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RECURRENT DIMENSIONS AND DIOPHANTINE CONDITIONS OF DISCRETE DYNAMICAL SYSTEMS GIVEN BY CIRCLE MAPPINGS

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ABSTRACT. In this paper we study recurrent dimensions of discrete dynamical systems given by circle diffeomorphisms, using a renormalization method. We estimate the upper and the lower recurrent dimensions according to some algebraic properties of irrational rotation numbers of the circle mappings and we show that the gap values between the upper and the lower dimensions, which measure unpredictability levels of orbits, take positive values if the rotation numbers have good approximation properties by rational numbers.

1. INTRODUCTION

In this paper we study recurrent dimensions of discrete dynamical systems given by a circle diffeomorphism $f : S^1 \to S^1$. The rotation number of f is defined by

$$\rho(f) = \lim_{n\to\infty} \frac{\hat{f}(x) - x}{n}$$

where $\hat{f} : \mathbf{R} \to \mathbf{R}$ is a lift of f such that $\pi \circ \hat{f} = f \circ \pi$, $\pi : \mathbf{R} \to \mathbf{R}/\mathbf{Z}(= S^1)$ is a covering map. Our purpose of this paper is to estimate the recurrent dimensions of the discrete orbits $\Sigma_x = \{f^n(x) : n \in \mathbf{N}_0\}$ according to the algebraic properties of $\rho(f)$.

The following theorem by Poincaré is well known.

Theorem 1.1 (Poincaré,1885). *If $f : S^1 \to S^1$ is a homeomorphism without periodic points, then there exist a rotation $R_\alpha(x) := x + \alpha(mod\, 1)$ and a continuous surjective monotone map $h : S^1 \to S^1$, which satisfies*

$$h \circ f = R_\alpha \circ h$$

and α is an irrational number and equal to the rotation number of f. Consequently, $\rho(f)$ is independent of x.

In the case of Theorem 1.1 we say that f is semi-conjugate to the rotation R_α or h is a semi-conjugacy between f and R_α. Furthermore, if h is strictly monotone (one-to-one), we say that f is conjugate to the rotation R_α or h is a conjugacy between f and R_α.

If f is sufficiently smooth, f is conjugate to a rotation. The following theorem was given by Denjoy.

Theorem 1.2 (Denjoy,1932). *If $f : S^1 \to S^1$ is C^2-diffeomorphism without periodic points, then f is topologically conjugate to a rotation. That is, the conjugacy h between f and the rotation is a homeomorphism.*

The regularity of the conjugacy was studied by so many authors; Herman (1979), Yoccoz(1984), Khanin and Sinai(1987), Stark(1988). Here we introduce the estimate by Katznelson and Ornstein [1].

We say that g is $C^{m+\delta}$-class where $m \geq 1$ is an iteger and $0 \leq \delta < 1$, if g is C^m and its m-th derivative is Hölder continuous with its exponent δ.

Theorem 1.3 (Katznelson and Ornstein, 1989). *Let $f : S^1 \to S^1$ be a C^k-diffeomorphism, $k > 0$, without periodic points and its rotation number α satisfies the Diophantine condition for $\beta \geq 0$:*

$$|\alpha - \frac{p}{q}| > \frac{C}{q^{2+\beta}} \qquad (*)$$

for all $p/q \in \mathbf{Q}$. Then, if $\beta + 2 < k$, the conjugacy h between f and the rotation R_α is of class $C^{k-1-\beta-\varepsilon}$ for all $\varepsilon > 0$.

In our previous papers ([7], [8], [9]) we introduce the gaps of recurrent dimensions, which are differences between the upper and the lower recurrent dimensions, as the index parameters, which measure unpredictability levels of the orbits.

In view of Theorem 1.2 and 1.3 we estimate the gaps of recurrent dimensions of the discrete orbit Σ_x, given by a C^k-class function f, in the following cases.

(I) The rotation number satisfies the assumption $\beta + 2 < k$ and the conjugacy h is smooth : C^γ-class, $\gamma \geq 1$.
(II) The rotation number satisfies $2 \leq k \leq \beta + 2$ and h is a homeomorphism.

Our plan of this paper as follows. In section 2 we introduce the classifications of irrational numbers to parametrize the Diophantine condition $(*)$ and give defintions of recurrent dimensions. In section 3 we estimate the gaps of recurrent dimensions in the case (I) and in section 4 we treat the case (II). In section 5, introducing a renormalization technique and showing some fractal structures of the intervals given by the circle mapping, we prove some Lemmas, which are used to estimate the recurrent dimensions in section 4.

2. CLASSIFICATION OF IRRATIONAL NUMBERS

Let τ be an irrational number. In our previous papers ([5], [6], [8]) we introduce the following classifications according to (good or bad) levels of approximation by rational numbers.

We say that τ is an α-order Roth number if there exists $\alpha \geq 0$ such that, for every $\beta : \beta > \alpha$, there exists a constant $c_\beta > 0$, which satisfies

$$|\tau - \frac{q}{p}| \geq \frac{c_\beta}{p^{2+\beta}}$$

for all rational numbers $q/p \in \mathbf{Q}$.

Let $\{n_i/m_i\}$ be the Diophantine approximation of τ. Then we call τ an α-order weak Liouville number if there exists a subsequence $\{m_{k_j}\} \subset \{m_j\}$, which satisfies

$$|\tau - \frac{n_{k_j}}{m_{k_j}}| < \frac{c}{m_{k_j}^{2+\alpha}}, \quad \forall j$$

for some constants $c, \alpha > 0$.

Furthermore, we can parametrize the Diophantine condition $(*)$ as follows (see [8] for details).

Let $R(\alpha)$ be the set of α-order Roth numbers and $wL(\beta)$ the set of β-order weak Liouville numbers. Then we can show

$$R(\alpha) \subset R(\alpha'), \ \alpha \leq \alpha', \ \ wL(\beta) \subset wL(\beta'), \ \beta \geq \beta',$$

$$R(\alpha) \subset R(\alpha'), \ \alpha \leq \alpha', \ wL(\beta) \subset wL(\beta'), \ \beta \geq \beta',$$
$$R(\alpha)^c \subset \bigcap_{\beta<\alpha} wL(\beta), \quad wL(\beta) \subset \bigcap_{\beta>\alpha} R(\alpha)^c,$$
$$R(0)^c = \bigcup_{\beta>0} wL(\beta)$$

where the complements are considered in the set of all irrational numbers. Thus, for each irrational number τ, there exists a constant d_0, which specifies the levels of (bad or good) approximations by rational numbers:

$$\begin{aligned}(2.1) \qquad & \inf\{\alpha : \tau \text{ is an } \alpha\text{-order Roth number}\} \\ & = \sup\{\beta : \tau \text{ is a } \beta\text{-order weak Liouville number}\} := d_0.\end{aligned}$$

In our previous paper [7] we introduced a d_0-(D) condition for a pair of irrational numbers (For more than two irratinal numbers, see [9]). For a single irrational case, let us say that τ satisfies a d_0-(D) condition if (2.1) holds.

Definitions of recurrent dimensions:

Define the first ε-recurrent time by

$$M_\varepsilon(x) = \min\{m \in \mathbf{N} : |f^m(x) - x| < \varepsilon\}.$$

and the upper and lower recurrent dimensions by

$$\overline{D}_x = \limsup_{\varepsilon \to 0} \frac{\log M_\varepsilon(x)}{-\log \varepsilon}, \qquad \underline{D}_x = \liminf_{\varepsilon \to 0} \frac{\log M_\varepsilon(x)}{-\log \varepsilon}.$$

Then we can define the gaps of recurrent dimensions by $G_x = \overline{D}_x - \underline{D}_x$. (See [7] or [8] for further details.)

If the gap value G_x takes a positive value, we cannot exactly determine or predict the ε-recurrent time of the orbits. Thus we propose the value G_x as the parameter, which measures the unpredictability level of the orbit.

3. SMOOTH CONJUGACY CASE

In this section we consider the case where the conjugacy h between the circle map f and the rotaion is C^γ-class, $\gamma \geq 1$. First we note that the metric in S^1 is induced by the covering (quotient) map $\pi : \mathbf{R} \to S^1$ such that

$$|x - y| := \inf_{m \in \mathbf{Z}} |x - y - m|, \quad x, y \in S^1$$

where we use the same notation as that for usual absolute values as far as not being confused.

Theorem 3.1. *Let $f : S^1 \to S^1$ be a C^3-diffeomorphism without periodic points and its rotation number α satisfy the d_0-(D) condition for $0 \leq d_0 < 1$. Then, for each $x \in S^1$, we have*

$$\underline{D}_x \leq \frac{1}{1 + d_0}, \quad \overline{D}_x \geq 1.$$

Consequently, we have

$$G_x \geq 1 - \frac{1}{1 + d_0}.$$

Proof. Since the Diophantine condition $(*)$ in Theorem 1.3 is satisfied with $\beta = 1 - \varepsilon_0 > d_0$ for some sufficiently small $\varepsilon_0 > 0$, the conjugacy h is $C^{1+\varepsilon_0-\varepsilon}$-class for every $\varepsilon > 0$. Thus we can admit C^1-conjugacy $h : h \circ f = R_\alpha \circ h$. Since $f^n(x) = h^{-1} \circ R_\alpha^n \circ h$ and Lipschitz continuity conditins of h and h^{-1}, which are given by the Mean Value Theorem, such that

$$C_1 |x - y| \leq |h(x) - h(y)| \leq C_2 |x - y|, \quad x, y \in S^1 : |x - y| \leq \frac{1}{2}$$

for some $C_2 > C_1 > 0$, we can take an integer m :

$$(3.1) \qquad \begin{aligned} |f^n(x) - x| &= |h^{-1} \circ R_\alpha^n \circ h(x) - (h^{-1} \circ h)(x)| \\ &\leq C_1^{-1} |\alpha n - m|, \\ |\alpha n - m| &\leq \frac{1}{2}, \end{aligned}$$

and also an integer m' :

$$(3.2) \qquad \begin{aligned} |f^n(x) - x| &= |h^{-1} \circ R_\alpha^n \circ h(x) - (h^{-1} \circ h)(x)| \\ &\geq C_2^{-1} |\alpha n - m'|, \\ |\alpha n - m'| &\leq \frac{1}{2}. \end{aligned}$$

Let $\{q_k/p_k\}$ be the Diophantine sequence of the rotation number α of f. It follows from d_0-(D) condition that for every $\varepsilon > 0$ there exists a subsequence

$\{p_{k_j}\}$ such that

$$|\alpha p_{k_j} - q_{k_j}| \le \frac{c}{p_{k_j}^{1+d_0-\varepsilon}}, \quad \forall j.$$

Thus by (3.1) and (3.2) we have

$$|f^{k_j}(x) - x| \le \frac{cC_1^{-1}}{p_{k_j}^{1+d_0-\varepsilon}} := \varepsilon_j.$$

It follows from the definition that we can estimate lower recurrent dimension.

$$\begin{aligned} \underline{D}_x &= \liminf_{\varepsilon\to\infty} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\ &= \liminf_{j\to\infty} \inf_{\varepsilon_{j+1}\le\varepsilon\le\varepsilon_j} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\ &\le \liminf_{j\to\infty} \frac{\log M(\varepsilon_j)}{-\log\varepsilon_j} \\ &\le \lim_{j\to\infty} \frac{\log p_{k_j}}{-\log c + \log C_1 + (1+d_0-\varepsilon)\log p_{k_j}} \\ &= \frac{1}{1+d_0-\varepsilon} \end{aligned}$$

for every $\varepsilon > 0$.

Next we show the lower estimate. Here we use the following elementary property of the Dophantine sequence that

$$\frac{1}{p_k(p_{k+1}+p_k)} < |\alpha - \frac{q_k}{p_k}| < \frac{1}{p_k p_{k+1}} < \frac{1}{p_k^2} \tag{3.3}$$

and

$$\inf_{r\in\mathbf{N}} |\alpha n - r| \ge |\alpha p_k - q_k| \tag{3.4}$$

holds for every $n : 1 \le n < p_{k+1}$. It follows from (3.2) that we have

$$|f^n(x) - x| \ge C_2^{-1}|\alpha p_k - q_k| \ge \frac{1}{2C_2 p_{k+1}} := \varepsilon_k$$

for every $n : 1 \le n < p_{k+1}$. Thus we can estimate the upper recurrent dimension

$$\begin{aligned} \overline{D}_x &= \limsup_{\varepsilon\to\infty} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\ &= \limsup_{k\to\infty} \sup_{\varepsilon_{k+1}\le\varepsilon\le\varepsilon_k} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\ &\ge \limsup_{k\to\infty} \frac{\log M(\varepsilon_k)}{-\log\varepsilon_k} \\ &\ge \lim_{k\to\infty} \frac{\log p_{k+1}}{\log 2C_2 + \log p_{k+1}} = 1 \end{aligned}$$

and from the definition of the gap values we obtain the conclusion. □

4. TOPOLOGICAL CONJUGATE CASE

(Hereafter we show some Lemmas and Theorems without proofs, which will be given in a forthcoming paper.)

Next we consider the case (II). f has a unique invariant probability measure μ, defined by $\mu(A) = \lambda(h(A))$ where h is the conjugacy between f and the rotation and λ is a Lebesgue measure.

Let $\{q_k/p_k\}$ be the Diophantine sequence of the rotation number α of f and denote

$$m_k(x) = |f^{p_k}(x) - x|,$$
$$\alpha_k = |p_k\alpha - q_k|,$$

then we consider the subsets $A,\ B$ of S^1, defined by

$$A = \{x \in S^1 : \limsup_{k\to\infty} \frac{m_k(x)}{\alpha_k} > 0\},$$
$$B = \{x \in S^1 : \liminf_{k\to\infty} \frac{\alpha_k}{m_k(x)} > 0\}.$$

We note that

$$\alpha_k = \int_{S^1} m_k(x)d\mu(x) \tag{4.1}$$

(see [3]).

We can estimate the measure of these subsets:

Lemma 4.1. *Let $f : S^1 \to S^1$ be a C^2-diffeomorphism. Then we have*

$$\lambda(A) = \lambda(B) = 1. \tag{4.2}$$

Remark 4.2. It is known that the circle mapping f is conjugate to an irrational rotation if and only if its minimal invariant set (a non-empty compact invariant set which is minimal) is equal to S^1. Thus we can easily show that the invariant subsets A, B are dense in S^1.

Theorem 4.3. *Let $f : S^1 \to S^1$ be a C^2-diffeomorphism without periodic points and its rotation number α. Then we have*

$$\overline{D}_x \geq 1, \quad a.e.\ \ x \in S^1. \tag{4.3}$$

Theorem 4.4. *Let $f : S^1 \to S^1$ be a C^2-diffeomorphism without periodic points and its rotation number α satisfy the d_0-(D) condition for $d_0 > 0$. Then we have*

$$\underline{D}_x \leq \frac{1}{1+d_0}, \quad a.e.\ \ x \in S^1. \tag{4.4}$$

Consequently, we can estimate the gap values by

$$G_x \geq \frac{d_0}{1+d_0}, \quad a.e.\ \ x \in S^1. \tag{4.5}$$

5. Renormalization method

In this section, applying some renormalizatin techniques (cf. [3] or [10], here we use the notations in [3]), we prove some Lemmas, one of which is used in the proof of Theorem 4.3 and we show some fractal structures of the intervals given by the circle mapping.

Here, let $g : S^1 \to S^1$ be an orientation preserving homeomorphism without fixed points, then we note that g can be identified with a map $\bar{g} : [0,1] \to [0,1]$ such that $\bar{g}(0) = \bar{g}(1)$ and there exists a unique point $c \in (0,1)$, which satisfies that $\bar{g}$ is continuous and monotone increasing on $[0,c)$ and $(c,1]$ and

$$\lim_{t \uparrow c} \bar{g}(t) = 1, \quad \lim_{t \downarrow c} \bar{g}(t) = 0.$$

Denote $[a,b]$, a closed interval in $[0,1]$, by J and define the space $\mathcal{S}(J)$ of maps $g : J \to J$ such that $g(a) = g(b) \in (a,b)$ and there exists a unique discontinuity point $c \in (a,b)$, which satisfies that g is continuous and monotone increasing on $[a,c)$ and $(c,b]$ and

$$\lim_{t \uparrow c} g(t) = b, \quad \lim_{t \downarrow c} g(t) = a.$$

For $J = [a,b]$, denote $J' = (a,c)$, $J'' = (c,b)$, then we can admit the following two cases: (i) $g(J') \subset J''$, $J' \subset g(J'')$, (ii) $g(J'') \subset J'$, $J'' \subset g(J')$.

First we consider the case (i). For $g \in \mathcal{S}(J)$, since g does not have a fixed point and g is monotone increasing on J', J'', we define

$$a(g) = \max\{k \in \mathbf{N} : \ g^i(J') \subset J'' \ \text{ for all } \ i = 1, \dots, k\}.$$

Then we have the ordered intervals $J', g(J'), \dots, g^{a(g)}(J')$, each of which has one common boundary point with the next one, and $g^{a(g)+1}(J') \cap J' \neq \emptyset$. Furthermore, the closure of $J' \cup g(J') \cup \cdots \cup g^{a(g)+1}(J')$ covers the closed interval J.

We define the first return map $\mathcal{R}(g) : K \to K$ of g to K for an interval $K \subset J$ by $\mathcal{R}(g)(x) = g^k(x)$ where $k = k(x) = \min\{i \in \mathbf{N} : \ g^i(x) \in K\}$.

Denote $J(g) = \overline{J' \cup g^{a(g)+1}(J')}$, then we can see that $\mathcal{R}(g)$ of g to $J(g)$ is in $\mathcal{S}(J(g))$ and

$$\mathcal{R}(g)(J'' \cap J(g)) \subset J' = J' \cap J(g) \tag{5.1}$$

and

$$\mathcal{R}(g)|_{J'} = (g|_{J''})^{a(g)} \circ (g|_{J'}), \quad \mathcal{R}(g)|_{J'' \cap J(g)} = g|_{J''}. \tag{5.2}$$

For the case (ii) we can define the number $a(g)$ similarly.

Now, using the circle mapping $f : S^1 \to S^1$, which has no periodic points, we inductively define the renormalization sequences of intervals $\{J_n\}$ and of the return maps $\{\varphi_n\} : \varphi_n \in \mathcal{S}(J_n)$ and of the numbers $\{a_n\}$, which determine the continued fraction expantion for the rotation number of f.

Define $J_0 = J$, $\varphi_0 : J_0 \to J_0$, $\varphi_0 = f$ and denote the interioir of the right component of $J_0 \backslash \{c\}$ by J_0' and the interioir of the other component of $J_0 \backslash \{c\}$ by J_0'' and

$$a_1 = \begin{cases} a(f) + 1 & \text{if } \ f(J_0') \subset J_0'' \\ 1 & \text{if } \ J_0' \supset f(J_0''), \end{cases}$$

$$J_1 = \begin{cases} J(\varphi_0) & \text{if } f(J_0') \subset J_0'' \\ J & \text{if } J_0' \supset f(J_0''), \end{cases}$$

and

$$\varphi_1 = \begin{cases} \mathcal{R}(f) & \text{if } f(J_0') \subset J_0'' \\ f & \text{if } J_0' \supset f(J_0''). \end{cases}$$

Now suppose that $n \geq 2$ and $J_1, ..., J_{n-1}, \varphi_1, ..., \varphi_{n-1}$ are defined and that $\varphi_{n-1} : J_{n-1} \to J_{n-1}$ has no fixed points, then we inductively define the interval J_n, the return map φ_n to J_n and the integer a_n by

$$J_n = J(\varphi_{n-1}), \quad \varphi_n = \mathcal{R}(\varphi_{n-1}) : J_n \to J_n, \quad a_n = a(\varphi_{n-1}).$$

On the other hand, if $\varphi_{n-1} : J_{n-1} \to J_{n-1}$ has fixed points, then we must let $a_n = +\infty$ and stop the inductive process, but, since we assume that f has no periodic points, that is, φ_n has no fixed points, we can define each sequence infinitely.

Thus, if $f(J') \subset J''$,

$$a_1 = a(f) + 1, \quad \varphi_1 = \mathcal{R}(f),$$
$$a_n = a(\mathcal{R}^{n-1}(f)), \quad \varphi_n = \mathcal{R}^n(f), \quad n = 2, 3, ...$$

and, if $J' \supset f(J'')$,

$$a_1 = 1, \quad \varphi_1 = f,$$
$$a_n = a(\mathcal{R}^{n-2}(f)), \quad \varphi_n = \mathcal{R}^{n-1}(f), \quad n = 2, 3, ...$$

Now we use the following notations: Let J_n' be the interioir of the left component of $J_n \backslash \{c\}$ if n is odd and of the right component if n is even. Denote the interioir of the other component of $J_n \backslash \{c\}$ by J_n''. Then we have

$$J_n' = J_{n-1}'' \cap J_n, \quad J_n'' = J_{n-1}' \cap J_n = J_{n-1}'$$

and $\varphi_n(J_n') \subset J_n''$ for all $n \geq 1$. Also we have $\varphi_1|_{J_1'} = f, \ \varphi_1|_{J_1''} = f^{a_1}$ and it follows from (5.1) and (5.2) that we have

$$\varphi_n|_{J_n'} = \varphi_{n-1}|_{J_{n-1}''},$$
$$\varphi_n|_{J_n''} = (\varphi_{n-1}|_{J_{n-1}''})^{a(\varphi_{n-1})} \circ (\varphi_{n-1}|_{J_{n-1}'}).$$

Therefore by induction we have

$$\varphi_n|_{J_n'} = f^{p_{n-1}}, \quad \varphi_n|_{J_n''} = f^{p_n}$$

where p_n is defined inductively by

$$p_0 = 1, \quad p_1 = a_1,$$
$$p_{n+1} = a_{n+1}p_n + p_{n-1} \quad \text{for} \quad n \geq 1.$$

Since $\varphi_n : J_n \to J_n$ is in $\mathcal{S}(J_n)$, we have

$$J_n = [f^{p_{n-1}}(c), f^{p_n}(c)], \quad J_n' = (c, f^{p_n}(c)), \quad J_n'' = (f^{p_{n-1}}(c), c)$$

if n is even and similarly we obtain the intervals

$$J_n = [f^{p_n}, f^{p_{n-1}}(c)], \quad J_n' = (f^{p_n}(c), c), \quad J_n'' = (c, f^{p_{n-1}}(c))$$

if n is odd.

We can show the fractal tiling structures of the intervals by using the renormalization method.

Lemma 5.1. *Under the above setting we obtain for* $0 \le j \le p_{n+1}$ *that*

$$f^j(c) \in J_n \Longleftrightarrow j = ip_n + p_{n-1}, \quad i \in \{0, ..., a_{n+1}\} \tag{5.3}$$

and also we obtain the fractal tiling structures of the interval J *such that*

$$J = cl[\{ \bigcup_{i=0}^{p_{n-1}-1} f^i(J'_n)\} \bigcup \{ \bigcup_{i=0}^{p_n-1} f^i(J''_n)\}] \tag{5.4}$$

where all intervals in the union are disjoint.

Applying Lemma 5.1 and also using the renormalization method, we can show the following lemma.

Lemma 5.2. *Under the same Hypotheses as Theorem 4.3 there exists a constant* $b_0 : 0 < b_0 < 1$ *such that*

$$|f^j(x) - x| \ge b_0 m_n(x)$$

holds for every $j < p_{n+1}$.

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