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Kyoto University
Movement of Hot Spots
on the Exterior Domain of a Ball

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1 Introduction

We consider the initial-boundary value problems of the heat equation in the exterior domain of a ball,

\begin{align}
\begin{cases}
\partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\
\partial_n u = 0 & \text{on } \partial\Omega \times (0, \infty), \\
\quad u(x, 0) = \phi(x) & \text{in } \Omega,
\end{cases}
\end{align}

and

\begin{align}
\begin{cases}
\partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\
\quad u(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\
\quad u(x, 0) = \phi(x) & \text{in } \Omega,
\end{cases}
\end{align}

where

\[ \Omega = \mathbb{R}^N \setminus \overline{B(0, L)}, \quad L > 0, \quad N \geq 2. \]

Here \( \partial_t = \partial/\partial t, \partial_n = \partial/\partial \nu, \nu = \nu(x) \) is the outer unit normal vector to \( \partial\Omega \) at \( x \in \partial\Omega \), and \( B(0, L) = \{ x \in \mathbb{R}^N : |x| < L \} \). Throughout this paper we assume that

\[ \phi \in L^2(\Omega, e^{\lambda |x|^2} \, dx) \]

for some \( \lambda > 0 \). For any \( t > 0 \), we may denote by \( H(t) \) the set of the maximum points of \( u(\cdot, t) \), that is,

\[ H(t) = \left\{ x \in \overline{\Omega} : u(x, t) = \max_{y \in \overline{\Omega}} u(y, t) \right\}, \]
and call $H(t)$ the set of hot spots of the solution $u$ at the time $t$. In this paper we study the movement of hot spots $H(t)$ of the solution $u$ of (1.1) or (1.2) as $t \to \infty$.

Chavel and Karp [3] studied the heat equation $\partial_t u = \Delta u$ in several Riemannian manifolds, and obtained some asymptotic properties of solutions concerning the movement of hot spots of the solution. In particular, for the Euclidean space $\mathbb{R}^N$, they proved that, for any nonzero, nonnegative initial data $\phi \in L_{c}^{\infty}(\mathbb{R}^N)$, the hot spots $H(t)$ of the solution at each time $t > 0$ are contained in the closed convex hull of the support of $\phi$, and $H(t)$ tends to the center of mass of $\varphi$ as $t \to \infty$. Subsequently, Jimbo and Sakaguchi [11] studied the movement of hot spots of the solution of the heat equation in the half space $\mathbb{R}_{+}^N$ and in the exterior domain of a ball, under boundary conditions. In particular, for the Cauchy-Neumann problem (1.1) in the exterior domain $\Omega = \mathbb{R}^N \setminus B(0, L)$ with the nonzero, nonnegative, radially symmetric initial data $\phi \in L_{c}^{\infty}(\Omega)$, they proved that the hot spots $H(t)$ satisfies

\begin{equation}
H(t) \subset \partial \Omega = \partial B(0, L)
\end{equation}

for all sufficiently large $t$. Furthermore, for the Cauchy-Dirichlet problem in the exterior domain $\Omega = \mathbb{R}^3 \setminus B(0, L)$ with the nonzero, nonnegative, radially symmetric initial data $\phi \in L_{c}^{\infty}(\Omega)$, they proved that there exist a positive constant $T$ and a continuous function $r = r(t) \in C([T, \infty) : (L, \infty))$ such that

\begin{equation}
\lim_{t \to \infty} r(t)^3 t^{-1} = 2
\end{equation}

and

$H(t) = \{x \in \mathbb{R}^N : |x| = r(t)\}, \quad t \geq T.$

Their proofs of (1.3) and (1.4) heavily depend on the radially symmetry of the solutions and the properties of zero sets of the heat equation in $\mathbb{R}$, and it seems so difficult to apply their proofs to the solutions without the radially symmetry. (For the movement of hot spots of the solution for the Cauchy-Neumann problem in bounded domains, see [1], [2], [10], [12], and [14].)

In this paper we study the movement of hot spots $H(t)$ of the solutions of the Cauchy-Neumann problem (1.1) or the Cauchy-Dirichlet problem (1.2) in the exterior domain $\Omega$ of a ball, without the radially symmetry of the solutions. In Sections 2 and 3, we give the results on the movement of the set of hot spots $H(t)$ for the problems (1.1) and (1.2), respectively.
2 On the Cauchy-Neumann Problem (1.1)

In this section we assume

\[(2.1) \quad \phi \in L^2(\Omega, e^{N|\mathbf{x}|^2} \, dx), \quad \int_{\Omega} \phi(x) \, dx > 0, \]

and give some results on the movement of the hot spots \(H(t)\) for the solution of (1.1) as \(t \to \infty\). We first give a sufficient condition for the hot spots \(H(t)\) to exist only on the boundary \(\partial \Omega\) for all sufficiently large \(t\).

**Theorem 2.1** (See Theorem 1.1 in [8].) Let \(u\) be a solution of the Cauchy-Neumann problem (1.1) under the condition (2.1). Put

\[ A_{\phi}^N = \int_{\Omega} x\phi(x) \left(1 + \frac{L^N}{N-1} |\mathbf{x}|^{-N} \right) \, dx / \int_{\Omega} \phi(x) \, dx. \]

Assume

\[(2.2) \quad A_{\phi}^N \in B(0, L) = \mathbb{R}^N \setminus \overline{\Omega}. \]

Then there exists a positive constant \(T\) such that

\[(2.3) \quad H(t) \subset \partial \Omega = L \{ x \in \mathbb{R}^N : |x| = L \}

for all \(t \geq T\).

In particular, we see that, under the condition (2.1), the hot spots \(H(t)\) of the radial solution of (1.1) exists only on the boundary of the domain \(\Omega\) for all sufficiently large \(t\).

**Remark 2.1** Let \(u\) be a solution of the Cauchy-Neumann problem (1.1) under the condition (2.1). Let \(C(t)\) a center of mass of \(u(t)\), that is,

\[ C(t) = \int_{\Omega} xu(x, t) \, dx / \int_{\Omega} u(x, t) \, dx. \]

Then it does not necessarily hold that \(C(t) = C(0)\) for all \(t > 0\). On the other hand, we put

\[ A(t)^N(t) \equiv \int_{\Omega} xu(x, t) \left(1 + \frac{L^N}{N-1} |\mathbf{x}|^{-N} \right) \, dx / \int_{\Omega} u(x, t) \, dx, \quad t > 0. \]

Then we have \(A_{\phi}^N(t) = A_{\phi}^N\) for all \(t > 0\), and \(\lim_{t \to \infty} C(t) = A_{\phi}^N\).
Next we give a result on the limit of the set $H(t)$ as $t \to \infty$.

**Theorem 2.2** (See Theorem 1.2 in [8].) Let $u$ be a solution of the Cauchy-Neumann problem (1.1) under the condition (2.1). Assume $A^N_\phi \neq 0$. Put

\[ x_\infty = L \frac{A^N_\phi}{|A^N_\phi|} \quad \text{if} \quad A^N_\phi \in B(0, L) \quad \text{and} \quad x_\infty = A^N_\phi \quad \text{if} \quad A^N_\phi \in \overline{\Omega} \]

Then

\[ \lim_{t \to \infty} \sup \{|x_\infty - y| : y \in H(t)\} = 0. \]

By Theorem 2.2, we see that the hot spots $H(t)$ tends to one point $x_\infty$ as $t \to \infty$ if $A_\phi \neq 0$, and see that (1.3) does not hold if $A_\phi \in \Omega$.

Next we will explain the outline of the proofs of Theorems 2.1 and 2.2. As in stated in [11], it is difficult to know the sign of differential of the Neumann heat kernel even for the case that $\Omega$ is the exterior of a ball, and so it seems difficult to obtain Theorems 2.1 and 2.2 by using the fundamental properties of the Neumann heat kernel. We consider the following two eigenvalue problems,

\[
\begin{aligned}
\{P_0 \varphi \equiv \frac{1}{\rho} \text{div} (\rho \nabla \varphi) &= -\lambda \varphi \quad \text{in} \quad \mathbb{R}^N, \\
\varphi &\in H^1(\mathbb{R}^N, \rho dy), \quad \rho(y) = \exp\left(\frac{|y|^2}{4}\right),
\end{aligned}
\]

and

\[(2.4) \quad -\Delta_{\mathbb{S}^{N-1}} Q = \omega Q \quad \text{on} \quad \mathbb{S}^{N-1}, \]

such that $0 = \omega_0 < \omega_1 = N - 1 < \omega_2 = 2N < \omega_3 < \cdots$, where $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. Let $l_k$ be the dimension of the eigenspace of the eigenvalue problem (2.4) corresponding to $\omega = \omega_k$ and $\{Q_{k,i}\}_{i=1}^{l_k}$ the eigenfunctions of (2.4) corresponding to $\omega = \omega_k$ such that $(Q_{k,i}, Q_{k,j})_{L^2(\mathbb{S}^{N-1})} = \delta_{ij}$, $i, j = 1, \ldots, l_k$. In particular we may take

\[(2.5) \quad Q_{1,i}\left(\frac{x}{|x|}\right) = c_q \frac{x_i}{|x|^2}, \quad i = 1, \ldots, N, \]

for some positive constant $c_q = c_q(N) > 0$. Furthermore we have the following lemma on the eigenfunctions of (E) (see [5] and [13]).
Lemma 2.1  Let \( k = 0, 1, 2, \ldots \). Let \( \{\lambda_{k,i}\}_{i=0}^{\infty} \) be the eigenvalues of

\[
(E_k) \quad \begin{cases}
P_k \varphi \equiv P_0 \varphi - \frac{\omega_k}{|y|^2} \varphi = -\lambda \varphi & \text{in } \mathbb{R}^N, \\
\varphi \text{ is a radial function in } \mathbb{R}^N, \\
\varphi \in L^2(\mathbb{R}^N, \rho dy),
\end{cases}
\]

such that \( \lambda_{k,0} < \lambda_{k,1} < \lambda_{k,2} < \ldots \) and \( \varphi_{k,i} \) the eigenfunction corresponding to \( \lambda_{k,i} \) such that \( \|\varphi_{k,i}\|_{L^2(\Omega, \rho dx)} = 1 \). Then

\[
\lambda_{k,i} = \frac{N + k}{2} + i, \quad \varphi_{k,0}(y) = c_k |y|^k \exp \left( -\frac{|y|^2}{4} \right)
\]

for some constants \( c_k \). Furthermore \( \{\lambda_{k,i}\}_{i=0}^{\infty} \) give all eigenvalue of \( (E) \), and the eigenspace of \( (E) \) corresponding to \( \lambda \) are spanned by the eigenfunctions \( \{\varphi_{k,i}(y)Q_{k,j}(y/|y|)\}_{j=1}^{l_k} \) with \( \lambda = \lambda_{k,i} \).

In order to prove Theorems 2.1 and 2.2, we may assume, without loss of generarily, that \( \phi \in L^2(\Omega, \rho dx) \). Then, by Lemma 2.1, there exist radial functions \( \{\phi_{k,j}\}_{k \in \mathbb{N} \cup \{0\}, j=1, \ldots, l_k} \), such that \( \phi_{k,i} \in L^2(\Omega, \rho dx) \) and

\[
\phi = \sum_{k=0}^{\infty} \sum_{j=1}^{l_k} \phi_{k,j}(|x|)Q_{k,j} \left( \frac{x}{|x|} \right) \text{ in } L^2(\Omega, \rho dx),
\]

Furthermore let \( v_{k,j} \) be the radial solution of the Cauchy-Neumann problem

\[
(L_k^N) \quad \begin{cases}
\partial_t v = L_k v \equiv \Delta v - \frac{\omega_k}{|x|^2} v \varepsilon & \text{in } \Omega \times (0, \infty), \\
\partial_\nu v = 0 & \text{on } \partial \Omega \times (0, \infty), \\
v(x, 0) = \phi_{k,j}(x) & \text{in } \Omega.
\end{cases}
\]

Then the function

\[
v_{k,j}(x, t)Q_{k,j} \left( \frac{x}{|x|} \right)
\]

is a solution of (1.1) with the initial data \( \phi_{k,j}(x)Q_{k,j}(x/|x|) \). Furthermore we see that

\[
u(x, t) = \sum_{k=0}^{\infty} \sum_{j=1}^{l_k} u_{k,j}(x, t) \text{ in } C^2(\Omega),
\]
for all $t > 0$. Therefore we have only to study the asymptotic behavior of the radial solution of the Cauchy-Neumann problem $(L_k^N)$ in order to study the one of the solution $u$ of (1.1).

Let $v_k$ be the solution of the Cauchy-Neumann problem $(L_k^N)$ with the initial data $\varphi$, where $\varphi$ is a radial function belonging to $L^2(\Omega, \rho dx)$. In order to study the asymptotic behavior of the solution $v_k$, we define a rescaled function $w_k$ of the solution $v_k$ as follows:

$$w_k(y, s) = (1+t)^{\frac{N+k}{2}} v_k(x, t), \quad y = (1+t)^{-\frac{1}{2}} x, \quad s = \log(1+t).$$

Then the function $w_k$ satisfies

$$(P_k^N) \begin{cases} \partial_s w_k = P_k w_k + \frac{N+k}{2} w_k & \text{in } W, \\ \partial_{\nu} w_k = 0 & \text{on } \partial W, \\ w_k(y, 0) = \varphi(y) & \text{in } \Omega, \end{cases}$$

where

$$\Omega(s) = e^{-s/2} \Omega, \quad W = \bigcup_{0<s<\infty} (\Omega(s) \times \{s\}), \quad \partial W = \bigcup_{0<s<\infty} (\partial \Omega(s) \times \{s\}).$$

We study the asymptotic behavior of the first eigenvalue and the first eigenfunction of the operator $P_k$, and obtain the asymptotic behavior of the solution $w_k$ in the space $L^2$ with weight $\rho$. Furthermore, for $k = 0, 1, 2$, by using the radially symmetry of $v_k$, the equations $(L_k^N)$ and $(P_k^N)$, and the Ascoli-Arzera theorem, we study the asymptotic behavior of $v_k$, $\partial_r v_k$, and $\partial_{rr} v_k$ as $t \to \infty$.

For the case $k = 0$, we extend the domain of $w_0$ to $\mathbb{R}^N$, and apply the Ascoli-Arzera theorem to $w_0$. Then, by using the results on the asymptotic behavior of $w_0$ in the space $L^2$ with weight $\rho$, we obtain a result on the asymptotic behavior of $v_0$ and $\partial_r v_0$, where $r = |x|$. Furthermore we obtain a result on the asymptotic behavior of $\partial_{rr} v_0$ as $t \to \infty$ by using the ones of $v_0$ and $\partial_r v_0$.

**Proposition 2.1** Let $\varphi$ be a radial function in $\Omega$ satisfying (2.1). Let $v_0$ be a radial solution of $(L_0^N)$ with the initial data $\varphi$. Then

$$\lim_{t \to \infty} t^{\frac{N}{2}} v_0(x, t) = (4\pi)^{-\frac{N}{2}} \int_{\Omega} \varphi(x) dx$$
uniformly on any compact set in $\overline{\Omega}$. Furthermore, for any positive constants $\epsilon$, there exist positive constants $C$, $R$, and $T$ such that

$$\partial_r v_0(x, t) \leq -C t^{-\frac{N+1}{2}} \int_{\Omega} \varphi(x) dx$$

for all $x \in \Omega$ with $\epsilon(1 + t)^{1/2} \leq |x| \leq R(1 + t)^{1/2}$ and all $t \geq T$.

**Proposition 2.2** Let $\varphi$ be a radial function in $\Omega$ satisfying (2.1). Let $v_0$ be a radial solution of $(L_{0}^{N})$ with the initial data $\varphi$. Then there exist positive constants $R$ and $T$ such that

$$\partial_r v_0(x, t) \leq -\frac{1}{4} (4\pi)^{-\frac{N}{2}} t^{-\frac{N+2}{2}} (|x| - L) \int_{\Omega} \varphi(x) dx$$

for all $x \in \Omega$ with $|x| \leq L + R(1 + t)^{1/2}$ and $t \geq T$, where $r = |x|$. Furthermore, for any $R > L$,

$$\partial_r v_0(x, t) = -\frac{1}{2} (4\pi)^{-\frac{N}{2}} (1 + o(1)) |x| (1 - L^N |x|^{-N}) t^{-\frac{N+2}{2}} \int_{\Omega} \varphi(x) dx,$$

$$\partial_r^2 v_0(x, t) = -\frac{1}{2} (4\pi)^{-\frac{N}{2}} (1 + o(1)) (1 + (N - 1)L^N r^{-N}) t^{-\frac{N+2}{2}} \int_{\Omega} \varphi(x) dx$$

as $t \to \infty$, uniformly on $\Omega \cap B(0, R)$.

On the other hand, for the case $k = 1$, the inequality

$$\sup_{s > 1} \|\nabla_{y}^{2} w_1(\cdot, s)\|_{C(\Omega(s))} < \infty$$

does not necessarily holds, and $w(y, s)$ tends to 0 uniformly for all $y$ with $|y| \leq Re^{-s/2}$ with any $R > L$. So it is not useful to apply the Ascoli-Arzela theorem to $w_1$ for the aim at studying the asymptotic behavior of $w_1$ and $\partial_r w_1$ in the domain $\{y \in \Omega(s) : |y| \leq Re^{-s/2}\}$, as $s \to \infty$. To overcome this difficulty, we may apply the Ascoli-Arzela theorem $w_1$ in the any annulus $D(\epsilon, R) = \{y \in \mathbb{R}^N : \epsilon \leq |y| \leq R\}$ with $0 < \epsilon < R$, and obtain the asymptotic behavior of $w_1$ in the annulus $D(\epsilon, R)$. Furthermore, by using the equation $(L_1)$ effectively, we study the asymptotic behavior of $v_1$, $\partial_r v_1$ and $\partial_r^2 v_1$ as $t \to \infty$, and obtain the following proposition.
Proposition 2.3 Let $\varphi$ be a radial function in $\Omega$ satisfying (2.1). Let $v_1$ be a radial solution of $(L_1^N)$ with the initial data $\varphi$. Put

$$U_l^N(r) = c_1 r \left(1 + \frac{L^N}{N-1}r^{-N}\right), \quad a_\varphi^N = \int_{\Omega} \varphi(x) U_l^N(|x|)dx.$$ 

Then there exists a positive constant $C$ such that

$$||\nabla v_1(x,t)||_{L^\infty(\Omega)} \leq C \left(|a_\varphi^N| + o(1)\right)t^{-\frac{N+2}{2}}$$

for sufficiently large $t$. Furthermore, for any $R > L$,

$$v_1(x,t) = (a_\varphi^N + o(1)) U_l^N t^{-\frac{N+2}{2}},$$

$$\partial_r v_1(x,t) = c_1 (a_\varphi^N + o(1)) \left(1 - L^N r^{-N}\right) t^{-\frac{N+2}{2}},$$

$$\partial_r^2 v_1(x,t) = c_1 (a_\varphi^N + o(1)) NL^N r^{-(N+1)} t^{-\frac{N+2}{2}},$$

as $t \to \infty$, uniformly on $\Omega \cap B(0,R)$.

Similarly we study the asymptotic behavior of $v_2$, $\partial_r v_2$ and $\partial_r^2 v_2$ as $t \to \infty$, and obtain the following proposition.

Proposition 2.4 Let $\varphi$ be a radial function in $\Omega$ satisfying (2.1). Let $v_2$ be a radial solution of $(L_2^N)$ with the initial data $\varphi$. Then there exists a positive constant $C_1$ such that

$$||v_2(\cdot,t)||_{L^\infty(\Omega)} \leq C_1 t^{-\frac{N+2}{2}},$$

$$||\partial_T v_2(\cdot,t)||_{L^\infty(\Omega)} \leq C_1 t^{-\frac{N+3}{2}},$$

for sufficiently large $t$. Furthermore, for any $R > L$, there exists a constant $C_2$ such that

$$|\partial_r^2 v_2(x,t)| \leq C_2 t^{-\frac{N+3}{2}}$$

for all $x \in \Omega$ with $|x| \leq R$ and all sufficiently large $t$.

By Propositions 2.1–2.4, we may obtain the asymptotic behavior of the solutions $u_{k,j}$, $k = 0, 1, 2$, $j = 1, \ldots, l_k$. Finally, by (2.6), we put

$$\phi_3 = \phi - \sum_{k=0}^2 \sum_{j=1}^{l_k} \phi_{k,j}(|x|)Q_{k,j}\left(\frac{x}{|x|}\right),$$

and study the solution of (1.1) with the initial data $\phi_3$. Then we have
Proposition 2.5 Assume (2.1). Let $\phi_3$ be a function defined by (2.6) and (2.7). Let $u_3$ be a function of (1.1) with the initial data $\phi_3$. Then there exists a constant $C$ such that
\[
\|\nabla_{x}^{k}u_3(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N+3}{2}}, \quad k = 0, 1, 2,
\]
for all sufficiently large $t$.

By Propositions 2.1–2.5, we obtain the asymptotic behavior of $u$, $\nabla_x u$, and $\nabla_x^2 u$ as $t \to \infty$, and may obtain Theorems 2.1 and 2.2.

3 On the Cauchy-Neumann Problem (1.2)

In this section we assume that
\[
(3.1) \quad \phi \in L^2(\Omega, \rho dx), \quad m_\phi > 0,
\]
where $\rho(x) = \exp(|x|^2/4)$ and
\[
m_\phi = \begin{cases} 
\int_{\Omega} \phi(x) \left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right) dx & \text{if } N \geq 3, \\
\int_{\Omega} \phi(x) \log \frac{|x|}{L} dx & \text{if } N \geq 2.
\end{cases}
\]

We first give the following results on the asymptotic behavior of the solution $u$ of (1.2), which imply that the hot spots $H(t)$ run away from the boundary $\partial\Omega$ as $t \to \infty$.

Theorem 3.1 (See Theorem 1.1 in [9].) Let $u$ be a solution of the Cauchy-Dirichlet problem (1.2) under the condition (3.1) and $N \geq 3$. Then
\[
(3.2) \quad \lim_{t \to \infty} \int_{\Omega} u(x, t) dx = m_\phi > 0
\]
and
\[
(3.3) \quad \lim_{t \to \infty} t^{\frac{N}{2}} u(x, t) = (4\pi)^{-\frac{N}{2}} m_\phi \left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right)
\]
uniformly for all $x$ on any compact set in $\overline{\Omega}$. 
Theorem 3.2 (See Theorem 1.2 in [9].)
Let $u$ be a solution of the Cauchy-Dirichlet problem (1.2) under the condition (3.1) and $N = 2$. Then there exists a constant $C$ such that

$$
\|u(\cdot, t)\|_{L^1(\Omega)} \leq C (\log t)^{-1} \|\phi\|_{L^2(\Omega, \rho dx)}
$$

for all $t \geq 1$. Furthermore

$$
\lim_{t \to \infty} (\log t) \int_{\Omega} u(x, t) dx = 2m_{\phi}
$$

and

$$
\lim_{t \to \infty} t (\log t)^2 u(x, t) = \frac{1}{\pi} m_{\phi} \log \frac{|x|}{L}
$$

uniformly for all $x$ on any compact set in $\overline{\Omega}$.

Remark 3.1 Collet, Martínez, and Martín [4] used the probability method to prove the asymptotic behavior of the Dirichlet heat kernel $G = G(x, y, t)$ on the exterior domain of a compact set as $t \to \infty$. In particular, for the exterior domain $\mathbb{R}^N \setminus B(0, L)$, they obtained that

$$
\lim_{t \to \infty} t^{\frac{N}{2}} G(x, y, t) = (4\pi)^{-\frac{N}{2}} \left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right) \left(1 - \frac{L^{N-2}}{|y|^{N-2}}\right) \quad \text{if } N \geq 3,
$$

$$
\lim_{t \to \infty} t (\log t)^2 G(x, y, t) = \frac{1}{\pi} \log \frac{|x|}{L} \log \frac{|y|}{L} \quad \text{if } N = 2,
$$

for all $x, y \in \Omega$ (see also [6]). By (3.3) and (3.6), we may obtain (3.7) and (3.8), and the proof of this paper is complete different from the one of [4]. Furthermore we remark that Herraz [7] applied the comparison method to the Cauchy-Dirichlet problem (1.2) in the exterior domain of a compact set, and obtained the similar results to Theorems 3.1 and 3.2 for nonnegative initial data $\phi$.

Next we give a result on the rate for the hot spots $H(t)$ to run away from the boundary $\Omega$ as $t \to \infty$.

Theorem 3.3 (See Theorem 1.3 in [9].)
Let $u$ be a solution of the Cauchy-Dirichlet problem (1.2) under the condition (3.1). Put

$$
\zeta(t) = 2(N - 2)L^{N-2} t \quad \text{if } N \geq 3, \quad \zeta(t) = 2t (\log t)^{-1} \quad \text{if } N = 2.
$$
Then
\begin{equation}
\lim_{t \to \infty} \sup_{x \in H(t)} |\zeta(t)^{-1}|x|^N - 1| = 0.
\end{equation}

Furthermore there exists a positive constant $T$ such that, if $x \in H(t)$ and $t \geq T$, then
\begin{equation}
H(t) \cap l_x = \{x\},
\end{equation}
where $l_x = \{kx/|x| : k \geq 0\}$.

Next we give a sufficient condition for the hot spots $H(t)$ to consist of one point $x(t)$ after a finite time. Furthermore we give the limit of $x(t)/|x(t)|$ as $t \to \infty$.

**Theorem 3.4** (See Theorem 1.4 in [9].) Let $u$ be a solution of the Cauchy-Dirichlet problem (1.2) under the condition (3.1). Assume that
\[ A_\phi^D \equiv \int_{\Omega} x\phi(x) \left(1 - \frac{L^N}{|x|^N}\right) dx \neq 0. \]

Then there exist a positive constant $T$ and a smooth curve $x = x(t) \in C^\infty([T, \infty) : \Omega)$ such that $H(t) = \{x(t)\}$ for all $t \geq T$ and
\begin{equation}
\lim_{t \to \infty} \frac{x(t)}{|x(t)|} = \frac{A_\phi^D}{|A_\phi^D|}.
\end{equation}

Therefore, by Theorems 3.3 and 3.4, we see that, under the assumptions (3.1) and $A_\phi^D \neq 0$, the set of hot spots $H(t)$ consists of one point $x(t)$ after a finite time, and
\[ \lim_{t \to \infty} \zeta(t)^{-1/N}|x(t)| = 1, \quad \lim_{t \to \infty} x(t)/|x(t)| = A_\phi^D/|A_\phi^D|. \]

Next we explain the outline of the proofs of Theorems 3.1–3.3. In the similar way to the Cauchy-Neumann problem (1.1), we have only to study the asymptotic behavior of the radial solutions $v_k$ of the Cauchy-Dirichlet problem
\[ (L_k^D) \quad \begin{cases} \partial_t v = L_k v \equiv \Delta v - \frac{\omega_k}{|x|^2} v_k & \text{ in } \Omega \times (0, \infty), \\ v = 0 & \text{ on } \partial\Omega \times (0, \infty), \\ v(x, 0) = \varphi(x) & \text{ in } \Omega, \end{cases} \]
where \( \varphi \) is a radial function belonging to \( L^2(\Omega, \rho dx) \) and \( k = 0, 1, 2, \ldots \). Furthermore, by the same argument with in the Cauchy-Neumann problem (1.1), we introduce a rescaled function \( w_k \) of \( v_k \), and study the asymptotic behavior of the rescaled functions \( w_k \) as \( s \to \infty \). For the case \( N \geq 3 \), we study the asymptotic behavior of \( v_0 = v_0(y, s) \) in the space \( L^2 \) with weight \( \rho \), and obtain the one of \( v_0 = v_0(x, t) \) for all \( x \in \Omega \) with \( |x| \sim t^{1/2} \) as \( t \to \infty \). Furthermore, by using the radially symmetry of \( v_0 \) and \( (L_0^\varnothing) \), we obtain the asymptotic behavior of \( \partial_r v_0, \partial_r^2 v_0, \text{ and } \partial_t v_0 \) for all \( x \in \Omega \) with \( |x| = O(t^{1/2}) \) as \( t \to \infty \).

**Proposition 3.1** Let \( \varphi \) be a radial function in \( \Omega \) satisfying (2.1). Let \( v_0 \) be a radial solution of \( (L_0^D) \) with the initial data \( \varphi \) and \( N \geq 3 \). Put

\[
U_L^{D,0}(r) = c_0 \left(1 - \frac{L^{N-2}}{r^{N-2}}\right), \quad a_\varphi^{D,0} = \int_{\Omega} \varphi(x) U_L^{D,0}(|x|) dx.
\]

Then there hold that

\[
v_0^*(r, t) = t^{-\frac{N}{2}}(a_\varphi^{D,0} + o(1)) U_L^{0}(r) + \frac{N}{2} t^{-\frac{N+2}{2}}(a_0 + o(1)) O(r^2) + O(t^{-\frac{N+4}{2}})O(r^4),
\]

\[
(\partial_r v_0^*)(r, t) = t^{-\frac{N}{2}}(a_\varphi^{D,0} + o(1)) \partial_r U_L^{0}(r) - \frac{Nc_0}{4} rt^{-\frac{N+2}{2}}(a_\varphi^{D,0} + o(1))(1 + O(r^{-1})) + O(t^{-\frac{N+4}{2}})O(r^3),
\]

\[
(\partial_r^2 v_0^*)(r, t) = t^{-\frac{N}{2}}(a_\varphi^{D,0} + o(1)) \partial_r^2 U_L^{0}(r) - U_L^{0}(r) \frac{N}{2} t^{-\frac{N+2}{2}}(a_\varphi^{D,0} + o(1)) + O(t^{-\frac{N+4}{2}})O(r^2),
\]

\[
(\partial_t v_0^*)(r, t) = -\frac{N}{2} t^{-\frac{N+2}{2}}(a_\varphi^{D,0} + o(1)) U_L^{0}(r) + O(t^{-\frac{N+4}{2}})O(r^2)
\]

for all \( r \geq L \) and \( t \geq 1 \).

For the case \( N = 2 \), the behavior of \( v_0 \) is different from the one for the case \( N \geq 3 \). By the similar way to in the case \( N \geq 3 \), we first obtain \( \max_{x \in \partial \Omega} |\partial_r v_0(x, t)| = O(t^{-1}(\log t)^{-1}) \) as \( t \to \infty \). This gives that \( ||v_0(\cdot, t)||_{L^1(\Omega)} = O((\log t)^{-1}) \) as \( t \to \infty \). By using the similar argument to in the case \( N \geq 3 \) again, we have \( \max_{x \in \partial \Omega} |\partial_r v_0(x, t)| = O(t^{-1}(\log t)^{-2}) \) as \( t \to \infty \), and obtain the following proposition.
Proposition 3.2 Let \( \varphi \) be a radial function in \( \Omega \) satisfying (2.1). Let \( v_0 \) be a radial solution of \((L_0^D)\) with the initial data \( \varphi \) and \( N = 2 \). Put

\[
\tilde{a}_{\varphi}^{D,0} = 4c_0^2 \int_{\Omega} \varphi(x) \log \frac{|x|}{L} dx.
\]

Then there exists a function \( \zeta_1 = \zeta_1(t) \) and \( \zeta_2(t) \) with

\[
\lim_{t \to \infty} t(\log t)^2 \zeta_1(t) = \tilde{a}_{\varphi}^{D,0}, \quad \lim_{t \to \infty} t^2(\log t)^2 \zeta_2(t) = \tilde{a}_{\varphi}^{D,0},
\]

such that

\[
\begin{align*}
v_0(r, t) &= \zeta_1(t) \log \frac{r}{L} + O(r^2 \log r) \zeta_1(t) + O(r^4) O(t^{-3}(\log t)^{-1}), \\
(\partial_r v_0)(r, t) &= \frac{\zeta_1(t)}{r} - \zeta_1(t) r \log r (1 + o(1)) + O(r^3) O(t^{-3}(\log t)^{-1}), \\
(\partial_r^2 v_0)(r, t) &= -\frac{\zeta_1(t)}{r^2} - U_{L}^{0}(r) \zeta_1(t) + O(r^2) O(t^{-3}(\log t)^{-1}), \\
(\partial_t v_0)(r, t) &= -\left(\log \frac{r}{L}\right) \zeta_2(t) + O(r^2) O(t^{-3}(\log t)^{-1})
\end{align*}
\]

for all \( r \geq L \) and \( t \geq 2 \).

Furthermore, by the similar argument to the problem (1.1), we obtain the asymptotic behavior of the solutions \( v_1 \) and \( v_2 \).

Proposition 3.3 Let \( \varphi \) be a radial function in \( \Omega \) satisfying (2.1). Let \( v_1 \) be a radial solution of \((L_1^D)\) with the initial data \( \varphi \) and \( N \geq 2 \). Put

\[
U_{L}^{D,1}(r) = c_1 r \left(1 - \frac{L^N}{r^N}\right), \quad a_{\varphi}^{D,1} = \int_{\Omega} \varphi(x) U_{L}^{D,1}(|x|) dx.
\]

Then there hold that

\[
\begin{align*}
v_1^*(r, t) &= t^{-\frac{N+2}{2}} (a_{\varphi}^{D,1} + o(1)) U_L^1(r) + O(r^2) O(t^{-\frac{N+3}{2}}), \\
(\partial_r v_1^*)(r, t) &= t^{-\frac{N+2}{2}} (a_{\varphi}^{D,1} + o(1)) \partial_r U_L^1(r) + O(r) O(t^{-\frac{N+3}{2}}), \\
(\partial_r^2 v_1^*)(r, t) &= t^{-\frac{N+2}{2}} (a_{\varphi}^{D,1} + o(1)) \partial_r^2 U_L^1(r) + O(t^{-\frac{N+3}{2}})
\end{align*}
\]

for all \( r \geq L \) and \( t > 1 \).
Proposition 3.4 Let $\varphi$ be a radial function in $\Omega$ satisfying (2.1). Let $v_2$ be a radial solution of $(L^2_D)$ with the initial data $\varphi$ and $N \geq 2$. Then there hold that

$$v_2^*(r, t) = O(t^{-\frac{N+4}{2}} \log t)U_L^{D,2}(r) + O(t^{-\frac{N+4}{2}})O(r^{2\log r}),$$

$$\partial_r v_2^*(r, t) = O(t^{-\frac{N+4}{2}} \log t)\partial_r U_L^{D,2}(r) + O(t^{-\frac{N+4}{2}})r\log\frac{r}{L},$$

$$\partial_r^2 v_2^*(r, t) = O(t^{-\frac{N+4}{2}} \log t)\partial_r^2 U_L^{D,2}(r) + O(t^{-\frac{N+4}{2}})\log\frac{r}{L},$$

for all $r \geq L$ and $t > 1$, where

$$U_L^{D,2}(r) = c_2r^2 \left(1 - \frac{L^{N+2}}{r^{N+2}}\right).$$

Therefore, by the similar argument to the problem (1.1) and Propositions 3.1-3.4, we may prove Theorems 3.1-3.3. In order to prove Theorem 3.4, we study the asymptotic behavior of $x/|x|$ for all $x \in H(t)$ and all sufficiently large $t$, by using the asymptotic behavior of $v_0$ and $v_1$. Furthermore we compare the hot spots $H(t)$ with the radial solution of (1.2) with the initial data $\varphi \in L^2_0(\Omega, \rho dx)$ with $m_\varphi = m_\phi$. Then we may prove that, if $t$ is sufficiently large, then the matrix $\{-\partial_x, \partial_x, u(x, t)\}_{i,j=1}^N$ is positive definite for all points near the hot spots $H(t)$, and complete the proof Theorem 3.4.

References


