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PROPERLY EFFICIENT POINTS IN SET-VALUED ANALYSIS

集合値解析における真性有効点

黒岩 大史 (DAISHI KUROIWA)

島根大学総合理工学部数理・情報システム学科

1. INTRODUCTION AND PRELIMINARIES

Let E be a locally convex topological vector space over the real number field $\mathbb{R}$, K be a convex cone of E, and assume that K is pointed and closed. By using K, we can define a vector ordering $\leq_K$ on E;

$$x \leq_K y \quad \overset{\text{def}}{\Longleftrightarrow} \quad y - x \in K.$$

In this situation, we can consider notions of efficiency and proper efficiency in vector optimization; for a nonempty subset A of E,

- $x \in A$ is said to be a minimal point of A with respect to $\leq_K$ if $a \leq_K x$ for some $a \in A$, then $x \leq_K a$; the set of all minimal points of A with respect to $\leq_K$ is denoted by $\mathrm{Min}(A \mid \leq_K)$;
- $x \in A$ is said to be a properly minimal point of A with respect to $\leq_K$ if there exists a convex cone $L \subsetneq E$ such that $K \subset \mathrm{int}L \cup \{\theta\}$ and x is a minimal point of A with respect to $\leq_L$; the set of all properly minimal points of A with respect to $\leq_K$ is denoted by $\mathrm{PrMin}(A \mid \leq_K)$,

where θ means the null vector of E, and $\mathrm{int}L$ the set of all interior points of L.

When we consider efficiency in set-valued optimization, there are two criteria; one is for vector optimization, which is the typical one, see [2], and the other is for set optimization, which is defined and researched recently, see [3, 4].

In this paper we introduce notions of proper efficiency for set optimization, and investigate them by an embedding idea. In section 2, we consider two binary relations on certain families, and define notions proper efficiency based on these relations. Also we characterize these relations by using positive polar cone. In section 3, to show an embedding theorem, we construct a vector space, and introduce a metric on the space which consists an adequate topology.

2. A NOTION OF PROPER EFFICIENCY IN SET-VALUED OPTIMIZATION

We consider notions of efficiency for set-valued optimization in the sense of set optimization. Let $\mathcal{C}(E)$ be the family of all nonempty compact convex sets

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in E; we define two binary relations $\leq_K^l$ and $\leq_K^u$ on $\mathcal{C}(E)$ as follows: for A, $B \in \mathcal{C}(E)$,

$$A \leq_K^l B \overset{\text{def}}{\iff} A + K \supset B, \qquad A \leq_K^u B \overset{\text{def}}{\iff} A \subset B - K$$

see [3, 4]. We have a result concerned with these relations by using the positive polar cone of K; let K^+ be the positive polar cone, that is,

$$K^+ = \{x^* \in E^* \mid \langle x^*, x\rangle \geq 0, \forall x \in K\}.$$

Proposition 1. *For each A, $B \in \mathcal{C}(E)$, the following three assertions are equivalent:*

(1) $A \leq_K^l B$,
(2) $A + C \leq_K^l B + C$ *for some* $C \in \mathcal{C}(E)$,
(3) $\inf \langle y^*, A\rangle \leq \inf \langle y^*, B\rangle$ *for all* $y^* \in K^+$,

and also the following three assertions are equivalent:

(4) $A \leq_K^u B$,
(5) $A + C \leq_K^u B + C$ *for some* $C \in \mathcal{C}(E)$,
(6) $\sup \langle y^*, A\rangle \leq \sup \langle y^*, B\rangle$ *for all* $y^* \in K^+$.

Note that Proposition 1 holds when K is a nontrivial closed convex cone of E.

Now we introduce notions of efficiency on $\mathcal{C}(E)$, see [3, 4]. Let $\mathcal{A}$ be a nonempty subfamily of $\mathcal{C}(E)$, that is $\emptyset \neq \mathcal{A} \subset \mathcal{C}(E)$. $X \in \mathcal{A}$ is said to be a minimal point of $\mathcal{A}$ with respect to $\leq_K^l$ if $A \leq_K^l X$ for some $A \in \mathcal{A}$, then $X \leq_K^l A$; the set of all minimal points of $\mathcal{A}$ with respect to $\leq_K^l$ is denoted by $\mathrm{Min}(\mathcal{A} \mid \leq_K^l)$. Also efficiency with respect to $\leq_K^u$ and the set $\mathrm{Min}(\mathcal{A} \mid \leq_K^u)$ and defined.

Next we define proper efficiency on $\mathcal{C}(E)$.

Definition 1. *Let $\mathcal{A}$ be a nonempty subfamily of $\mathcal{C}(E)$. $X \in \mathcal{A}$ is said to be a properly minimal point of $\mathcal{A}$ with respect to $\leq_K^l$ if there exists a convex cone $L \subsetneq E$ such that $K \subset \mathrm{int} L \cup \{\theta\}$ and X is a minimal point of $\mathcal{A}$ with respect to $\leq_L^l$; the set of all properly minimal points of $\mathcal{A}$ with respect to $\leq_K^l$ is denoted by* $\mathrm{PrMin}(\mathcal{A} \mid \leq_K^l)$.

Also the proper efficiency with respect to $\leq_K^u$ and the set $\mathrm{PrMin}(\mathcal{A} \mid \leq_K^u)$ are defined.

These notions of efficiency and proper efficiency are generalizations of ones in vector-valued optimization.

Proposition 2. *Let $\mathcal{A}$ be a nonempty subfamily of $\mathcal{C}(E)$, and assume that A is singleton for any $A \in \mathcal{A}$, then the following three assertions are equivalent:*

(1) $x \in \mathrm{Min}(\cup\mathcal{A} \mid \leq_K)$,
(2) $\{x\} \in \mathrm{Min}(\mathcal{A} \mid \leq_K^l)$,
(3) $\{x\} \in \mathrm{Min}(\mathcal{A} \mid \leq_K^u)$,

and also the following three assertions are equivalent:

(4) $x \in \mathrm{PrMin}(\cup\mathcal{A} \mid \leq_K)$,
(5) $\{x\} \in \mathrm{PrMin}(\mathcal{A} \mid \leq_K^l)$,
(6) $\{x\} \in \mathrm{PrMin}(\mathcal{A} \mid \leq_K^u)$,

where $\cup\mathcal{A} = \{a \in E \mid \exists A \in \mathcal{A}$ *such that* $a \in A\}$.

Example 1. *Let* $E = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $\mathcal{A} = \{A_t \mid t \in [-\sqrt{2}, \sqrt{2}]\}$ *where* $A_t = \{(x, y) \in E \mid x^2 + y^2 \leq 1, \ x + y = t\}$. *Then we have*

$$\mathrm{Min}(\mathcal{A} \mid \leq^l_K) = \{A_t \mid t \in [-\sqrt{2}, -1]\},$$

$$\mathrm{PrMin}(\mathcal{A} \mid \leq^l_K) = \{A_t \mid t \in [-\sqrt{2}, -1)\},$$

$$\mathrm{Min}(\mathcal{A} \mid \leq^u_K) = \mathrm{PrMin}(\mathcal{A} \mid \leq^u_K) = \{A_t \mid t = -\sqrt{2}\}.$$

In vector optimization, we have $\mathrm{Min}(\cup\mathcal{A} \mid \leq_K) = \{(x, y) \in E \mid x^2 + y^2 = 1, \ x \leq 0, \ y \leq 0\}$, *and* $\mathrm{PrMin}(\cup\mathcal{A} \mid \leq_K) = \{(x, y) \in E \mid x^2 + y^2 = 1, \ x < 0, \ y < 0\}$.

3. An investigation of proper efficiency in set-valued optimization

In this section, only binary relation $\leq^l_K$ will be used. The similar argument will be available for relation $\leq^u_K$.

To study proper efficiency in set optimization, we consider an embedding; we will construct a vector space $\mathcal{V}$ in which $\mathcal{C}(E)$ is embedded, c.f. [1].

Theorem 1. *Let a binary relation* $\simeq$ *on* $\mathcal{C}(E)^2$ *be defined by*

$$(A_1, B_1) \simeq (A_2, B_2) \overset{\text{def}}{\iff} A_1 + B_2 + K = A_2 + B_1 + K,$$

for (A_1, B_1), $(A_2, B_2) \in \mathcal{C}(E)^2$. *Then* $\simeq$ *is an equivalence relation on* $\mathcal{C}(E)^2$. *We denote the quotient space* $\mathcal{C}(E)^2/\simeq$ *as* $\mathcal{V}$, *that is*

$$\mathcal{V} = \{[(A, B)] \mid (A, B) \in \mathcal{C}(E)^2\},$$

where $[(A, B)] = \{(A', B') \in \mathcal{C}(E)^2 \mid (A, B) \simeq (A', B')\}$. *Let addition and scalar multiplication in the quotient space* $\mathcal{V}$ *as follows:*

$$[(A_1, B_1)] + [(A_2, B_2)] = [(A_1 + A_2, B_1 + B_2)],$$

$$\lambda \cdot [(A, B)] = \begin{cases} [(\lambda A, \lambda B)] & \text{if } \lambda \geq 0 \\ [((-\lambda)B, (-\lambda)A)] & \text{if } \lambda < 0. \end{cases}$$

Then $(\mathcal{V}, +, \cdot)$ *is a vector space over* $\mathbb{R}$.

The null vector in $\mathcal{V}$ is $[(\{\theta\}, \{\theta\})]$, and denote Θ. We define the following notation: Let L be a convex cone in E, and let

$$\mu(L) := \{[(A, B)] \in \mathcal{V} \mid B + L \supset A\},$$

then we can check $\mu(L)$ is a convex cone in $\mathcal{V}$, and especially, $\mu(K)$ is a pointed convex cone in $\mathcal{V}$. Generally, we can induce order relations in $\mathcal{V}$ for an arbitrary convex cone $\mathcal{K}$ in $\mathcal{V}$ since $\mathcal{V}$ is regarded as a general ordered vector space. A binary relation $\leq_{\mathcal{K}}$ in $\mathcal{V}$ is defined as follows: for $[(A_1, B_1)], [(A_2, B_2)] \in \mathcal{V}$,

$$[(A_1, B_1)] \leq_{\mathcal{K}} [(A_2, B_2)] \overset{\text{def}}{\iff} [(A_2, B_2)] - [(A_1, B_1)] \in \mathcal{K},$$

and also an efficiency in $\mathcal{V}$ is defined as follows: for $\mathcal{U} \subset \mathcal{V}$

$$\mathrm{Min}(\mathcal{U} \mid \leq_{\mathcal{K}}) = \{U \in \mathcal{U} \mid U' \in \mathcal{U}, \ U' \leq_{\mathcal{K}} U \Rightarrow U \leq_{\mathcal{K}} U'\}.$$

As a consequence of the embedding, we have an important result for research of efficiency in set optimization.

Proposition 3. *Let a function $\varphi : \mathcal{C}(E) \to \mathcal{V}$ be defined by $\varphi(A) = [(A, \{\theta\})]$ for any $A \in \mathcal{C}(E)$. Then $A \in \mathrm{Min}(\mathcal{A} \mid \leq_K^l)$ if and only if $\varphi(A) \in \mathrm{Min}(\varphi(\mathcal{A}) \mid \leq_{\mu(K)}$).*

To consider a notion of proper efficiency in $\mathcal{V}$, we introduce a topology in $\mathcal{V}$. To our purpose, we would like to find a topology in $\mathcal{V}$ satisfying the following condition:

> If $L \subsetneq E$ be a convex cone with $K \subset \mathrm{int}L \cup \{\theta\}$, then $\mu(K) \subset \mathrm{int}\mu(L) \cup \{\Theta\}$ holds, where $\mathrm{int}\mu(L)$ is the set of all interior points of $\mu(L)$ in the topology on $\mathcal{V}$.

We usually consider the following norm; assume that W is a weak*-compact base of K^+, a functional $\|\cdot\|$ on $\mathcal{V}$ is defined by, for each $[(A, B)] \in \mathcal{V}$,

$$\|[(A,B)]\| = \sup_{y^* \in W} |\inf \langle y^*, A\rangle - \inf \langle y^*, B\rangle| .$$

Then this is a norm on $\mathcal{V}$, c.f. [4]. However, this norm is inadequate.

Example 2. *Let $E = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(0,0)\}$, and $B = [(1,-1),(-1,1)]$, the line segment from $(1,-1)$ to $(-1,1)$. In this situation, for any convex cone $L \subsetneq E$ with $K \subset \mathrm{int}L \cup \{\theta\}$, $[(A,B)] \notin \mathrm{int}\mu(L) \cup \{\Theta\}$ though $[(A,B)] \in \mu(K)$. Indeed, if $[(A,B)] \in \mathrm{int}\mu(L) \cup \{\Theta\}$, then we can choose a positive number δ such that $[(A,B)] + \delta\mathcal{N}$ is included in $\mu(L)$, where $\mathcal{N} = \{[(C,D)] \in \mathcal{V} \mid \|[(C,D)]\| \leq 1\}$. Also we can show $[(N,\{\theta\})] \in \mathcal{N}$ where $N = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ when $W = \{(x,y) \in \mathbb{R}^2_+ \mid x^2 + y^2 = 1\}$. However $[(A,B)] + \delta[(N,\{\theta\})] \notin \mu(L) \cup \{\Theta\}$ because $B + L \not\supset A + \delta N$.*

Now we consider the following metric d on $\mathcal{V}$; this consists an adequate topology.

Theorem 2. *Let P be a compact convex base of K, p an element of P, and assume that P does not meet $\{\lambda p \mid \lambda \in [0,1)\}$. For $\lambda \in [0,1)$, let $K_\lambda = \mathrm{cone}(-\lambda p + P)$. For $[(A,B)]$, $[(C,D)] \in \mathcal{V}$,*

$$d([(A,B)],[(C,D)]) = \min\{1, e(A+D, B+C)\}$$

where

$$e(A,B) = \inf\{\lambda \in [0,1) \mid A + K_\lambda = B + K_\lambda\},$$

then d is a metric on $\mathcal{V}$.

Lemma 1. *Under the same assumptions in Theorem 2,*

(1) *K_λ is a closed convex cone of E for each $\lambda \in [0,1)$,*
(2) *$K_\lambda \subset K_\mu$ if $0 \leq \lambda < \mu < 1$,*
(3) *$\bigcap_{\mu \in (\lambda,1)} K_\mu = K_\lambda$ for any $\lambda \in [0,1)$,*
(4) *$A + K_{e(A,B)} = B + K_{e(A,B)}$, for any A, $B \in \mathcal{C}(E)$ with $e(A,B) < 1$.*

Definition 2. *Let $\mathcal{K}$ be a pointed convex cone in $\mathcal{V}$ and $\mathcal{U}$ a nonempty subset of $\mathcal{V}$. U is a properly minimal point of $\mathcal{U}$ with respect to $\leq_\mathcal{K}$ if there exists a convex cone $\mathcal{L} \subsetneq \mathcal{V}$ such that $\mathcal{K} \subset \mathrm{int}\mathcal{L} \cup \{\Theta\}$ and U is a minimal point of $\mathcal{U}$ with respect to $\leq_\mathcal{L}$, where $\mathrm{int}\mathcal{L}$ is the set of all interior points of $\mathcal{L}$ on the topology defined by the metric d in $\mathcal{V}$. The set of all properly minimal points of $\mathcal{U}$ with respect to $\leq_\mathcal{K}$ is denoted by $\mathrm{PrMin}(\mathcal{U} \mid \leq_\mathcal{K})$.*

Then we have the following results.

Lemma 2. *Assume that there exists a convex cone* $L \subsetneq E$ *satisfying* $K \subset \mathrm{int}L \cup \{\theta\}$. *Then* $\mu(K) \subset \mathrm{int}\mu(L) \cup \{\Theta\}$, *where* $\mu(L) = \{[(A, B)] \in \mathcal{V} \mid B \leq^l_L A\}$.

Theorem 3. *If* $\mathrm{PrMin}(\mathcal{A} \mid \leq^l_K)$ *is nonempty, then* $\mathrm{PrMin}(\varphi(\mathcal{A}) \mid \leq_{\mu(K)})$ *is also nonempty.*

Theorem 4. *Assume that* $\varphi(\mathcal{A})$ *is sequentially compact in* $(\mathcal{V}, d)$, *that is for each* $\{A_n\}_{n\in\mathrm{N}} \subset \mathcal{A}$ *there exists a subsequence* $\{A_{n'}\}$ *of* $\{A_n\}$ *and* $A_0 \in \mathcal{A}$ *such that* $e(A_{n'}, A) \to 0$. *Then* $\mathrm{PrMin}(\varphi(\mathcal{A}) \mid \leq_{\mu(K)})$

References

[1] H. Røadström, An embedding theorem for spaces of convex sets. *Proc. Amer. Math. Soc.* **3** (1952), 165–169.

[2] D. T. Luc, Theory of vector optimization. Lecture Notes in Economics and Mathematical Systems, **319**. *Springer-Verlag, Berlin,* 1989.

[3] D. Kuroiwa, Existence Theorems of Set Optimization with Set-Valued Maps. *Journal of Informations & Optimization Sciences* **24** (2003), 73–84.

[4] D. Kuroiwa, Existence of efficient points of set optimization with weighted criteria. *Journal of Nonlinear and Convex Analysis* **4** (2003), 117–124.

[5] D. Kuroiwa and T. Nuriya, Proper efficiency and embedding theorem in set optimization, preprint.

Department of Mathematics and Computer Science, Interdisciplinary Faculty of Science and Engineering, Shimane University, Matsue, 690-8504, JAPAN.