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Asymptotic behavior of solutions of anisotropic curvature motions

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1 Introduction

If the two chemical or physical states coexists, the interfaces are often observed as the boundaries of two states. The dynamics of interfaces is one of the interesting problems in applied mathematics. If the interfaces between the two states are moved by the local forces, they are often controlled by the surface free energy and the energy difference between two phases. The surface free energy usually depends on the orientations which represents $\Psi(\theta)$ is a function of θ with period π where θ is the angle between the x axis and the normal vector. Let Γ_t be the interface, V_n a normal velocity of the interface and κ a curvature. In this note we consider the following moving boundary problem in the two-dimensional space $(N=2)$:

$$\begin{cases} V_n = -\Psi(\theta)\left(\Psi(\theta)+\Psi''(\theta)\right)\kappa + a\Psi(\theta) \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases} \tag{1.1}$$

where a is a constant which corresponds to the energy difference between the two states. This equation was introduced independently by Angenent and Gurtin [1] (also see [2, 3, 9] for instance).

We always assume

(H1) $\Psi \in C^2(\mathbb{R})$ and Ψ'' is a globally Lipschitz function,

(H2) there exists positive constants λ_i $(i=1,2,3,4)$ such that for all $\theta \in \mathbb{R}$

$$\lambda_1 \le \Psi(\theta) \le \lambda_2, \qquad \lambda_3 \le \Psi(\theta)+\Psi''(\theta) \le \lambda_4.$$

If the interface Γ_t is represented by the level set of U, that is,

$$\Gamma_t = \{(x,y) \mid U(x,y,t)=0\},$$

then U satisfies the following degenerate parabolic equation:

$$U_t = -|\nabla U|\Psi(\theta)\left\{-\sum_{i,j}\frac{\partial^2\Phi}{\partial p_i \partial p_j}(\nabla U)\frac{\partial^2 U}{\partial x_i \partial x_j} + a\right\}. \tag{1.2}$$

If Γ_t is a graph, then we may set $U(x,y,t) = y - u(x,t)$. Define

$$\theta(p) := \mathrm{Arccos}\left(\frac{-p}{\sqrt{1+p^2}}\right),$$

and then $\theta'(p) = -1/(1+p^2)$. Denoting the angle between the normal vector $(-u_x, 1)$ and the x axis by $\theta(u_x)$ and setting

$$G_1(u_x) := \Psi(\theta(u_x))\left(\Psi(\theta(u_x)) + \Psi''(\theta(u_x))\right), \qquad G_2(u_x) := a\Psi(\theta(u_x))\sqrt{1+u_x^2},$$

we see that u satisfies the following parabolic equation:

$$\begin{cases} u_t = \dfrac{G_1(u_x)}{1+u_x^2}u_{xx} + G_2(u_x) & \text{in } \mathbb{R}\times(0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}. \end{cases} \tag{1.3}$$

The existence of solutions to (1.3) is guaranteed by Barles, Biton and Ley [8, Chapter 4] when the initial function u_0 has a polynomial growth at infinity and satisfies:

(H3) There exist $\nu \in [0, (1+\sqrt{5})/2)$ and a modulus of continuity m such that

$$|u_0(x) - u_0(y)| \le m\left((1+|x|+|y|)^\nu |x-y|\right) \qquad \text{for all } x, y \in \mathbb{R}.$$

The comparison principle also holds for (1.3). For the detail, see [10].

2 The traveling curved fronts

Consider the solution of

$$u_t = \frac{G_1(u_x)}{1+u_x^2}u_{xx} + G_2(u_x). \tag{2.1}$$

Definition. *We say that a solution u of (2.1) is a traveling curved front if it holds that $u(x,t) = \varphi(x - c_1 t) + c_2 t$ for all $(x,t) \in \mathbb{R}\times[0,+\infty)$ where there exist $0 < \theta_- < \theta_+ < \pi$ such that the function φ has two asymptotic lines $y = \tan(\theta_\pm - \pi/2)x$ as $x \to \pm\infty$.*

The function φ is called the profile of the front and the vector $c := {}^t(c_1, c_2)$ is the velocity of the front.

If u is a traveling curved front, then its profile φ satisfies

$$c_2 - c_1\varphi'(x) = \frac{G_1(\varphi'(x))\varphi''(x)}{1+\varphi'(x)^2} + G_2(\varphi'(x)). \tag{2.2}$$

Let $\theta(x)$ be the angle between the x-axis and the normal vector to the graph of φ at the point x. Then we have

$$\varphi'(x) = \tan\left(\theta - \frac{\pi}{2}\right), \tag{2.3}$$

and (2.2) reduces to

$$\begin{aligned} \theta'(x) &= f(\theta), && (2.4)\\ \theta(-\infty) &= \theta_-, && (2.5)\\ \theta(\infty) &= \theta_+, && (2.6) \end{aligned}$$

where

$$f(\theta) := \frac{c_1 \cos\theta + c_2 \sin\theta - a\Psi(\theta)}{\Psi(\theta)(\Psi(\theta) + \Psi''(\theta))\sin\theta}.$$

By (2.5) and (2.6), c_1 and c_2 are uniquely determined as follows:

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= a \begin{pmatrix} \cos\theta_+ & \sin\theta_+ \\ \cos\theta_- & \sin\theta_- \end{pmatrix}^{-1} \begin{pmatrix} \Psi(\theta_+) \\ \Psi(\theta_-) \end{pmatrix} \\ &= -\frac{a}{\sin(\theta_+ - \theta_-)} \begin{pmatrix} \sin\theta_- & -\sin\theta_+ \\ -\cos\theta_- & \cos\theta_+ \end{pmatrix} \begin{pmatrix} \Psi(\theta_+) \\ \Psi(\theta_-) \end{pmatrix}. \end{aligned} \tag{2.7}$$

First we state the following lemma.

Lemma 2.1. *For any $\theta_\pm$ $(0 < \theta_- < \theta_+ < \pi)$, there exist a unique pair of constants (c_1, c_2) such that*

$$f(\theta_\pm) = 0.$$

Moreover, if $a \neq 0$, then

$$\begin{cases} af(\theta) > 0 & \text{for } \theta_- < \theta < \theta_+, \\ af(\theta) < 0 & \text{for } 0 \leq \theta < \theta_-, \quad \theta_+ < \theta \leq \pi, \\ af'(\theta_-) > 0, & af'(\theta_+) < 0. \end{cases} \tag{2.8}$$

See [10] for the detail. As a consequence of this Lemma, we can easily see that there is a connecting orbit from θ_- to θ_+ satisfying (2.4) and then a unique traveling curved front. Note that Angenent and Gurtin in Section 6.3 of [1] already proved the lemma in the context of the Finsler metric and the existence of the traveling curved fronts was already shown in [1, Theorem on steady motions, p. 349] or [9, Section 9.2, p. 65]. The advantage of our proof is that it also gives the global stability of the traveling curved front.

Theorem 2.2. *Let $u(x,t)$ be a solution of* (2.1) *with $u(x,0) = u_0(x)$ satisfying*

$$\lim_{x\to\pm\infty} |u_0(x) - x\tan(\theta_\pm - \pi/2)| = 0. \tag{2.9}$$

Then,

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} |u(x,t) - \varphi(x - c_1 t) - c_2 t| = 0.$$

This theorem can be proved using similar arguments as in [12]. To construct the supersolutions and the subsolutions, we also need an other kinds of traveling curved fronts (see [10]).

Next, we consider any interfaces which may not be represented by the graph. Let Γ_0 be a curve in $\mathbb{R}^2$ possessing asymptotic lines $y = x\tan(\theta_\pm - \pi/2)$ as $x \to \pm\infty$. More precisely, there are two interfaces $\Gamma_0^\pm = \{(x,y) \mid y = u_0^\pm(x)\}$ where

$$u_0^-(x) \le \inf_{(x,y)\in\Gamma_0} y \le \sup_{(x,y)\in\Gamma_0} y \le u_0^+(x),$$
$$\lim_{x\to\pm\infty} |u_0^\pm(x) - x\tan(\theta_\pm - \pi/2)| = 0.$$

Let U_0 be a continuous function such that

$$\{(x,y)\in\mathbb{R}^2 \mid U_0(x,y) = 0\} = \Gamma_0,$$
$$\liminf_{y\to\infty} U_0(x,y) > 0, \qquad \limsup_{y\to-\infty} U_0(x,y) < 0 \qquad \text{for all } x\in\mathbb{R}.$$

Then, we obtain the following result.

Theorem 2.3. *Let Γ_0 be as above and U be the unique solution of* (1.2) *with $U(x,y,0) = U_0(x,y)$. Set*

$$\Gamma_t := \{(x,y)\in\mathbb{R}^2 \mid U(x,y,t) = 0\}.$$

Then, for any $\varepsilon > 0$, there exists $T > 0$ such that for all $t \ge T$

$$\Gamma_t \subset \{(x,y)\in\mathbb{R}^2 \mid |y - \varphi(x - c_1 t) - c_2 t| \le \varepsilon\}.$$

3 Singular limit of traveling curved fronts and crystalline motions

In this section we consider the profile of the traveling waves when Ψ includes the small parameter $\varepsilon > 0$.

We assume that $\Psi = \Psi(\theta,\varepsilon)$ belongs to $C^2(\mathbb{R},\mathbb{R})$ and satisfies (H2) where λ_1 and λ_2 are independent of ε; λ_3 and λ_4 depend on ε. We use $f(\theta,\varepsilon)$ instead of $f(\theta)$ to emphasize the dependence of ε.

(H4) There exist $0 \le \theta_1 < \theta_2 < \cdots < \theta_m < 2\pi$ and positive constants m_j such that

$$\Psi(\theta,\varepsilon) + \Psi_{\theta\theta}(\theta,\varepsilon) \to \sum_{j=1}^{m} m_j \delta(\theta - \theta_j) \qquad \text{in the distribution sense as } \varepsilon \downarrow 0.$$

(H5) There are positive integers j_1 and j_2 such that $1 \le j_1 < j_2 \le m$ and $\theta_- < \theta_{j_1} < \theta_{j_2} < \theta_+$ and $\theta_{j_1 - 1} < \theta_-$, if $j_1 \ge 2$, and $\theta_+ < \theta_{j_2+1}$, if $j_2 \le m-1$.

Using (2.4) and the definition of f, we have

$$\int_{\theta_0}^{\theta} \frac{ds}{f(\theta,\varepsilon)} = \int_{x_0}^{x} dx.$$

Putting

$$d_j := \frac{m_j \Psi(\theta_j) \sin\theta_j}{c_1 \cos(\theta_j) + c_2 \sin(\theta_j) - a\Psi(\theta_j)}, \tag{3.1}$$

we see that θ converges to the step function and that the traveling wave φ converges to the segment with the slope $\tan(\theta_j - \pi/2)$ in the interval $[x_j, x_j + d_j]$ $(j = j_1, \cdots j_2)$ where $x_{j+1} = x_j + d_j$ and x_{j_1} is chosen appropriately. The traveling front converges to a faceting which moves the constant velocity. The length of the each facet and its normal vector are

$$L_j := \frac{d_j}{\sin\theta_j}, \quad n_j := \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}$$

respectively. We note that the length of the facet does not depend on t because it is a traveling front. The normal velocity is

$$V_j := n_j \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{3.2}$$

By (3.1), we have

$$\begin{aligned} V_j &= c_1 \cos\theta_j + c_2 \sin\theta_j \\ &= \left(\frac{m_j \sin\theta_j}{d_j} + a \right) \Psi(\theta_j) \\ &= \left(\frac{m_j}{L_j} + a \right) \Psi(\theta_j). \end{aligned} \tag{3.3}$$

This shows that the traveling front of (1.1) converges to the traveling faceting governed by the crystalline motion (3.3) (see [1, Section 10.3] or [9, Section 12.5]).

We consider the following example (see Fig. 1). Set

$$\Psi(\theta,\varepsilon) := \sqrt{\cos^2\left(\theta-\frac{\pi}{4}\right)+\varepsilon} + \sqrt{\sin^2\left(\theta-\frac{\pi}{4}\right)+\varepsilon}.$$

Then,

$$\Psi(\theta,\varepsilon)+\Psi_{\theta\theta}(\theta,\varepsilon) = \frac{\varepsilon(1+\varepsilon)}{(\cos^2(\theta-\pi/4)+\varepsilon)^{3/2}} + \frac{\varepsilon(1+\varepsilon)}{(\sin^2(\theta-\pi/4)+\varepsilon)^{3/2}}.$$

Setting

$$\theta_j := \frac{\pi+2(j-1)\pi}{4}, \qquad (j=1,2,3,4),$$

we get

$$\Psi(\theta_j,\varepsilon)+\Psi_{\theta\theta}(\theta_j,\varepsilon) = \frac{\varepsilon(1+\varepsilon)}{(1+\varepsilon)^{3/2}} + \frac{\varepsilon(1+\varepsilon)}{\varepsilon^{3/2}} \to \infty \qquad \text{as } \varepsilon \to 0.$$

Using the change of variables $\sin s = \sqrt{\varepsilon}\tan\eta$, we can check that

$$\int_{-\delta}^{\delta} \frac{\varepsilon(1+\varepsilon)}{(\sin^2 s+\varepsilon)^{3/2}}ds = \int_{-\arctan(\sin\delta/\sqrt{\varepsilon})}^{\arctan(\sin\delta/\sqrt{\varepsilon})} \frac{(1+\varepsilon)\cos\eta}{\sqrt{1-\varepsilon\tan^2\eta}}d\eta \to 2 \qquad \text{as } \varepsilon \to 0.$$

We see that (H4) and (H5) hold and that $m_j = 2$.

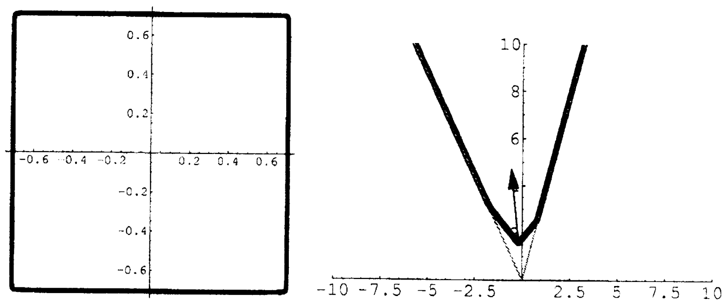

Figure 1: The profiles of the frank diagram and the traveling curved front

4 Expanding solutions of the anisotropic mean curvature flow

Hereafter, we assume $a < 0$, which means $G_2 < 0$. In an isotropic case (i.e., $\Psi \equiv 1$), Deckelnick et al [7] proved that the solution $u(x,t)$ of (1.3) with $u_0(x) = |x|\tan(\theta_0 - \pi/2)$ behaves

$$\frac{1}{t}\left|u(x,t) - tQ\left(\frac{x}{t}\right)\right| \to 0 \qquad \text{as } t \to \infty$$

where

$$Q(s) = \begin{cases} -\sqrt{a^2 - s^2} & (|s| \le |a|\cos\theta_0), \\ -|s|\tan(\theta_0 - \pi/2) - \dfrac{|a|}{\sin\theta_0} & (|s| > |a|\cos\theta_0) \end{cases}$$

(see Fig. 2). Since $s^2 + Q(s)^2 = a^2$ on $[-|a|\cos\theta_0, |a|\cos\theta_0]$, the solution looks like an arc after an appropriate rescaling.

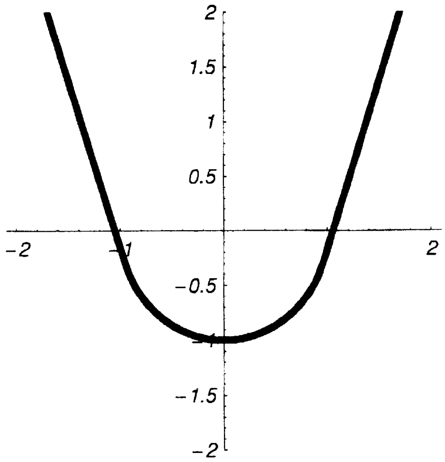

Figure 2: The graph of Q with the case where $a = 1$ and $\theta_0 = \pi/10$

Next consider the anisotropic case. Set

$$\widehat{G}_3(\theta) := a\left\{\Psi'(\theta)\cos(\theta - \pi/2) + \Psi(\theta)\sin(\theta - \pi/2)\right\}.$$

By (H2), we see that $\widehat{G}_3'(\theta) = a(\Psi'' + \Psi) < 0$. Thus we can define

$$\Theta(s) := \widehat{G}_3^{-1}(-s).$$

For the anisotropic case we can show that the limiting profile after the rescaling is

$$\widetilde{Q}(s) := \begin{cases} s\tan(\theta_- - \pi/2) + \dfrac{a\Psi(\theta_-)}{\cos(\theta_- - \pi/2)} \\ \qquad\qquad \text{for } s \le -\widehat{G}_3(\theta_-), \\ -a\Psi'(\Theta(s))\sin(\Theta(s) - \pi/2) + a\Psi(\Theta(s))\cos(\Theta(s) - \pi/2) \\ \qquad\qquad \text{for } -\widehat{G}_3(\theta_-) < s < -\widehat{G}_3(\theta_+), \\ s\tan(\theta_+ - \pi/2) + \dfrac{a\Psi(\theta_+)}{\cos(\theta_+ - \pi/2)} \\ \qquad\qquad \text{for } s \ge -\widehat{G}_3(\theta_+). \end{cases} \tag{4.1}$$

We can check that $(s, \widetilde{Q}(s))$ is a portion of a circle on the Finsler metric. This result is also applicable to the Allen-Cahn equation. See [14] for the detail.

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