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Convergence of Ergodic Nets and Approximate Solutions of Linear Functional Equations

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I. Introduction

Let X be a Banach space, let $A: D(A) \subset X \to X$ be a closed linear operator, and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets of linear operators on X satisfying:
(1) $\|A_\alpha\| \leq M$ for all α;
(2) $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subset AB_\alpha = I - A_\alpha$ for all α;
(3) $R(A_\alpha) \subset D(A)$ for all α, and $AA_\alpha \to 0$ strongly (resp. uniformly).

Then $\{A_\alpha\}$ is called a *strong (resp. uniform) A-ergodic net* and $\{B_\alpha\}$ its *companion net.* For some $x \in X$, if $\{A_\alpha x\}$ converges, the limit is called the ergodic limit at x. For given $y \in X$, if $\{B_\alpha y\}$ converges, the limit x is a solution of the linear functional equation $Ax = y$; thus $\{B_\alpha y\}$ are approximate solutions of $Ax = y$.

In this talk, we discuss results concerning convergence of A_α and B_α, including strong convergence theorems, uniform convergence theorems, theorems on rates of optimal and non-optimal convergence, and the sharpness of non-optimal convergence. The general results provide unified approaches to investigation of strong convergence, uniform convergence, and convergence rates of ergodic limits of various operator families and of the approximate solutions of the associated linear functional equations.

II. Results on $\mathcal{A}$-ergodic Nets

Let X be a Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X.

Definition. Given a family $\mathcal{A}$ of closed linear operators on X, a net $\{A_\alpha\}$ in $B(X)$ is called an *$\mathcal{A}$-ergodic net* if the following conditions hold:
(a) There is an $M > 0$ such that $\|A_\alpha\| \leq M$ for all α;
(b) $\|(A_\alpha - I)x\| \to 0$ for all $x \in \bigcap_{A \in \mathcal{A}} N(A)$, and there is α_0 such that $R(A_\alpha - I) \subset \overline{\sum_{A \in \mathcal{A}} R(A)}$ for all $\alpha \geq \alpha_0$;
(c) For every $A \in \mathcal{A}$, there is a α_A such that $R(A_\alpha) \subset D(A)$ for all $\alpha \geq \alpha_A$ and $\text{w-}\lim_\alpha AA_\alpha x = 0$ for all $x \in X$, and $\|A_\alpha Ax\| \to 0$ for all $x \in D(A)$.

Note that when $\mathcal{A} = \{T - I; T \in \mathcal{S}\}$ for some semigroup $\mathcal{S} \subset B(X)$, $\{A_\alpha\}$ becomes the so-called a *right, weakly left $\mathcal{S}$-ergodic net* in [7, p. 75], which was first studied by Eberlein [5].

Theorem. [8] *Let $\{A_\alpha\}$ be an $\mathcal{A}$-ergodic net. Then the operator P, defined by*

$$\begin{cases} D(P) := \{x \in X; s\text{-}\lim_\alpha A_\alpha x \text{ exists}\}, \\ Px = s\text{-}\lim_\alpha A_\alpha x, x \in D(P), \end{cases}$$

is a bounded linear projection with norm $\|P\| \leq M$, range $R(P) = \cap_{A \in \mathcal{A}} N(A)$, and null space $N(P) = \overline{\sum_{A \in \mathcal{A}} R(A)}$.

II-1. Strong Ergodic Theorems

In the following, we consider $\mathcal{A}$-ergodic nets for the case where $\mathcal{A}$ consists of a single closed operator A.

Definition II-1.1. Let $A: D(A) \subset X \to X$ be a closed linear operator, and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $B(X)$ satisfying:

(C1) $\|A_\alpha\| \le M$ for all α;
(C2) $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subset AB_\alpha = I - A_\alpha$ for all α;
(C3) $R(A_\alpha) \subset D(A)$ for all α, and $\|AA_\alpha\| = O(e(\alpha))$;
(C4) $B_\alpha^* x^* = \varphi(\alpha)x^*$ for all $x^* \in R(A)^\perp$, and $|\varphi(\alpha)| \to \infty$;
(C5) $\|A_\alpha x\| = O(f(\alpha))$ (*resp.* $o(f(\alpha))$) implies $\|B_\alpha x\| = O(\frac{f(\alpha)}{e(\alpha)})$ (resp. $o(\frac{f(\alpha)}{e(\alpha)})$),

where e and f are positive functions satisfying $0 < e(\alpha) \le f(\alpha) \to 0$. We shall call $\{A_\alpha\}$ a *uniform A-ergodic net* and $\{B_\alpha\}$ its *companion net.*

The functions $e(\alpha)$ and $f(\alpha)$ are to act as estimators of the convergence rates of $\{A_\alpha x\}$ and $\{B_\alpha y\}$, which, in practical applications, approximate the ergodic limit and the solution x of $Ax = y$, respectively. The assumptions (C4) and (C5) play key roles in the proofs of our theorems and prevail among practical examples.

$\{A_\alpha\}$ is said to be strongly (resp. uniformly) ergodic if $D(P) = X$ and $A_\alpha x \to Px$ for all $x \in X$ (resp. $\|A_\alpha - P\| \to 0$).

The following strong convergence theorems for the systems $\{A_\alpha\}$ and $\{B_\alpha\}$ are proved in [20].

Theorem II-1.2 (Strong Ergodic Theorem). *Under conditions* (C1) - (C4), *P is a bounded linear projection with range $R(P) = N(A)$, null space $N(P) = \overline{R(A)}$, and domain*

$$D(P) = N(A) \oplus \overline{R(A)} = \{x \in X; \{A_\alpha x\} \text{ has a weak cluster point}\}.$$

Theorem II-1.3. *Under conditions* (C1) - (C4), *the following conditions are equivalent:*
(i) $y \in A(D(A) \cap \overline{R(A)})$;
(ii) $x = \text{s-}\lim_\alpha B_\alpha y$ *exists;*
(iii) *There is a subnet* $\{B_\beta\}$ *of* $\{B_\alpha\}$ *such that* $x = \text{w-}\lim_\alpha B_\alpha y$ *exists.*
The x in (ii) is the unique solution of $Ax = y$ in $\overline{R(A)}$.

Let B_1 be the operator defined by $\begin{cases} D(B_1) := \{y \in X; \lim_\alpha B_\alpha y \text{ exists}\}; \\ B_1 x := \lim_\alpha B_\alpha y \text{ for } y \in D(B_1). \end{cases}$

Theorem II-1.4 *Under conditions* (C1) - (C4), *B_1 is the inverse operator A_1^{-1} of the restriction $A_1 := A|\overline{R(A)}$ of A to $\overline{R(A)}$; it has range $R(B_1) = D(A_1) = D(A) \cap \overline{R(A)}$ and domain $D(B_1) = R(A_1) = A(D(A) \cap \overline{R(A)})$. Moreover, for each $y \in D(B_1)$, $B_1 y$ is the unique solution of the functional equation $Ax = y$ in $\overline{R(A)}$.*

Theorem II-1.5 *Under conditions* (C1) - (C4), *$\{A_\alpha\}$ is strongly ergodic if and only if $N(A)$ separates $R(A)^\perp$, if and only if $R(A) = D(B_1) = A(D(A) \cap \overline{R(A)})$, if and only if $\{A_\alpha x\}$ has a weak cluster point for each $x \in X$. These are true in particular when X is reflexive.*

Theorem II-1.6. *The following relations hold:*

$$R(A_1) = \{y \in X; \lim_\alpha B_\alpha y \text{ exists}\} = \{y \in X; \{B_\alpha y\} \text{ has a weak cluster point}\}$$
$$\subset R(A) \subset \{x \in X; \sup_\alpha \|B_\alpha x\| < \infty\} \subset \overline{R(A)}.$$

It is known [20, Remarks 1.5 and 1.7] that the first inclusion in Theorem II-1.6 is an equality, i.e., $R(A) = R(A_1)$, if (and only if, when A is densely defined) $\{A_\alpha\}$ is strongly ergodic. As the following Uniform Ergodic Theorem (II-2.1) shows, the last inclusion is an equality if and only if $\{A_\alpha\}$ is uniformly ergodic, and, in this case, the other two inclusions are also equalities.

II-2. Uniform Ergodic Theorems

The next two theorems are proved in [22].

Theorem II-2.1. (Uniform Ergodic Theorem). *Under conditions* (C1) - (C4), *the following are equivalent:*
(i) $\{A_\alpha\}$ *is uniformly ergodic, i.e.,* $D(P) = X$ *and* $\|A_\alpha - P\| \to 0$.
(ii) $R(A)$ (*or* $R(A_1)$) *is closed.*
(iii) $R(A^2)$ (*or* $R(A_1^2)$) *is closed.*
(iv) $X = N(A) \oplus R(A)$.
(v) $\{B_\alpha|_{R(A)}\}$ *is uniformly bounded.*
(vi) B_1 *is bounded.*
(vii) $\{x \in X; \sup_\alpha \|B_\alpha x\| < \infty\} = \overline{R(A)}$.
(viii) $\{x \in X; \sup_\alpha \|B_\alpha x\| < \infty\}$ *is closed.*
Moreover, in this case, we have $D(B_1) = R(A_1) = R(A)$, $\|A_\alpha - P\| \leq (M+1)\|B\|\|AA_\alpha\| = O(e(\alpha))$, $\|B_\alpha|_{R(A)} - B_1\| \leq (M+1)\|B\|^2\|AA_\alpha\| = O(e(\alpha))$.

The equivalence of the first six conditions is proved in [22]. Because of Theorem II-1.6, (ii) obviously implies (vii), and (viii) implies (v) by the uniform boundedness principle.

Theorem II-2.2. *Let X be a Grothendieck space with the Dunford-Pettis property. Then under conditions* (C1) - (C3), $\{A_\alpha\}$ *is uniformly ergodic if and only if it is strongly ergodic.*

II-3. Condition (*) and Uniform Ergodicity

It follows from Theorem II-2.1 that if $\{A_\alpha\}$ is uniformly ergodic, then the following solvability condition for the functional equation $Ax = y$ holds:

$$(*) \qquad R(A) = \{x \in X; \sup_\alpha \|B_\alpha x\| < \infty\}.$$

But the converse implication is in general not true. In this section we first give some conditions which are equivalent to or sufficient for (*), and then discuss when (*) and uniform ergodicity are equivalent and when they are not. The results in this section are proved in [26].

Theorem II-3.1. *Under conditions* (C1) - (C4), *the following three conditions are equivalent:*
(i) $\{x \in X; \sup_\alpha \|B_\alpha x\| < \infty\} = R(A)$;
(ii) $\overline{A(D(A) \cap U)} \subset R(A)$.
(iii) $R(A)$ *is an* F_σ *set.*
When $A \in B(X)$, *we also have the next equivalent condition:*
(iv) *There is an equivalent norm in* X, *with closed unit ball* U', *such that* $A(U')$ *is closed.*

In view of the equivalence of (i) and (ii) in Theorem II-3.1, the closedness of $A(D(A) \cap U)$ is a sufficient condition for (*) to hold. The following are some examples with this property.

Corollary II-3.2. *If, in Theorem II-3.1, X is a dual space (with its dual norm), say* $X = Y^*$, *and A is the dual operator of a closed operator B, i.e.,* $A = B^*$, *then* $A(D(A) \cap U)$ *is closed and* (*) *holds.*

In particular, the conclusion of Corollary II-3.2 holds when A is a densely defined closed operator on a reflexive space.

Corollary II-3.3. *In* Theorem II-3.1, *if* $I + A$ *is either a contraction of* $X = L_1(\mu)$, *with* μ *a* σ*-finite measure, or an irreducible Markov operator on* $X = C(K)$, *with K a compact Hausdorff space, then* $A(U)$ *is closed and* (*) *holds.*

Theorem II-3.4. *Let* $\{A_\alpha\}$ *be a strongly ergodic A-ergodic net on a Banach space X, and suppose all operators in* $\{A, A_\alpha, B_\alpha; \alpha\}$ *are commutative. If* $A \in B(X)$, (*) *is satisfied, and* $\{A_\alpha\}$ *is not uniformly ergodic, then* $\overline{R(A)}$ *contains a separable infinite dimensional closed subspace isomorphic to a dual Banach space.*

Corollary II-3.5. *Let X be a Banach space which does not contain any infinite dimensional separable closed subspace isomorphic to a dual Banach space, let $\{A_\alpha\}$ be an A-ergodic net, with $A \in B(X)$, and suppose all operators in $\{A, A_\alpha, B_\alpha; \alpha\}$ are commutative. Then $\{A_\alpha\}$ is uniformly ergodic if and only if it is strongly ergodic and satisfies (*).*

II-4. Rates of Ergodic Limits

We first specify the required notations. Let $X_1 := \overline{R(A)}$ and $X_0 := D(P) = N(A) \oplus X_1$. Since the operator $B_1 : D(B_1) \subset X_1 \to X_1$ is closed, its domain $D(B_1)\,(= R(A_1))$ is a Banach space with respect to the norm $\|x\|_{B_1} := \|x\| + \|B_1 x\|$.

Let $B_0 : D(B_0) \subset X_0 \to X_0$ be the operator $B_0 := 0 \oplus B_1$. Then its domain

$$D(B_0)\,(= N(A) \oplus D(B_1) = N(A) \oplus A(D(A) \cap \overline{R(A)}))$$

is a Banach space with norm $\|x\|_{B_0} := \|x\| + \|B_0 x\|$, and $[D(B_0)]\widetilde{\ }_{X_0} = N(A) \oplus [D(B_1)]\widetilde{\ }_{X_1}$.

Now we can state the following theorem from [24], which is concerned with optimal convergence and non-optimal convergence rates of ergodic limits and approximate solutions.

Theorem II-4.1. *Under conditions* (C1) - (C5) *the following statements hold.*

(i) *For $x \in X_0 = N(A) \oplus \overline{R(A)}$, one has:*

$$\begin{aligned}\|A_\alpha x - Px\| = O(f(\alpha)) &\Leftrightarrow K(e(\alpha), x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(f(\alpha)) \\ &\Leftrightarrow x \in [D(B_0)]\widetilde{\ }_{X_0} \;\; (\text{in case } f = e).\end{aligned}$$

(ii) *For $x \in \overline{R(A)}$, one has:*

$$\begin{aligned}\|A_\alpha x\| = O(f(\alpha)) &\Leftrightarrow K(e(\alpha), x, X_1, D(B_1), \|\cdot\|_{B_1}) = O(f(\alpha)) \\ &\Leftrightarrow x \in [D(B_1)]\widetilde{\ }_{X_1} \;\; (\text{in case } f = e).\end{aligned}$$

(iii) *For $y \in D(B_1) = R(A_1)$ one has:*

$$\begin{aligned}\|B_\alpha y - B_1 y\| = O(f(\alpha)) &\Leftrightarrow K(e(\alpha), B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(f(\alpha)) \\ &\Leftrightarrow y \in A(D(A) \cap [D(B_1)]\widetilde{\ }_{X_1}) \;\; (\text{in case } f = e).\end{aligned}$$

The saturation case ($f = e$) was proved in [23]. It was also shown there that

(1) for $x \in X_0$, $\|A_\alpha x - Px\| = o(e(\alpha)) \Leftrightarrow x \in N(A)$;
(2) for $x \in X$ $\|B_\alpha x\| = o(1) \Leftrightarrow x = 0$;
(3) for $y \in D(B_1) = R(A_1)$, $\|B_\alpha y - B_1 y\| = o(e(\alpha)) \Leftrightarrow y = 0$.

Thus, when $A \neq 0$, the rate of optimal convergence of $\|A_\alpha y\| = O(e(\alpha))$ is sharp everywhere on $[D(B_1)]\widetilde{\ }_{X_1} \setminus \{0\}$.

The sharpness of non-optimal convergence rate: $\|A_\alpha y\| = O(f(\alpha))$ with f satisfying $f(\alpha)/e(\alpha) \to \infty$ is shown in the following theorem.

Theorem II-4.2. *Suppose that A, $\{A_\alpha\}$, and $\{B_\alpha\}$ satisfy conditions* (C1) - (C5), *with $f(\alpha)/e(\alpha) \to \infty$. Then $R(A)$ is not closed if and only if there exists an element $y_f \in X_1$ such that* $\|A_\alpha y_f\| \begin{cases} = O(f(\alpha)); \\ \neq o(f(\alpha)). \end{cases}$

III. Specializations to Discrete Semigroups

In this section we deduce from the general results in the previous section their specializations for discrete semigroups.

Let T be a power bounded operator. It is routine to verify that $A := T - I$, $A_n := n^{-1}\sum_{k=0}^{n-1} T^k$, $B_n := -n^{-1}\sum_{k=1}^{n-1}\sum_{j=0}^{k-1} T^j$ satisfy conditions (C1) - (C5) with $e(n) = n^{-1}$, $\phi(n) =$

$(n-1)/2$, and $f(n) = n^{-\beta}$, $0 < \beta \le 1$. Therefore Theorems II-4.1 and II-4.2 yield the following theorem.

Theorem III-1. *Let T be a power bounded operator. Then we have:*

(i) *The mapping $P : x \to \lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1} T^k x$ is a bounded linear projection with $R(P) = N(T-I)$, $N(P) = \overline{R(T-I)}$, and $D(P) = N(T-I) \oplus \overline{R(T-I)}$. For $0 < \beta \le 1$ and $x \in D(P)$, we have*

$$\|n^{-1}\sum_{k=0}^{n-1} T^k x - Px\| = O(n^{-\beta}) \Leftrightarrow K(n^{-1}, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(n^{-\beta}).$$

Moreover, $\|n^{-1}\sum_{k=0}^{n-1} T^k x - Px\| = O(n^{-1}$ (resp. $o(n^{-1})$) if and only if $x \in N(T-I) \oplus [(T-I)\overline{(T-I)X}]^{\sim}_{\overline{(T-I)X}}$ (resp. $x \in N(T-I)$).

(ii) *The mapping $B_1 : y \to -\lim n^{-1}\sum_{k=1}^{n-1}\sum_{j=0}^{k-1} T^j y$ is the inverse operator of $(T-I)|_{\overline{(T-I)X}}$; for each $y \in (T-I)\overline{(T-I)X}$, $B_1 y$ is the unique solution of the functional equation $(T-I)x = y$ in $\overline{(T-I)X}$. For $0 < \beta \le 1$ we have $\|n^{-1}\sum_{k=1}^{n-1}\sum_{j=0}^{k-1} T^j y + B_1 y\| = O(n^{-\beta})$ $\Leftrightarrow K(n^{-1}, B_1 y, \overline{(T-I)X}, D(B_1), \|\cdot\|_{B_1}) = O(n^{-\beta})$. Moreover, $\|n^{-1}\sum_{k=1}^{n-1}\sum_{j=0}^{k-1} T^j y + B_1 y\| = O(n^{-1})$ (resp. $o(n^{-1})$) $\Leftrightarrow y \in [(T-I)\overline{(T-I)X}]^{\sim}_{\overline{(T-I)X}}$ (resp. $y = 0$).*

(iii) *$(T-I)X$ is not closed if and only if for every $0 < \beta < 1$ there is an element $y_\beta \in \overline{(T-I)X}$ such that $\|n^{-1}\sum_{k=0}^{n-1} T^k y_\beta\| \begin{cases} = O(n^{-\beta}) \\ \ne o(t^{-\beta}) \end{cases}$ $(n \to \infty)$.*

Remark. (i) was originally proved by Butzer and Westphal [3].

Let $\{\lambda_n\}$ be a sequence of numbers satisfying $0 < \lambda_n \le 1$ and $\sum_{n=1}^{\infty} \lambda_n(1-\lambda_n) = \infty$. Let $A_n := \prod_{i=1}^{n}[(1-\lambda_i) + \lambda_i T]$, $B_1 = \lambda_1 I$, $B_{n+1} = \lambda_{n+1} I + [(1-\lambda_{n+1}) + \lambda_{n+1}T]B_n$, $n = 1, 2, \ldots$. It is easy to see that $B_n(T-I) = A_n - I$ for $n \ge 1$ (cf. [20]).

If T is power bounded, then $\{A_n\}$ is uniformly bounded and $\|A_n(T-I)\| \to 0$ as $n \to \infty$.

For $x, y \in X$ define $f_0(x) = x$, $g_0(y) = 0$, $f_n(x) = [(1-\lambda_n) + \lambda_n T]f_{n-1}(x)$, and $g_n(y) = \lambda_n y + [(1-\lambda_n) + \lambda_n]g_{n-1}(y)$, $n = 1, 2, \ldots$. Applying Theorems II-4.1 and II-4.2 we obtain the following theorem.

Theorem III-2. *Let T be a power bounded operator. Then we have:*

(i) *The mapping $P : x \to \lim_{n\to\infty} f_n(x)$ is a bounded linear projection with $R(P) = N(T-I)$, $N(P) = \overline{R(T-I)}$, and $D(P) = N(T-I) \oplus \overline{R(T-I)}$. For $0 < \beta \le 1$ and $x \in D(P)$, we have*

$$\|f_n(x) - Px\| = O(n^{-\beta}) \Leftrightarrow K(n^{-1}, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(n^{-\beta}).$$

Moreover,

$$\|f_n(x) - Px\| = O(n^{-1}) \text{ (resp. } o(n^{-1})) \Leftrightarrow x \in N(T-I) \oplus [(T-I)\overline{(T-I)X}]^{\sim}_{\overline{(T-I)X}} \text{ (resp. } x \in N(T-I)).$$

(ii) *The mapping $B_1 : y \to -\lim_{n\to\infty} g_n(y)$ is the inverse operator of $(T-I)|_{\overline{(T-I)X}}$;*

for each $y \in (T-I)\overline{(T-I)X}$, $B_1 y$ is the unique solution of the functional equation $(T-I)x = y$ in $\overline{(T-I)X}$.

For $0 < \beta \le 1$ we have

$$\|g_n(y) + B_1 y\| = O(n^{-\beta}) \Leftrightarrow K(n^{-1}, B_1 y, \overline{(T-I)X}, D(B_1), \|\cdot\|_{B_1}) = O(n^{-\beta}).$$

Moreover,

$$\|g_n(y) + B_1 y\| = O(n^{-1}) \text{ (resp. } o(n^{-1})) \Leftrightarrow y \in [(T-I)\overline{(T-I)X}]^{\sim}_{\overline{(T-I)X}} \text{ (resp. } y = 0).$$

(iii) *$(T-I)X$ is not closed if and only if for every $0 < \beta < 1$ there is an element $y_\beta \in \overline{(T-I)X}$ such that* $\|g_n(y)\| \begin{cases} = O(n^{-\beta}) \\ \neq o(t^{-\beta}) \end{cases} (n \to \infty).$

IV. Specialization to Pseudoesolvents

A $B(X)$-valued function $J : \lambda \to J_\lambda$, defined on a subset $D(J)$ of the complex plane C, is called a pseudo-resolvent on X if it satisfies the resolvent equation:

$$J_\lambda - J_\mu = (\mu - \lambda) J_\lambda J_\mu \text{ for all } \lambda, \mu \in D(J). \tag{5.1}$$

J has a unique maximal extension $\hat{J}$, which is also a pseudo-resolvent on X; $\hat{J}$ has an open domain $D(\hat{J})$ over which $\hat{J}$ is analytical. We assume that J is already maximal. The following lemma is well-known ([27, p. 216]).

Lemma IV-1. (i) *The subspaces $N(J_\lambda)$, $R(J_\lambda), R(J_\lambda^2), N(\lambda J_\lambda - I), R(\lambda J_\lambda - I)$, and $R((\lambda J_\lambda - I)^2)$ are independent of the parameter λ.*

(ii) *The pseudo-resolvent J is the resolvent of a closed linear operator A (i.e., $J_\lambda = (\lambda - A)^{-1}$) if and only if $N(J_\lambda) = 0$. In this case we have $A := \lambda - J_\lambda^{-1}$, $R(J_\lambda) = D(A), R(J_\lambda^2) = D(A^2)$, $N(\lambda J_\lambda - I) = N(A)$, $R(\lambda J_\lambda - I) = R(A)$, and $R((\lambda J_\lambda - I)^2) = R(A^2)$.*

Let $X_1 := \overline{R(\lambda J_\lambda - I)}$, and let $B_1^{(\lambda)}$ and B_1 be operators defined by $B_1^{(\lambda)} y = \lim B_\alpha^{(\lambda)} y = \lim \lambda^{-1}[(\alpha - \lambda)J_\alpha - I]y$ and $B_1 y = -\lim J_\alpha y$, respectively. We also define $D(B_0) := N(\lambda J_\lambda - I) \oplus D(B_1)$ and $B_0 := 0 \oplus B_1$.

Lemma IV-2. *We have $D(B_1) = D(B_1^{(\lambda)}) = (\lambda J_\lambda - I)X_1$ and $B_1 y = B_1^{(\lambda)} y + \lambda^{-1} y$ for all $y \in D(B_1)$ and $\lambda \in D(J)$; the graph norms $\|\cdot\|_{B_1}$ and $\|\cdot\|_{B_1^{(\lambda)}}$ are equivalent on $D(B_1)$, and the graph norms $\|\cdot\|_{B_0}$ and $\|\cdot\|_{B_0^{(\lambda)}}$ are equivalent on $D(B_0)$.*

Noting these facts, we can apply the general results in Sections II and III to $(A^{(\lambda)}, A_\alpha, B_\alpha^{(\lambda)})$ to deduce the following results. They follow from Theorems II-1.2 and II-1.5, and Theorem II-4.1.

Theorem IV-3. [26] *Let J be a pseudo-resolvent on X such that $0 \in \overline{D(J)}$ and $\|\alpha J_\alpha\| = O(1)\,(\alpha \to 0, \alpha \in D(J))$. Let P be the operator defined by $Px := \lim_{\alpha \to 0} \alpha J_\alpha x$. Then*

(i) *P is a bounded linear projection with range $R(P) = N(\lambda J_\lambda - I)$, null space $N(P) = \overline{R(\lambda J_\lambda - I)}$, and domain*

$$D(P) = N(\lambda J_\lambda - I) \oplus \overline{R(\lambda J_\lambda - I)} = \{x \in X; \{\alpha J_\alpha x\}_{\alpha \to 0} \text{ has a weak cluster point}\}.$$

(ii) *$\{\alpha J_\alpha\}$ is strongly ergodic if and only if $N(\lambda J_\lambda - I)$ separates $R(\lambda J_\lambda - I)^\perp$, if and only if $\{\alpha J_\alpha x\}_{\alpha \to 0}$ has a weak cluster point for each $x \in X$. These conditions are satisfied in particular when X is reflexive.*

(iii) *For $x \in X_0 = N(\lambda J_\lambda - I) \oplus \overline{R(\lambda J_\lambda - I)}$, one has:*

$$\|\alpha J_\alpha x - Px\| = o(\alpha) \Leftrightarrow x \in N(\lambda J_\lambda - I);$$

$$\begin{aligned} \|\alpha J_\alpha x - Px\| = O(\alpha^\theta) &\Leftrightarrow K(\alpha, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(\alpha^\theta) (\text{for } 0 < \theta \le 1) \\ &\Leftrightarrow x \in [D(B_0)]^{\sim}_{X_0} \text{ (in case } \theta = 1). \end{aligned}$$

Theorem IV-4. [26] *Under the assumption of Theorem IV-3 we have:*

(i) $\{J_\alpha y\}_{\alpha\to 0}$ *converges strongly if and only if it contains a weakly convergent subnet.*

(ii) *For each* $y \in D(B_1) = (\lambda J_\lambda - I)X_1$, $B_1 y$ *is the unique solution of the functional equation* $(\lambda J_\lambda - I)x = J_\lambda y$ *in* X_1 *for every* $\lambda \in D(J)$.

(iii) *For* $y \in D(B_1)$, *one has:*

$$\|J_\alpha y + B_1 y\| = O(\alpha^\theta) \Leftrightarrow K(\alpha, B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(\alpha^\theta) (for\ 0 < \theta \le 1)$$
$$\Leftrightarrow y \in (\lambda J_\lambda - I)[D(B_1)]^{\sim}_{X_1})\ (when\ \theta = 1).$$

By applying Theorem II-2.1 to J and Lemma IV-2 we obtain the next theorem.

Theorem IV-5. [26] *Let* J *be a pseudo-resolvent on* X *such that* $0 \in \overline{D(J)}$ *and* $\|\alpha J_\alpha\| = O(1)\,(\alpha \to 0, \alpha \in D(J))$. *The following are equivalent:*

(i) $\{\alpha J_\alpha\}$ *is uniformly ergodic, i.e.,* $D(P) = X$ *and* $\|\alpha J_\alpha - P\| \to 0$.

(ii) $R(\lambda J_\lambda - I)$ *(or* $(\lambda J_\lambda - I)(X_1)$*) is closed.*

(iii) $R((\lambda J_\lambda - I)^2)$ *(or* $(\lambda J_\lambda - I)^2(X_1)$*) is closed.*

(iv) $\sup\{\|J_\alpha|_{X_1}\|; \alpha \in D(J), |\alpha| \le \delta\} < \infty$ *for some* $\delta > 0$.

(v) B_1 *is bounded.*

(vi) $\{x \in X; \sup\{\|J_\alpha x\|; \alpha \in D(J), |\alpha| \le \delta\} < \infty\}$ *is closed for some* $\delta > 0$.

Moreover, in this case, we have $D(B_1) = X_1 = R(\lambda J_\lambda - I)$, $\|\alpha J_\alpha - P\| = O(\alpha)\,(\alpha \to 0)$ *and* $\|J_\alpha|_{X_1} + B_1\| = O(\alpha)\,(\alpha \to 0)$.

From Corollaries II-3.2 and II-3.3 we can deduce the following result for pseudo-resolvents.

Theorem IV-6. [26] *Let* J *be a pseudo-resolvent on* X *such that* $0 \in \overline{D(J)}$ *and* $\|\alpha J_\alpha\| = O(1)\,(\alpha \to 0, \alpha \in D(J))$. *In each of the following cases, we have that* $(\lambda J_\lambda - I)U$ *is closed and (**):*

$$R(\lambda J_\lambda - I) = \{x \in X; \sup\{\|J_\alpha x\|; \alpha \in D(J), |\alpha| \le 1\} < \infty\}.$$

(1) X *is a dual space and* $J_\alpha, \alpha \in D(J)$, *are dual operators.*

(2) $X = L_1(\mu)$, *with* μ *a* σ*-finite measure and* $\|\lambda J_\lambda\| \le 1$.

(3) $X = C(K)$, *with* K *a compact Hausdorff space, and* λJ_λ *is an irreducible Markov operator.*

From results in Section II-3 we deduce the next theorem.

Theorem IV-7. [26] *Let* J *be a pseudo-resolvent on* X *such that* $0 \in \overline{D(J)}$ *and* $\|\alpha J_\alpha\| = O(1)\,(\alpha \to 0, \alpha \in D(J))$.

(i) *If* αJ_α *does not converge in operator norm as* $\alpha \to 0$ *and satisfies (**), and if either* X *is separable, or* αJ_α *converges strongly, then* $\overline{R(\lambda J_\lambda - I)}$ *contains a separable infinite-dimensional closed subspace isomorphic to a dual Banach space.*

(ii) *If* X *does not contain any infinite-dimensional separable closed subspace isomorphic to a dual Banach space, then* $\{\alpha J_\alpha\}$ *converges in operator norm as* $\alpha \to 0$ *if and only if it converges strongly and (**) holds.*

(iii) *If* X *does not contain any infinite-dimensional closed subspace isomorphic to a dual Banach space, and if* X *is separable or* $\lambda J_\lambda - I$ *is injective, then* $\{\alpha J_\alpha\}$ *converges in operator norm as* $\alpha \to 0$ *if and only if (**) holds.*

If A is a closed operator such that $0 \in \overline{\rho(A)}$ and $\|\alpha(\alpha - A)^{-1}\| = O(1)\,(\alpha \to 0)$ (i.e., a generalized Hille-Yosida operator), then $\{J_\alpha = (\alpha - A)^{-1},\ \alpha \in \rho(A)\}$ is a pseudo-resolvent. We have $A_\alpha = \alpha(\alpha - A)^{-1}$, $B_\alpha^{(\lambda)} = \lambda^{-1}(A - \lambda)(\alpha - A)^{-1}$, $A^{(\lambda)} = \lambda A(\lambda - A)^{-1}$, $N(\lambda J_\lambda - I) = N(A)$, $R(\lambda J_\lambda - I) = R(A)$, $R((\lambda J_\lambda - I)^2) = R(A^2)$, $X_1 = \overline{R(A)}$, $X_0 = N(A) \oplus \overline{R(A)}$, $A_1^{(\lambda)} = \lambda A(\lambda - A)^{-1}|_{X_1}$, and $B_1^{(\lambda)} = (A_1^{(\lambda)})^{-1} = \lambda^{-1}(\lambda - A)(A|_{X_1})^{-1}$ with $D(B_1^{(\lambda)}) = R(A_1^{(\lambda)}) = A(\lambda - A)^{-1}(X_1)$. Also we have $D(B_1) = D(B_1^{(\lambda)})$ and $B_1 y = B_1^{(\lambda)} y + \lambda^{-1} y = (A|_{X_1})^{-1} y = A_1^{-1} y$ for all $y \in D(B_1)$.

In this case, (iii) of Theorem IV-3 reduces to (i) of Theorem 3 in [24], (i) and (ii) of Theorem IV-4 reduce to Theorem 3.1 in [20], and (iii) of Theorem IV-4 leads to (ii) of Theorem 3 in [24]. From Theorems IV-5, IV-6, and IV-7 we deduce the following three corollaries.

Corollary IV-8. [26] *For a generalized Hille-Yosida operator A, the following conditions are equivalent:*
(i) $\alpha(\alpha - A)^{-1}$ *converges in operator norm as* $\alpha \to 0$.
(ii) $R(A)$ *is closed.*
(iii) $R(A^2)$ *is closed.*
(iv) $\sup\{\|(\alpha - A)^{-1}|_{R(A)}\|; \alpha \in \rho(A), |\alpha| \leq \delta\} < \infty$ *for some* $\delta > 0$.
(v) $B_1 = (A_1)^{-1}$ *is bounded.*
(vi) $\{x \in X; \sup\{\|(\alpha - A)^{-1}x\|; \alpha \in \rho(A), |\alpha| \leq \delta\} < \infty\}$ *is closed for some* $\delta > 0$.
Moreover, in this case, we have $X_1 = R(A)$, $\|\alpha(\alpha - A)^{-1} - P\| = O(\alpha)\,(\alpha \to 0)$ *and* $\|(\alpha - A)^{-1}|_{X_1} + A_1^{-1}\| = O(\alpha)\,(\alpha \to 0)$.

Corollary IV-9. *Let A be a generalized Hille-Yosida operator. In each of the following cases, we have*

$$(***) \qquad R(A) = \{y \in X; \|(\alpha - A)^{-1}y\| = O(1)(\alpha \to 0, \alpha \in \rho(A))\}.$$

(1) *X is a dual space and A is a dual operator.*
(2) $X = L_1(\mu)$, *with μ a σ-finite measure and* $\|\lambda(\lambda - A)^{-1}\| \leq 1$.
(3) $X = C(K)$, *with K a compact Hausdorff space, and* $\lambda(\lambda - A)^{-1}$ *is an irreducible Markov operator.*

Corollary IV-10. *Let A be a generalized Hille-Yosida operator.*

(i) *If A satisfies (***) and* $\alpha(\alpha - A)^{-1}$ *does not converge in operator norm as* $\alpha \to 0$, *and if either X is separable, or* $\alpha(\alpha - A)^{-1}$ *converges strongly, then* $\overline{R(A)}$ *contains a separable infinite-dimensional closed subspace isomorphic to a dual Banach space.*

(ii) *If X does not contain any infinite-dimensional separable closed subspace isomorphic to a dual Banach space, then* $\alpha(\alpha - A)^{-1}$ *converges in operator norm as* $\alpha \to 0$ *if and only if it converges strongly and (***) holds.*

(iii) *If X does not contain any infinite-dimensional closed subspace isomorphic to a dual Banach space, and if X is separable or A is injective, then* $\alpha(\alpha - A)^{-1}$ *converges in operator norm as* $\alpha \to 0$ *if and only if (***) holds.*

Then the following theorem follows from Theorems II-3.1 and II-3.2 immediately.

Theorem IV-11. *Let A be a closed operator such that* $0 \in \overline{\rho(A)}$ *and* $\|\lambda(\lambda - A)^{-1}\| = O(1)(\lambda \to 0)$. *Then the following are true for* $0 < \beta \leq 1$:

(i) *For* $x \in X_0$, *one has* $\|\lambda(\lambda - A)^{-1}x - Px\| = O(|\lambda|^\beta)(\lambda \to 0) \Leftrightarrow K(|\lambda|, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(|\lambda|^\beta)\,(\lambda \to 0)$.

(ii) *For* $y \in D(B_1) = R(A_1)$, *one has* $\|(A - \lambda)^{-1}y - B_1 y\| = O(|\lambda|^\beta)\,(\lambda \to 0) \Leftrightarrow K(|\lambda|, B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(|\lambda|^\beta)\,(\lambda \to 0)$.

(iii) $R(A)$ *is not closed if and only if for each (some)* $0 < \beta < 1$ *there exists an element* $y_\beta \in \overline{R(A)}$ *such that* $\|\lambda(\lambda - A)^{-1}y_\beta\| \begin{cases} = O(|\lambda|^\beta) \\ \neq o(|\lambda|^\beta) \end{cases} (\lambda \to 0)$.

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