*$, where $\bar{j}$ is the upper bound for automated industries. Managers in $j \in (\bar{j}, j^*]$ who want to introduce automation cannot do so because of this technological limitation. Then, this section examines the effects on the wage of the technological changes, in particular, an improvement of automation productivity γ and an increment of automation frontier $\bar{j}$. The assumption on $\bar{j}$ is similar to the model of Acemoglu and Restrepo (2018b). This analysis highlights the differences from their model.

In this section, the range of intermediate good industries is the same as Section 2, $j \in [0, 1]$. Managers in $j \in [0, \bar{j}]$ choose automation technology. Thus, the gross

[*18] When γ is large, the decrease in Y is relatively gentle, and the negative effect is weak. Thus, the total effect of N on the wage becomes positive when N is small. However, an increase in N magnifies a decrease in Y and the negative effect. Therefore, even in this case, the wage decreases with N when N is not small.

interest rate (20), the wage (23), and the final goods (24) are rewritten as:

$$R_t = \frac{\alpha Y_t}{K_t} \cdot \bar{j}, \tag{37}$$

$$w_t = (1 - \bar{j}) Y_t, \tag{38}$$

$$Y_t = \left[\gamma \left(\frac{K_t}{\bar{j}}\right)^{\alpha}\right]^{\bar{j}} \left(\frac{1}{1-\bar{j}}\right)^{1-\bar{j}}. \tag{39}$$

From (37), (38), savings (3), and the profits from automation technology (16), capital accumulation equation (27) becomes:

$$K_{t+1} = \frac{\beta}{1+\beta}\bar{j}(1-\bar{j})Y_t - \frac{K_{t+1}}{\alpha(1+\beta)}\left[(1-\alpha) - \frac{C\bar{j}}{2Y_{t+1}}\right]. \tag{40}$$

At first, the following proposition examines the effect of γ on the wage.

Proposition 6. *When there exists a technological frontier $\bar{j} < j^*$, the effect of an increase in γ on the wage is positive in the long run.*

The proof of this proposition is given in Appendix B. This proposition shows that an improvement of automation technology γ increases the wage. Since an increase in γ does not influence the number of industries using automation technology, the displacement effect does not emerge, and the productivity effect raises the wage.

Let us consider the effect of the increment of the automation frontier on the wage. This effect is represented as the following equation from (38):

$$\frac{dw}{d\bar{j}} = -Y + (1-\bar{j})\frac{dY}{d\bar{j}}. \tag{41}$$

As in the previous section, we call the first term the displacement effect and the second term the productivity effect.

To analyze the productivity effect, we differentiate (39) with $\bar{j}$:

$$\begin{aligned} \frac{dY}{d\bar{j}} &= \frac{\partial Y}{\partial K}\frac{dK}{d\bar{j}} + \frac{\partial Y}{\partial \bar{j}}, \\ &= \alpha\bar{j}\frac{Y}{K}\frac{dK}{d\bar{j}} + Y\left[(1-\alpha) + \log\gamma\left(\frac{K}{\bar{j}}\right)^{\alpha}(1-\bar{j})\right]. \end{aligned} \tag{42}$$

The first term of (42) reflects the indirect effect through capital accumulation, and the second term reflects the direct effect. From (40), the effect of an increase in $\bar{j}$ on capital accumulation is:

$$\frac{dK}{d\bar{j}} = \left(1 + \frac{1-\alpha}{\alpha(1+\beta)} - \frac{C\bar{j}}{2(1+\beta)\alpha Y}\right)^{-1} \times \left[\frac{\beta}{1+\beta}(1-2\bar{j})Y + \frac{CK}{2(1+\beta)\alpha Y} + \left(\frac{\beta}{1+\beta}\bar{j}(1-\bar{j}) - \frac{1}{Y}\frac{CK\bar{j}}{2(1+\beta)\alpha Y}\right)\frac{dY}{d\bar{j}}\right]. \tag{43}$$

Equations (42) and (43) determine $dK/d\bar{j}$ and $dY/d\bar{j}$. Because the signs of them are not clear analytically, we conduct a numerical analysis.

(Figure 3 around here)

Figure 3 illustrates how the wage depends on γ and $\bar{j}$.[*19] The x-axis is γ, the y-axis is $\bar{j}$, and the z-axis is the wage. The parameters are $\alpha = 0.3,\ \beta = 0.82,$ and $C = 4.4$. The figure shows that when the level of $\bar{j}$ is large (small) enough, the effect of an increase in $\bar{j}$ is negative (positive) when γ is sufficiently small or large. In the model of Acemoglu and Restrepo (2018b), the wage always increases with $\bar{j}$. In contrast, the wage decreases in some ranges of γ and $\bar{j}$ in this model.

(Figure 4 around here)

To understand why the wage decreases with $\bar{j}$ when $\bar{j}$ is sufficiently large, we focus on the productivity effect. As described previously, the first term of (42) corresponds to the indirect effect through capital accumulation, and the second term reflects the direct effect. Figure 4 shows how the amount of aggregate capital K depends on γ and $\bar{j}$. Since K is increasing with $\bar{j}$ from this numerical analysis, the indirect effect is positive, which is identical to Acemoglu and Restrepo (2018a).

By using (37) and (38), the direct effect is rewritten as:

$$\frac{\partial Y}{\partial \bar{j}} = \left[\left(\log\frac{1}{w} - \log\frac{\gamma}{R^{\alpha}}\right) + \alpha\log\alpha - (1-\alpha)\log Y + 1 - \alpha\right]Y. \tag{44}$$

The part depending on effective factor prices is the effect pointed out by Acemoglu and Restrepo (2018a) and many other models based on them. The remaining part in

[*19] This figure focuses on the range of γ and $\bar{j}$ assuring the existence of a unique steady state.

(44) (when $\alpha = 1$, the additional part disappears) emerges because the aggregate final good production function (39) is decreasing return to scale, and this part decreases with Y.

In Acemoglu and Restrepo (2018b), the negative displacement effect is dominated by the positive productivity effect because of the indirect effect through capital accumulation. In this model, the wage could decrease with $\bar{j}$, as shown in Figure 3, which means that the sum of the displacement effect and the productivity effect is negative. Because the displacement effect is the same as Acemoglu and Restrepo (2018b), the productivity effect is smaller or even may be negative. One of the reasons for this difference is the decreasing return to scale of the final good production function.

6 Subsidy and Tax

Sirkin et al. (2015) suggest that a high initial cost prevents the introduction of automation technology. This section introduce a subsidy to fixed costs that encourage the diffusion of automation technology, and examines the effects of such a policy. Since this model utilizes OLG setting, over-accumulation might occurs. It might be possible that a subsidy to introduce automation technology increases the aggregate output and ultimately leads to an improvement in overall economic welfare. Let us consider that a subsidy that compensates for a certain ratio of fixed costs, which is financed by a lump-sum tax on young households.

This policy changes the budget constraint for young households such that:

$$c_t(j) = w_t - s_t(j) - \tau_t, \tag{45}$$

where τ represents the lump-sum tax. The profit from automation technology is:

$$\pi_t^k(j) = \max_{k_t(j)} \; p_t(j)y_t^k(j) - R_t k_t(j) - (1-\sigma)C(j), \tag{46}$$

where $\sigma \in [0, 1]$ is the subsidy rate. A policymaker's budget constraint is:

$$\tau_t = \int_0^{j_t^*} \sigma C(j)dj. \tag{47}$$

This section examines the effects of this subsidy numerically. The parameters are the same as in the previous numerical example except for γ ($\gamma = 4.65$). The values of γ and C ensure that a unique steady state exists.

(Figure 5 around here)

Figure 5 illustrates the effects of the subsidy on the wage, the final good production Y, the extent of automation technology diffusion j^*, aggregate capital K, interest rate R, and economic welfare at the steady state. The welfare is measured by the aggregation of each household's lifetime utility. Figure 5 shows that this subsidy encourages the adoption of automation technology because it reduces the associated fixed costs. Thus, the automation technology diffusion level gets high when the subsidy rate increases. The automation diffusion leads to a decline in $1 - j^*$, but an increase in the final goods. Therefore, as in the previous section, this subsidy has both positive and negative effects on the wage. Figure 5 shows that the negative effect dominates the positive effect.[*20]

Figure 5 also shows that as the subsidy rate σ increases, welfare increases at first; however, it subsequently decreases. The welfare of each household is determined by the lifetime income and R. The lifetime income is:

$$(1 - \tau_t)w_t + \frac{\pi^i_{t+1}(\sigma_{t+1})}{R_{t+1}}, \quad for\ i = l, k. \tag{48}$$

Increases in R, w and τ push down the lifetime income, and increases in the return from production and the subsidy pull up the lifetime income. The increase (decrease) in the lifetime income pulls up (pushes down) the welfare. R has not only the indirect effect through lifetime income, but also a direct effect on the welfare because R increases the value of savings. As shown in Figure 5, R increases with σ, which coincides with a decrease in K. Recalling that the profit in industry j is $(1 - \alpha)Y - (1 - \sigma)Cj$, the profits for managers increase with Y and σ. The increase in the profits has a positive effect on the lifetime income. On the other hand, the increase

[*20] If we set a different parameter set, for example, larger γ, the positive effect is stronger, and the region, in which the wage increases with σ, emerges.

in τ and the decrease in w have a negative effect on the lifetime income. When the positive effects are larger (smaller) than the negative effects, each household's welfare increases (decreases). Figure 5 shows that aggregate welfare increases when σ is low and decreases when σ is high.

This model employs the OLG model's setup, and households have different types. Some generations may not derive benefits from the subsidy policy, particularly the households that do not choose automation technology in response to the subsidy policy because of the high fixed costs. They receive no direct gain from the policy. In addition, households that face zero fixed costs also receive no direct gains.

Let us consider the dynamics of lifetime utility to examine the effects of the policy on economic welfare and whether all households of all generations benefit from the policy. At first, the economy is at a steady state, and the subsidy policy is enforced in Period 3.

(Figure 6 around here)

Figure 6 illustrates the dynamics of the lifetime utility of households who are old in period t. The upper left panel illustrates the dynamics of the average lifetime utility of households. The upper right panel shows the lifetime utility of households that choose non-automation technology, even if the subsidy policy is operating. The lower left panel shows the lifetime utility of households that face zero fixed costs. In each panel, the blue line (round), the green line (square), and the red line (diamond) indicate the dynamics in cases in which the subsidy rate, σ, is 0.01, 0.015, and 0.035, respectively. The black (dotted) line in each panel indicates the welfare level in the previous steady state. Figure 7 demonstrates the dynamics of households' lifetime income focused on in Figure 6 and the interest rate. The blue (round), green (square), and red (diamond) lines indicate the dynamics in cases in which the subsidy rate, σ, is 0.01, 0.015, and 0.035, respectively. The black (dotted) lines indicate the level of R and lifetime income in the previous steady state.

The upper left panel of Figure 6 shows that the average welfare is improved by this policy compared with the economy in the previous steady state, at any time.

However, not all households benefit from the policy. Figure 6 focuses on the two types of households, who seem to gain little benefit from the policy. One type of households is that they choose non-automation technology, even if the subsidy policy is operating, and the other type of households is that they face zero fixed costs to introduce automation technology.[*21][*22]

The upper right panel of Figure 6 indicates that the lifetime utility of the households who choose non-automation technology increases even in Period 4. They only pay the tax during the young period and receive no subsidy directly. From Figure 7, their lifetime income decreases in all cases. However, since the policy increases R from Figure 7, their income during the old period, which is funded by savings, increases. The positive effect is dominant, and their welfare increases.

The lower left panel of Figure 6 indicates the decrease in the welfare of Period 4 households with zero fixed costs when σ are 0.01 and 0.015. From Figure 7, their lifetime income decreases, and R increases. They pay the tax during the young period, and the increase in R discounts the value of the profits during the old period, but they directly gain no benefit. When the subsidy rate is low, the negative effect of the decrease in lifetime income on the welfare is larger than the positive effect of R in Period 4. However, when $\sigma = 0.035$, their welfare increase even in Period 4.[*23] Since the lifetime income of households who introduce automation technology

[*21] The households who introduce automation technology with positive fixed costs gain larger benefits than the two types of households. Particularly, the old households in Period 3 that choose automation technology receive the transfer from the young generation without paying tax.

[*22] Their welfare in Period 3 are the same with previous periods. Figure 7 shows that R and their lifetime income in Period 3 are also the same as in previous periods. Since the aggregate capital in Period 3 K_3 is determined in Period 2, and the subsidy for fixed costs does not directly influence each manager's demand for capital, R remains unchanged in Period 3. Thus, in Period 3, while the subsidy policy affects the lifetime income of households who introduce automation technology and face positive fixed costs, it does not affect other households' lifetime income. Note that the figures indicate the welfare and lifetime income of households who are old in Period t.

[*23] Until σ is around 0.07, their welfare of Period 4 and subsequent generations increase. If σ is over 0.07, their welfare decrease in Period 4 and increase in Period 5 by larger level than in Period 3. As shown in Figure 5, if σ is over around 0.1, the welfare level in Period 5 is also smaller than in Period 3.

by paying positive fixed costs is larger than households with zero fixed costs, their welfare increases by the subsidy policy. Thus, when $\sigma = 0.035$, all households of all generations benefit from the policy. However, as shown in Figure 5, when σ is large enough, an increase in σ decreases the average welfare, and thus, all households do not benefit from the marginal increase in the subsidy rate.

7 Conclusion

This paper focuses on the effects of fixed costs on automation technology diffusion and examines the relationship between the diffusion and the wage. In this model, households choose either automation technology or non-automation technology for production. When they choose the automation technology, they pay fixed costs. Thus, their choice determines the extent of automation technology diffusion and fixed costs play an essential role in their decision.

The analysis in this paper yields two significant results. First, the wage can decrease with an improvement in automation productivity or decreases in fixed costs, both of which accelerate the automation diffusion. The diffusion has both positive and negative effects on the wage. The positive (negative) effect dominates in the long-run when the extent of automation technology diffusion is small (large) enough. In Acemoglu and Restrepo (2018b), the wage increases in the long-run because of the positive effect through capital accumulation. In contrast to their result, the wage decreases when the level of diffusion is high enough, even if capital accumulation exists.

Second, the labor income share, which is the sum of the income share for workers and managers, decreases with the automation diffusion. This result is consistent with the recent empirical studies, which point out the decline of the labor income share for the last several decades (Karabarbounis and Neiman, 2014; Grossman et al., 2017; Alvarez-Cuadrado et al., 2018).

Then, this paper examines two extensions. First, we introduce an automation technology frontier. This setting is the same as in Acemoglu and Restrepo (2018b).

Because of the technology frontier, automation is not available for some industries. In contrast to Acemoglu and Restrepo (2018b), in which an expansion of the frontier always increases the wage in the long-run, it can decrease the wage in the long-run. This setting is the same as in Acemoglu and Restrepo (2018b), in which the wage always increases in the long-run. Second, this paper examines the long-run and short-run effects of a subsidy policy for fixed costs on the economy. This analysis demonstrates that such a subsidy policy has both positive and negative effects on the long-run wage and welfare. This paper shows that there exist subsidy rates, which improve the welfare of all households of all generations.

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Appendix A. Existence and Stability of Unique Steady State

A.1 Existence of Unique Steady State

In this section, we will show Proposition 1 in a more general way. In the main part of the paper, the two types of capital dynamics are described as:

$$K_t = \Phi(j_t^*) \tag{A1}$$
$$K_{t+1} = \Psi(j_t^*) \tag{A2}$$
$$where\ \ \Phi(j_t^*) = \left(\frac{Cj_t^*}{1-\alpha}\right)^{\frac{1}{\alpha j_t^*}} (1-j_t^*)^{\frac{1-j_t^*}{\alpha j_t^*}} j_t^* \gamma^{-\frac{1}{\alpha}} \tag{A3}$$
$$\Psi(j^*) = \frac{2\alpha\beta C}{(1-\alpha)\{2\alpha(1+\beta)+(1-\alpha)\}}(1-j_t^*)(j_t^*). \tag{A4}$$

Let us express these dynamics in logarithmic form.

$$\phi(j_{t+1}^*) \equiv \log\Phi(j_{t+1}^*) = \log(\frac{Cj_{t+1}^*}{1-\alpha})^{\frac{1}{\alpha j_{t+1}^*}} (1-j_{t+1}^*)^{\frac{1-j_{t+1}^*}{\alpha j_{t+1}^*}} j_{t+1}^* \gamma^{-\frac{1}{\alpha}}, \tag{A5}$$
$$\psi(j_t^*) \equiv \log\psi(j_t^*) = \log AC(1-j_t^*)(j_t^*), \tag{A6}$$
$$A \equiv \frac{2\alpha\beta}{(1-\alpha)\{2\alpha(1+\beta)+(1-\alpha)\}}. \tag{A7}$$

Taking the limits of $\phi(j_{t+1}^*)$ and $\psi(j_t)$ to zero and one;

$$\lim_{j\to 0}\psi(j_t^*) = -\infty, \quad \lim_{j\to 1}\psi(j_t^*) = -\infty, \tag{A8}$$
$$\lim_{j\to 0}\phi(j_{t+1}^*) = -\infty, \quad \lim_{j\to 1}\phi(j_{t+1}^*) = \frac{1}{\alpha}\log\left(\frac{C}{\gamma(1-\alpha)}\right). \tag{A9}$$

For simplicity, we ignore the time scripts. The gap between ϕ and ψ is

$$\phi(j^*) - \psi(j^*) =$$
$$\frac{1}{j^*\alpha}\left(\log(\frac{C}{1-\alpha}) - j^*\log(\gamma(AC)^\alpha) + \log j^* + (1-j^*(1+\alpha))\log(1-j^*)\right). \tag{A10}$$

Thus, the limit of this gap approaching zero is negative infinity. Therefore, if the slope of ϕ is higher than ψ in $[0,1]$, the fixed point is unique. The gap of the slope between ϕ and ψ is

$$\frac{\partial \phi(j^*)}{\partial j^*} - \frac{\partial \psi(j^*)}{\partial j^*} = \frac{1}{\alpha(j^*)^2(1-j^*)} \left\{ (1-j^*) \left(\log(\frac{1-\alpha}{Cj^*(1-j^*)}) + 1 - j^* \right) + (j^*)^2\alpha \right\} \tag{A11}$$

$$\equiv \frac{1}{\alpha(j^*)^2(1-j^*)} Q(j^*)$$

Thus, if $C < 1-\alpha$, the slope of ϕ is higher than ψ in $[0,1]$. The limits of $Q(j^*)$ approaching zero is positive infinity and to one is α. We take the differential with respect to j^*,

$$\frac{\partial Q(j^*)}{\partial j} = Q'(j^*) = \log(\frac{Cj^*(1-j^*)}{1-\alpha}) + 2(1+\alpha)j^* - \frac{1}{j^*}. \tag{A12}$$

And differentiate $Q(j^*)$ with respect to j^* one more time,

$$\frac{\partial^2 Q(j^*)}{\partial (j^*)^2} = \frac{-2(1+\alpha)(j^*)^3 + 2\alpha(j^*)^2 + 1}{(j^*)^2(1-j^*)}. \tag{A13}$$

The smaller extreme point of the numerator of the above equation is zero and its other extreme point is in the range of zero to one. In addition, the limit of it approaching zero is positive, and the limit approaching one is negative. Thus, the numerator of this equation crosses zero in $[0,1]$ only once. Therefore, $Q'(j^*)$ has an inverse-U shape. When the maximized value of $Q'(j^*)$ from zero to one is negative or when the value of $Q(j^*)$ evaluated at the smaller solution for $Q'(j^*)$ is positive, the slope of ϕ is higher than ψ in $[0,1]$ and a unique steady state exist. This is summarized by following proposition:

Proposition A.1.

(1) If the following condition holds, the maximized value of $Q'(j^*)$ from zero to one is negative and the slope of ϕ is higher than ψ in $j \in [0,1]$ and a unique steady state exists:

$$C < C_1 \tag{A14}$$

$$\textit{where}\ \ C_1 = \frac{(1-\alpha)e^{[-2(1+\alpha)j^{s1}+\frac{1}{j^{s1}}]}}{j^{s1}[1-j^{s1}]},$$
$$j^{s1} = \frac{1}{1+\alpha}\left(\sqrt[3]{\nu+\frac{1+\alpha}{2}\sqrt{\nu}}+\sqrt[3]{\nu-\frac{1+\alpha}{2}\sqrt{\nu}}+\frac{\alpha}{3}\right),$$
$$\nu = \frac{(1+\alpha)^2}{4}+\frac{\alpha^3}{27}.$$

(2) Suppose that (A14) does not hold. Then, if the following condition is satisfied, a unique steady state exists:

$$C_1 < C < C_2, \tag{A15}$$

$$\textit{where}\ \ C_2 = \exp\left\{\left[\frac{1}{j^{s2}} - 2(1+\alpha)j^{s2}\right] - \log\left[\frac{j^{s2}(1-j^{s2})}{1-\alpha}\right]\right\}.$$

$j^{s2} \in [0,1]$ *holds the following equation:*

$$-(1+\alpha)(j^{s2})^3 + 2\alpha(j^{s2})^2 + 2(j^{s2}) - 1 = 0.$$

(3) If inequality (33) is not satisfied, when γ *is in the following range, there is a unique steady state:*

$$\gamma < \left(\frac{Cj^{s4}(1-j^{s4})^{1-j^{s4}(1+\alpha)}}{1-\alpha}\right)^{\frac{1}{j^{s4}}} A\cdot C^{\alpha}$$
$$\gamma > \left(\frac{Cj^{s3}(1-j^{s3})^{1-j^{s3}(1+\alpha)}}{1-\alpha}\right)^{\frac{1}{j^{s3}}} A\cdot C^{\alpha},$$

where j^{s3} *is the smaller solution of* $Q(j^*) = 0$ *and* j^{s4} *is the other solution of* $Q(j^*) = 0$.

Proof .

From (A13), the limits of the molecule of Q'' *are*

$$\lim_{j^*\to 0}[-2(1+\alpha)(j^*)^3 + 2\alpha(j^*)^2 + 1] = 1 \tag{A16}$$
$$\lim_{j^*\to 1}[-2(1+\alpha)(j^*)^3 + 2\alpha(j^*)^2 + 1] = -1, \tag{A17}$$

and the extremums are at $j^* = 0$ *and* $j^* \in (0,1)$*. Thus,* Q' *has a single peak between zero and one.*

We can divide three cases to show the conditions for the existence of a unique steady state. Figure A.1 illustrates the cases graphically.

(Figure A.1 around here)

(1) *When Q' has no solution, Q, $j^* \in [0,1]$ is positive and a unique steady state exists. Let j^{s1} be the summit of Q'. Since Q' has no solution, $Q'(j^{s1}) < 0$. Thus, Q is decreasing with j^*. Because the limits of Q to zero is positive infinity and to one is 1, $\phi - \psi$ is increasing with j^*. Therefore, there exists a unique steady state since $\phi - \psi$ goes from negative infinity to positive infinity and it crosses zero in $[0,1]$ only once.*

(2) *In this case, $Q'(j^{s1}) > 0$ and Q' has two solutions. Let j^m be the smaller solution. When $Q(j^m) > 0$, $Q(j^*)$, $j^* \in [0,1]$ is positive and a unique steady state exists. Since $Q'(j^m) = 0$, from (A12),*

$$\log \frac{1-\alpha}{Cj^m(1-j^m)} = 2(1+\alpha)j^m - \frac{1}{j^m}.$$

Using the equation above, the condition for the uniqueness of the solution is:

$$Q(j^m) = \frac{1}{\alpha(j^m)^2(1-j^m)}\left\{(1-j^m)\left(\log(\frac{1-\alpha}{Cj^*(1-j^m)}) + 1 - j^m\right) + (j^m)^2\alpha\right\} > 0$$
$$\Leftrightarrow -(1+\alpha)(j^m)^3 + 2\alpha(j^m)^2 + 2j^m - 1 > 0. \tag{A18}$$

When $j^m = 0$, the left-hand side is -1, and when $j^m = 1$, it is α. Moreover, the derivative of (A18) is positive in the range of $j^m \in [0,1]$. Thus, the left-hand side of (A18) has one solution in the range of $[0,1]$. Let j^{s2} be the solution. Since $j^m < j^{s1}$ by definition, $j^{s2} < j^m < j^{s1}$ is the condition for the uniqueness of the steady state.

We first show $j^{s2} < j^{s1}$. Since j^{s2} is the solution for (A18),

$$\alpha = \frac{-(j^{s2})^3 + 2j^{s2} - 1}{(j^{s2})^3 - 2(j^{s2})^2}. \tag{A19}$$

The right hand side is decreasing in j at the range from zero to one over $j \in$

$[0,1]$. *Thus, j^{s2} decreases as α increases. On the other hand, since $Q''(j^{s1}) = 0$,*

$$\alpha = \frac{2(j^{s1})^3 - 1}{2(j^{s1})^2(1 - j^{s1})}. \tag{A20}$$

The right-hand side is increasing in j at the range from zero to one. Thus, j^{s1} increases as α increases. Because $j^{s1} > j^{s2}$ when $\alpha = 0$, $j^{s1} > j^{s2}$ for $\alpha \in [0,1]$.

Then, we show that there exists C_2 such that $j^{s2} = j^m$. j^{s2} does not depend on C from (A18). On the other hand, j^m is decreasing in C. This is because $Q'(j^{s1})$ increases in C from (A12), and j^{s1} does not depend on C, so then, the smaller solution of Q', that is j^m, decreases as the extreme value $Q'(j^{s1})$ increases. Moreover, $j^m \to 0$ as $C \to \infty$. Therefore, there exists C_2 such that $j^{s2} = j^m$ and thus,

$$\log\left[\frac{C_2 j^{s2}(1 - j^{s2})}{1 - \alpha}\right] + 2(1+\alpha)j^{s2} - \frac{1}{j^{s2}} = 0$$

$$\Leftrightarrow C_2 = \exp\left\{\left[\frac{1}{j^{s2}} - 2(1+\alpha)j^{s2}\right] - \log\left[\frac{j^{s2}(1 - j^{s2})}{1 - \alpha}\right]\right\}.$$

Since $j^{s2} < j^m$ corresponds to $C < C_2$, $C_1 < C < C_2$ is the condition for the existence of a unique steady state in this case.

(3) In this case, $Q(j^{s1}) > 0$ and $Q(j^{s2}) < 0$. Let j^{s3} be the smaller solution of Q and j^{s4} be the larger one. When $\phi(j^{s3}) - \psi(j^{s3}) < 0$, a unique steady state exists. Moreover, when $\phi(j^{s4}) - \psi(j^{s4}) > 0$, a unique steady state also exists.

□

From this proposition, the following corollary is easily given:

Corollary A.1.
If inequalities (A14) and (A15) are not satisfied, when γ is in the following range, there are two or three steady states:

$$\left(\frac{Cj^{s4}(1 - j^{s4})^{1 - j^{s4}(1+\alpha)}}{1 - \alpha}\right)^{\frac{1}{j^{s4}}} AC^\alpha \leq \gamma \leq \left(\frac{Cj^{s3}(1 - j^{s3})^{1 - j^{s3}(1+\alpha)}}{1 - \alpha}\right)^{\frac{1}{j^{s3}}} AC^\alpha.$$

When an equality holds, there are two steady states.

When the conditions of Proposition A.1(3) or Corollary 1 are satisfied, multiple equilibria can occur. However, this paper focus on the case of a unique equilibrium for simplicity.

Obviously, $\psi(j_t^*)$ is maximized at 1/2 and increases for $[0, 1/2)$ and decreases for $(1/2, 1]$. If we differentiate $\phi(j_{t+1}^*)$ with respect to: j_{t+1}^*,

$$\frac{\partial \phi(j_{t+1}^*)}{\partial j_{t+1}^*} = \frac{1}{\alpha (j_{t+1}^*)^2} \left[\log \left(\frac{1-\alpha}{C j_{t+1}^* (1 - j_{t+1}^*)} \right) + 1 - (1-\alpha) j_{t+1}^* \right]. \tag{A21}$$

We then define the second bracket of this equation as $\tilde{\phi}(j_{t+1}^*)$. Both the limits of $\tilde{\phi}(j_{t+1}^*)$ to one and zero are positive infinity. Differentiate $\tilde{\phi}(j_{t+1}^*)$ with respect to j_{t+1}^*:

$$\frac{\partial \tilde{\phi}(j_{t+1}^*)}{\partial j_{t+1}^*} = \frac{1}{j_{t+1}^* (1 - j_{t+1}^*)} \left[(j_{t+1}^*)^2 (1-\alpha) + j_{t+1}^* (1+\alpha) - 1 \right]. \tag{A22}$$

The solution of the second bracket in $[0, 1]$ is

$$j^s = \frac{-(1+\alpha) + \sqrt{(1+\alpha)^2 + 4(1-\alpha)}}{2(1-\alpha)}. \tag{A23}$$

Thus, the following proposition is given:

Proposition A.2.
If $C < \frac{(1-\alpha)e^{1-(1-\alpha)j^s}}{j^s(1-j^s)}$ *where* $j^s = \frac{-(1+\alpha)+\sqrt{(1+\alpha)^2+4(1-\alpha)}}{2(1-\alpha)}$ ϕ *increases monotonically in* $[0, 1]$.

When ϕ monotonically increases, there are two cases of dynamics; the monotonic-convergence case and the cyclical-convergence case. When the slope of $\psi(j^*)$ in the steady state is positive (e.g., $j^* < 1/2$), monotonic-convergence occurs On the other hand, when the slope of $\psi(j^*)$ in the steady state is negative (e.g., $j^* > 1/2$), cyclical-convergence occurs. However, when the slope of $\psi(j^*)$ at the steady state is negative and the degree of the slope of ϕ at the steady state is smaller than ψ, dynamics are not convergent. The gap of the degree for slopes between ϕ and ψ in [1/2, 1] is:

$$\frac{1}{\alpha (j^*)^2 (1-j^*)} \left\{ (1-j^*) \left(\log(\frac{1-\alpha}{C j^* (1-j^*)}) + 1 - j^* + 2\alpha j^* \right) - (j^*)^2 \alpha \right\}. \tag{A24}$$

The limit of this equation to one is $-\alpha$. Thus, when j^* is sufficiently large at steady state, the dynamics diverge.

A.2 Stability of Unique Steady State

Since (A5) and (A6) have the same value at equilibrium,

$$\begin{aligned}\Omega &\equiv \phi(j^*_{t+1}) - \psi(j^*_t) \\ &= \frac{1}{\alpha j^*_{t+1}}\left[\log\frac{Cj^*_{t+1}}{1-\alpha}\right] + \frac{1-j^*_{t+1}}{\alpha j^*_{t+1}}\log(1-j^*_{t+1}) + \log j^*_{t+1} - \frac{1}{\alpha}\log\gamma \\ &\quad - [\log AC + \log(1-j^*_t) + \log j^*_t] = 0. \end{aligned} \tag{A25}$$

By applying the Implicit Function Theorem to the equation above,

$$\begin{aligned}\frac{dj^*_{t+1}}{dj^*_t} &= -\frac{d\Omega/dj^*_t}{d\Omega/dj^*_{t+1}} \\ &= -\frac{-d\psi(j^*_t)/dj^*_t}{d\phi(j^*_{t+1})/dj^*_{t+1}} \\ &= \frac{-\left[\frac{1}{1-j^*_t} - \frac{1}{j^*_t}\right]}{\frac{1}{\alpha(j^*_{t+1})^2}\left\{\left[1-\log\frac{Cj^*_{t+1}}{1-\alpha}\right] - \log(1-j^*_{t+1}) - (1-\alpha)j^*_{t+1}\right\}}\end{aligned} \tag{A26}$$

To assure stability, the following assumption is necessary:

$$\left|\frac{dj^*_{t+1}}{dj^*_t}\right| < 1. \tag{A27}$$

This equation is equivalent to:

$$-1 < -\frac{-d\psi(j^*_t)/dj^*_t}{d\phi(j^*_{t+1})/dj^*_{t+1}} < 1 \tag{A28}$$

$$\Leftrightarrow \frac{d\phi(j^*_{t+1})}{dj^*_{t+1}} + \frac{d\psi(j^*_t)}{dj^*_t} > 0 \quad and \quad \frac{d\phi(j^*_{t+1})}{dj^*_{t+1}} - \frac{d\psi(j^*_t)}{dj^*_t} > 0 \tag{A29}$$

Evaluating the conditions above at a steady state, $d\phi(j^*_{t+1})/dj^*_{t+1} - d\psi(j^*_t)/dj^*_t$ is equivalent to (A11). Thus, the latter condition is satisfied when $Q > 0$, that is (1) or (2) of Proposition A.1 holds. When $d\psi(j^*_t)/dj^*_t > 0$, that is $j^* < 1/2$, the former condition is also satisfied. Since (1) or (2) of Proposition A.1, a unique steady state exists. Thus, when $j^* < 1/2$, the steady state is globally stable. Then, we consider the case of $j^* > 1/2$. Since $d\psi/dj^* < 0$ in this case, we focus on the former condition. The condition evaluated in the steady state is:

$$\left.\frac{d\phi(j^*_{t+1})}{dj^*_{t+1}}\right|_{j^*_{t+1}=j^*} + \left.\frac{d\psi(j^*_t)}{dj^*_t}\right|_{j^*_t=j^*} = \frac{1}{\alpha(j^*)^2(1-j^*)}Z(j^*), \tag{A30}$$

$$where \ \ Z(j^*) \equiv (1-j^*)\left(\log(\frac{1-\alpha}{Cj^*(1-j^*)}) + 1 - j^*\right) + (1-j^*)2\alpha j^* - (j^*)^2\alpha.$$

When $j = 1/2$, $Q(j^*) = Z(j^*)$. This relationship implies that

$$\left.\left[\frac{d\phi(j^*)}{dj^*} + \frac{d\psi(j^*)}{dj^*}\right]\right|_{j^*=1/2} = \left.\left[\frac{d\phi(j^*)}{dj^*} - \frac{d\psi(j^*)}{dj^*}\right]\right|_{j^*=1/2}. \tag{A31}$$

Around $j^* = 1/2$, $\frac{d\phi(j^*)}{dj^*} + \frac{d\psi(j^*)}{dj^*}$ is positive, since under Proposition 1's (2), $\frac{d\phi(j^*)}{dj^*} - \frac{d\psi(j^*)}{dj^*}$ is positive. Taking the limit of $Z(j^*)$ to one, this value of $Z(j^*)$ is $-\alpha$. Thus, the steady state is unstable around $j^* = 1$. This means that both stable areas and unstable areas exist in $j^* \in [1/2, 1]$. Then, we derive the sufficient conditions that there exists unique threshold, which divide the stable area with the unstable area. Before presenting proposition, we show following lemma.

Lemma A.1. *Suppose that the conditions Proposition A.1's (1) or (2) hold. Then, in $j^* \in [1/2, 1]$, $Z''(j^*)$ has only one solution j^{z1}, and it is positive under j^{z1} and negative over j^{z1}.*

Proof .

From (A30), the second derivative of $Z(j^*)$ is:

$$\frac{\partial^2 Z(j^*)}{\partial(j^*)^2} = \frac{2(3\alpha-1)(j^*)^3 - 6\alpha(j^*)^2 + 1}{(j^*)^2(1-j^*)}. \tag{A32}$$

The molecule determines the sign of $Z''(j^*)$. When $j^* = 0, \alpha/(3\alpha - 1)$, this function achieves the vertexes. Taking the limits to $1/2$ and 1, the molecule of $Z''(j^*)$ are:

$$\lim_{j^*\to 1/2}[2(3\alpha-1)(j^*)^3 - 6\alpha(j^*)^2 + 1] = \frac{3}{4}(1-\alpha) > 0$$
$$\lim_{j^*\to 1}[2(3\alpha-1)(j^*)^3 - 6\alpha(j^*)^2 + 1] = -1.$$

Taking it into account that the molecule is the cubic function, $Z''(j^*)$ has only one solution j^{z1} in $j^* \in [1/2, 1]$, and it is positive under j^{z1} and negative over j^{z1}. $\square$

Figure A.2 sketches graphically this proof.

(Figure A.2. around here)

Next, we consider when there is a unique threshold dividing the area featuring stable steady state with the area featuring unstable steady state. As we described, this is satisfied when $Z(j^*)$ has a unique solution. To examine the shape of $Z(j^*)$, we focus on the relationship between $Z(j^*)$ and $Z'(j^*)$, which is illustrated in Figure A.3.

Lemma A.1 implies that $Z'(j^*)$ has a single peak. The derivative of $Z(j^*)$, $Z'(j^*)$ is:

$$Z'(j^*) = \frac{1}{j^*}\left[j^* \log\left\{\frac{Cj^*(1-j^*)}{1-\alpha}\right\} + 2(1-3\alpha)(j^*)^2 + 2\alpha j^* - 1\right]. \tag{A33}$$

Taking the limits to 1/2 and 1, the values of $Z'(j^*)$ are:

$$\lim_{j^* \to 1/2} = \log\left\{\frac{C}{4(1-\alpha)}\right\} - (1-\alpha), \tag{A34}$$

$$\lim_{j^* \to 1} = -\infty. \tag{A35}$$

If its limit value to 1/2 is positive, since $Z'(j^*)$ has a single peak, $Z'(j^*)$ has a unique solution in $j^* \in [1/2, 1]$. This implies that $Z(j^*)$ has also a unique solution, because $Z(j^*)$ is positive around $j^* = 1/2$ and negative around $j^* = 1$ when the conditions of Proposition A.1's (1) or (2) hold.

If the limit value is negative, making division into two cases: $Z'(j^{z1}) \leq 0$ or $Z'(j^{z1}) > 0$. Since $Z'(j^{z1})$ is extreme value of $Z'(j^*)$, $Z'(j^*)$ is negative (or zero) in $j^* \in [1/2, 1]$ when $Z'(j^{z1}) \leq 0$. Thus, $Z(j^*)$ is weakly decreasing in j^*, and it has a unique solution. When $Z'(j^{z1}) > 0$, $Z'(j^*)$ has two solutions. Let the solutions be j^{z2} and j^{z3}, and assume that $j^{z2} < j^{z3}$. The extreme values, $Z(j^{z2})$ and $Z(j^{z3})$, determine the shape of $Z(j^*)$. When $Z(j^{z2}) > 0$, $Z(j^*)$ has a unique steady state between j^{z3} and 1. When $Z(j^{z2}) < 0$, if $Z(j^{z3}) < 0$, $Z(j^*)$ has a unique steady state between 1/2 and j^{z2}.

Following proposition summarizes the above discussion.

Proposition A.3. *Suppose that the conditions of Proposition A.1's (1) or (2) hold,* *that is, a unique steady state exists. Following conditions assure that there is a unique threshold dividing the area featuring stable steady state with the area featuring unstable steady state.*

1. $$C \geq C^{z1}, \text{ where } C^{z1} \equiv 4(1-\alpha)e^{1+\alpha}. \tag{A36}$$

2. *Suppose that* $C < C^{z1}$.

   - $$Z'(j^{z1}) \leq 0 \Leftrightarrow C \leq C^{z2} \tag{A37}$$
     $$\text{where } C^{z2} \equiv \frac{1-\alpha}{j^{z1}(1-j^{z1})} \exp\left[2(3\alpha-1)j^{z1} - 2\alpha + \frac{1}{j^{z1}}\right] \tag{A38}$$

   - *When* $Z'(j^{z1}) > 0$*, that is,* $C > C^{z2}$,
     $$Z(j^{z2}) > 0, \tag{A39}$$
     $$Z(j^{z2}) < 0 \text{ and } Z(j^{z3}) < 0 \tag{A40}$$

□

(Figure A.3 around here)

Following figures show graphically the cases of dynamics. In each figure, the solid line and the dot line represents $\Phi(j^*_{t+1})$ and $\Psi(j^*_t)$, respectively. Since $j^*_t = j^*_{t+1}$ in the steady states, the intersections correspond to the steady states. Figure A.1 demonstrates the case of the unique steady state with monotonic convergence, and Figure A.2 does the case of the unique steady state with cyclical convergence. Figure A.3 shows the case of the multiple steady state. Since $\Phi(j^*_{t+1})$ is not monotonic, this case have potentially multiple equilibria.

Appendix B. Proofs of Propositions

B.1. The Proof of Proposition 3

At first, we propose a lemma that shows the relationship between γ and j^*.

Lemma B.1.

Suppose that the conditions (1) or (2) of Proposition A.1 holds. Then:

$$\frac{dj^*}{d\gamma} > 0$$
$$\frac{dj^*}{dC} < 0$$

Proof B.1.

When conditions (1) or (2) presented in Proposition A.1 holds, $\phi(j^*) - \psi(j^*)$ *in the unique steady state is increasing in* j^* *since* $Q(j^*) > 0,\ j^* \in [0,1]$*. From (A11), in the steady state,*

$$\frac{d\Omega(j^*)}{dj^*} = \frac{1}{\alpha(j^*)^2(1-j^*)}Q(j^*) > 0, \tag{B1}$$
$$where \ \ \Omega(j^*) \equiv \phi(j^*) - \psi(j^*). \tag{B2}$$

Since $\phi(j^*) - \psi(j^*) = 0$ *in the steady state, by applying the Implicit Function Theorem to (A10),*

$$\frac{dj^*}{d\gamma} = \frac{1}{\gamma\alpha}\left(\frac{d\Omega(j^*)}{dj^*}\right)^{-1} > 0. \tag{B3}$$
$$\frac{dj^*}{dC} = -\frac{1-j^*}{Cj^*}\left(\frac{d\Omega(j^*)}{dj^*}\right)^{-1} < 0. \tag{B4}$$

□

By using this lemma, Propositions 3 and 4 are proved.

Proof B.2 (Proof of Proposition 3)**.** *From (17) and (23) in the main script, the wage in a steady state are characterized by* j^**,*

$$w = j^*(1-j^*)\frac{C}{1-\alpha}. \tag{B5}$$

Thus, the first order differentiation with respect to γ *is*

$$\frac{dw}{d\gamma} = \frac{dw}{dj^*}\frac{dj^*}{d\gamma} = \frac{C(1-2j^*)}{1-\alpha}\frac{dj^*}{d\gamma}. \tag{B6}$$

Since $dj^*/d\gamma > 0$ *from Lemma B.1, (B6) implies that if the threshold at equilibrium* j^* *is lower than* $1/2$*, the productivity improvement of automation technology pushes the wage up. On the other hand, if* $j^* > 1/2$*, it pulls the wage down.* □

Proof B.3 (Proof of Proposition 4). *Suppose that the condition (1) of Proposition 1 holds. From (B5),*

$$\begin{aligned}\frac{dw}{dC} &= \frac{\partial w}{\partial C} + \frac{\partial w}{\partial j^*}\frac{dj^*}{dC} \\ &= j^*(1-j^*)\frac{1}{1-\alpha} + \frac{C(1-2j^*)}{1-\alpha}\frac{dj^*}{dC}. \end{aligned} \tag{B7}$$

By considering the same as Proposition 3, the threshold j^c satisfies following equation:

$$j^c(1-j^c) + (1-2j^c)(\alpha j^c - 1)\left(\frac{d\Omega(j^c)}{dj^*}\right)^{-1} = 0. \tag{B8}$$

The derivative of the second term of the above equation is:

$$(-4j+\alpha+2)\left(\frac{d\Omega(j)}{dj}\right)^{-1} - (1-2j)(\alpha j - 1)\left(\frac{d\Omega(j)}{dj}\right)^{-2}\frac{Q'\alpha(j)^2(1-j) - Qj(2-3j)}{\alpha[(j)^2(1-j)]^2}. > 0 \tag{B9}$$

Because the condition of (1) of Proposition 1 holds, $Q' < 0$. Thus, the second term of the above equation is positive. The first term is positive when $j = 1/2$. What the derivative of the second term of (B8) is positive implies that it shifts up the derivative of $j(1-j)$ around $j = 1/2$. Thus, $j^c > 1/2$. □

B.2. The Proof of Proposition 5

From (17) and (11) in the main script, the aggregate profit of managers at the steady state is:

$$\Pi \equiv \int_0^{j^*} \pi^k(j)dj + \int_{j^*}^1 \pi^l(j)dj = \frac{1}{2}C(j^*)^2 + 0 = \frac{1}{2}C(j^*)^2. \tag{B10}$$

The aggregate fixed cost is:

$$\mathcal{C} \equiv \int_0^{j^*} C \cdot j dj = \frac{1}{2}C(j^*)^2. \tag{B11}$$

Thus, the income share of managers is

$$\frac{\Pi}{Y - \mathcal{C}} = \frac{(1-\alpha)j^*}{2-(1-\alpha)j^*}. \tag{B12}$$

The marginal effect of of an improvement of automation technology on the labor share of income is:

$$\frac{d\Pi/(Y-\mathcal{C})}{d\gamma} = \frac{1-\alpha}{[2-(1-\alpha)j^*]^2} \cdot \frac{dj^*}{d\gamma} > 0 \tag{B13}$$

When the conditions of (2) or (3) in Proposition A.1 holds, (B3) assures the inequality above.

On the other hand, the income share of labor is given by (B5):

$$\frac{w}{Y-\mathcal{C}} = \frac{2(1-j^*)}{2-(1-\alpha)j^*} \tag{B14}$$

Thus, since the marginal effect is

$$\frac{dw/(Y-\mathcal{C})}{d\gamma} = -\frac{2(1+\alpha)}{[2-(1-\alpha)j^*]^2}\frac{dj^*}{d\gamma} < 0, \tag{B15}$$

The wage decreases as automation technology improves. As for total income share:

$$\frac{d}{d\gamma}\left(\frac{\Pi}{Y-\mathcal{C}} + \frac{w}{Y-\mathcal{C}}\right) = \frac{-(1+3\alpha)}{[2-(1-\alpha)j^*]^2} < 0. \tag{B16}$$

Thus, the total income share decreases with γ. The effect of a change in C on income share is the same discussion.

B.2. The Proof of Proposition 6

In the steady state, (40) is:

$$H \equiv \frac{\beta}{1+\beta}\bar{j}(1-\bar{j})Y - \frac{K}{\alpha(1+\beta)}\left[1-\alpha+\frac{C\bar{j}}{2Y}\right] - K = 0. \tag{B17}$$

By totally differentiating the above equation, the effect of an increase in γ on K is:

$$\frac{dK}{d\gamma} = -\frac{\partial H}{\partial Y}\left(\frac{\partial H}{\partial K}\right)^{-1}\frac{dY}{d\gamma}, \tag{B18}$$

$$where \quad \frac{\partial H}{\partial Y} = \frac{\beta}{1+\beta}\bar{j}(1-\bar{j}) - \frac{1}{Y}\frac{K\cdot C\bar{j}}{2(1+\beta)\alpha Y}, \tag{B19}$$

$$\frac{\partial H}{\partial K} = -\frac{1-\alpha}{\alpha(1+\beta)} + \frac{C\bar{j}}{2(1+\beta)\alpha Y} - 1. \tag{B20}$$

By using (B17), (B20) is:

$$\frac{\partial H}{\partial K} = -\frac{\beta}{1+\beta}\bar{j}(1-\bar{j})\frac{Y}{K}. \tag{B21}$$

From (39), the effect of an increase in γ on Y is:

$$\frac{dY}{d\gamma} = \bar{j}\frac{Y}{\gamma} + \alpha\bar{j}\frac{Y}{K}\frac{dK}{d\gamma}. \tag{B22}$$

By substituting (B18) into (B22):

$$\frac{dY}{d\gamma} = \bar{j}\frac{Y}{\gamma}\left[1 + \alpha\bar{j}\frac{Y}{K}\left(\frac{\partial H}{\partial K}\right)^{-1}\frac{\partial H}{\partial Y}\right]^{-1}. \tag{B23}$$

From (B19) and (B21),

$$\left(\frac{\partial H}{\partial K}\right)^{-1}\frac{\partial H}{\partial Y} = \frac{K}{Y}\left[1 - \frac{2\alpha\beta(1-\bar{j})Y^2}{CK}\right]. \tag{B24}$$

Substituting (B24) into (B23):

$$\frac{dY}{d\gamma} = \bar{j}\frac{Y}{\gamma}\left[(1-\alpha\bar{j}) + \alpha\bar{j}\frac{CK}{2\alpha\beta(1-\bar{j})Y^2}\right] > 0. \tag{B25}$$

Thus, effect of an increase in γ on the wage as the following equation is positive:

$$\frac{dw}{d\gamma} = (1-\bar{j})\frac{dY}{d\gamma} > 0. \tag{B26}$$

□

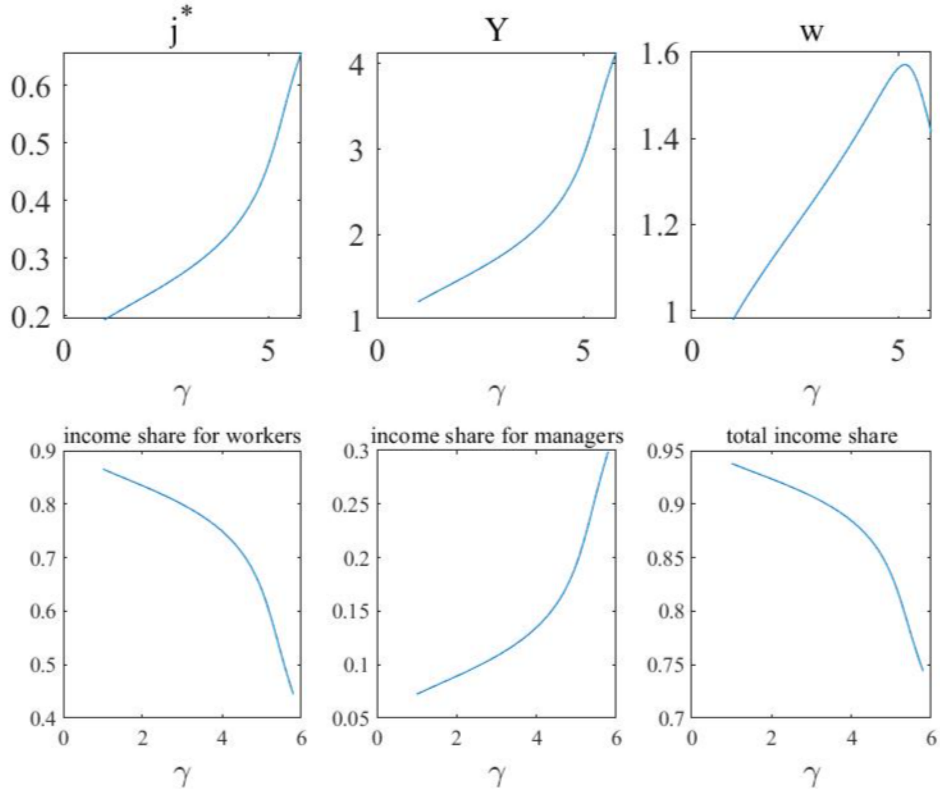


Figure 1   The effects of an improvement of automation productivity in the steady state.

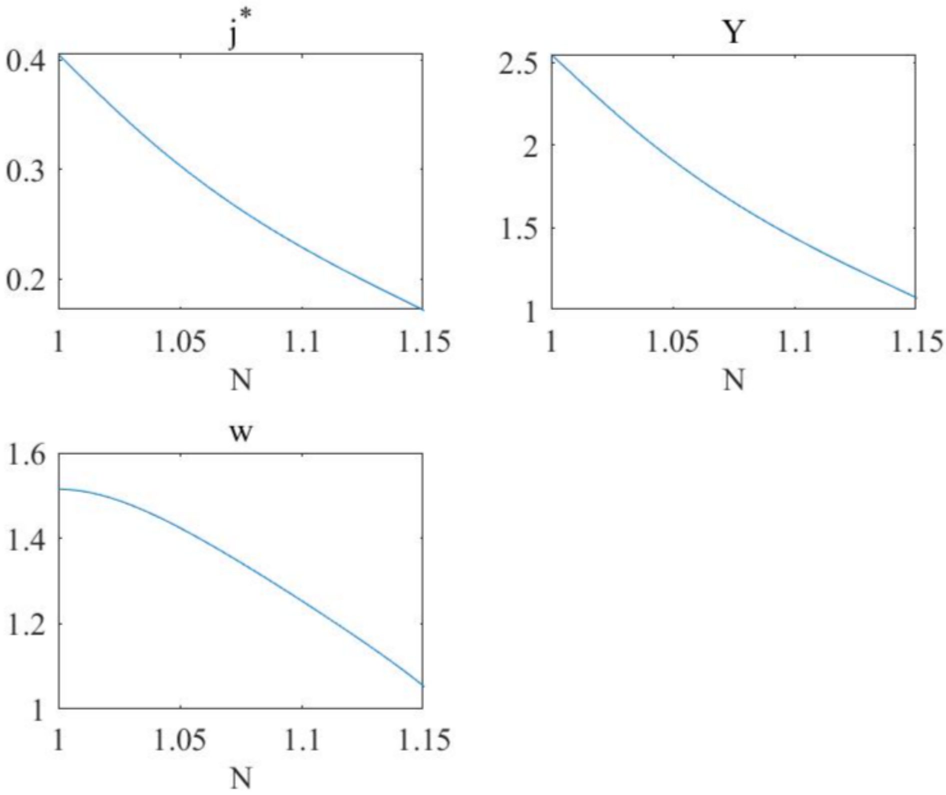


Figure 2 The effects of a creation of new industries in the steady state.

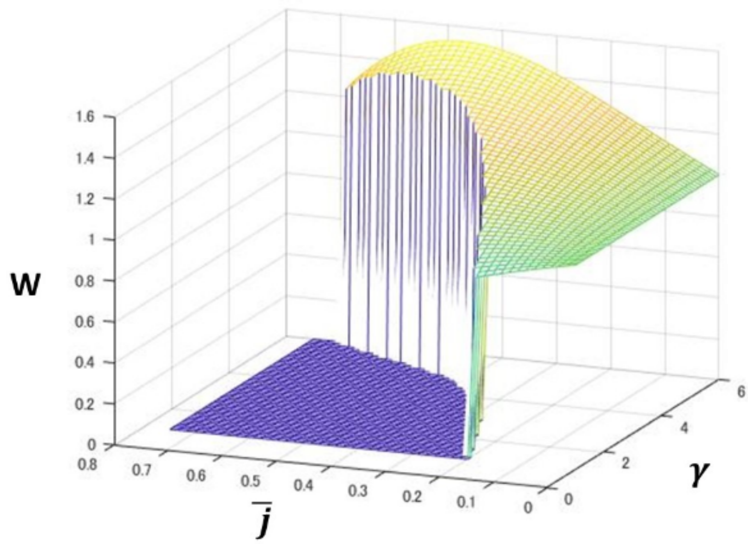


Figure 3 Wage when there is technological limitation $\overline{j}$. This figure illustrates the case of $\overline{j} < j^*$.

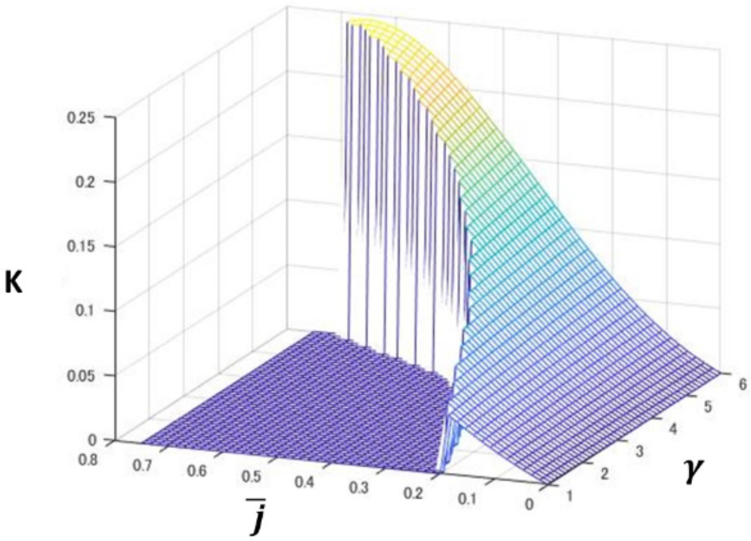


Figure 4 Aggregate capital when there is technological limitation $\overline{j}$. This figure illustrates the case of $\overline{j} < j^*$.

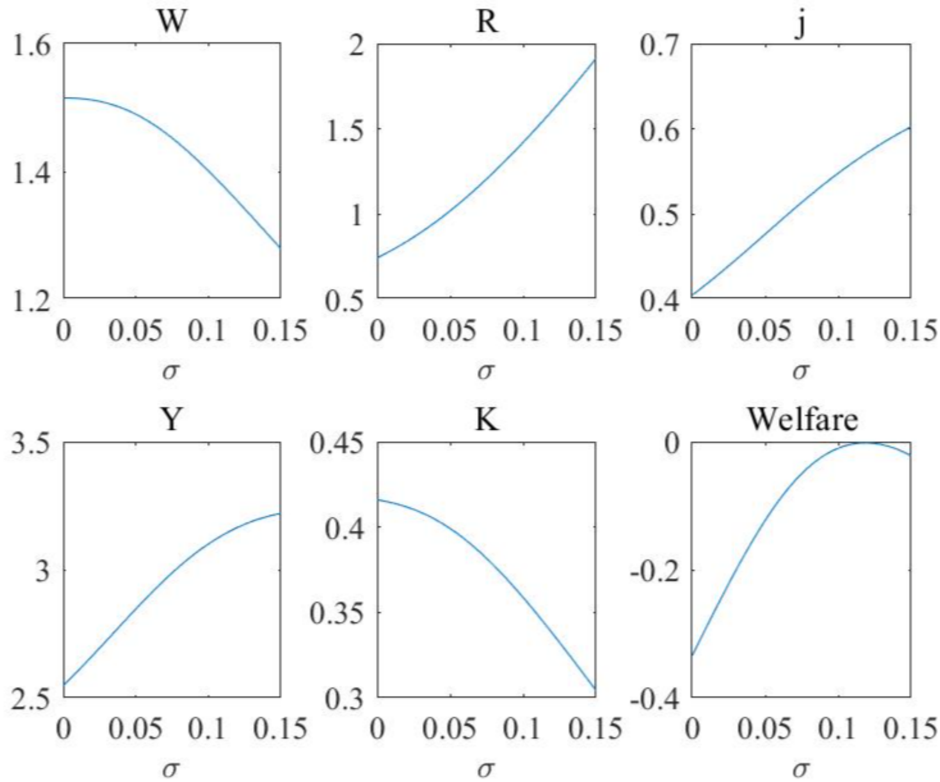


Figure 5 Effects of subsidy in the steady state.

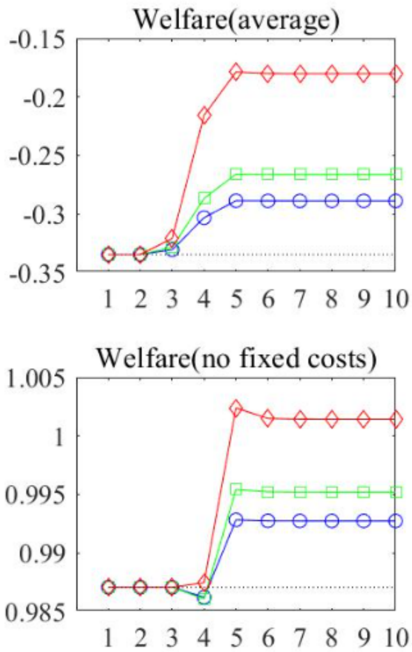


Figure 6 Dynamics of welfare: aggregate welfare of all households, welfare of households who choose non-automation technology and welfare of households who face zero fixed costs. In each panel, the blue line (round), green line (square), and red line (diamond) indicate the dynamics in cases in which the subsidy rate, σ, is 0.01, 0.015, and 0.035, respectively. The black (dotted) line in each panel indicates the level of lifetime utility in the previous steady state.

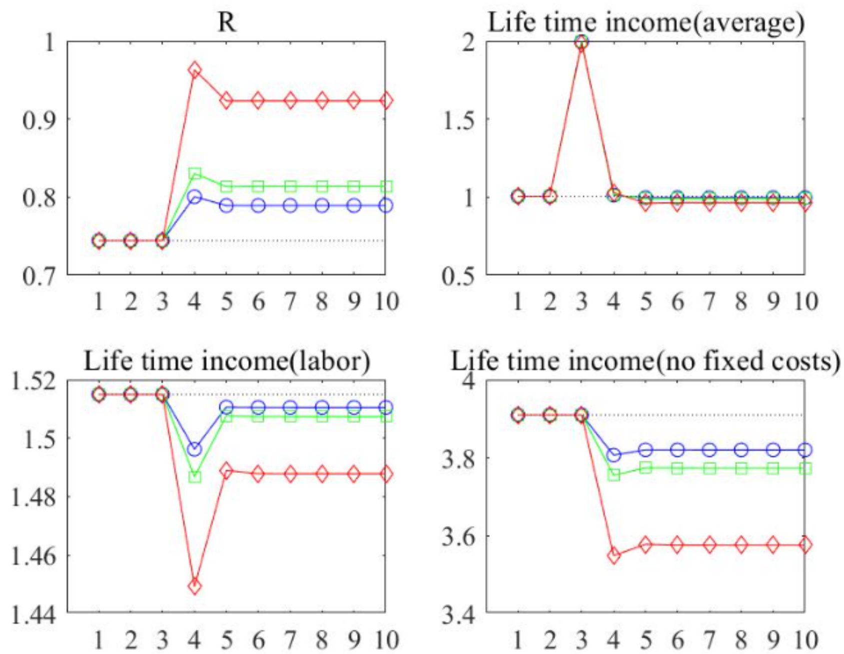


Figure 7 Dynamics of interest rate and lifetime income of each type of household proposed in Figure 6. The meaning of each line is the same as Figure 6

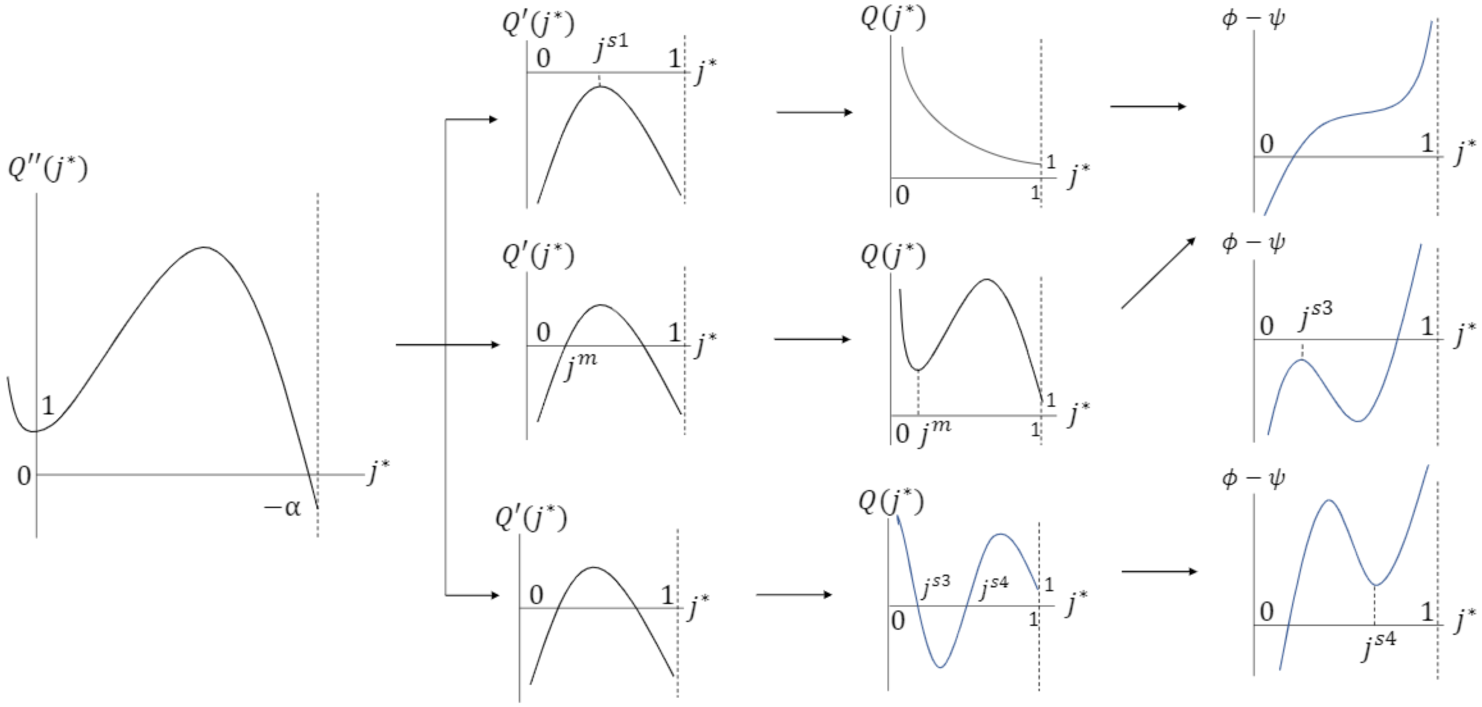
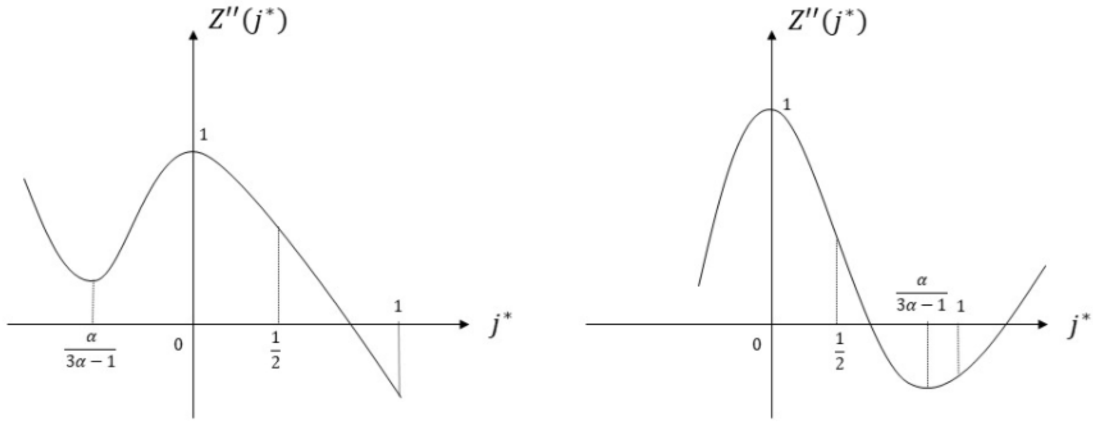


Figure A.1 Cases of Proposition A.1.



$3\alpha - 1 < 0$ $\qquad$ $3\alpha - 1 > 0$

Figure A.2 Shape of $Z''(j^*)$ in Lemma A.1.

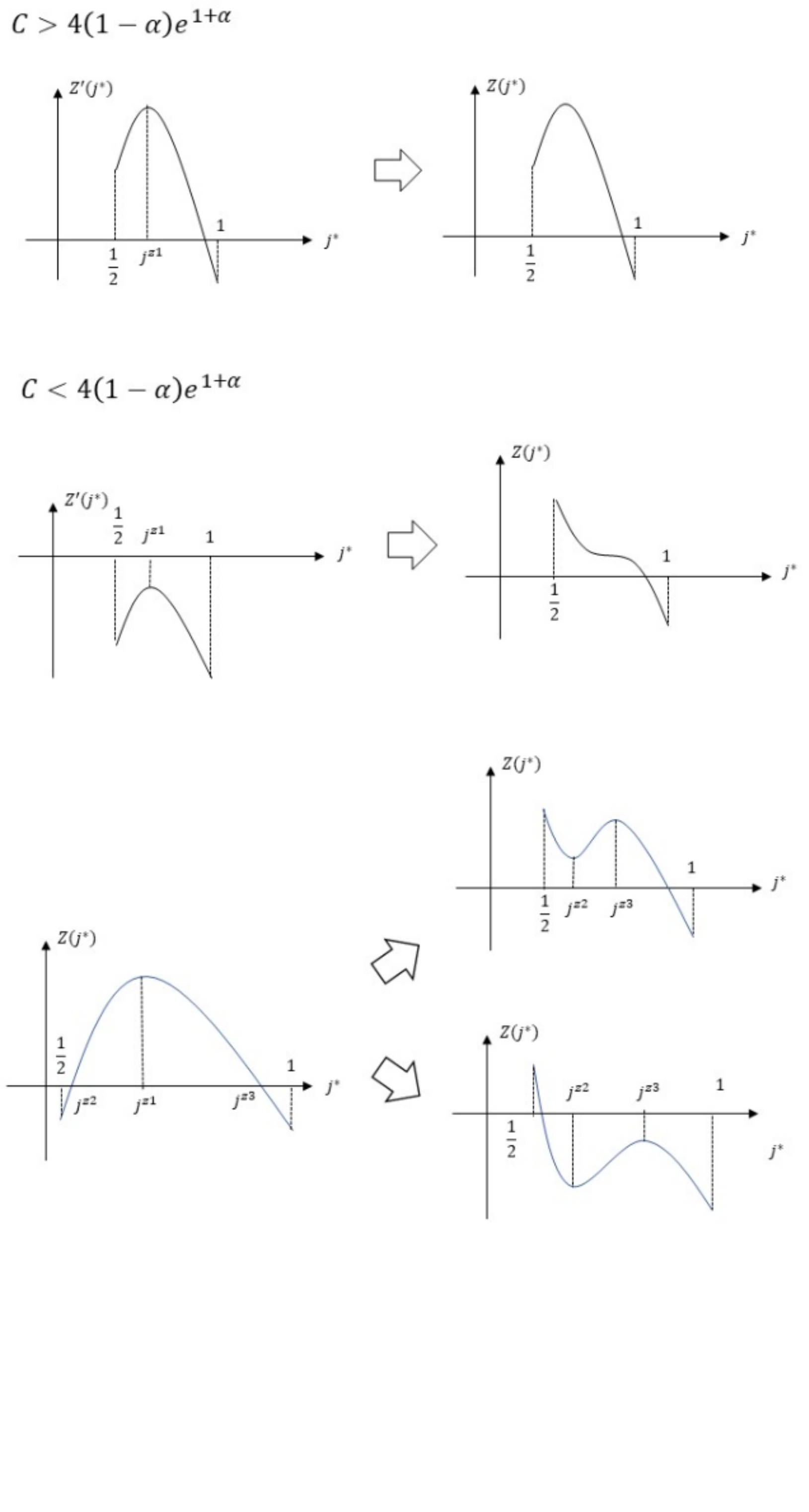


Figure A.3 Shape of $Z''(j^*)$ in Lemma A.1.

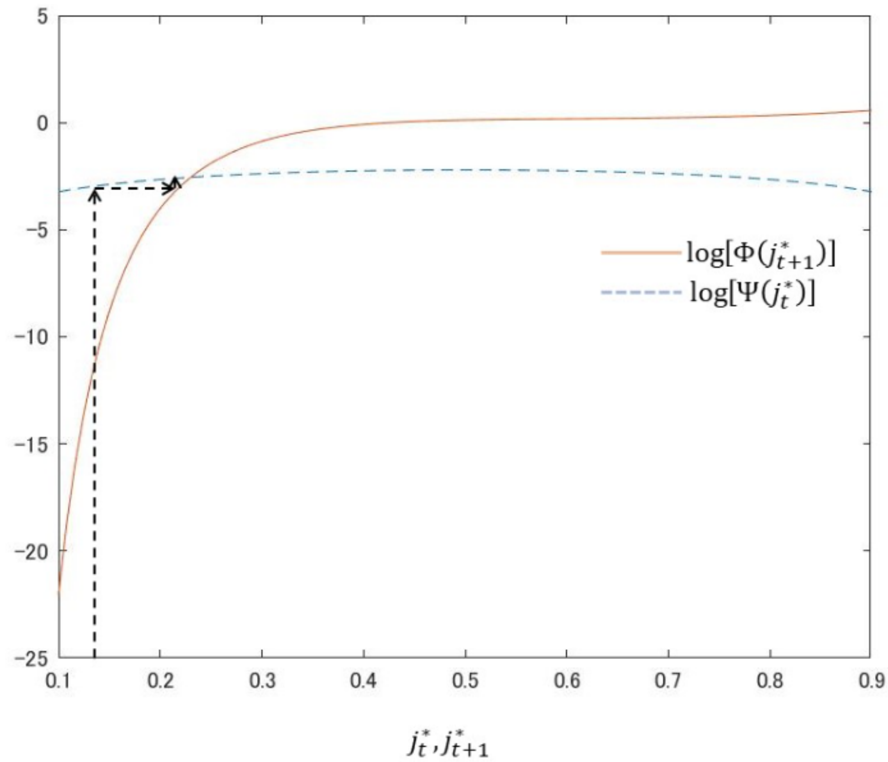


Figure A.4 Case of the unique steady state with monotonic convergence.

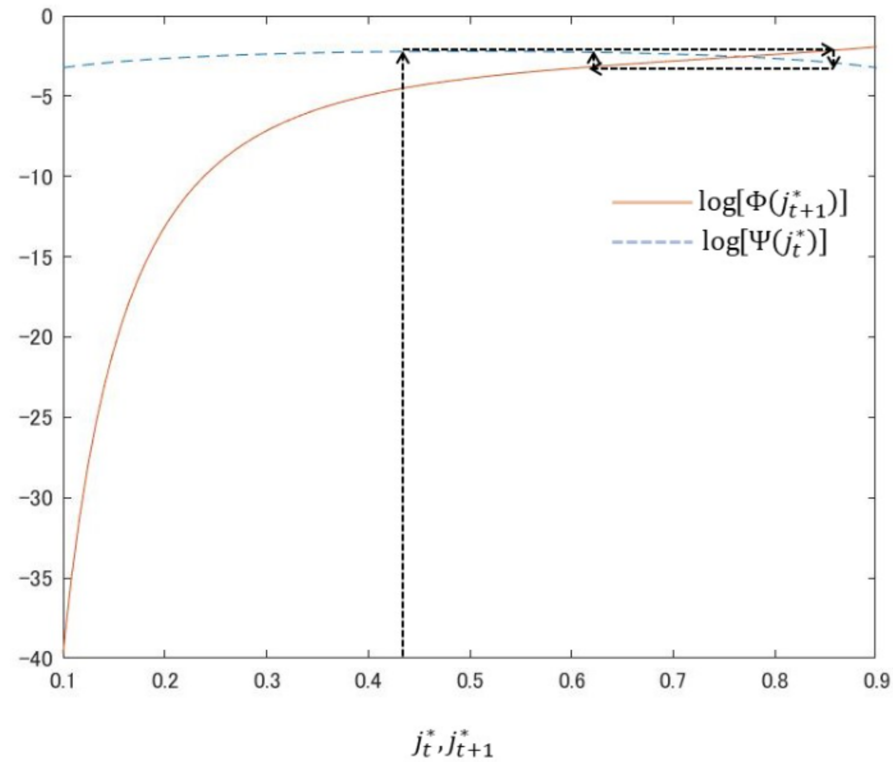


Figure A.5 Case of the unique steady state with cyclical convergence.

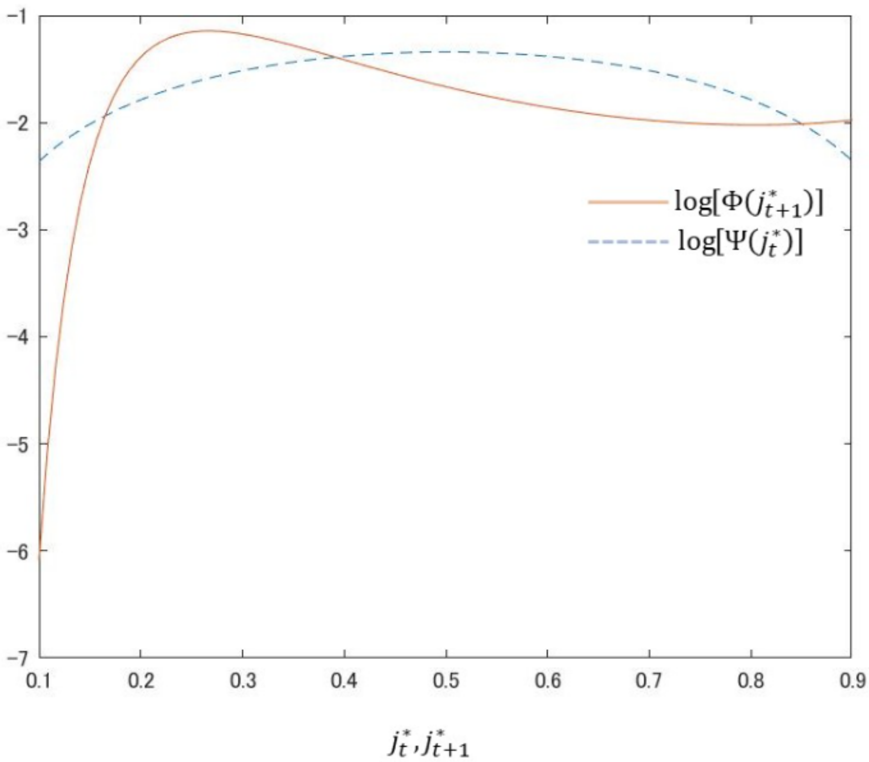


Figure A.6 Case of the multiple steady states .