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CONVERGENCE OF DYNAMICAL SYSTEMS WITH CONVEX LYAPUNOV FUNCTIONS

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ABSTRACT. This is a survey of recent results regarding the convergence of several classes of dynamical systems with convex Lyapunov functions in general Banach spaces. For each class we define an appropriate complete metric space of dynamical systems and show that most of them (in the sense of Baire category) are convergent. In some cases the set of divergent systems is not only of the first category, but also σ-porous.

INTRODUCTION

The study of minimization methods for convex functions is a central topic in optimization theory. In this survey, we are given a continuous convex function f defined on a bounded, closed and convex subset K of a Banach space X, and a minimization algorithm is a self-mapping $A: K \to K$ such that $f(Ax) \le f(x)$ for all $x \in K$. We show that for most of these algorithms A, the sequences $\{f(A^n x)\}_{n=1}^{\infty}$ tend to the infimum of f for all initial values $x \in K$. When we say that most of the elements of a complete metric space X enjoy a certain property, we mean that the set of points which have this property contains a G_δ everywhere dense subset of X. In other words, this property holds generically. Such an approach, when a certain property is investigated for the whole space X and not just for a single point in X, has already been successfully applied in many areas of Analysis [1-6, 8, 11, 15, 16]. We now recall the concept of porosity [6, 13, 16] which will enable us to obtain even more refined results.

Let (Y, d) be a complete metric space. We denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. We say that a subset $E \subset Y$ is porous in (Y, d) if there exist $\alpha \in (0,1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B_d(z, \alpha r) \subset B_d(y, r) \setminus E.$$

A subset of the space Y is called σ-porous in (Y, d) if it is a countable union of porous subsets in (Y, d).

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure 0 and are of the first category.

To point out the difference between porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there are a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0,1)$ is a constant which depends only on E.

Our paper is organized as follows. In Section 1 we review the minimization methods studied in [7, 12, 13], where the convex function f is assumed to be uniformly continuous. In the second section f is assumed to be merely continuous [14]. The third section is devoted to some examples.

1. UNIFORMLY CONTINUOUS LYAPUNOV FUNCTIONS

Assume that $(X, ||\cdot||)$ is a Banach space with norm $||\cdot||$, $K \subset X$ is a nonempty, bounded, closed and convex subset of X, and $f: K \to R^1$ is a convex uniformly continuous function. Set

$$\inf(f) = \inf\{f(x) : x \in K\}.$$

Observe that this infimum is finite because K is bounded and f is uniformly continuous. We consider the topological subspace $K \subset X$ with the relative topology. Denote by $\mathfrak{A}$ the set of all self-mappings $A : K \to K$ such that

$$f(Ax) \leq f(x) \text{ for all } x \in K, \tag{1.1}$$

and by $\mathfrak{A}_c$ the set of all continuous mappings $A \in \mathfrak{A}$.

In Example 2 of Section 3 we construct many such mappings.

For the set $\mathfrak{A}$ we define a metric $\rho : \mathfrak{A} \times \mathfrak{A} \to R^1$ by

$$\rho(A, B) = \sup\{||Ax - Bx|| : x \in K\}, \ A, B \in \mathfrak{A}. \tag{1.2}$$

Clearly, the metric space $\mathfrak{A}$ is complete and $\mathfrak{A}_c$ is a closed subset of it. Denote by $\mathfrak{M}$ (respectively, $\mathfrak{M}_c$) the set of all sequences $\{A_t\}_{t=1}^{\infty} \subset \mathfrak{A}$ (respectively, $\mathfrak{A}_c$). A member of $\mathfrak{M}$ will occasionally be denoted by a boldface $\mathbf{A}$. For the set $\mathfrak{M}$ we consider the uniformity determined by the following base:

$$E(N, \epsilon) = \{(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \in \mathfrak{M} \times \mathfrak{M} : \rho(A_t, B_t) \leq \epsilon, \ t = 1, \ldots N\},$$

where N is a natural number and $\epsilon > 0$. Clearly, the uniform space $\mathfrak{M}$ is metrizable (by a metric $\rho_w : \mathfrak{M} \times \mathfrak{M} \to R^1$) and complete.

From the point of view of the theory of dynamical systems each element of $\mathfrak{M}$ describes a nonstationary dynamical system with a Lyapunov function f. Also, some optimization procedures in Hilbert and Banach spaces can be represented by elements of $\mathfrak{M}$ (see the first example in Section 3 and [9, 10]).

In [12] we show that for a generic sequence taken from the space $\mathfrak{M}_c$ the value of the Lyapunov function along all trajectories tends to its infimum. More precisely, we obtain the following two theorems.

Theorem 1.1. *There exists a set $\mathcal{F} \subset \mathfrak{M}_c$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}_c$ such that for each $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathcal{F}$ the following assertion holds:*

For each $\epsilon > 0$, there exists a neighborhood U of $\mathbf{B}$ in $\mathfrak{M}_c$ and a natural number N such that for each $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in U$ and each $x \in K$,

$$f(C_N \ldots C_1 x) \leq \inf(f) + \epsilon.$$

Theorem 1.2. *There exists a set $\mathcal{G} \subset \mathfrak{A}_c$ which is a countable intersection of open everywhere dense sets in $\mathfrak{A}_c$ such that for each $B \in \mathcal{G}$ the following assertion holds:*

For each $\epsilon > 0$, there exists a neighborhood U of B in $\mathfrak{A}_c$ and a natural number N such that for each $C \in U$ and each $x \in K$,

$$f(C^N x) \leq \inf(f) + \epsilon.$$

The key auxiliary result which is used in the proofs of these theorems is the following proposition.

Proposition 1.1. *There exists a mapping $A_* \in \mathfrak{A}_c$ with the following property:*

Given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \epsilon$, the inequality

$$f(A_* x) \leq f(x) - \delta(\epsilon)$$

is true.

Remark 1.1. If there is $x_{min} \in K$ for which $f(x_{min}) = \inf(f)$, then we can set $A_*(x) = x_{min}$ for all $x \in K$.

In the sequel we continue to study the metric space $(\mathfrak{A}, \rho)$ and its closed subset $\mathfrak{A}_c$. For the set $\mathfrak{M}$ we will consider two uniformities and the topologies induced by them. The first one has already been defined. The topology it induces will be called weak and denoted by τ_w. Clearly, $\mathfrak{M}_c$ is a closed subset of $\mathfrak{M}$ with the weak topology.

For the set $\mathfrak{M}$ we also define a metric $\rho_s : \mathfrak{M} \times \mathfrak{M} \to R^1$ by

$$\rho_s(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) = \sup\{\rho(A_t, B_t) : t = 1, 2, \ldots\}, \ \{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty} \in \mathfrak{M}.$$

Clearly, the metric space $(\mathfrak{M}, \rho_s)$ is complete and $\mathfrak{M}_c$ is a closed subset of $(\mathfrak{M}, \rho_s)$. In the sequel we will also study the metric space $(\mathfrak{M}_c, \rho_s)$.

Denote by τ_s the topology induced by the metric ρ_s on $\mathfrak{M}$. Since τ_s is clearly stronger than τ_w, it will be called strong. We consider the topological subspace $\mathfrak{M}_c \subset \mathfrak{M}$ with the relative weak and strong topologies.

The following notion of normality was introduced in [7].

A mapping $A \in \mathfrak{A}$ is called normal if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \epsilon$, the inequality

$$f(Ax) \leq f(x) - \delta(\epsilon)$$

is true.

A sequence $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ is called normal if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \epsilon$ and each integer $t \geq 1$, the inequality

$$f(A_t x) \leq f(x) - \delta(\epsilon)$$

holds.

In [7] we show that a generic element taken from the spaces $\mathfrak{A}$, $\mathfrak{A}_c$, $\mathfrak{M}$ and $\mathfrak{M}_c$ is normal. This is important because it turns out that the sequence of values of the Lyapunov function f along any (unrestricted) trajectory of such an element tends to the infimum of f on K.

For $\alpha \in (0,1)$, $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$ and $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathfrak{M}$, define $\alpha\mathbf{A} + (1-\alpha)\mathbf{B} = \{\alpha A_t + (1-\alpha)B_t\}_{t=1}^{\infty} \in \mathfrak{M}$.

We can easily prove the following fact.

Proposition 1.2. *Let $\alpha \in (0,1)$, $\mathbf{A}, \mathbf{B} \in \mathfrak{M}$ and let $\mathbf{A}$ be normal. Then $\alpha\mathbf{A} + (1-\alpha)\mathbf{B}$ is also normal.*

We now state the main results of [7].

Theorem 1.3. *Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ be normal and let $\epsilon > 0$. Then there exists a neighborhood U of $\mathbf{A}$ in $\mathfrak{M}$ with the strong topology and a natural number N such that for each $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in U$, each $x \in K$, and each $r : \{1,2,\dots\} \to \{1,2,\dots\}$,*

$$f(C_{r(N)} \dots C_{r(1)} x) \leq \inf(f) + \epsilon.$$

Theorem 1.4. *Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ be normal and let $\epsilon > 0$. Then there exists a neighborhood U of $\mathbf{A}$ in $\mathfrak{M}$ with the weak topology and a natural number N such that for each $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in U$ and each $x \in K$,*

$$f(C_N \dots C_1 x) \leq \inf(f) + \epsilon.$$

Theorem 1.5. *There exists a set $\mathcal{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ with the strong topology and a set $\mathcal{F}_c \subset \mathcal{F} \cap \mathfrak{M}_c$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}_c$ with the strong topology such that each $\mathbf{A} \in \mathcal{F}$ is normal.*

Theorem 1.6. *There exists a set $\mathcal{F} \subset \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ and a set $\mathcal{F}_c \subset \mathcal{F} \cap \mathfrak{A}_c$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}_c$ such that each $\mathbf{A} \in \mathcal{F}$ is normal.*

In [13] we prove two theorems. The first one extends Theorem 1.3 to perturbed trajectories of a normal sequence. The study of such trajectories is obviously of considerable practical significance [9, 10].

Theorem 1.7. *Let $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ be normal and let ϵ be positive. Then there exist a natural number n_0 and a number $\gamma > 0$ such that for each integer $n \geq n_0$, each mapping $r : \{1,\dots,n\} \to \{1,2,\dots\}$, and each sequence $\{x_i\}_{i=0}^{n} \subset K$ which satisfies*

$$||x_{i+1} - A_{r(i+1)} x_i|| \leq \gamma, \; i = 0, \dots, n-1,$$

the inequality $f(x_i) \leq \inf(f) + \epsilon$ holds for $i = n_0, \dots, n$.

The second result of [13] improves upon Theorems 1.5 and 1.6. For each of the spaces $\mathfrak{M}$, $\mathfrak{M}_c$, $\mathfrak{A}$ and $\mathfrak{A}_c$, these theorems establish the existence of an everywhere dense G_δ subset such that each one of its elements is normal. In [13] we show that if the function f is Lipschitzian, then for each of the above spaces, the complement of the subset of all normal elements is not only of the first category, but also a σ-porous set.

Theorem 1.8. *Let $\mathcal{F}$ be the set of all normal sequences in the space $\mathfrak{M}$ and let*

$$F = \{A \in \mathfrak{A} : \{A_t\}_{t=1}^{\infty} \in \mathcal{F} \text{ where } A_t = A,\ t = 1, 2, \dots\}.$$

Assume that the function f is Lipschitzian. Then the complement of the set $\mathcal{F}$ is a σ-porous subset of $(\mathfrak{M}, \rho_s)$ and the complement of the set $\mathcal{F} \cap \mathfrak{M}_c$ is a σ-porous subset of $(\mathfrak{M}_c, \rho_s)$. Moreover, the complement of the set F is a σ-porous subset of $(\mathfrak{A}, \rho)$ and the complement of the set $F \cap \mathfrak{A}_c$ is a σ-porous subset of $(\mathfrak{A}_c, \rho)$.

2. Continuous Lyapunov functions

In this section we continue to use the notation introduced in the previous sections, but the convex function $f : K \to R^1$ is assumed to be merely continuous and bounded from below.

We also consider the space $K \times \mathfrak{A}_c$ equipped with the product topology and the space $K \times \mathfrak{M}_c$ which is equipped with a pair of topologies. One of them (which is called the weak topology) is the product of the topology of K and the weak topology of $\mathfrak{M}_c$, and the second one (which is called the strong topology) is the product of the topology of K and the strong topology of $\mathfrak{M}_c$.

In [14], assuming that f is merely continuous, we are stil able to obtain two results in the direction of the previous sections. To achieve this, we change our point of view and consider a new framework. The main feature of this new framework is that the initial point of a trajectory of our dynamical system may also vary.

We now state the two main results of [14].

Theorem 2.1. *There exists a set $\mathcal{F} \subset K \times \mathfrak{M}_c$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $K \times \mathfrak{M}_c$ such that for each $(x, \{A_t\}_{t=1}^{\infty}) \in \mathcal{F}$, the following property holds:*

For each $\epsilon > 0$, there exists a neighborhood $\mathcal{U}$ of $(x, \{A_t\}_{t=1}^{\infty})$ in $K \times \mathfrak{M}_c$ with the weak topology and a natural number N such that for each $(y, \{B_t\}_{t=1}^{\infty}) \in \mathcal{U}$,

$$f(B_N \dots B_1 y) \le \inf(f) + \epsilon.$$

Theorem 2.2. *There exists a set $\mathcal{F} \subset K \times \mathfrak{A}_c$ which is a countable intersection of open everywhere dense subsets of $K \times \mathfrak{A}_c$ such that for each $(x, A) \in \mathcal{F}$, the following property holds:*

For each $\epsilon > 0$, there exists a neighborhood $\mathcal{U}$ of (x, A) in $K \times \mathfrak{A}_c$ and a natural number N such that for each $(y, B) \in \mathcal{U}$,

$$f(B^N y) \le \inf(f) + \epsilon.$$

3. Examples

Let $(X, \|\cdot\|)$ be a Banach space. In this section we present examples of continuous mappings $A : K \to K$ satisfying $f(Ax) \le f(x)$ for all $x \in K$, where K is a bounded, closed and convex subset of X and $f : K \to R^1$ is a convex function [12].

Example 1. Let $f : X \to R^1$ be a convex, uniformly continuous function satisfying

$$f(x) \to \infty \text{ as } \|x\| \to \infty.$$

Evidently, the function f is bounded from below. For each real number c, let $K_c = \{x \in X : f(x) \le c\}$. Fix a real number c such that $K_c \ne \emptyset$. Clearly, the set K_c is bounded, closed and convex. We suppose that the function f is strictly convex on K_c, namely

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in K_c$, $x \ne y$, and all $\alpha \in (0,1)$.

Let $V : K_c \to X$ be any continuous mapping. For each $x \in K_c$, there is a unique solution of the following minimization problem:

$$f(z) \to \min, \quad z \in \{x + \alpha V(x) : \alpha \in [0,1]\}.$$

This solution will be denoted by Ax. Since $f(Ax) \le f(x)$ for all $x \in K_c$, we conclude that $A(K_c) \subset K_c$.

It is shown in [12] that the mapping $A : K_c \to K_c$ is continuous.

Example 2. Let K be a bounded, closed and convex subset of X, and let $f : K \to R^1$ be a convex continuous function which is bounded from below. For each $x_0, x_1 \in K$ satisfying $f(x_0) > f(x_1)$, we will construct a continuous mapping $A : K \to K$ such that $f(Ax) \le f(x)$ for all $x \in K$ and $Ax = x_1$ for all x in a neighborhood of x_0.

Indeed, let $x_0, x_1 \in K$ with $f(x_0) > f(x_1)$. There are numbers r_0 and ϵ_0 such that

$$f(x) - \epsilon_0 > f(x_1) \text{ for all } x \in K \text{ satisfying } ||x - x_0|| \le r_0.$$

Now we define an open covering $\{V_x : x \in K\}$ of K. Let $x \in K$. If $||x - x_0|| < r_0$, we set

$$V_x = \{y \in K : ||y - x_0|| < r_0\} \text{ and } a_x = x_1.$$

If $||x - x_0|| \ge r_0$, then there is $r_x \in (0, 4^{-1}r_0)$ and $a_x \in K$ such that

$$f(a_x) \le f(y) \text{ for all } y \in \{z \in K : ||z - x|| \le r_x\}.$$

In this case, we set

$$V_x = \{y \in K : ||y - x|| < r_x\}.$$

Clearly, $\cup\{V_x : x \in K\} = K$. There is a continuous partition of unity $\{\phi_x\}_{x \in K}$ on K subordinated to $\{V_x\}_{x \in K}$ (namely, supp $\phi_x \subset V_x$ for all $x \in K$). For $y \in K$ we define

$$Ay = \sum_{x \in K} \phi_x(y) a_x.$$

Evidently, the mapping A is well defined, $A : K \to X$, and it is continuous. Since $\sum_{x \in K} \phi_x(y) = 1$ for all $y \in K$ and K is convex, we see that $A(K) \subset K$.

It is shown in [12] that $f(Ay) \le f(y)$ for all $y \in K$ and that $Ay = x_1$ if $||y - x_0|| \le 4^{-1}r_0$.

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