Schauder’s Fixed Point Theorems
Concerning Complete Metric Spaces of Fuzzy Numbers

Seiji SAITO
Graduate School of Information Science and Technology,
Osaka University
E-mail: saito@ist.eng.osaka-u.ac.jp
Osaka, Japan, 565-0871

Abstract
Two aims of our study are follows: One is to prove that a complete metric space of fuzzy numbers becomes a Banach space under a condition that the metric has a homogeneous property. Another is to give sufficient conditions that a subset in the complete metric space and an into continuous mapping on the subset have at least one fixed point by applying Schauder’s fixed point theorem.

1 Introduction
Fuzzy numbers are characterized by membership functions which have three properties: normality, compact convex support and upper semi-continuity. Membership functions are described by α-cut sets, i.e., level sets for $0 \leq \alpha \leq 1$, which are compact convex subsets of $\mathbb{R}^n$ under the above assumptions of membership functions hold. In [6] the author discussed an embedding theorem where metric spaces of compact convex sets are complete. There are so many results on completeness of metric spaces of various kinds of fuzzy numbers and metrics in [2, 4].

In analyzing qualitative properties of differential equations Schauder’s fixed point in complete linear spaces is very useful, because it guarantees the existence of solutions for integral equations corresponding to the differential equations etc. Schauder’s fixed point is as follows: let $S$ be a bounded convex and closed subset in a Banach space. If an into mapping $V$ on $S$ is continuous and the closure of $d(V(S))$ is compact, then $V$ has at least one fixed point in $S$. (See e.g., [9].) It can be easily seen that various sets of fuzzy numbers are complete metric spaces with suitable metrics, but it is not possible to discuss the qualitative properties of solutions in the complete metric spaces by applying Schauder’s fixed point theorem rather than the contraction principle and the comparison method (e.g., [2, 3, 4, 7, 8]).

In this paper we introduce a parametric representation of fuzzy numbers, which are strictly fuzzy convex, then the fuzzy numbers can be identified by bounded closed curves in the two-dimensional metric space. Moreover we show that if the set of all the fuzzy numbers becomes a complete linear space and establish sufficient conditions for fixed points to exist in the complete metric space by applying Schauder’s fixed point in the induced Banach space.

2 Complete Metric Space of Fuzzy Numbers
Denote $I = [0, 1]$. The following definition means that a fuzzy number can be identified with a membership function.

Definition 1 Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by

$F_b^I = \{ \mu : \mathbb{R} \to I \text{ satisfying (i)-(iv) below} \}$

(i) $\mu$ has a unique number $m \in \mathbb{R}$ such that $\mu(m) = 1$ (normality);
(ii) $\text{supp}(\mu) = \{ \xi \in \mathbb{R} : \mu(\xi) > 0 \}$ is bounded in $\mathbb{R}$ (bounded support);
(iii) \( \mu \) is strictly fuzzy convex on supp(\( \mu \)) as follows:

(a) if supp(\( \mu \)) \( \neq \{m\} \), then

\[
\mu(\lambda \xi_1 + (1 - \lambda) \xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]
\]

for \( \xi_1, \xi_2 \in \text{supp}(\mu) \) with \( \xi_1 \neq \xi_2 \) and 0 < \( \lambda < 1 \);

(b) if supp(\( \mu \)) = \{m\}, then \( \mu(m) = 1 \) and \( \mu(\xi) = 0 \) for \( \xi \neq m \);

(iv) \( \mu \) is upper semi-continuous on \( \mathbb{R} \) (upper semi-continuity).

It follows that \( \mathbb{R} \subset \mathcal{F}^u_\mu \). Because \( m \) has a membership function as follows:

\[
\mu(m) = 1, \quad \mu(\xi) = 0 (\xi \neq m)
\]

Then \( \mu \) satisfies the above (i)-(iv).

In usual case a fuzzy number \( x \) satisfies fuzzy convex on \( \mathbb{R} \), i.e.,

\[
\mu(\lambda \xi_1 + (1 - \lambda) \xi_2) \geq \min[\mu(\xi_1), \mu(\xi_2)]
\]

for 0 \( \leq \lambda \leq 1 \) and \( \xi_1, \xi_2 \in \mathbb{R} \). Denote \( \alpha \)-cut sets by

\[
\mathcal{L}_\alpha(\mu) = \{\xi \in \mathbb{R} : \mu(\xi) \geq \alpha\}
\]

for \( \alpha \in I \). When the membership function is fuzzy convex, then we have the following remarks.

Remark 1 The following statements (1) - (4) are equivalent each other, provided with (i) of Definition 1.

(1) (2.2) holds;

(2) \( \mathcal{L}_\alpha(\mu) \) is convex with respect to \( \alpha \in I \);

(3) \( \mu \) is non-decreasing in \( \xi \in (-\infty, m) \), non-increasing in \( \xi \in [m, +\infty) \), respectively;

(4) \( \mathcal{L}_\alpha(\mu) \subset \mathcal{L}_\beta(\mu) \) for \( \alpha > \beta \).

Remark 2 The above condition (iiiia) is stronger than (2.2). From (iiiia) it follows that \( \mu(\xi) \) is strictly monotonously increasing in \( \xi \in [\min \text{supp}(\mu), m] \). Suppose that \( \mu(\xi_1) \geq \mu(\xi_2) \) for \( \xi_1 < \xi_2 \leq m \). From Remark 1(ii), it follows that \( \mu(\xi_1) = \mu(\xi_2) \) for some \( \xi_1 < \xi_2 \), so we get \( \mu(\xi) = \mu(\xi_1) = \mu(\xi_2) \) for \( \xi \in [\xi_1, \xi_2] \). This contradicts with Definition 1(iiiia). Thus \( \mu \) is strictly monotonously increasing. In the similar way \( \mu \) is strictly monotonously decreasing in \( \xi \in [m, \max \text{supp}(\mu)] \). This condition plays an important role in Theorem 1.

We introduce the following parametric representation of \( \mu \in \mathcal{F}^u_\mu \) as

\[
x_1(\alpha) = \min \mathcal{L}_\alpha(\mu), \quad x_2(\alpha) = \max \mathcal{L}_\alpha(\mu)
\]

for 0 \( \leq \alpha \leq 1 \) and

\[
x_1(0) = \min \text{supp}(\mu), \quad x_2(0) = \max \text{supp}(\mu).
\]

In the following example we illustrate typical types of fuzzy numbers.

Example 1 Consider the following \( L - R \) fuzzy number \( x \in \mathcal{F}^u_\mu \) with a membership function as follows:

\[
\mu(\xi) = \begin{cases} 
L(\frac{|m - \xi|}{\rho}) & (\xi \leq m) \\
R(\frac{|\xi - m|}{\rho}) & (\xi > m)
\end{cases}
\]

Here it is said that \( m \in \mathbb{R} \) is a center and \( \rho > 0 \) are spreads. \( L, R \) are I-valued functions. Let \( \mathcal{L}(\xi) = \max(L(\xi), 0) \) etc. We identify \( \mu \) with \( x = (x_1, x_2) \). As long as there exist \( L^{-1} \) and \( R^{-1} \), we have \( x_1(\alpha) = m - L^{-1}(\alpha) \) and \( x_2(\alpha) = m + R^{-1}(\alpha) \).

Let \( \mathcal{L}(\xi) = -c_1\xi + 1 \), where \( c_1 > 0 \) and \( |x_1 - m| \leq \ell \).

We illustrate the following cases (i)-(iv).

(i) Let \( \mathcal{L}(\xi) = -c_2\xi + 1 \), where \( c_2 > 0 \). Then \( c_2(2x_2 - m) = c_1(2x_2 - m) \).

(ii) Let \( \mathcal{L}(\xi) = c_2\xi^2 + 1 \), where \( c_2 > 0 \). Then \( c_2(2x_2 - m) = c_2(2x_2 - m) \).

(iii) Let \( \mathcal{L}(\xi) = -c_2\xi^2 + 1 \), where \( c_2 > 0 \). Then \( c_2(2x_2 - m) = c_2(2x_2 - m) \).

(iv) Let \( c \) be a real number such that \( 0 < c < 1 \). Denote

\[
\mathcal{L}(\xi) = \begin{cases} 
1 & (\xi = 0) \\
-c_2 + c & (0 < \xi \leq 1)
\end{cases}
\]

and let \( \mathcal{R}(\xi) = \mathcal{L}(\xi) \). Then we have \( \ell(x_2 - m) = r(m - x_1) \) for \( |x_1 - m| \leq \ell \). The representation of \( x = (x_1, x_2) \) is as follows:

\[
x_1(\alpha) = m - (1 - \frac{c}{\ell})\ell, \quad x_2(\alpha) = m + (1 - \frac{c}{\ell})\ell
\]

\[
x_1(0) = m, \quad x_2(0) = m
\]

The membership function is given by as follows:

\[
\mu(\xi) = \begin{cases} 
0 & (\xi < x_1(0), \xi > x_2(0)) \\
x_1^{-1}(\xi) & (x_1(0) \leq \xi \leq x_2(0)) \\
1 & (\xi = m) \\
x_2^{-1}(\xi) & (m \leq \xi \leq x_2(0))
\end{cases}
\]
Denote by $C(I)$ the set of all the continuous functions on $I$ to $R$. The following theorem shows a membership function is characterized by $x_1, x_2$.

**Theorem 1** Denote the left-, right-end points of the $\alpha$-cut set of $\mu \in \mathcal{F}_b$ by $x_1(\alpha), x_2(\alpha)$, respectively. Here $x_1, x_2 : I \rightarrow R$. The following properties (i)-(iii) hold:

(i) $x_1, x_2 \in C(I)$;

(ii) $\max_{\alpha \in I} x_1(\alpha) = x_1(1) = m = \min_{\alpha \in I} x_2(\alpha) = x_2(1)$;

(iii) $x_1, x_2$ are non-decreasing, non-increasing on $I$, respectively, as follows:

(a) there exists a positive number $c \leq 1$ such that $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in [0, c]$ and that $x_1(\alpha) = m$ for $\alpha \in [c, 1]$;

(b) $x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in I$;

Conversely, under the above conditions (i) - (iii), if we denote

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\}$$

for $\xi \in R$, then $\mu \in \mathcal{F}_b^I$.

**Remark 3** From the above Condition (i) a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve over $R^2$ and $x_1(\alpha) \leq x_2(\alpha)$ for $\alpha \in I$.

**Proof.** (i) Let $x = (x_1, x_2) \notin R$. Let $\lim_{n \rightarrow \infty} \alpha_n = \alpha_0$ for $\alpha_0 \in I$. Denote $A_1 = \liminf_{n \rightarrow \infty} x_1(\alpha_n)$. We shall prove that $A_1 \geq x_1(\alpha_0)$. Suppose that $A_1 < x_1(\alpha_0)$. Then for any sufficiently small $e > 0$ there exist a number $n$ such that $A_1 - e < x_1(\alpha_n) < A_1 + e < x_1(\alpha_0)$. Denote

$$M = \{\alpha \in I : x_1(\alpha) = x_2(\alpha) = m\},$$

$$S(c) = \{\alpha \in I : x_1(\alpha) = c \text{ on some interval}\}$$

for $c \in \mathcal{C}$.

There are the three cases as follows;

(a) $\alpha_0 \in M$; (b) $\alpha_0 \in S(c)$ for some $c$; (c) $\alpha_0 \notin M \cup S(c)$ for any $c$.

In case of (a) we consider two cases: (a1) $\alpha_0$ is an interior point of $M$, i.e., there exists a sufficiently small number $\delta$ such that the neighborhood $U_\delta(\alpha_0) \subset M$; (a2) $\alpha_0$ is a isolated point. In (a1) it follows that $m < A_1 + e < m$, which leads a contradiction. In (a2) there exist two integers $p < q$ such that

$$|x_1(\alpha_p) - A_1| < 1/q < |x_1(\alpha_q) - A_1| < 1/p.$$
Let $x, y \in F^*_b$ to $g(x,y) : F^*_b \times F^*_b \rightarrow F^*_b$ be calculated by the extension principle of Zadeh. The membership function $\mu_{g(x,y)}(\xi)$ of $g$ is as follows:

$$
\mu_{g(x,y)}(\xi) = \sup_{\xi = \xi_1 = \xi_2} \min(\mu_1(\xi_1), \mu_2(\xi_2))
$$

Here $\xi, \xi_1, \xi_2 \in R$ and $\mu_1, \mu_2$ are membership functions of $x, y$, respectively. From the extension principle, it follows that, in case where $g(x,y) = x+y$,

$$
\mu_{x+y}(\xi) = \max_{\xi = \xi_1 + \xi_2} \min(\mu_1(\xi_1), \mu_2(\xi_2)) = \max_{\xi = \xi_1 + \xi_2, \xi \in L}(\mu_1(\xi_1), i=1,2)
$$

Thus we get $x+y = (x_1 + y_1, x_2 + y_2)$. In the similar way $x-y = (x_1 - y_1, x_2 - y_2)$. Denote a metric by

$$
d_{\infty}(x, y) = \sup_{\alpha \in I} \max(\|x_1(\alpha) - y_1(\alpha)\|, \|x_2(\alpha) - y_2(\alpha)\|)
$$

for $x = (x_1, x_2), y = (y_1, y_2) \in F^*_b$.

Theorem 2 $F^*_b$ is a complete metric space in $C(I)^2$.

Proof. Let a Cauchy sequence $\{x_k = (x^{(k)}_1, x^{(k)}_2) \in F^*_b : k = 1, 2, \ldots \}$, it suffices to show that there is an fuzzy number $x_0 \in F^*_b$ such that $\lim_{n \to \infty} d(x_n, x_0) = 0$. Since the well-known Cauchy's theorem in Calculus, there exists $\lim_{n \to \infty} d(x_n, x_0) = 0$ from the parametric representation, the relation (3.4) means that the following equations hold

1. $x_1 + y_1 = u_1 + v_1 = x_1 + y_1$, $i=1,2$ (i)
2. $x_2 \in \mathcal{F}^*_{b}$ (ii)
3. $x_1(\alpha) \leq x_2(\alpha)$ for $\alpha \in I$ and $x_1(1) = m = x_2(1)$ (iii)
4. $x_1(\alpha) \neq x_2(\alpha)$ for $\alpha \in (0, c)$ (iv)

Suppose that there exists a number $n \neq m$ such that $x_1(1) = x_2(1) = n$. Then $x_1(\alpha) = m$, $x_2(\alpha) = m$. If $x_1(\alpha) = m$, $x_2(\alpha) = m$ then $\mu_{x}(\xi) = \mu_{y}(\xi)$ for $\xi \in \{\alpha \in I : \xi_1 + \xi_2, \xi_1 \in L\}$, $i=1,2$.

Thus the Cauchy limit is unique.

3 Induced Linear Spaces of Fuzzy Numbers

According to the extension principle of Zadeh, for respective membership functions $\mu_1, \mu_2$ of $x, y \in F^*_b$ and $\lambda \in R$, the following addition and a scalar product are given as follows:

$$
\mu_{x+y}(\xi) = \sup_{\lambda \in \mathcal{F}^*_{b}}(\mu_1(\xi)/\lambda) = \sup_{\eta \in \mathcal{F}^*_{b}}(\mu_2(\eta)) = \in [c, 1]
$$

Putting $x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)$ by the parametric representation, the relation (3.4) means that the following equations hold

$$
\mu_{x+y}(\xi) = \mu_{x+y}(\eta) = \in [c, 1]
$$

such that one of the following cases (i) and (ii) hold:

1. if $(x, y) \sim (u, v)$, then $[x, y] = [u, v]$ (i)
2. if $(x, y) \neq (u, v)$, then $[x, y] \cap [u, v] = \emptyset$ (ii)
Then $\mathcal{F}_{b}^{a}$/ $\sim$ is a linear space with the following addition and scalar product

$$[x, y] + [u, v] = [x + u, y + v]$$  \hfill (3.5)

$$\lambda [x, y] = \begin{cases} \{ (\lambda x, \lambda y) \} & (\lambda \geq 0) \\ \{ ((-\lambda)y, (-\lambda)x) \} & (\lambda < 0) \end{cases}$$  \hfill (3.6)

for $\lambda \in \mathbb{R}$ and $[x, y], [u, v] \in \mathcal{F}_{b}^{a}$/ $\sim$. They denote a norm in $\mathcal{F}_{b}^{a}$/ $\sim$ by

$$\| [x, y] \| = \sup_{\alpha \in I} d_{H}(L_{\alpha}(\mu_{x}), L_{\alpha}(\mu_{y})).$$

Here $d_{H}$ is the Hausdorff metric is as follows:

$$d_{H}(L_{\alpha}(\mu_{x}), L_{\alpha}(\mu_{y})) = \max(\sup_{\epsilon \in L_{\alpha}(\mu_{x})} \inf_{\eta \in L_{\alpha}(\mu_{y})} |\xi - \eta|, \sup_{\eta \in L_{\alpha}(\mu_{y})} \inf_{\epsilon \in L_{\alpha}(\mu_{x})} |\xi - \eta|)$$

It can be easily seen that $\| [x, y] \| = d_{\infty}(x, y)$.

Note that $\| [x, y] \| = 0$ in $\mathcal{F}_{b}^{a}$/ $\sim$ if and only if $x = y$ in $\mathcal{F}_{b}^{a}$.

4 Schauder's Fixed Point Theorem in Complete Metric Spaces

In the following theorem we show that the complete metric space $\mathcal{F}_{b}^{a}$/ $\sim$ has an induced Banach space.

Theorem 3 Let $S$ be a bounded closed subset in $\mathcal{F}_{b}^{a}$/ $\sim$. Assume that $S$ contains any segments of $x, y \in S$, i.e., $\lambda x + (1 - \lambda)y \in S$ for $\lambda \in I$. Let $V$ be an into continuous mapping on $S$. Assume that the closure $coll(V(S))$ is compact in $\mathcal{F}_{b}^{a}$. Then $V$ has at least one fixed point $x$ in $S$, i.e., $V(x) = x$.

Proof. Denote $X = \{ [x, 0] \in \mathcal{F}_{b}^{a}$/ $\sim$ : $x \in \mathcal{F}_{b}^{a} \}$. We shall prove that $X$ is a Banach space. Let $\{ [x_{0}, n] : i = 1, 2, \cdot \cdot \cdot \}$ be a Cauchy sequence in $X$. Without loss of generality we denote $[x_{0}, n] = [x_{m}, 0]$ for $x_{m} \in \mathcal{F}_{b}^{a}$ with $\| [x_{0}, n] - [x_{m}, 0] \| = 0$ as $n, m \to \infty$, which means that $\lim_{n, m \to \infty} d_{\infty}(x_{n}, x_{m}) = 0$. By the completeness of $\mathcal{F}_{b}^{a}$, there exists an element $x_{0} \in \mathcal{F}_{b}^{a}$ such that $\lim_{n \to \infty} d_{\infty}(x_{n}, x_{0}) = 0$ from Theorem 2. This means that $\lim_{n \to \infty} \| [x_{0}, n] - [x_{0}, 0] \| = 0$ and $X$ is a Banach space.

Put a subset $S_{1} = \{ [x, 0] \in X : x \in S \}$. Then $S_{1}$ is clearly bounded in $X$. Denote a mapping on $S_{1}$ by

$$V_{1}([x, 0]) = [V(x), 0] \text{ for } [x, 0] \in S_{1}.$$  \hfill (3.7)

It follows that for $[u, 0] = [v, 0] \in X$ we have $u = v$ and $V_{1}([u, 0]) = [V(u), 0] = [V(v), 0] = V_{1}([v, 0])$. We get $V_{1}$ is into an onto mapping on $S_{1}$.

Let $x \in \mathcal{F}_{b}^{a}$ be a limit of a sequence $\{ y_{n} \in S \}$ such that $d(y_{n}, x) \to 0$ as $n \to \infty$. From the closedness of $S$, it follows that $x \in S$ and as long as $d(y_{n}, x) = \| y_{n}, 0 - x, 0 \| = 0$ (n $\to \infty$) with $[y_{n}, 0] \in S_{1}$ and $[x, 0] \in X$, we have $[x, 0] \in S_{1}$. Thus $S_{1}$ is closed. For $x, y \in S$ it follows that $\lambda x + (1 - \lambda)y \in S$ and

$$\| [x, 0] + (1 - \lambda)[y, 0] \| = \| [\lambda x + (1 - \lambda)y, 0] \|$$

Therefore $S_{1}$ is convex in the Banach space $X$. When $y \to x$ in $\mathcal{F}_{b}^{a}$, by the continuity of $V$, we have

$$\| V_{1}([y, 0]) - V_{1}([x, 0]) \| = \| V(y, 0) - V(x, 0) \|$$

Thus $V_{1}$ is continuous on $S_{1}$.

Finally, we shall prove the relative compactness of $V_{1}(S_{1})$. Let $\{ V_{1}([x_{n}, 0]) : n = 1, 2, \cdot \cdot \cdot \}$ be a sequence in $S_{1}$. Because of the relative compactness of $V(S)$, there exists a subsquence $\{ V_{1}([\overline{x}_{n}, 0]) \} \subset \{ V_{1}([x_{n}, 0]) \}$ such that

$$\lim_{n \to \infty} d_{\infty}(V(\overline{x}_{n}), y) = 0, \text{ where } y \in d(V(S)).$$

Since

$$d_{\infty}(V(\overline{x}_{n}), y) = \| V(\overline{x}_{n}), 0 - y, 0 \| = \| V(\overline{x}_{n}), 0 \| - |y, 0|,$$

we have $[y, 0] \in d(V(S))$. Thus $d(V(S))$ is compact in $X$.

Therefore the mapping $V_{1} : S_{1} \to S_{1}$, where $S_{1}$ is a bounded closed and convex subset in the Banach space $X$, is continuous. Here $d(V_{1}(S_{1}))$ is relatively compact in $X$. By Schauder's fixed point theorem in Banach spaces, there exists a fixed point of $V_{1}$ in $S_{1}$, i.e., $[V(x), 0] = [x, 0]$, which means that $V(x) = x$ in $S$.

Q.E.D.

In the following theorem complete metric spaces have at least one fixed point of the induced Banach space.

Theorem 4 Let $\mathcal{F}$ be a complete metric space with a metric $d$. Assume that $\mathcal{F}$ is closed under addition and scalar product, and that $d(\lambda x, 0) = |\lambda|d(x, 0)$ for the scalar product $\lambda x$ and $\lambda \in \mathbb{R}, x \in \mathcal{F}$. Denote $X = \{ [x, 0] : x \in \mathcal{F} \}$. Here $x, y$ for $x, y \in \mathcal{F}$ are equivalence classes of (3.4) and 0 is the origin. Then $X$ is a Banach space concerning addition (3.5), scalar product (3.6) and norm $\| [x, 0] \| = d(x, 0)$ for $[x, 0] \in X$.

Moreover let $S$ be a bounded closed subset in $\mathcal{F}$. Assume that $S$ contains any segments of $x, y \in S$ in the same meaning of Theorem 3. Let $V$ be an into continuous mapping on $S$. Assume that the closure $d(V(S))$
is compact in $F$. Then $V$ has at least one fixed point in $S$.

**Proof.** It can be seen that $X$ is a linear space. $\| [x, 0] \|$ is a norm in $X$. For $[x, 0], [y, 0] \in X$ it follows that

$$
\| [x, 0] + [y, 0] \| = \| [x + y, 0] \| = d(x + y, 0) \\
\leq d(x, y) + d(y, 0) \\
= d(x, y) + d(y, 0) \\
= \| [x, 0] \| + \| [y, 0] \|
$$

since we have $[x + y, y] = [x, 0]$ and $d(x, y) = d(x, 0)$ for $x, y \in F$. It is clear that $\| [x, 0] \|$ is positive definite and for $\lambda \in \mathbb{R}$

$$
\| \lambda x, 0 \| = d(\lambda x, 0) = |\lambda| \| [x, 0] \|
$$

In the same way as the discussion of Theorem 3, $X$ is complete.

Denote a subset $S_1 = \{[x, 0] \in X : x \in S\}$ and a mapping $V_1$ such that $V_1([x, 0]) = [V(x), 0]$ for $[x, 0] \in S_1$. The following properties (i)-(iii) can be proved in the similar way of the proof of Theorem 3.

(i) $S_1$ is bounded closed and convex in $X$;

(ii) $V_1$ is an into continuous mapping on $S_1$;

(iii) $cl(V_1(S_1))$ is relatively compact in $X$.

Then, by Schauder’s fixed point theorem, there exists at least one fixed point $[x_0, 0]$ of $V_1$ in $S_1$, i.e., $V(x_0) = x_0$ in $S$.

Q.E.D.

**Example 2 (1)** Let $(\mathbb{R}, d)$ be the discrete metric space with

$$
d(x, y) = 0 (x = y) \quad ; \quad d(x, y) = 1 (x \neq y).
$$

It follows that $d([x, 0]) = 1 \neq |\lambda|d([x, 0]) = |\lambda|$ for $x \neq 0, |\lambda| \neq 0, 1$. Then $X = \{[x, 0] : x \in \mathbb{R} \}$ cannot be a normed space concerning $\| [x, 0] \| = d(x, 0)$ for $x \in \mathbb{R}$, because $\| [x, 0] \|$ is not homogeneous.

(2) Let $K_C(\mathbb{R}^n)$ be the set of all compact convex subsets in $\mathbb{R}^n$. Assume that $d_H$ is the Hausdorff metric in $\mathbb{R}^n$ as follows:

$$
d_H(A, B) = \max(\sup_{\xi \in A} \inf_{\eta \in B} \| \xi - \eta \|, \sup_{\eta \in B} \inf_{\xi \in A} \| \xi - \eta \|)
$$

Here $A, B \in K_C(\mathbb{R}^n)$ and $\| \cdot \|$ is a norm in $\mathbb{R}^n$. Then we have $d_H(\lambda A, \theta) = |\lambda|d_H(A, \theta)$ for $A \in K_C(\mathbb{R}^n), \lambda \in \mathbb{R}$ where $\lambda A = \{\lambda a : a \in K_C(\mathbb{R}^n)\}$. By Theorem 4 it follows that the set of equivalence classes $X = \{[A, 0] \in K_C(\mathbb{R}^n) / \sim : A \in K_C(\mathbb{R}^n) \}$ is a linear space with a norm $\| [A, 0] \| = d_H(A, \theta)$. Here the equivalence relation $\sim$ is given in (3.4). It can be seen that $X$ is a Banach space by the embedding theorem in [6].

Let $S$ be a bounded closed subset in $K_C(\mathbb{R}^n)$. Assume that $S$ contains any segments of $A, B \in S$ in the same meaning of Theorem 4. Let $V$ be an into continuous set-valued mapping on $S$. Assume that the closure $cl(V(S))$ is compact in $K_C(\mathbb{R}^n)$. Then $V$ has at least one fixed point $A_0 \in S$, i.e., $V(A_0) = A_0$.

5 **Applications to FBVP**

Consider the following boundary value problems of fuzzy differential equations

$$
x''(t) = f(t, x, x'), \quad x(a) = A, x(b) = B. \quad (5.7)
$$

Here $t \in J = [a, b] \subset \mathbb{R} = (-\infty, +\infty)$ and fuzzy numbers $A, B \in \mathcal{F}^+_b$, which is a set of fuzzy numbers with compact supports and strict quasi-concavity, and $f : J \times \mathcal{F}^+_b \times \mathcal{F}^+_b \to \mathcal{F}^+_b$ is a continuous function.

In order to discuss the qualitative properties of solutions to (5.7) we consider the following Fredholm equation

$$
x(t) = w(t) + \int_a^b G(t, s)f(s, x(s), x'(s))ds
$$

for $t \in J$. Let $A, B \in \mathcal{F}^+_b$ be in the same fuzzy numbers of (5.7). Here a fuzzy function $w \in C(J, \mathcal{F}^+_b)$ and an $\mathbb{R}$-valued function $G \in C(\mathbb{R}^2; \mathbb{R})$ with $G(t, s) \geq 0$ such that

$$
w(t) = \frac{A(b - t) + B(t - a)}{b - a}, \quad (5.8)
$$

$$
G(t, s) = \begin{cases} 
\frac{(b - t)(s - a)}{b - a} & (a \leq t \leq s \leq b) \\
\frac{(b - s)(t - a)}{b - a} & (a \leq s \leq t \leq b) 
\end{cases}, \quad (5.9)
$$

In the same way as in the discussion concerning boundary value problems of ordinary differential equation the following lemma is shown immediately.

**Lemma 1** A fuzzy function $x$ is a continuously differentiable solution of (5.7) if and only if $x$ is a fixed point of $T : C^1(J, \mathcal{F}^+_b) \to C^2(J, \mathcal{F}^+_b)$ such that

$$
[T(x)](t) = w(t) + \int_a^b G(t, s)f(s, x(s), x'(s))ds.
$$

Assume that the following properties (i)-(iii) hold.
A function \( f = (f_1, f_2) : J \times \mathcal{F}_b^{s} \times \mathcal{F}_b^{s} \to \mathcal{F}_b^{s} \) is continuous. Here \((f_1, f_2)\) is the parametric representation of \( f \).

Let \( r_i > 0 \) for \( i = 1, 2 \). Then there exists a function \( h_i : [0, \infty) \to [0, \infty) \) such that
\[
|f_i(t, x, y, \alpha)| \leq h_i(|u_i(\alpha) - y_i|)
\]
for \( t \in J, x \in \mathcal{S}_w(r, N) \), \( i = 1, 2 \), and \( |x_i(\alpha)| \leq r \), \( y = (y_1, y_2) \in \mathcal{F}_b^{s} \). Here \( x = (x_1, x_2) \) is the parametric representation of \( x, y \), respectively.

Assume that \( h_i, i = 1, 2 \), satisfy
\[
\int_{0}^{\infty} \frac{\eta^{2}}{h_i(\eta)} d\eta > 2r_i.
\]

We say that the above conditions (i)-(iii) are a fuzzy type of Nagumo’s conditions and they are applied to the fuzzy boundary value problem (5.7) in the same way as [1].

**Lemma 2** Assume that \( f = (f_1, f_2) \) satisfies fuzzy type of Nagumo’s conditions. Let \( r_i > 0, i = 1, 2 \), be in fuzzy type of Nagumo’s conditions and a solution \( x = (x_1, x_2) \in C^2(J; \mathcal{F}_b^{s}) \) of (5.7) satisfy \( |x_i(t, \alpha)| \leq r_i \) for \( i = 1, 2, t \in J, \alpha \in I \).

Then, there exist numbers \( N_i > 0, i = 1, 2 \) such that \( |x_i(t, \alpha)| \leq N_i \) for \( t \in J, \alpha \in I \).

**Proof.**

From the above lemmas we get the following existence theorem on the Fredholm equation by the Schauder’s fixed point theorem in Section 4. We have an existence theorem of (5.7) by the Schauder’s fixed point theorem in Section 4. Here \( z = (x_1, x_2, x_1', x_2') \), \( u \in C^1(J; \mathbb{R})^3 \times C(J; \mathbb{R})^3, c = (A_1, A_2, B_1, B_2) \in C(J; \mathbb{R})^4 \),
\[
X(t) = e^{tM} = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 t \\ 0 & 1 & 0 \end{pmatrix}
\]
with \( X(0) = E \),

where \( E \) is the identity matrix, \( M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)

and \( L \) denotes a bounded linear operator from \( C^1(J; \mathbb{R})^3 \times C(J, \mathbb{R})^3 \) to \( C(J; \mathbb{R})^4 \) by
\[
L(z) = (x_1(a, \alpha), x_2(a), x_1(b), x_2(b))^T.
\]

Then, in [8], we get the existence theorem on the Volterra type of (5.7) as follows.

**Theorem 6** Assume that positive numbers \( R, r \) satisfy \( R < e^{-c/(b-a)} \) and \( r > \frac{Q(L(t, x))}{e^{-c/(b-a)}} \). Let \( f \) satisfy
\[
\int_{a}^{b} 1_{[0,r]}(f(s, z)) ds \leq r \cdot R.
\]

If \( A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{F}_b^{s} \), \( d(A, 0) + d(B, 0) \leq \frac{R}{(b-a)}(b-a) - \frac{c}{(b-a)} \cdot R - \| L \| \), then (5.7) has at least one solution in \( S \). Here
\[
Q = \int_{a}^{b} 1_{[0,r]}(f(s, z), 0) ds.
\]

**References**


