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Kyoto University
A conjugate-set game induced from the conjugate point

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Abstract In some nonlinear diffusive phenomena, e.g., grain growth in annealing pure metal and segregation between biological species, the systems have three or more stable states, see the left picture of Fig.1. Sternberg and Zeimer(Ref. 1) established the existence of local minimizers to the problem of partitioning certain domain $\Omega \subset R^2$ into three subdomains having least interfacial area. Further, Ikota and Yanagida investigated stability for stationary curves with one triple junction in Ref. 2 and stability for stationary binary-tree type interfaces in Ref. 3. In this paper, we consider a static version of their diffusion problem, which is formulated as an unconstrained nonlinear programming problem. We consider second-order optimality conditions and discuss stability and instability for stationary curves with one or two triple junctions. The great difference between the previous researches and ours is that our main concern is not stability but instability. We give a new insight to this problem from the viewpoint of game theory.

1 Introduction

In this paper, we consider a static version of the diffusion problem, see the right picture of Fig.1. It is formulated as follows. Let $\Omega$ be a bounded domain in $R^2$ with a smooth boundary, $X_i = (x_i, y_i)$ ($i = 1, 2$) be in the interior of $\Omega$, and $X_i$ ($i = 3, 4, 5, 6$) be on the boundary $\partial \Omega$. Then our problem is to minimize the sum of the lengths of five line segments. We call this extremal problem the 4-phase partition problem.

Figure 1: Dynamic version and static version.

By using arclength parameters, it can be formulated as an unconstrained nonlinear programming problem with eight variables. The main purposes of this paper are to discuss stability and instability of stationary solutions for the extremal problem in terms

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of the curvatures of the boundary and to give a new insight to the instable case from the viewpoint of the cooperative game.

This paper is organized as follows. In Section 2, we give a first-order optimality condition for the 4-phase partition problem. In Section 3, we analyze stability and instability of the stationary solution for the 4-phase partition problem. They are stated in terms of the curvature of the boundary. In Section 4, we define strict conjugate sets and a cooperative game that is called the conjugate-set game. In Section 5, we compute the Shapley value and the core for the 4-phase partition problem.

2 First-order optimality conditions

In this section, we give first-order necessary optimality conditions for the 4-phase partition problem. First, we deal with the nondegenerate case $X_1 \neq X_2$. Since $X_i$'s on the boundary can be locally represented as $X_i(s_i) = (x_i(s_i), y_i(s_i))$, where $s_i$ denotes an arclength parameter, the objective function is given by

$$f(x_1, y_1, x_2, y_2, s_3, \ldots, s_6) := ||X_2 - X_1|| + \sum_{i=3}^{4} ||X_i(s_i) - X_1|| + \sum_{i=5}^{6} ||X_i(s_i) - X_2||,$$

where $||\cdot||$ denotes the Euclidean norm.

**Theorem 2.1** If $(X_1, \ldots, X_6)$ is a nondegenerate local minimizer of the 4-phase partition problem, then, (a) the angle between any two adjacent line segments equals $2\pi/3$ and (b) any line segment $X_iX_j$ with $X_i \in \text{int} \Omega$ and $X_j \in \partial \Omega$ orthogonally intersects the boundary.

By using the following lemma, we see that $X_1 \neq X_2$ for any local minimizer $(X_1, \ldots, X_6)$.

**Lemma 2.1** Let $f^1(x_1, y_1, x_2, y_2) := ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}$. Then

$$\partial f^1(x_1, y_1, x_1, y_1) = \{(-\xi, -\eta, \xi, \eta); \xi^2 + \eta^2 \leq 1\}.$$

3 Stability and instability

In this section, we discuss stability and instability of stationary solutions for the 4-phase partition problem. By virtue of Theorem 2.1, there is no loss of generality if we assume that the stationary solution is given by

$$X_1 = (0, 0), \quad X_2 = \ell_2(-1, 0), \quad X_3 = \ell_3(1/2, -\sqrt{3}/2), \quad X_4 = \ell_4(1/2, \sqrt{3}/2),$$

$$X_5 = \ell_5(-1/2, \sqrt{3}/2) + X_2, \quad X_6 = \ell_6(-1/2, -\sqrt{3}/2) + X_2.$$  \hspace{1cm} (2)

Hence, from the transversality condition (b) of Theorem 2.1, we may assume that the tangent vectors at $X_k$'s, say $X_k'$, are given by

$$X_1' = (\sqrt{3}/2, 1/2), \quad X_4' = (-\sqrt{3}/2, 1/2),$$
\[ X'_5 = (-\sqrt{3}/2, -1/2), \quad X'_6 = (\sqrt{3}/2, -1/2). \]

As is well known, if the Hesse matrix of the objective function \( A := f''(x_1, \ldots, s_6) \) is positive definite, then \((X_1, \ldots, X_6)\) is a local minimizer. According to Sylvester's criterion, \( A \) is positive definite if and only if its leading principal minors are all positive. Let \( A_k \) denote the \( k \)-th leading principal submatrix of \( A \). Then the first four leading principal minors are always positive, and

\[
|A_3| = 9(1 + h_3 \ell)/16L, \quad |A_6| = 9(h_3 + h_4 + h_3h_4 \ell)/16L, \\
|A_7| = 9(h_4h_5 + h_3h_5 + h_3h_4 + h_3h_4h_5 \ell)/16L, \\
|f''| = |A_8| = 9(h_4h_3h_6 + h_3h_5h_6 + h_3h_4h_5 + h_3h_4h_5h_6 \ell)/16L,
\]

where

\[
\ell_i := \begin{cases} ||X_i - X_1||, & i = 2, 3, 4, \\ ||X_i - X_2||, & i = 1, 5, 6, \end{cases}
\]

\( h_k := X'_k X''_k / ||X_k|| \) denotes the curvature of the boundary at \( X_k, \ell := \ell_2 + \ell_3 + \ldots + \ell_6 \) and \( L := \ell_2 \ell_3 \cdots \ell_6 \).

**Theorem 3.1** The Hesse matrix of \( f \) at a stationary solution is positive definite if and only if \( D_5 := 1 + h_3 \ell, \ D_6 := h_3 + h_4 + h_3h_4 \ell, \ D_7 := h_4h_5 + h_3h_5 + h_3h_4 + h_3h_4h_5 \ell, \) and \( D_8 := h_4h_3h_6 + h_3h_5h_6 + h_3h_4h_6 + h_3h_4h_5 + h_3h_4h_5h_6 \ell \) are positive.

**Theorem 3.2** If at least two of \( h_k \)'s are negative, then the stationary solution is instable, that is, at least one of the leading principal minors is negative.

**Theorem 3.3** When just one of \( h_k \)'s is zero, \( f'' \) is positive definite if and only if three other \( h_k \)'s are positive.

**Theorem 3.4** When one \( h_k \) is negative and others are positive, \( f'' \) is positive definite if and only if

\[
h_k > - \prod_{j \neq k} h_j \left( \sum_{i \neq k, j \neq k} h_i h_j + \ell \prod_{j \neq k} h_j \right). \tag{3}
\]

**Theorem 3.5** When at least two of \( h_k \)'s are zero and others are positive, \( f'' \) is nonnegative definite.

Table 1 summarizes Theorem 3.1- Theorem 3.5. The numbers 0, 1, \ldots, 4 in the first column (row) stand for the number of negative (positive) curvatures, respectively. "S" means stable, that is, \( f'' \) is positive definite. "IS" means instable, that is, a certain principal minor of \( f'' \) is negative. "+0" means \( f'' \) is nonnegative definite. * corresponds to Theorem 3.4.
Table 1: Stability and instability of the 4-phase partition problem.

<table>
<thead>
<tr>
<th>$h_k &lt; 0$</th>
<th>$h_k &gt; 0$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>+0</td>
<td>+0</td>
<td>+0</td>
<td>$S$</td>
<td>$S$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$IS$</td>
<td>$IS$</td>
<td>$IS$</td>
<td>$*$</td>
<td>$x$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$IS$</td>
<td>$IS$</td>
<td>$IS$</td>
<td>$x$</td>
<td>$x$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$IS$</td>
<td>$IS$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
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</tr>
</tbody>
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Game theory provides mathematical tools to analyze cooperative games. In this section, we use the Shapley value and the core to evaluate the contribution of each variable to decreasing the objective function. Namely, they answer the question: How much does each variable of the strict conjugate set contribute to decrease the objective function?

The Shapley value of player $i$ for the cooperative game with the characteristic function $v$ is defined by

$$\phi_i(v) = \sum_{i\in S} \{v(S) - v(S - \{i\})\} n!/(n - s)!/(s - 1)!,$$  \hfill (4)

where $s$ denotes the cardinal number of $S$. Then it holds that $\phi_1 + \cdots + \phi_n = v(N)$, see e.g. Aumann and Hart (Ref. 11). Since the conjugate-set game is superadditive, the core is defined as follows.

$$\{x \in R^n \mid \sum_{i \in S} x_i \geq v(S) \forall S \subset N, \ x \geq 0\}.$$

In the next two sections, we compute the Shapley value and the core for the 4-phase and 3-phase partition problems.

5 Game theoretic aspect of the 4-phase partition problem

In this section, we consider a special case of the 4-phase partition problem that $l_k \equiv 1$ and $h := h_3 = h_4 = h_5 = h_6 < 0$. We first list off the strict conjugate sets. Next, we compute the Shapley value (4) and the core (5). In the following examples, for a vector $(v_1, \ldots, v_m)$ and a constant $c \neq 0$, we simply denote $(v_1/c, \ldots, v_m/c)$ by $(v_1, \ldots, v_m)/c$.

Figure 2: $l_k$'s and $h_k$'s are constant.

Although this problem has eight variables $x_1, y_1, \ldots, s_5$ and $s_6$, we regard two variables $x_k$ and $y_k$ ($k = 1, 2$) as a pair $X_k$ in this section. Because, the conclusion delivered from the former way depends on the coordinates. Then, the (instable) stationary point (2) reduces to

$$X_1 = (0, 0), \ X_2 = (-1, 0), \ X_3 = (1/2, -\sqrt{3}/2), \ X_4 = (1/2, \sqrt{3}/2), \ X_5 = (-3/2, \sqrt{3}/2), \ X_6 = (-3/2, -\sqrt{3}/2).$$
Example 5.1 We consider the case of $h = -1/4$. Then, since the principal minors of the Hesse matrix corresponding to two points $X_i$ and $X_j$ ($i \neq j$) are positive, any subset $\{X_i, X_j\}$ is not strict conjugate set. It is easily seen by computing principal minors that minimal strict conjugate sets have forms of $\{X_1, X_2, X_3\}$ or $\{X_2, X_5, X_6\}$ up to symmetry. Then the Shapley value is given by

$$ \phi := (\phi_{X_1}, \phi_{X_2}, \phi_{X_3}, \phi_{X_4}, \phi_{X_5}, \phi_{X_6}) = (7, 7, 4, 4, 4, 4)/15. $$  \(6\)

Next, let us compute the core. Following the standard notation on the core, we denote by $x_k$ the imputation of player $X_k$ $(k = 1, \ldots, 6)$. Since $x_1 + \cdots + x_6 = v(\{X_1, \ldots, X_6\}) = 2$, $x_1 + x_3 + x_4 \geq 1$, and $x_2 + x_5 + x_6 \geq 1$, we have $x_1 + x_3 + x_4 = x_2 + x_5 + x_6 = 1$. On the other hand, since $x_1 + x_2 + x_3 \geq 1$, we get $x_2 \geq x_4$. Similarly, we have $x_2 \geq x_3$ and $x_1 \geq \max\{x_5, x_6\}$. So the core is given by

$$ \{x \geq 0 | x_1 + x_3 + x_4 = x_2 + x_5 + x_6 = 1, \ x_1 \geq x_5 \lor x_6, \ x_2 \geq x_3 \lor x_4\}, $$

where $a \lor b := \max\{a, b\}$, and the Shapley value $\phi$ belongs to the core.

6 3-phase partition problem

It is easy to discuss stability and instability of stationary solutions for the 3-phase partition problem, see Fig. 4. So, we only describe the conclusion.

In the following, let $f := \sum_{k=2}^{4} ||X_k - X_1||$, $\ell_k := ||X_k - X_1||$, and $\ell := \ell_2 + \ell_3 + \ell_4$. 

Theorem 6.1 (Kawasaki, Ref. 12) If \((X_1, \ldots, X_4)\) is a local minimizer of the 3-phase partition problem, then (a) the angle between any two adjacent line segments equals \(2\pi/3\) and (b) any line segment \(X_1X_3\) with \(X_3 \in \partial \Omega\) orthogonally intersects the boundary.

Theorem 6.2 (a) The Hesse matrix of \(f\) at a stationary solution is positive definite if and only if \(1 + h_2\ell, h_2 + h_3 + h_2h_3\ell, h_3h_4 + h_2h_4 + h_2h_3h_4\ell\) are positive. (b) If at least two of \(h_k\)'s are negative, then the stationary solution is instable. (c) When just one of \(h_k\)'s is zero, \(f''\) is positive definite if and only if other \(h_k\)'s are positive. (d) When \(h_k\) is negative, and \(h_i\) and \(h_j\) are positive, \(f''\) is positive definite if and only if

\[ h_k > -h_ih_j/(h_i + h_j + h_ih_j\ell). \]  

(7) 

(e) When at least two of \(h_k\)'s are zero and the other is positive, \(f''\) is nonnegative definite.

References