数理解析研究所講究録 1415 巻 2005 年 11-19

Weak and Strong Convergence Theorems for Nonexpansive Semigroups in Banach Spaces

Wataru Takahashi (高橋 渉)

Department of Mathematical and Computing Sciences
Tokyo Institute of Technology

1 Introduction

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A mapping T of C into itself is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for every} \quad x, y \in C.$$

For a mapping T of C into itself, we denote by $F(T)$ the set of fixed points of T. We also denote by $\mathbb{N}$ and $\mathbb{R}_+$ the sets of positive integers and nonnegative real numbers, respectively. A family $\{S(t) : t \in \mathbb{R}_+\}$ of mappings of C into itself is called a *one-parameter nonexpansive semigroup* on C if it satisfies the following conditions: (1) $S(t+s)x = S(t)S(s)x$ for every $t, s \in \mathbb{R}_+$ and $x \in C$; (2) $S(0)x = x$ for every $x \in C$; (3) for each $x \in C$, $t \mapsto S(t)x$ is continuous; (4) $\|S(t)x - S(t)y\| \leq \|x - y\|$ for every $t \in \mathbb{R}_+$ and $x, y \in C$. Consider the initial value problem:

$$\begin{cases} \frac{du(t)}{dt} + Au(t) \ni 0 & \text{for every } t > 0, \\ u(0) = x, \end{cases} \tag{1}$$

where A is an m-accretive operator in H and x is an element of $\overline{D(A)}$. It is well-known that (1) has a unique strong solution $u : \mathbb{R}_+ \to H$ and $\overline{D(A)}$ is closed and convex. Putting $S(t)x = u(t)$, we have that the family $\{S(t) : t \in \mathbb{R}_+\}$ of mappings of $\overline{D(A)}$ into itself is a one -parameter nonexpansive semigroup on $\overline{D(A)}$; see [7] for more details.

Baillon and Brézis [6] proved the following nonlinear ergodic theorem for a one-parameter nonexpansive semigroup:

Theorem 1.1. *Let C be a nonempty closed convex subset of H and let $\{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C such that $\bigcap_{t\in\mathbb{R}_+} F(S(t))$ is nonempty. Then, for each $x \in C$,*

$$\frac{1}{\lambda}\int_0^\lambda S(s)x\, ds \rightharpoonup z \in \bigcap_{t\in\mathbb{R}_+} F(S(t))$$

as $\lambda \to \infty$, where $\rightharpoonup$ denotes the weak convergence.

Shimizu and Takahashi [13] also introduced the first iterative scheme for finding a common fixed point of a one-parameter nonexpansive semigroup and proved the following strong convergence theorem of Halpern's type:

Theorem 1.2. *Let C be a nonempty closed convex subset of H and let $\{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C such that $\bigcap_{t\in\mathbb{R}_+} F(S(t))$ is nonempty. Suppose that $\{\alpha_n\} \subset [0,1]$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, for each $x \in C$, the sequence $\{x_n\}$ generated by $x_1 = x$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds \quad \text{for every } n = 1, 2, \dots$$

converges strongly to a common fixed point Px of $S(t), t \in \mathbb{R}_+$ as $t_n \to \infty$, where P is the metric projection of C onto $\bigcap_{t\in\mathbb{R}_+} F(S(t))$.

Motivated by Shimizu and Takahashi [13], Atsushiba and Takahashi [3] also obtained the following weak convergence theorem of Mann's type:

Theorem 1.3. *Let C be a nonempty closed convex subset of H and let $\{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C such that $\bigcap_{t\in\mathbb{R}_+} F(S(t))$ is nonempty. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds$$

for every $n \in \mathbb{N}$, where $t_n \to \infty$ as $n \to \infty$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $0 < \alpha_n \le a < 1$, then $\{x_n\}$ converges weakly to a common fixed point of $\bigcap_{t\in\mathbb{R}_+} F(S(t))$.

In this article, we deal with weak and strong convergence theorems for general nonexpansive semigroups in Banach spaces which are strongly connected with Theorems 1.1, 1.2 and 1.3. In Section 3, we first discuss nonlinear ergodic theorems in a uniformly convex Banach space whose norm is Fréchet differentiable. Then, we consider nonlinear ergodic theorems in the case when a Banach space is strictly convex and the domains of the nonexpansive semigroups are compact. In Section 4, we deal with weak and strong convergence theorems of Halpern's type and Mann's type for nonexpansive semigroups in Banach spaces.

2 Preliminaries

Let C be a nonempty closed convex subset of a Banach space E and let T be a mapping of C into C. Then we denote by $R(T)$ the range of T. Let D be a subset of C and let P be a mapping of C into D. Then P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \ge 0$. A mapping P of C into C is said to be a *retraction* if $P^2 = P$. If a mapping P of C into C is a retraction, then $Pz = z$ for every $z \in R(P)$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D.

Let E be a Banach space. Then, for every ε with $0 \le \varepsilon \le 2$, the *modulus $\delta(\varepsilon)$ of convexity* of E is defined by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}.$$

A Banach space E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. E is also said to be *strictly convex* if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A uniformly convex Banach space is strictly convex.

Let E be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals x^* on E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. With each $x \in E$, we associate the set $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$. Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \phi$ for any $x \in E$. Then the multi-valued operator $J : E \to E^*$ is called the *duality mapping* of E. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. When this is the case, the norm of E is said to be *Gâteaux differentiable*. It is said to be *Fréchet differentiable* if for each x in U, this limit is attained uniformly for y in U. The space E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in U$, the limit is attained uniformly for $x \in U$. It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E has a Fréchet differentiable norm, then J is norm to norm continuous; see [21, 22] for more details.

Let S be a semitopological semigroup, i.e., a semigroup with Hausdorff topology such that for each $s \in S$, the mappings $t \mapsto ts$ and $t \mapsto st$ of S into itself are continuous. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let X be a subspace of $B(S)$ containing constants. Then, an element μ of X^* is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. We know that $\mu \in X^*$ is a mean on X if and only if

$$\inf\{f(s) : s \in S\} \leq \mu(f) \leq \sup\{f(s) : s \in S\}$$

for every $f \in X$. For a mean μ on X and $f \in X$, sometimes we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s \in S$ and $f \in B(S)$, we define elements $\ell_s f$ and $r_s f$ of $B(S)$ given by $(\ell_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S$. Let X be a subspace of $B(S)$ containing constants which is invariant under ℓ_s, $s \in S$ (resp. r_s, $s \in S$). Then a mean μ on X is said to be *left invariant* (resp. *right invariant*) if $\mu(f) = \mu(\ell_s f)$ (resp. $\mu(f) = \mu(r_s f)$) for all $f \in X$ and $s \in S$. An *invariant mean* is a left and right invariant mean. Let S be a semitopological semigroup and let C be a nonempty subset of a Banach space E. Then a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following: (i) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$; (ii) for each $x \in C$, the mapping $s \mapsto T_s x$ is continuous; (iii) for each $s \in S$, T_s is a nonexpansive mapping of C into itself. For a nonexpansive semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C, we denote by $F(\mathcal{S})$ the set of common fixed points of $T_s, s \in S$. We also denote by $C(S)$ the Banach space of all bounded continuous functions on S.

3 Nonlinear Ergodic Theorems

In this section, we deal with nonlinear ergodic theorems for nonexpansive semigroups in a Banach space. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on $C(S)$. Then $\{\mu_\alpha \in A\}$ is said to be *asymptotically invariant* if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(\ell_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

If C is a nonempty closed convex subset of a reflexive Banach space E and $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Then since for each $x \in C$ and $y^* \in E^*$, the real valued function $t \mapsto \langle T_t x, y^* \rangle$ is in $C(S)$, we can define the value $\mu_t \langle T_t x, y^* \rangle$ of μ at this function. So, by the Riesz theorem, there exists an $x_0 \in E$ such that $\mu_t \langle T_t x, y^* \rangle = \langle x_0, y^* \rangle$ for every $y^* \in E^*$. We write such an x_0 by $T_\mu x$ or $\int T_t x d\mu(t)$; see [17, 21] for more details.

Now, we can state a nonlinear ergodic theorem for nonexpansive semigroups in a Banach space. Before stating it, we give a definition. A net $\{\mu_\alpha\}$ of continuous linear functionals on $C(S)$ is called *strongly regular* if it satisfies the following conditions: (i) $\sup_\alpha \|\mu_\alpha\| < +\infty$; (ii) $\lim_\alpha \mu_\alpha(1) = 1$; (iii) $\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in S$.

Theorem 3.1 ([9]). *Let S be a commutative semitopological semigroup and let E be a unifromly convex Banach space with a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty. Then there exists a unique nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$. Further, if $\{\mu_\alpha\}$ is a strongly regular net of continuous linear functionals on $C(S)$, then for each $x \in C$, $T_{\mu_\alpha} T_t x$ converges weakly to Px uniformly in $t \in S$.*

In 1999, Lau, Shioji and Takahashi[10] extended Hirano, Kido and Takahashi's result to an amenable semigroup of nonexpansive mappings on a uniformly convex Banach space whose norm is Fréchet differentiable.

Theorem 3.2 ([10]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let S be a semitopological semigroup. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \phi$. Suppose that $C(S)$ has an invariant mean. Then there exists a unique nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$. Further, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on $C(S)$, then for each $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to Px.*

Atsushiba and Takahashi [4] proved a nonlinear strong ergodic theorem for a one-parameter semigroup in a strictly convex Banach space which is connected with Dafermos and Slemrod [8].

Theorem 3.3 ([4]). *Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Let $\mathcal{S} = \{S(t) : 0 \leqq t < \infty\}$ be a one-parameter nonexpansive semigroup on C and let $x \in C$. Then, $(1/t)\int_0^t S(\tau + h)x d\tau$ converges strongly to a common fixed point of $S(t)$, $t \in [0, \infty)$ uniformly in $h \in [0, \infty)$.*

Further, Atsushiba, Lau and Takahashi [1] obtained the following theorem which generalizes Theorem 3.3.

Theorem 3.4 ([1]). *Let E be a strictly convex Banach space, let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C, where S is commutative. Let X be a subspace of $B(S)$ such that $1 \in X$, X is r_s−invariant for each $s \in S$ and the function $t \mapsto \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on X and let $x \in C$. Then, $\int T_{h+t} x d\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of T_t, $t \in S$ uniformly in $h \in S$.*

4 Weak and Strong Convergence Theorems

Atsushiba, Shioji and Takahashi [2] established a weak convergence theorem of Mann's type for a nonexpansive semigroup in a Banach space.

Theorem 4.1 ([2]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \phi$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that $\|\mu_n - \ell_s^* \mu_n\| = 0$ for every $s \in S$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad n = 1, 2, \dots,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0, a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to an element $x_0 \in F(\mathcal{S})$.

Using Theorem 4.1, we can prove a weak convergence theorem of Mann's type for a one-parameter nonexpansive semigroup.

Theorem 4.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a closed convex subset of E. Let $\mathcal{S} = \{S(t) : t \in [0, \infty)\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \phi$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{s_n} \int_0^{s_n} S(t) x_n dt, \quad n = 1, 2, \dots,$$

where $s_n \to \infty$ as $n \to \infty$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0, a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point $z \in F(\mathcal{S})$.

Shioji and Takahashi [14] also established the following strong convergence theorem of Halpern's type for a nonexpansive semigroup in a Banach space.

Theorem 4.3 ([14]). *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \phi$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that $\|\mu_n - \ell_s^* \mu_n\| = 0$ for every $s \in S$. Suppose that $x, y_1 \in C$ and $\{y_n\}$ is given by*

$$y_{n+1} = \beta_n x + (1 - \beta_n) T_{\mu_n} y_n, \quad n = 1, 2, \dots,$$

where $\{\beta_n\}$ is in $[0,1]$. If $\{\beta_n\}$ is chosen so that $\lim_{n \to \infty} \beta_n = 0$ and $\Sigma_{n=1}^{\infty} \beta_n = \infty$, then $\{y_n\}$ converges strongly to an element of $F(\mathcal{S})$.

Recently, Suzuki and Takahashi[16] established a strong convergence theorem of Mann's type for a one-parameter nonexpansive semigroup in a Banach space without strict convexity. For proving the result, they used the following lemmas:

Lemma 4.4 **([15]).** *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that*

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1.$$

Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n)z_n$ for all $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty} (\|w_n - w_{n+k}\| - \|z_n - z_{n+k}\|) \le 0$$

for all $k \in \mathbb{N}$. Then $\liminf_{n\to\infty} \|w_n - z_n\| = 0$.

Lemma 4.5 **([16]).** *Let A and B be measurable subsets of $[0,\infty)$ and let $\{t_n\}$ be a sequence in $(0,\infty)$ with $\lim_{n\to\infty} t_n = \infty$. Suppose that*

$$\lim_{n\to\infty} \frac{\mu([0,t_n)\cap A)}{t_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mu([0,t_n)\cap B)}{t_n} = 1,$$

where μ is the Lebesgue measure. Then

$$\lim_{n\to\infty} \frac{\mu([0,t_n)\cap A \cap B)}{t_n} = 1$$

and $[t,\infty)\cap A\cap B \ne \phi$ for all $t > 0$.

Theorem 4.6 **([16]).** *Let C be a compact convex subset of a Banach space E and let $\mathcal{S} = \{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C. Let $x_1 \in C$ and define a sequence in C by*

$$x_{n+1} = \frac{\alpha_n}{t_n}\int_0^{t_n} S(s)x_n\, ds + (1-\alpha_n)x_n$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{t_n\} \subset (0,\infty)$ satisfy the following conditions:

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1, \quad \lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Miyake and Takahashi[11] extended Suzuki and Takahashi's result to a general commutative nonexpansive semigroup in a Banach space.

Theorem 4.7 **([11]).** *Let C be a compact convex subset of a Banach space E and let S be a commutative semigroup with identity 0. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ containing 1 such that $\ell_s X \subset X$ for each $s \in S$ and the functions $s \mapsto \langle T_s x, x^*\rangle$ and $s \mapsto \|T_s x - y\|$ are contained in X for each*

$x, y \in C$ *and* $x^* \in E^*$ *and let* $\{\mu_n\}$ *be an asymptotically invariant sequence of means on* X *such that* $\lim_{n\to\infty} \|\mu_n - \mu_{n+1}\| = 0$. *Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *such that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Let $x_1 \in C$ *and let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = \alpha_n T_{\mu_n} x_n + (1 - \alpha_n) x_n$$

for every $n = 1, 2, \ldots$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* $\mathcal{S}$.

Miyake and Takahashi[12] also obtained the following strong convergence theorem of Halpern's type for a general commutative nonexpansive semigroup.

Theorem 4.8 ([12]). *Let* C *be a compact convex subset of a smooth and strictly convex Banach space* E, *let* S *be a commutative semigroup with identity* 0. *Let* $\mathcal{S} = \{T_t : t \in S\}$ *be a nonexpansive semigroup on* C, *let* X *be a subspace of* $B(S)$ *containing* 1 *such that* $\ell_s X \subset X$ *for each* $s \in S$ *and the functions* $s \mapsto \langle T_s x, x^* \rangle$ *and* $s \mapsto \|T_s x - y\|$ *are contained in* X *for each* $x, y \in C$ *and* $x^* \in E^*$ *and let* $\{\mu_n\}$ *be a strongly regular sequence of means on* X. *Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$. *Let* $x \in C$ *and let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_{\mu_n} x_n$$

for every $n = 1, 2, 3, \ldots$. *Then* $\{x_n\}$ *converges strongly to* Px, *where* P *is a unique sunny nonexpansive retraction of* C *onto* $F(\mathcal{S})$.

Using Theorem 4.8, we can obtain the following strong convergence theorem for a one-parameter nonexpansive semigroup.

Theorem 4.9. *Let* C *be a compact convex subset of a smooth and strictly convex Banach space* E *and let* $\mathcal{S} = \{S(t) : t \in \mathbb{R}_+\}$ *be a one-parameter nonexpansive semigroup on* C. *Let* $x_1 = x \in C$ *and let* $\{x_n\}$ *be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} S(s) x_n \, ds$$

for every $n = 1, 2, 3, \ldots$, *where* $\{\alpha_n\}$ *is a sequnece in* $[0,1]$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\{t_n\}$ *is an increasing sequence in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. *Then* $\{x_n\}$ *converges strongly to* Px, *where* P *is a unique sunny nonexpansive retracton of* C *onto* $F(\mathcal{S})$.

References

[1] S. Atsushiba, A.T. Lau and W. Takahashi, *Nonlinear strong ergodic theorems for commutative nonexpansive semigroups on strictly convex Banach spaces*, J. Nonlinear Convex Anal. 1 (2000), 213–231.

[2] S. Atsushiba, N. Shioji and W. Takahashi, *Approximating common fixed points by the Mann iteration procedure in Banach spaces*, J. Nonlinear Convex Anal. **1** (2000), 351–361

[3] S. Atsushiba and W. Takahashi, *Approximating common fixed points of nonexpansive semigroups by the Mann iteraton process*, Ann. Univ. Marie Curie-Sklodowska, **51** (1997), 1–16.

[4] S. Atsushiba and W. Takahashi, *Strong convergence theorems for one-parameter nonexpansive semigroups with compact domains*, in Fixed Point Theory and Applications, Volume 3 (Y. J. Cho, J. K. Kim and S. M. Kang Eds.), Nova Scientific Publishers, New York, 2002, pp.15–31.

[5] J. B. Baillon, *Quelques propriétès de convergence asymptotique pour les semigroupes de contractions impaires*, C. R. Acad. Sci. Paris **283** (1976), 75–78.

[6] J. B. Baillon and H. Brézis, *Une remarque sun le compertement asymptotique des semigroupes nonlinéaires*, Houston J. Math. **2** (1976), 5–7.

[7] H. Brèzis, *Opérateurs maximaux monotones*, Mathematics Studies No.5, North-Holland, Amsterdam, 1973.

[8] C.M. Dafermos and M. Slemrod, *Asymptotic behavior of nonlinear contraction semigroups*, J. Func. Anal. **13** (1973), 97–106.

[9] N. Hirano, K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269–1281.

[10] A. T. Lau, N. Shioji and W. Takahashi, *Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces*, J. Func. Anal. **161** (1999), 62–75.

[11] H. Miyake and W. Takahashi, *Storong convergence theorems for commutative nonexpansive semigroups in general Banach spaces*, Taiwanese J. Math., to appear.

[12] H. Miyake and W. Takahashi, *Storong convergence theorems and sunny nonexpansive retractions in Banach spaces*, to appear.

[13] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), 71–83.

[14] N. Shioji and W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive semigroups in Banach spaces*, J. Nonlinear Convex Anal. **1** (2000), 73–87.

[15] T. Suzuki, *Strong convergence theorem to common fixed points of two nonexpansive mappings in general Banach spaces*, J. Nonlinear Convex Anal. **3** (2002), 381–391.

[16] T. Suzuki and W. Takahashi, *Strong convergence of Mann's type sequences for one-parameter nonexpansive semigroups in general Banach spaces*, J. Nonlinear Convex Anal. **5** (2004), 209–216.

[17] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[18] W. Takahashi, *Fixed point theorems for families of nonexpansive mappings on unbounded sets*, J. Math. Soc. Japan **36** (1984), 543–553.

[19] W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **97** (1986), 55–58.

[20] W. Takahashi, *Fixed point theorem and nonlinear ergodic theorem for nonexpansive semigroups without convexity*, Canad. J. Math. **44** (1992), 880–887.

[21] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[22] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).