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# Nonlinear mappings and the theory of reproducing kernels

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**Abstract**

In this lecture, the author gave a survey on *nonlinear* from the viewpoint of the theory of reproducing kernels based on his works.

*Keywords:* Nonlinear ordinary differential equation, Pythagorean theorem, inverse of a family of matrices, inverse of a family of bounded linear operators, tensor product of reproducing kernel Hilbert spaces, nonlinear mapping, reproducing kernel, representation of inverse function, operator equation, generalized inverse, Tikhonov regularization
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## 1 Nonlinear ordinary differential equations

At first, as a news in nonlinear ordinary differential equations, we referred to the recent paper [15]. In general, for nonlinear ordinary differential equations with variable coefficients, we can give analytical and general solutions for very restricted equations only. In [15], we found a large class of nonlinear ordinary differential equations with variable coefficients of the first order for which we can give analytical and general solutions by simple transforms. Furthermore, we can determine such class of differential equations. For further generalizations and for the case of the second order ordinary differential equations, see [17,16].

We shall state typical examples from [17]:

$$T_y(x,y)y' = -T_x(x,y) \tag{1.1}$$

$\Leftrightarrow$

$$T(x,y) = C. \tag{1.2}$$

This will mean that our theory is a generalization of exact differential equations.

$$T_y(x,y)y' = F(x) - T_x(x,y) \tag{1.3}$$

$\Leftrightarrow$

$$T(x,y) = \int F(x)dx + C. \tag{1.4}$$

$$T_y(x,y)y' = F(x)T(x,y) - T_x(x,y) \tag{1.5}$$

$\Leftrightarrow$

$$T(x,y) = C\exp\left\{\int F(x)dx\right\}. \tag{1.6}$$

$$T_y(x,y)y' = F(x)T(x,y)^2 - T_x(x,y) \tag{1.7}$$

$\Leftrightarrow$

$$T(x,y) = \frac{-1}{\int F(x)dx + C}. \tag{1.8}$$

$$T_y(x,y)y' = F(x)e^{\alpha T(x,y)} - T_x(x,y), \quad \alpha \neq 0 \tag{1.9}$$

$\Leftrightarrow$

$$T(x,y) = -\frac{1}{\alpha}\ln\left\{-\alpha\int F(x)dx + C\right\}. \tag{1.10}$$

$$T_y(x,y)y' = F(x)\left[a^2 - T(x,y)^2\right] - T_x(x,y), \quad a > 0 \tag{1.11}$$

$\Leftrightarrow$

$$\frac{1}{2a}\log\left|\frac{a+T(x,y)}{a-T(x,y)}\right| = \int F(x)dx + C. \tag{1.12}$$

$$T_y(x,y)y' = F(x)\left[T(x,y)^2 - a^2\right] - T_x(x,y), \quad a > 0 \tag{1.13}$$

$\Leftrightarrow$

$$\frac{1}{2a}\log\left|\frac{a - T(x,y)}{a + T(x,y)}\right| = \int F(x)dx + C. \tag{1.14}$$

$$T_y(x,y)y' = F(x)\left[T(x,y)^2 + a^2\right] - T_x(x,y), \quad a > 0 \tag{1.15}$$

$\Leftrightarrow$

$$T(x,y) = a\tan\left(a\int F(x)dx + C\right). \tag{1.16}$$

$$T_y(x,y)y' = F(x)\sin T(x,y) - T_x(x,y) \tag{1.17}$$

$\Leftrightarrow$

$$\tan\frac{1}{2}T(x,y) = C\exp\left\{\int F(x)dx\right\}. \tag{1.18}$$

$$T_y(x,y)y' = F(x)\cos T(x,y) - T_x(x,y) \tag{1.19}$$

$\Leftrightarrow$

$$\tan\left(\frac{1}{2}T(x,y) + \frac{\pi}{4}\right) = C\exp\left\{\int F(x)dx\right\}. \tag{1.20}$$

$$T_y(x,y)y' = F(x)\tan T(x,y) - T_x(x,y) \tag{1.21}$$

$\Leftrightarrow$

$$\sin T(x,y) = C\exp\left\{\int F(x)dx\right\}. \tag{1.22}$$

Of course, we can easily solve these nonlinear differential equations, however, for the general form $y' = f(x,y)$ we can determine such class of differential equations and we can look for the *Tada* transform $z = T(x,y)$ in order to derive such normal form. So, following our general theory, we can give the following examples:

$$y' = \frac{x^3y - y + x^2y^2}{x + 2y} \quad (z = xy + y^2) \tag{1.23}$$

$\Leftrightarrow$

$$xy + y^2 = C \exp\left\{\frac{1}{3}x^3\right\}. \tag{1.24}$$

$$y' = \frac{x^2y^2 - y^2 + xy}{2xy + 1} \quad (z = xy^2 + y) \tag{1.25}$$

$\Leftrightarrow$

$$xy^2 + y = C \exp\left\{\frac{1}{2}x^2\right\}. \tag{1.26}$$

$$y' = \frac{x(xy^2 + xy)^2 - (y^2 + y)}{2xy + x} \quad (z = xy^2 + xy) \tag{1.27}$$

$\Leftrightarrow$

$$xy^2 + xy = \frac{1}{C - \frac{1}{2}x^2}. \tag{1.28}$$

$$y' = \frac{x^4y^4 + x^2 + x^2y^4 + 1 - y^2}{2xy} \quad (z = xy^2) \tag{1.29}$$

$\Leftrightarrow$

$$xy^2 = \tan\left(\frac{1}{3}x^3 + x + C\right). \tag{1.30}$$

$$y' = \frac{x^2(e^xy^2 + y + 3) - e^xy^2}{2e^xy + 1} \quad (z = e^xy^2 + y) \tag{1.31}$$

$\Leftrightarrow$

$$e^xy^2 + y + 3 = C \exp\left\{\frac{1}{3}x^3\right\}. \tag{1.32}$$

## 2 Generalizations of Pythagorean theorem

In a generalization of Pythagorean theorem, we found a very interesting non-linearity ([8]) and from there we found a concept of isometry between a Hilbert space and various Hilbert spaces by various bounded linear operators ([13]). As special cases, we got inverses of a family of matrices ([1]) which give full generalizations of Pythagorean theorem.

# 3 Nonlinear mappings of reproducing kernel Hilbert spaces

For general non-linear mappings of a reproducing kernel Hilbert space, by the restriction and by the products of the reproducing kernel, we can discuss the non-linear mappings in connection with linear mappings. Following a series of the papers, we discussed their essential ideas with very typical examples. See [11,12,9].
Let $E$ be an arbitrary nonvoid abstract set and let $H_K(E)$ be a Hilbert ( possibly finite-dimensional ) space admitting a reproducing kernel $K(p,q)$ on $E$. Then, the Hilbert space $H_K(E)$ is composed of complex-valued functions $f(p)$ on $E$ such that

$$K(\cdot,q) \in H_K(E) \quad \text{for any fixed } q \in E$$

and, for any member $f$ of $H_K(E)$ and for any fixed point $q$ of $E$,

$$(f(\cdot), K(\cdot,q))_{H_K} = f(q).$$

In general, a reproducing kernel $K(p,q)$ on $E$ is a positive matrix in the sense that for any points $\{p_j\}_j$ of $E$ and for any complex numbers $\{C_j\}_j$

$$\sum_{j,j'} C_j \overline{C_{j'}} K(p_{j'}, p_j) \geq 0.$$

Conversely, a positive matrix $K(p,q)$ on $E$ determines uniquely a functional Hilbert space ( for brevity a reproducing kernel Hilbert space is designated by RKHS ) $H_K(E)$. In general, for a Hilbert space $H$ comprising functions $f(p)$ on $E$, there exists a reproducing kernel $K(p,q)$ for $H$ if and only if for any point $q$ of $E$, the point evaluation $f(p)$ is a bounded linear functional on $H$. This nice property will show that reproducing kernel Hilbert spaces are very good Hilbert spaces.
In connection with the analytic function

$$\sum_{n=0}^{\infty} d_n z^n, \quad d_n \text{ are complex numbers,}$$

we shall consider the RKHS $H_K(E)$ as an input function space of the nonlinear transform

$$\varphi: \quad f \in H_K(E) \longrightarrow \sum_{n=0}^{\infty} d_n(p) f(p)^n,$$

where $\{d_n(p)\}$ is a sequence of *arbitrary given* complex-valued functions on $E$.
In this nonlinear transform $\varphi$, we can see that the images $\varphi(f)$, $f \in H_K(E)$, belong to a Hilbert space $\mathbf{H}$ which is naturally determined by the nonlinear transform $\varphi$ and there exists a natural norm inequality between the two norms $\|\varphi(f)\|_{\mathbf{H}}$ and $\|f\|_{H_K}$.
In order to see these facts we need the three basic ideas; that is, sums, products and restrictions of reproducing kernels.
For two positive matrices $K_1(p,q)$ and $K_2(p,q)$ on $E$, the sum $K_3(p,q) = K_1(p,q) + K_2(p,q)$ is, of course, a positive matrix on $E$. The RKHS $H_{K_3}$ admitting the reproducing kernel $K_3(p,q)$ on $E$ is composed of all functions

$$f = f_1 + f_2 \quad (f_j \in H_{K_j})$$

and the norm in $H_{K_3}$ is given by

$$\|f\|^2_{H_{K_3}} = \min\{\|f_1\|^2_{H_{K_1}} + \|f_2\|^2_{H_{K_2}}\},$$

where the minimum is taken over all the expressions for $f$.
The product $K_4(p_1,p_2;q_1,q_2) = K_1(p_1,q_1)K_2(p_2,q_2)$ on $(E\times E)\times(E\times E)$ is, of course, a positive matrix on $E \times E$. The RKHS $H_{K_4}$ admitting the reproducing kernel $K_4(p_1,p_2;q_1,q_2)$ on $E\times E$ is composed of all functions

$$f(p_1,p_2) = \sum_{n=1}^{\infty} f_{1,n}(p_1)f_{2,n}(p_2) \quad (f_{j,n} \in H_{K_j}) \tag{3.33}$$

having finite norms

$$\|f\|^2_{H_{K_4}} = \sum_{n=1}^{\infty} \|f_{1,n}\|^2_{H_{K_1}} \|f_{2,n}\|^2_{H_{K_2}} < \infty. \tag{3.34}$$

That is, the RKHS $H_{K_4}$ is the tensor product $H_{K_1} \otimes H_{K_2}$. In particular, note that for $f_1 \in H_{K_1}, f_2 \in H_{K_2}$, the product $f_1(p_1)f_2(p_2) \in H_{K_1} \otimes H_{K_2}$ and the product is a function on $E \times E$. It is not a function on $E$ but on $E \times E$. It is not a single but two variable function. In order to catch *nonlinear transforms*, we need the idea of the restriction of reproducing kernels.
The restriction $K_5(p,q) = K_4(p,p;q,q)$ to the diagonal set $E$ of $E \times E$ is a positive matrix and the RKHS $H_{K_5}$ admitting the reproducing kernel $K_5(p,q)$ on $E$ is composed of all functions $f(p) \equiv f(p,p)$ in (3.33) satisfying (3.34). The norm in $H_{K_5}$ is given by

$$\|f\|^2_{H_{K_5}} = \min \sum_{n=1}^{\infty} \|f_{1,n}\|^2_{H_{K_1}} \|f_{2,n}\|^2_{H_{K_2}},$$

where the minimum is taken over all the expressions satisfying for $f(p) = f(p,p)$ on $E$.
In particular, note that for $f_1 \in H_{K_1}$, in the typical nonlinear transform

$$f_1 \longrightarrow f_1^2,$$

$f_1^2$ belongs to the reproducing kernel Hilbert space $H_{K_1^2}$ admitting the reproducing kernel $K_1(p,q)^2$ and we have the norm inequality

$$\|f_1^2\|^2_{H_{K_1^2}} \leq (\|f_1\|^2_{H_{K_1}})^2.$$

This is a key idea to understand *nonlinear transforms,* because we were able to identify the images $f_1^2$; that is, we were able to find a space containing the images. Further, the space is a natural one in the sense that the reproducing kernel Hilbert space $H_{K_1^2}$ is spanned by the typical nonlinear images $K_1(p,q)^2$ of the typical members $K_1(p,q)$ of $H_{K_1}$ for $q \in E$. Furthermore, note that in the above inequality, equality holds for the functions $K_1(p,q)$ for any point $q$ of $E$.
For $n$-times sum and $n$-times product, the circumstances are similar. Hence, we have, in particular, for any $f_j \in H_{K_j}$ $(j = 1, 2, ..., N)$

$$\|\sum_{j=1}^{N} f_j\|^2_{H_{(\sum_{j=1}^{N} K_j)}} \leq \sum_{j=1}^{N} \|f_j\|^2_{H_{K_j}}$$

and

$$\|f^n\|^2_{H_{K^n}} \leq \|f\|^{2n}_{H_K}.$$

One typical example is given as follows:
For any absolutely continuous function $f$ on $[0,1]$ satisfying

$$0 < \int_0^1 f'(x)^2 dx < 1$$

and $f(0) = 0$,

$$\int_0^1 \left(\frac{f(x)}{1 - f(x)}\right)'^2 (1-x)^2 dx \leq \frac{\int_0^1 f'(x)^2 dx}{1 - \int_0^1 f'(x)^2 dx}.$$

It will be very pleasant to note that for functions $\min(x,y)(0 < y < 1)$, equality holds.

# 4 Representations of inverses of an arbitrary mapping

Of course, to represent the inverse of a nonlinear mapping in terms of the nonlinear mapping will be essentially involved and difficult, however, we discussed the general representation of inverse of an arbitrary mapping, by using the theory of reproducing kernels. Such challenge seems to be absured, however, surprisingly enough, in the procedure, we were able to obtain new, definite and concrete results. See [10].
One typical example is: For any positive real number $n$

$$x^{1/n} = \frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{\cos(\xi^n t)\ \sin xt}{t} dt d\xi.$$

# 5 Applications to the Tikhonov regularization

At the last part of the lecture, based on the recent research topics in [2-7, 14], we reported the applications of the general theory of reproducing kernels to the theory of Tikhonov regularization which has basic applications to various operator equations for numerical analysis and to many inverse problems. In particular, for the extremal functions in the Tikhonov regularization, we can obtain good and concrete representations by using the theory of reproducing kernels. We also gave numerical experiments for some concrete problems.

# References

[1] M. Asaduzzaman and S. Saitoh, *Inverse of a family of matrices and generalizations of Pythagorean theorem*, PanAmerican Math. J. **12**(2003), 45-53.

[2] T. Matsuura and S. Saitoh, *Analytical and numerical solutions of the inhomogeneous wave equation*, **1**(2004), Article 7, 1-18.

[3] T. Matsuura and S. Saitoh, *Dirichlet's principle using computers*, Applicable Analysis (to appear).

[4] T. Matsuura and S. Saitoh, *Numerical inversion formulas in the wave equation*, J. of Computational Mathematics and Optimization (to appear).

[5] T. Matsuura and S. Saitoh, *Analytical and numerical solutions of linear ordinary differential equations with constant coefficients*, J. of Analysis and Applications (to appear).

[6] T. Matsuura, S. Saitoh and D.D. Trong, *Numerical solutions of the Poisson equation*, **83**(2004), 1037-1051.

[7] T. Matsuura, S. Saitoh and D.D. Trong, *Approximate and analytical inversion formulas in heat conduction on multidimensional spaces*, J. of Inverse and Ill-posed Problems (to appear).

[8] T. M. Rassias and S. Saitoh, *The Pythagorean theorem and linear mappings*, PanAmerican Math. J. **12**(2002), 1-20.

[9] S. Saitoh, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Res. Notes in Math. Series **369**, Addison Wesley Longman Ltd (1997), UK.

[10] S. Saitoh, *Representations of inverse functions*, Proc. Amer. Math. Soc. **125**(1997), 3633-3639.

[11] S. Saitoh, *Natural norm inequalities in nonlinear transforms*, General Inequalities **7**(1997), 39-52.

[12] S. Saitoh, *Nonlinear transforms and analyticity of functions*, Nonlinear Mathematical Analysis and Applications, Hadronic Press. Palm Harbor, 1998, 223-234.

[13] S. Saitoh, *Inverse of a family of bounded linear operators, generalized pythagorean theorems and reproducing kernels*, Ill-Posed and Non-classical Problems of Mathematical Physics and Analysis (eds. M.M. Lavrent'ev and S.I. Kabanikhin), VSP(2003), 125-141.

[14] S. Saitoh, *Approximate Real Inversion Formulas of the Gaussian Convolution*, Applicable Analysis, **83**(2004), 727-733.

[15] T. Tada and S. Saitoh, *A method by separation of variables for the first order nonlinear ordinary differential equations*, J. Analysis and Applications, **2**(2004), 51-63.

[16] T. Tada and S. Saitoh, *A method by separation of variables for the second order ordinary differential equations*, International J. of Math. Sci. (to appear).

[17] T. Tada and S. Saitoh, *A method by transforms of variables for the first order nonlinear ordinary differential equations*, Far East J. of Math. Sci. (to appear).

(The author Takeo Tada of [15,16,17] has been concentrated only in such research topics over 30 years without other works and interest. I think he was able to obtain definite results that should be studied by almost all students in mathematical sciences and in the first course studying ordinary differential equations. If so, he will feel happily his long endurance and dream were fruitful. I am, indeed, studying a lot of things for human beings from my pure and lovely colleague.)