On nonlinear scalarization methods in set-valued optimization

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Abstract: Based on the relationship between two sets with respect to a convex cone, we introduce six different solution concepts on set-valued optimization problems. By using a nonlinear scalarization method, we obtain optimal sufficient conditions for efficient solutions of set-valued optimization problems.

Key words: Nonlinear scalarization, vector optimization, set-valued optimization, set-valued maps, optimality conditions.

1 Introduction

In recent study on set-valued optimization problems, some solution concepts are defined by the efficiency of vectors as elements of set-valued objective functions based on a preorder which is a comparison between vectors with respect to a convex cone; see, [4] and [6]. In this paper, based on the comparisons between two sets introduced in [2], we introduce six different solution concepts on the same problem but by defining six types of efficiency on images of set-valued objective functions directly. By using a nonlinear scalarization method involving $h_C(y; k) := \inf \{t : y \in tk - C\}$ where $C \neq Y$ is a convex cone with nonempty interior in a real topological vector space $Y$ and $k \in \text{int} C$, we obtain optimal sufficient conditions for efficient solutions of set-valued optimization problems.

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2 Relationships Between Two Sets

In this section, we introduce relationships between two sets in a vector space. Throughout this section, let $Z$ be a real ordered topological vector space with the vector ordering $\leq_C$ induced by a convex cone $C$: for $x, y \in Z$,

$$x \leq_C y \text{ if } y - x \in C.$$ 

First, we consider comparisons between two vectors. There are two types of comparable cases and in-comparable case. Comparable cases are as follows: for $a, b \in Z$,

1. $a \in b - C$ (i.e., $a \leq_C b$),
2. $a \in b + C$ (i.e., $b \leq_C a$).

When we replace a vector $b \in Z$ with a set $B \subset Z$, that is, we consider comparison between a vector and a set, there are four types of comparable cases and in-comparable case. Comparable cases are as follows: for $a \in Z, B \subset Z$,

1. $A \subset (b - C)$,
2. $A \cap (b - C) \neq \phi$,
3. $A \cap (b + C) \neq \phi$,
4. $A \subset (b + C)$.

By the same way, when we replace a vector $a \in Z$ with a set $A \subset Z$, that is, we consider comparison between two sets with respect to $C$, there are twelve types of some what comparable cases and in-comparable case. For two sets $A, B \subset Z$, $A$ would be inferior to $B$ if we have one of the following situations:

1. $A \subset (\cap_{b \in B}(b - C))$,
2. $A \cap (\cap_{b \in B}(b - C)) \neq \phi$,
3. $(\cup_{a \in A}(a + C)) \supset B$,
4. $(\cup_{a \in A}(a + C)) \cup B$,
5. $(\cap_{a \in A}(a + C)) \supset B$,
6. $(\cap_{a \in A}(a + C)) \cap B \neq \phi$,
7. $A \subset (\cap_{b \in B}(b - C))$,
8. $A \cap (\cap_{b \in B}(b - C)) \neq \phi$.

Also, there are eight converse situations in which $B$ would be inferior to $A$. Actually relationships (1) and (4) coincide with relationships (5) and (8), respectively. Therefore, we define the following six kinds of classification for set-relationships.

**Definition 2.1 (Set-relationships in [2])** Given nonempty sets $A, B \subset Z$, we define six types of relationships between $A$ and $B$ as follows:

1. $A \leq_C^1 B$ by $A \subset \cap_{b \in B}(b - C)$,
2. $A \leq_C^2 B$ by $A \cap (\cap_{b \in B}(b - C)) \neq \phi$,
3. $A \leq_C^3 B$ by $\cup_{a \in A}(a + C) \supset B$,
4. $A \leq_C^4 B$ by $(\cup_{a \in A}(a + C)) \cap B \neq \phi$,
5. $A \leq_C^5 B$ by $A \subset \cup_{b \in B}(b - C)$,
6. $A \leq_C^6 B$ by $A \cap (\cup_{b \in B}(b - C)) \neq \phi$.

**Proposition 2.1** For nonempty sets $A, B \in Z$ and a convex cone $C$ in $Z$, the following statements hold:

1. $A \leq_C^1 B$ implies $A \leq_C^2 B$;
2. $A \leq_C^2 B$ implies $A \leq_C^3 B$;
3. $A \leq_C^3 B$ implies $A \leq_C^4 B$;
4. $A \leq_C^4 B$ implies $A \leq_C^5 B$;
5. $A \leq_C^5 B$ implies $A \leq_C^6 B$;
6. $A \leq_C^6 B$ implies $A \leq_C^1 B$.
3 Nonlinear Scalarization

At first, we introduce a nonlinear scalarization for set-valued maps and show some properties on a characteristic function and scalarizing functions introduced in this section.

Let $X$ and $Y$ be a nonempty set and a topological vector space, $C$ a convex cone in $Y$ with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. We assume that $C \neq Y$, which is equivalent to

$$\text{int } C \cap (-\text{cl } C) = \emptyset \quad (3.1)$$

for a convex cone with nonempty interior, where $\text{int } C$ and $\text{cl } C$ denote the interior and the closure of $C$, respectively.

To begin with, we define a characteristic function

$$h_C(y; k) := \inf\{t : y \in tk - C\}$$

where $k \in \text{int } C$ and moreover $-h_C(-y; k) = \sup\{t : y \in tk + C\}$. This function $h_C(y; k)$ has been treated in some papers; see, [5] and [1], and it is regarded as a generalization of the Tchebyshev scalarization. Essentially, $h_C(y; k)$ is equivalent to the smallest strictly monotonic function with respect to $\text{int } C$ defined by Luc in [3]. Note that $h_C(\cdot; k)$ is positively homogeneous and subadditive for every fixed $k \in \text{int } C$, and hence it is sublinear and continuous.

Now, we give some useful properties of this function $h_C$.

**Lemma 3.1** Let $y \in Y$, then the following statements hold:

(i) If $y \in \text{int } C$, then $h_C(y; k) < 0$ for all $k \in \text{int } C$;

(ii) If there exists $k \in \text{int } C$ with $h_C(y; k) < 0$, then $y \in \text{int } C$.

**Proof.** First we prove the statement (i). Suppose that $y \in \text{int } C$, then there exists an absorbing neighborhood $V_0$ of 0 in $Y$ such that $y + V_0 \subset \text{int } C$. Since $V_0$ is absorbing, for all $k \in \text{int } C$, there exists $t_0 > 0$ such that $t_0k \in V_0$. Therefore, $y + t_0k \in y + V_0 \subset \text{int } C$. Hence, we have

$$\inf\{t : y \in tk - C\} \leq -t_0 < 0,$$

which shows that $h_C(y; k) < 0$.

Next we prove the statement (ii). Let $y \in Y$. Suppose that there exists $k \in \text{int } C$ such that $h_C(y; k) < 0$. Then, there exist $t_0 > 0$ and $c_0 \in C$ such that $y = -t_0k - c_0 = -(t_0k + c_0)$. Since $t_0k \in \text{int } C$ and $C$ is a convex cone, we have $y \in \text{int } C$. $\square$

**Remark 3.1** By combining statements (i) and (ii) above, we have the following: there exists $k \in \text{int } C$ such that $h_C(y; k) < 0$ if and only if $y \in \text{int } C$.

**Lemma 3.2** Let $y \in Y$, then the following statements hold:
(i) If \( y \in -\text{cl} \, C \), then \( h_C(y; k) \leq 0 \) for all \( k \in \text{int} \, C \);

(ii) If there exists \( k \in \text{int} \, C \) with \( h_C(y; k) \leq 0 \), then \( y \in -\text{cl} \, C \).

**Proof.** First we prove the statement (i). Suppose that \( y \in -\text{cl} \, C \). Then, there exist a net \( \{y_\lambda\} \subset -C \) such that \( y_\lambda \) converges to \( y \). For each \( y_\lambda \), since \( y_\lambda \in 0 \cdot k - C \) for all \( k \in \text{int} \, C \), \( h_C(y_\lambda; k) \leq 0 \) for all \( k \in \text{int} \, C \). By the continuity of \( h_C(\cdot; k) \), \( h_C(y; k) \leq 0 \) for all \( k \in \text{int} \, C \).

Next we prove the statement (ii). Let \( y \in Y \). Suppose that there exists \( k \in \text{int} \, C \) such that \( h_C(y; k) \leq 0 \). In the case \( h_C(y; k) < 0 \), from (ii) of Lemma 3.1, it is clear that \( y \in -\text{cl} \, C \). Then we assume that \( h_C(y; k) = 0 \) and show that \( y \in -\text{cl} \, C \). By the definition of \( h_C \), for each \( n = 1, 2, \ldots \), there exists \( t_n \in R \) such that

\[
h_C(y; k) \leq t_n < h_C(y; k) + \frac{1}{n}
\]  

and

\[
y \in t_n k - C.
\]  

From condition (3.2), \( \lim_{n \to \infty} t_n = 0 \). From condition (3.3), there exists \( c_n \in C \) such that \( y = t_n k - c_n \), that is, \( c_n = t_n k - y \). Since \( c_n \to -y \) as \( n \to \infty \), we have \( y \in -\text{cl} \, C \).

**Remark 3.2** By combining statements (i) and (ii) above, we have the following: there exists \( k \in \text{int} \, C \) such that \( h_C(y; k) \leq 0 \) if and only if \( y \in -\text{cl} \, C \).

**Lemma 3.3** Let \( y \in Y \), then the following statements hold:

(i) If \( y \in \text{int} \, C \), then \( h_C(y; k) > 0 \) for all \( k \in \text{int} \, C \);

(ii) If \( y \in -\text{cl} \, C \), then \( h_C(y; k) \geq 0 \) for all \( k \in \text{int} \, C \).

The following lemma shows (strictly) monotone property on \( h_C(\cdot; k) \).

**Lemma 3.4** Let \( y, \bar{y} \in Y \), then the following statements hold:

(i) If \( y \in \bar{y} + \text{int} \, C \), then \( h_C(y; k) > h_C(\bar{y}; k) \) for all \( k \in \text{int} \, C \);

(ii) If \( y \in \bar{y} + \text{cl} \, C \), then \( h_C(y; k) \geq h_C(\bar{y}; k) \) for all \( k \in \text{int} \, C \).

**Lemma 3.5** Let \( y, \bar{y} \in Y \) and \( k \in \text{int} \, C \), then the following statements hold:

(i) If \( h_C(y; k) > h_C(\bar{y}; k) \), then \( h_C(y - \bar{y}; k) > 0 \);

(ii) If \( h_C(y; k) \geq h_C(\bar{y}; k) \), then \( h_C(y - \bar{y}; k) \geq 0 \).

**Remark 3.3** In the above lemma, we note that each converse does not hold.
Now, we consider several characterizations for images of a set-valued map by the nonlinear and strictly monotone characteristic function $h_C$. We observe the following four types of scalarizing functions:

(1) $\psi^F_C(x; k) := \sup \{h_C(y; k) : y \in F(x)\},$
(2) $\varphi^F_C(x; k) := \inf \{h_C(y; k) : y \in F(x)\},$
(3) $-\varphi^F_C(x; k) = \sup \{-h_C(-y; k) : y \in F(x)\},$
(4) $-\psi^F_C(x; k) = \inf \{-h_C(-y; k) : y \in F(x)\}.$

Functions (1) and (4) have symmetric properties and then results for function (4) $-\psi^F_C$ can be easily proved by those for function (1) $\psi^F_C$. Similarly, the results for function (3) $-\varphi^F_C$ can be deduced by those for function (2) $\varphi^F_C$. By using these four functions we measure each image of set-valued map $F$ with respect to its 4-tuple of scalars, which can be regarded as standpoints for the evaluation of the image with respect to convex cone $C$.

**Proposition 3.1** Let $x \in X$, then the following statements hold:

(i) If $F(x) \cap (-\operatorname{int} C) \neq \emptyset$, then $\varphi^F_C(x; k) < 0$ for all $k \in \operatorname{int} C$;

(ii) If there exists $k \in \operatorname{int} C$ with $\varphi^F_C(x; k) < 0$, then $F(x) \cap (-\operatorname{int} C) \neq \emptyset$.

**Proof.** Let $x \in X$ be given. First we prove the statement (i). Suppose that $F(x) \cap (-\operatorname{int} C) \neq \emptyset$. Then, there exists $y \in F(x) \cap (-\operatorname{int} C)$. By (i) of Lemma 3.1, for all $k \in \operatorname{int} C$, $h_C(y; k) < 0$, and hence, $\varphi^F_C(x; k) < 0$.

Next we prove the statement (ii). Suppose that there exists $k \in \operatorname{int} C$ such that $\varphi^F_C(x; k) < 0$. Then, there exist $\varepsilon_0 > 0$ and $y_0 \in F(x)$ such that

$$h_C(y_0; k) \leq \inf_{y \in F(x)} h_C(y; k) + \varepsilon_0 < 0.$$

By (ii) of Lemma 3.1, we have $y_0 \in -\operatorname{int} C$, which implies that $F(x) \cap (-\operatorname{int} C) \neq \emptyset$.

**Remark 3.4** By combining statements (i) and (ii) above, we have the following: there exists $k \in \operatorname{int} C$ such that $\varphi^F_C(x; k) < 0$ if and only if $F(x) \cap (-\operatorname{int} C) \neq \emptyset$.

**Proposition 3.2** Let $x \in X$, then the following statements hold:

(i) If $F(x) \subset -\operatorname{int} C$ and $F(x)$ is a compact set, then $\psi^F_C(x; k) < 0$ for all $k \in \operatorname{int} C$;

(ii) If there exists $k \in \operatorname{int} C$ with $\psi^F_C(x; k) < 0$, then $F(x) \subset -\operatorname{int} C$.

**Proof.** Let $x \in X$ be given. First we prove the statement (i). Assume that $F(x)$ is a compact set and suppose that $F(x) \subset -\operatorname{int} C$. Then, for all $k \in \operatorname{int} C$,

$$F(x) \subset \bigcup_{t>0} (-tk - \operatorname{int} C).$$
By the compactness of $F(x)$, there exist $t_1, \ldots, t_m > 0$ such that

$$F(x) \subset \bigcup_{i=1}^{m} (-t_i k - \text{int } C).$$

Since $-t_p k - \text{int } C \subset -t_p k - \text{int } C$ for $t_p < t_q$, there exists $t_0 := \min\{t_1, \ldots, t_m\} > 0$ such that $F(x) \subset -t_0 k - \text{int } C$. For each $y \in F(x)$, we have

$$h_C(y; k) = \inf\{t : y \in tk - C\} \leq -t_0.$$

Hence,

$$\psi_C^F(x; k) = \sup_{y \in F(x)} h_C(y; k) \leq -t_0 < 0.$$

Next, we prove the statement (ii). Suppose that there exists $k \in \text{int } C$ such that $\psi_C^F(x; k) < 0$. Then, for all $y \in F(x)$, $h_C(y; k) < 0$. By (ii) of Lemma 3.1, we have $y \in -\text{int } C$, and hence $F(x) \subset -\text{int } C$.

Remark 3.5 By combining statements (i) and (ii) above, we have the following: there exists $k \in \text{int } C$ such that $\psi_C^F(x; k) < 0$. Then, for all $y \in F(x)$, $h_C(y; k) < 0$. By (ii) of Lemma 3.1, we have $y \in -\text{int } C$, and hence $F(x) \subset -\text{int } C$.

Moreover, we can replace (i) in Proposition 3.2 by another relaxed form.

Corollary 3.1 Let $x \in X$ and assume that there exists a compact set $B$ such that $B \subset -\text{int } C$. If $F(x) \subset B - C$, then $\psi_C^F(x; k) < 0$ for all $k \in \text{int } C$.

Proof. Let $x \in X$, and assume that there exists a compact set $B$ such that $B \subset -\text{int } C$ and $F(x) \subset B - C$. By applying (i) of Proposition 3.2 to $B$ instead of $F(x)$, for all $k \in \text{int } C$,

$$\sup_{y \in B} h_C(y; k) < 0.$$ 

Since $F(x) \subset B - C$, it follows from (i) of Lemma 3.1 and the subadditivity of $h_C(\cdot; k)$ that

$$h_C(y; k) \leq \sup_{z \in B} h_C(z; k)$$

for each $y \in F(x)$. Therefore, $\psi_C^F(x; k) < 0$ for all $k \in \text{int } C$.

Proposition 3.3 Let $x \in X$, then the following statements hold:

(i) If $F(x) \cap (-\text{cl } C) \neq \emptyset$, then $\varphi_C^F(x; k) \leq 0$ for all $k \in \text{int } C$;

(ii) If $F(x)$ is a compact set and there exists $k \in \text{int } C$ with $\varphi_C^F(x; k) \leq 0$, then $F(x) \cap (-\text{cl } C) \neq \emptyset$. 

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Proof. Let \( x \in X \) and we prove the statement (i). Suppose that \( F(x) \cap (-\text{cl } C) \neq \emptyset \). Then, there exists \( y \in F(x) \cap (-\text{cl } C) \). By (i) of Lemma 3.2, for all \( k \in \text{int } C \), \( h_C(y; k) \leq 0 \), and hence \( \varphi_C^F(x; k) \leq 0 \).

Next, we prove the statement (ii). Suppose that there exists \( k \in \text{int } C \) such that \( \varphi_C^F(x; k) \leq 0 \). In the case \( \varphi_C^F(x; k) < 0 \), from (ii) of Proposition 3.1, it is clear that \( F(x) \cap (-\text{cl } C) \neq \emptyset \). So we assume that \( \varphi_C^F(x; k) = 0 \) and show that \( F(x) \cap (-\text{cl } C) \neq \emptyset \). By the definition of \( \varphi_C^F \), for each \( n = 1, 2, \ldots \), there exist \( t_n \in R \) and \( y_n \in F(x) \) such that \( y_n \in t_n k - C \) and

\[
\varphi_C^F(x; k) \leq t_n < \varphi_C^F(x; k) + \frac{1}{n}.
\]

From (3.4), \( \lim_{n \to \infty} t_n = 0 \). Since \( F(x) \) is compact, we may suppose that \( y_n \to y_0 \) for some \( y_0 \in F(x) \) without loss of generality (taking subsequence). Therefore, \( y_n - t_n k \to y_0 \) and then \( y_0 \in -\text{cl } C \), which shows that \( F(x) \cap (-\text{cl } C) \neq \emptyset \).

Remark 3.5 By combining statements (i) and (ii) above, we have the following: under the compactness of \( F(x) \), there exists \( k \in \text{int } C \) such that \( \varphi_C^F(x; k) \leq 0 \) if and only if \( F(x) \cap (-\text{cl } C) \neq \emptyset \). Otherwise, there are counter-examples violating the statement (ii) such as an unbounded set approaching \(-\text{cl } C \) asymptotically or an open set whose boundary intersects \(-\text{cl } C \).

Proposition 3.4 Let \( x \in X \), then the following statements hold:

(i) If \( F(x) \subset -\text{cl } C \), then \( \psi_C^F(x; k) \leq 0 \) for all \( k \in \text{int } C \);

(ii) If there exists \( k \in \text{int } C \) with \( \psi_C^F(x; k) \leq 0 \), then \( F(x) \subset -\text{cl } C \).

Proof. Let \( x \in X \) be given. First we prove the statement (i). Suppose that \( F(x) \subset -\text{cl } C \). Then, for each \( y \in F(x) \), it follows from (i) of Lemma 3.2 that \( h_C(y; k) \leq 0 \) for all \( k \in \text{int } C \), and hence \( \psi_C^F(x; k) \leq 0 \) for all \( k \in \text{int } C \).

Next, we prove the statement (ii). Suppose that there exists \( k \in \text{int } C \) such that \( \psi_C^F(x; k) \leq 0 \). Then, for all \( y \in F(x) \), \( h_C(y; k) \leq 0 \). By (ii) of Lemma 3.2, we have \( y \in -\text{cl } C \), and hence \( F(x) \subset -\text{cl } C \).

Remark 3.6 By combining statements (i) and (ii) above, we have the following: there exists \( k \in \text{int } C \) such that \( \psi_C^F(x; k) \leq 0 \) if and only if \( F(x) \subset -\text{cl } C \).

4 Optimality Conditions

In this section, we introduce new definitions of efficient solution for set-valued optimization problems. Using the scalarization method introduced in Section 3, we obtain optimal sufficient conditions on such efficiency. Throughout this section, let \( X \) be a nonempty set, \( Y \) a real ordered topological vector space with convex cone \( C \). We assume that \( C \neq Y \) and \( \text{int } C \neq \emptyset \). Let \( F : X \to 2^Y \) be a set-valued map. A set-valued optimization problem is written as
(SVOP) \( \min F(x) \) subject to \( x \in V \), where \( V = \{ x \in X : F(x) \neq \phi \} \).

In this problem, we were defined an efficient solution as follows ever. Vector \( x_0 \in V \) is an efficient solution of (SVOP) if there exists \( y_0 \in F(x_0) \) such that \( F(x) \backslash \{ y_0 \} \cap (y_0 - C) = \phi \) for all \( x \in V \). This type of solution is defined based on a comparison between vectors. However \( F \) is a set-valued map, so it is natural to define efficient solution concepts based on direct comparisons between sets given in Definition 2.1.

**Definition 4.1** (Efficient solution of (SVOP)) \( x_0 \in V \) is said to be an efficient (resp. weakly efficient) solution for (SVOP) with respect to \( \leq^i_c \) for \( i = 1, \ldots, 6 \) if there exists no \( x \in V \backslash \{ x_0 \} \) satisfying \( F(x) \leq^i_c F(x_0) \) (resp. \( F(x) \leq^i_{\text{int}c} F(x_0) \)) for \( i = 1, \ldots, 6 \), respectively.

Using scalarization functions introduced in Section 3, we obtain the following optimal sufficient conditions for (SVOP).

**Theorem 4.1** Let \( x_0 \in V \). If there exists \( k \in \text{int} C \) such that either \( \varphi_{C}^{F}(x_0; k) \leq \varphi_{C}^{F}(x; k) \) or \( -\psi_{C}^{F}(x_0; k) \leq -\varphi_{C}^{F}(x; k) \) for any \( x \in V \), then \( x_0 \) is a weakly efficient solution for (SVOP) with respect to \( \leq^1_{\text{int}c} \).

**Proof.** Suppose that there exists \( k \in \text{int} C \) such that either \( \varphi_{C}^{F}(x_0; k) \leq \varphi_{C}^{F}(x; k) \) or \( -\psi_{C}^{F}(x_0; k) \leq -\varphi_{C}^{F}(x; k) \) for any \( x \in V \). Assume that \( x_0 \) is not a weakly efficient solution with respect to \( \leq^1_{\text{int}c} \). Then there exist \( \bar{x} \in V \) such that \( F(\bar{x}) \leq^1_{\text{int}c} F(x_0) \) (that is, \( \bar{y} \in \cap_{y_0 \in F(x_0)} (y_0 - \text{int} C) \) for any \( \bar{y} \in F(\bar{x}) \)). From condition (i) in Lemma 3.4, it follows that for any \( k \in \text{int} C \), \( h_{C}(\bar{y}; k) < h_{C}(y_0; k) \) and \( -h_{C}(-\bar{y}; k) < -h_{C}(-y_0; k) \) for \( \bar{y} \) and \( y_0 \) satisfying with \( \bar{y} \in F(\bar{x}) \) and \( y_0 \in F(x_0) \). Hence we get \( \psi_{C}^{F}(\bar{x}; k) < \varphi_{C}^{F}(x_0; k) \) and \( -\varphi_{C}^{F}(\bar{x}; k) < -\varphi_{C}^{F}(x_0; k) \), which are contradictions to the assumption.

**Theorem 4.2** Let \( x_0 \in V \).If there exist \( k \in \text{int} C \) such that either \( \varphi_{C}^{F}(x_0; k) \leq \varphi_{C}^{F}(x; k) \) or \( -\psi_{C}^{F}(x_0; k) \leq -\varphi_{C}^{F}(x; k) \) for any \( x \in V \), then \( x_0 \) is a weakly efficient solution for (SVOP) with respect to \( \leq^2_{\text{int}c} \).

**Theorem 4.3** Let \( x_0 \in V \).If there exist \( k \in \text{int} C \) such that either \( \varphi_{C}^{F}(x_0; k) \leq \varphi_{C}^{F}(x; k) \) or \( -\psi_{C}^{F}(x_0; k) \leq -\varphi_{C}^{F}(x; k) \) for any \( x \in V \backslash \{ x_0 \} \), then \( x_0 \) is a weakly efficient solution for (SVOP) with respect to \( \leq^3_{\text{int}c} \).

**Theorem 4.4** Let \( x_0 \in V \).If there exist \( k \in \text{int} C \) such that either \( \psi_{C}^{F}(x_0; k) \leq \psi_{C}^{F}(x; k) \) or \( -\varphi_{C}^{F}(x_0; k) \leq -\varphi_{C}^{F}(x; k) \) for any \( x \in V \), then \( x_0 \) is a weakly efficient solution for (SVOP) with respect to \( \leq^4_{\text{int}c} \).

**Theorem 4.5** Let \( x_0 \in V \).If there exist \( k \in \text{int} C \) such that either \( \psi_{C}^{F}(x_0; k) \leq \psi_{C}^{F}(x; k) \) or \( -\varphi_{C}^{F}(x_0; k) \leq -\varphi_{C}^{F}(x; k) \) for any \( x \in V \backslash \{ x_0 \} \), then \( x_0 \) is a weakly efficient solution for (SVOP) with respect to \( \leq^5_{\text{int}c} \).
Theorem 4.6 Let $x_0 \in V$. If there exist $k \in \text{int}C$ such that either $\psi_C^F(x_0; k) \leq \varphi_C^F(x; k)$ or $-\varphi_C^F(x_0; k) \leq -\psi_C^{-F}(x; k)$ for any $x \in V \setminus \{x_0\}$, then $x_0$ is a weakly efficient solution for (SVOP) with respect to $\leq_{\text{int}C}^{(6)}$.

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References


