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Singular perturbation problem for a model equation of phase separation

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概要

The behavior of steady state solutions of a model equation for phase separation is analyzed when the parameter, which depends on the ratio of the surface energy, tends to zero. Analytical convergence results are exhibited.

1 Introduction

Phase separations observed in a variety of material sciences are fascinating topics for research both from the mathematical and computational point of view [14]. The motion law introduced by Eguchi, Oki, and Matsumura (EOM), is derived from the first principles of thermodynamics of irreversible process; the system extends and connects the Cahn-Hilliard equation [1] and the Ginzburg-Landau equation. In one space dimension, under a suitable scaling of parameters, the EOM equation is expressed as follows [9][10].

$$\begin{cases} u_t = -\varepsilon^2 u_{xxxx} + ((a+v^2)u)_{xx} & \text{in } 0<x<l, \quad t>0 \\ v_t = v_{xx} + (b-u^2-v^2)v & \text{in } 0<x<l, \quad t>0 \\ u_x = u_{xxx} = v_x = 0 & \text{at } x=0 \text{ and } l, \quad t>0 \\ u|_{t=0} = u_0, \ v|_{t=0} = v_0 & \text{on } 0 \le x \le l, \end{cases} \tag{1}$$

where u, v denote the local concentration and the local degree of order, respectively. The constants a, b are assumed to be positive. The parameter ε depends on the ratio of the surface energy between the original u and v. It is easily noticed that the equation for u is Cahn-Hilliard type while the one for v is Ginzburg-Landau type. A direct computation shows that the total concentration of u is conserved under the evolution;

namely, there holds

$$\frac{1}{l}\int_0^l u(x,t)\,dx = m, \tag{2}$$

where m is a positive constant. Given initial data u_0, v_0 should satisfy required compatibility conditions:

$$(u_0)_x = (u_0)_{xxx} = (v_0)_x = 0 \qquad \text{at } x = 0,\, l, \quad \text{and } \frac{1}{l}\int_0^l u_0(x)\,dx = m.$$

For further information and other related issues, especially for the study of Cahn-Hilliard equations, we refer to [2][3][5][6][7] [11][12][13][15] and the references cited therein. We here just mention that the system (1) takes its rise in a gradient flow of the total free energy

$$F_\varepsilon[u,v] := \int_0^l (\frac{\varepsilon^2}{2}u_x^2 + \frac{1}{2}v_x^2 + \frac{a}{2}u^2 + \frac{1}{4}v^4 - \frac{b}{2}v^2 + \frac{1}{2}u^2v^2)\,dx. \tag{3}$$

Indeed one easily show that

$$\frac{d}{dt}F_\varepsilon[u,v](t) = -\int_0^l \{-\varepsilon^2 u_{xxx} + ((a+v^2)u)_x\}^2\,dx - \int_0^l v_t^2\,dx \leq 0,$$

for any solution (u,v) to (1); F_ε plays the role as Lyapunov functional of the system. Here and hereafter, the solution are understood to be classical.

In this article, we deal with the singular perturbation problem for (1). In particular, we are concerned with the behavior of steady state solutions as the parameter ε tends to 0. We recall that steady state solutions are given by $u_t = v_t \equiv 0$ in (1); that is,

$$\begin{cases} -\varepsilon^2 u_{xxxx} + ((a+v^2)u)_{xx} = 0 & \text{in } 0 < x < l \\ v_{xx} + (b - u^2 - v^2)v = 0 & \text{in } 0 < x < l \\ u_x = u_{xxx} = v_x = 0 & \text{at } x = 0 \text{ and } l \\ (1/l)\int_0^l u\,dx = m. \end{cases} \tag{4}$$

In [9], it is shown that every solution of (1) converges to a steady state solution as $t \to \infty$. Furthermore, the solution (u,v) to the system (4) is constructed as a critical point of the functional (3) in the function space

$$\mathcal{A} := \{(u,v) \in (H^1(0,l))^2 \,|\, \frac{1}{l}\int_0^l u\,dx = m\}.$$

We recall that $(u,v)=(m,0)$ and $(m,\pm\sqrt{b-m^2})$ if $b>m^2$ are refered to as trivial solutions. One of main achievements in [9] is that the structure of steady state solutions is rather rich enough; to be specific, the next properties is established in [8][9].

Theorem 1 ([9]) *There is at least one monotone non-trivial steady state solution of (4) if we assign suitably large b and m^2. Moreover, for any integer $k\geq 2$ and for appropriately chosen large b and m^2 depending on k, (4) has at least one non-monotone non-trivial steady state solution, each of whose derivatives changes sign exactly $(k-1)$-times.*

It is a challenging problem to study the behavior as $\varepsilon\to 0$, $t\to\infty$ for steady state solutions (u,v). This is the focus of this paper.

If we put $\varepsilon=0$ in (4) then we find that

$$(a+v^2)u=M \tag{5}$$

for some constant M. Invoking (2) we compute $m=l^{-1}\int_0^l M/(a+v^2)\,dx$ and hence

$$M=\frac{lm}{\int_0^l(a+v^2)^{-1}dx}>0. \tag{6}$$

Therefore the reduced equation of $\varepsilon=0$ for v is described as

$$\begin{cases} v_{xx}+(b-(\frac{M}{a+v^2})^2-v^2)v=0 & \text{in } 0<x<l \\ v_x=0 & \text{at } x=0 \text{ and } l, \end{cases} \tag{7}$$

where M is defined by (6). The equations (5)(6)(7) is analysed in §2.

Our intension is now to ask whether the solutions of (4) "converge" to the ones of (7) as $\varepsilon\to 0$ or not? If this really holds true, in what spaces or norms do they converge? Our answer to this question reads as follows.

Theorem 2 *Let $(u^\varepsilon,v^\varepsilon)$ denote solutions to (4) with positive parameters ε. The constants a, b, m are kept fixed. Then there exists a monotone decreasing sequence $\{\varepsilon_n\}_{n=1}^\infty$ tending to zero such that*

$$\begin{aligned} u^{\varepsilon_n}&\to u \quad \textit{strongly in } L^2(0,l) \\ v^{\varepsilon_n}&\to v \quad \textit{in } C[0,l]\cap C^1(0,l) \end{aligned} \tag{8}$$

as $n\to\infty$, where (u,v) solves the reduced equations (5)(6)(7).

2 Reduced equations

In this section, we investigate the reduced equation (7) with (6), which is provided by assigning $\varepsilon = 0$ in (4). As is carried out in our previous works [8][9], a variational method is exploited. We recall the functional (3) with $\varepsilon = 0$ over the function space $\mathcal{A}' := \{(u,v) \in L^2(0,l) \times H^1(0,l) \mid l^{-1}\int_0^l u\,dx = m\} \subset \mathcal{A}$; that is,

$$F_0[u,v] := \int_0^l (\frac{1}{2}v_x^2 + \frac{a}{2}u^2 + \frac{1}{4}v^4 - \frac{b}{2}v^2 + \frac{1}{2}u^2v^2)\,dx.$$

For any $\psi \in L^2(0,l)$ with $\int_0^l \psi\,dx = 0$ and $\varphi \in H^1(0,l)$, we derive

$$\frac{d}{d\sigma}F_0[(u+\sigma\psi, v]|_{\sigma=0} = \int_0^l (au+v^2)\psi\,dx,$$
$$\frac{d}{d\eta}F_0[u, v+\eta\varphi]|_{\eta=0} = \int_0^l \{v_x\varphi_x + (v^2+u^2-b)v\varphi\}\,dx,$$

which immediately implies that every critical point of F_0 over $\mathcal{A}'$ gives rise to the solutions (5)(7).

Taking account that the energy level of trivial solutions are

$$F_0[m,0] = \frac{lm^2a}{2}, \quad F_0[m,\pm\sqrt{b-m^2}] = \frac{lm^2a}{2} - \frac{l}{4}(b-m^2)^2 \text{ if } b > m^2,$$

we need to seek a test function $(\psi,\varphi) \in \mathcal{A}'$ such that

$$F_0[\psi,\varphi] < \frac{lm^2a}{2} - \frac{l}{4}(b-m^2)^2. \tag{9}$$

Then the standard minimization procedure yields a non-trivial solution for (5)(7). To this aim, as performed in our previous paper [9], we introduce

$$\psi(x) = m - \delta\cos\frac{\pi x}{l}, \quad \varphi(x) = \pm\sqrt{b - (m-\delta\cos\frac{\pi x}{l})^2},$$

where $\delta > 0$ with $b > (m+\delta)^2$ is determined later. We compute

$$\begin{aligned} F_0[\psi,\varphi] - \{\frac{lm^2a}{2} - \frac{l}{4}(b-m^2)^2\} \\ = \frac{la\delta^2}{4} + \frac{\delta^2\pi^2}{2l}\int_0^l \frac{(m-\delta\cos(\pi x/l))^2\sin^2(\pi x/l)}{b-(m-\delta\cos(\pi x/l))^2}\,dx \\ - \frac{lm^2\delta^2}{2} - \frac{3l}{32}\delta^4 + \frac{l\delta^2}{4}(b-m^2). \end{aligned}$$

If we further assume that $m^2 = b/2$ and $\delta = \sqrt{b}/4$, then we infer that

$$F_0[\psi,\varphi] - \{\frac{lm^2a}{2} - \frac{l}{4}(b-m^2)^2\}$$
$$= -\frac{67}{2^{13}}lb^2 + (\frac{la}{64} + \frac{\pi^2}{32l}\int_0^l \frac{(4-\sqrt{2}\cos(\pi x/l))^2\sin^2(\pi x/l)}{32-(4-\sqrt{2}\cos(\pi x/l))^2}\,dx)b,$$

from where we obtain (9) choosing larger b if necessary. The monotonicity of v as well as the existence of multi-bump solutions can be accomplished similarly as in [9]. To summarize we have proved the next proposition.

Proposition 3 *For suitably large b and m^2, there exists at least one monotone non-trivial solution of (5)(7). Moreover, for any integer $k \geq 2$ and for appropriately chosen large b and m^2 depending on k, (5)(7) has at least one non-monotone non-trivial solution, whose derivative changes sign exactly $(k-1)$ times, respectively.*

3 Proof of the convergence theorem

This section is devoted to the proof of Theorem 2. We begin with a priori estimates. An integration yields that the system (4) is equivalent to

$$\begin{cases} -\varepsilon^2 u_{xx} + (a+v^2)u = M_\varepsilon & \text{in } 0<x<l \\ v_{xx} + (b-u^2-v^2)v = 0 & \text{in } 0<x<l \\ u_x = v_x = 0 & \text{at } x=0 \text{ and } l \\ (1/l)\int_0^l u\,dx = m, & \end{cases} \tag{10}$$

for certain constant M_ε. The next proposition is in order, where the solution of (10) is simply written as (u,v) instead of $(u^\varepsilon, v^\varepsilon)$, without possible confusion.

Proposition 4 *For every solution (u,v) of (10), it follows that , independent of ε,*

$$\frac{a}{a+m}m \leq u(x) \leq \frac{a+b}{a}m \quad on\ 0 \leq x \leq l,$$
$$\max_{x\in[0,l]} |v(x)| \leq \sqrt{b}, \qquad am \leq M_\varepsilon \leq (a+b)m.$$

Furthermore, there exists a constant C independent of ε such that

$$\int_0^l \varepsilon^2 u_x^2\, dx + \int_0^l v_x^2\, dx \le C.$$

Proof. First we deal with a bound on v. If there happens to be $v(x) > \sqrt{b}$ over some interval, then at the maximum point x_M of v there holds

$$0 \ge v_{xx}(x_M) = v(x_M)(v(x_M)^2 + u(x_M)^2 - b) > 0,$$

a contradiction. The chance of being $v(x) < -\sqrt{b}$ is similarly discarded and we have proved $\max |v| \le \sqrt{b}$.

Next, at the minimum point x_m of u, we see that

$$0 \ge -\varepsilon^2 u_{xx}(x_m) = M_\varepsilon - (a + v(x_m)^2)u(x_m),$$
$$\min_{x\in[0,l]} u(x) = u(x_m) \ge \frac{M_\varepsilon}{a + v(x_m)^2} \ge \frac{M_\varepsilon}{a+b},$$

from where we obtain

$$m = \frac{1}{l}\int_0^l u\, dx \ge \frac{M_\varepsilon}{a+b}.$$

A similar consideration at the maximum point of u leads us to

$$\max_{x\in[0,l]} u(x) \le \frac{M_\varepsilon}{a+v^2} \le \frac{M_\varepsilon}{a} \le \frac{a+b}{a} m,$$
$$m = \frac{1}{l}\int_0^l u\, dx \le \frac{M_\varepsilon}{a}.$$

Finally we ascertain integral estimates. Multiply the equations in (10) by u and v, respectively, and integrate, we deduce that

$$\int_0^l \varepsilon^2 u_x^2\, dx + \int_0^l (a+v^2)u^2\, dx = \int_0^l M_\varepsilon u\, dx = lmM_\varepsilon,$$
$$\int_0^l v_x^2\, dx = \int_0^l (b - u^2 - v^2)v^2\, dx.$$

Combined with pointwise estimates above, we have finished the derivation of integral bounds.

Now we complete the proof Theorem 2. We appeal to the variation of constants formula.

$$\begin{aligned}
\varepsilon^2 u^\varepsilon(x) &= \frac{1}{2}\int_0^x \{(a+v^\varepsilon(y)^2)u^\varepsilon(y) - M_\varepsilon\}(x-y)\,dy \\
&\quad + \frac{1}{2}\int_x^l \{(a+v^\varepsilon(y)^2)u^\varepsilon(y) - M_\varepsilon\}(y-x)\,dy \\
v^\varepsilon(x) &= \frac{1}{2}\int_0^x (v^\varepsilon(y)^2 + u^\varepsilon(y)^2 - b)v^\varepsilon(y)(x-y)\,dy \\
&\quad + \frac{1}{2}\int_x^l (v^\varepsilon(y)^2 + u^\varepsilon(y)^2 - b)v^\varepsilon(y)(y-x)\,dy.
\end{aligned} \tag{11}$$

Thanks to Proposition 4, we infer that a monotone decreasing sequence $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots \to 0$ can be selected so that $M_\varepsilon \to M$ as well as

$$u^{\varepsilon_n} \to u, \quad v^{\varepsilon_n} \to v \quad \text{strongly in } L^2(0,l),$$

where $am \leq M \leq (a+b)m$ means for a constant and u, $v \in L^2(0,l)$ are bounded functions. Sending $\varepsilon_n \to 0$ in (11), we conclude that

$$\begin{aligned}
0 &= \frac{1}{2}\int_0^x \{(a+v(y)^2)u(y) - M\}(x-y)\,dy \\
&\quad + \frac{1}{2}\int_x^l \{(a+v(y)^2)u(y) - M\}(y-x)\,dy \\
v(x) &= \frac{1}{2}\int_0^x (v(y)^2 + u(y)^2 - b)v(y)(x-y)\,dy \\
&\quad + \frac{1}{2}\int_x^l (v(y)^2 + u(y)^2 - b)v(y)(y-x)\,dy.
\end{aligned}$$

which implies that (u, v) solves (5)(7). This finalizes the proof of Theorem 2.

4 Discussions

We investigate a singular perturbation problem for steady state solutions to a model equation of phase separation. Analytical convergence results are established as the parameter, which depends on the surface energy between the components, tends to zero. It is clarified that the rich structure of steady state solutions is preserved under this singular perturbation limit.

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