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MODIFIED WAVE OPERATORS TO THE NONLINEAR SCHRÖDINGER EQUATIONS IN ONE AND TWO SPACE DIMENSIONS

(空間1次元と2次元に於ける非線形SCHRÖDINGER方程式の修正波動作用素について)

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1. INTRODUCTION

We study the global existence and asymptotic behavior of solutions for the nonlinear Schrödinger equation

$$\mathcal{L}u = N_n(u) + G_n(u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.1}$$

in one or two space dimensions $n = 1$ and 2, where $\mathcal{L} = i\partial_t + \frac{1}{2}\Delta$ and

$$N_1(u) = \lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3,$$
$$N_2(u) = \lambda_1 u^2 + \lambda_2 \overline{u}^2,$$
$$G_n(u) = \lambda_0 |u|^{\frac{2}{n}} u$$

with $\lambda_0 \in \mathbb{R}$ and $\lambda_j \in \mathbb{C}$, $j = 1,2,3$. Following our paper [2], we construct a modified wave operator in L^2 to equation (1.1) for small final data $\phi \in H^{0,2} \cap \dot{H}^{-\delta}$ with $\frac{n}{2} < \delta < 2$, where the weighted Sobolev space is defined by

$$H^{m,s} = \{u \in \mathcal{S}'; \|u\|_{H^{m,s}} = \| \langle i\nabla \rangle^m \langle x \rangle^s u\|_{L^2} < \infty\},$$

where $\langle x \rangle = \sqrt{1+|x|^2}$ and the homogeneous Sobolev space is

$$\dot{H}^m = \left\{u \in \mathcal{S}'; \|u\|_{\dot{H}^m} = \|(-\Delta)^{\frac{m}{2}} u\|_{L^2} < \infty\right\}.$$

The nonlinearity is critical between the short range scattering and the long range one.

There are several results on the scattering theory for equation (1.1) in one or two space dimensions. In [4] it was shown the existence of the wave operator for equation (1.1) with $G_n(u) = 0$ by using the method by Hörmander [3], where he studied the life span of solutions of nonlinear Klein-Gordon equations and in [6] it was constructed the modified wave operator for equation (1.1) by combining the methods in [3] and [5]. More precisely, the following two propositions were obtained in [6]:

Proposition 1.1. *Let* $n = 1$, $\phi \in H^{0,3} \cap \dot{H}^{-4}$ *and* $\|\phi\|_{H^{0,3}} + \|\phi\|_{\dot{H}^{-4}}$ *be sufficiently small. Then there exists a unique global solution* u *of* (1.1) *such that* $u \in C(\mathbb{R}^+; L^2)$,

$$\sup_{t\geq 1} t^b \|u(t) - u_p(t)\|_{L^2} + \sup_{t\geq 1} t^b \Big(\int_t^\infty \|u(\tau) - u_p(\tau)\|_{L^\infty}^4 \, d\tau \Big)^{1/4} < \infty,$$

where $\frac{1}{2} < b < 1$, *and*

$$u_p(t) = \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}\big(\frac{x}{t}\big) \exp\big(- i\lambda_0 |\widehat{\phi}\big(\frac{x}{t}\big)|^{\frac{2}{n}} \log t \big).$$

Proposition 1.2. *Let* $n = 2$, $\phi \in H^{0,4} \cap \dot{H}^{-4}$, $x\phi \in \dot{H}^{-2}$ *and* $\|\phi\|_{H^{0,4}} + \|\phi\|_{\dot{H}^{-4}} + \|x\phi\|_{\dot{H}^{-2}}$ *be sufficiently small. Then there exists a unique global solution* u *of equation* (1.1) *such that* $u \in C(\mathbb{R}^+; L^2)$,

$$\sup_{t\geq 1} t^b \|u(t) - u_p(t)\|_{L^2} + \sup_{t\geq 1} t^b \Big(\int_t^\infty \|u(\tau) - u_p(\tau)\|_{L^4}^4 \, d\tau \Big)^{1/4} < \infty,$$

where $\frac{1}{2} < b < 1$.

Throughout this article, we denote the norm of a Banach space Z by $\|\cdot\|_Z$. Our purpose in this article is to improve the condition on a final data $\phi \in \dot{H}^{-4}$. In order to explain the reason why the previous proof by [4] and [3] requires such a condition, we give briefly the idea of paper [6] on the example of the Cauchy problem

$$\mathcal{L}u = u^2, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2. \tag{1.2}$$

If a solution u of (1.2) behaves like a free solution $U(t)\phi$ as $t \longrightarrow \infty$ for a given ϕ, then. $u_0(t,x) = \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}(\frac{x}{t})$ can be considered as an approximate solution of (1.2) since

$$U(t)\phi = \frac{1}{it} e^{\frac{ix^2}{2t}} \widehat{\phi}\big(\frac{x}{t}\big) + O\big(t^{-1-\alpha} \| |x|^{2\alpha} \phi \|_{L^1}\big).$$

By a direct calculation we find that $\mathcal{L}(u - u_0) = u^2 - \frac{1}{2it^3} e^{\frac{ix^2}{2t}} \widehat{|\cdot|^2 \phi}(\eta)$. with $\eta = \frac{x}{t}$. The last term of the right-hand side of the above equation is a remainder term which we denote by R. Hence the problem becomes

$$\mathcal{L}(u - u_0) = u^2 - u_0^2 + u_0^2 + R. \tag{1.3}$$

We find a solution in the neighborhood of u_0. however u_0^2 can not be considered as a remainder term since $\|u_0^2\|_{L^2} = t^{-1} \|\widehat{\phi}^2\|_{L^2}$. In order to cancel u_0^2 we try to find u_r such that $\mathcal{L}u_r - u_0^2$ is a remainder term. We put $u_r = t^{-b}P(\frac{x}{t})e^{\frac{iax^2}{2t}}$ to get $\mathcal{L}u_r = t^{-b}\frac{a(1-a)}{2}\frac{x^2}{t^2}P(\frac{x}{t})e^{\frac{iax^2}{2t}} + R_1$ which implies that we should take $P(\eta) = \frac{2}{a(a-1)}\frac{1}{\eta^2}\widehat{\phi}(\eta)^2$ and $a = b = 2$ to cancel u_0^2 in the right-hand side of (1.3) and we note that R_1 contains a term like $t^{-4}e^{\frac{ix^2}{t}}\frac{1}{\eta^4}\widehat{\phi}(\eta)^2$. Thus we get

$$\mathcal{L}(u - u_0 - u_r) = u^2 - u_0^2 + R + R_1.$$

This is the reason why we require a vanishing condition of $\widehat{\phi}(\eta)$ at the origin.

Our main result in the present article is the following.

Theorem 1.1. *Let $\phi \in H^{0,2} \cap \dot{H}^{-\delta}$ and $\|\phi\|_{H^{0,2}} + \|\phi\|_{\dot{H}^{-\delta}}$ be sufficiently small, where $\frac{n}{2} < \delta < 2$. Then there exists a unique global solution u of (1.1) such that $u \in C(\mathbb{R}^+; L^2)$,*

$$\sup_{t\geq 1} t^{\frac{\delta}{2}}\|u(t) - u_p(t)\|_{L^2} + \sup_{t\geq 1} t^{\frac{\delta}{2}}\Big(\int_t^\infty \|u(\tau) - u_p(\tau)\|_{X_n}^4 \, d\tau\Big)^{1/4} < \infty$$

where $X_1 = L^\infty, X_2 = L^4$,

$$u_p(t) = \frac{1}{(it)^{\frac{n}{2}}}e^{\frac{ix^2}{2t}}\widehat{\phi}\Big(\frac{x}{t}\Big)\exp\Big(-i\lambda_0\Big|\widehat{\phi}\Big(\frac{x}{t}\Big)\Big|^{\frac{2}{n}}\log t\Big).$$

Furthermore the modified wave operator

$$\widetilde{W}_+ : \phi \mapsto u(0)$$

is well-defined.

Similar result holds for the negative time.

Remark 1.1. If we consider the asymptotic behavior of solutions to the Cauchy problem for equation (1.1) with initial data $u(0,x) = \phi_0(x)$, $x \in \mathbb{R}^n$, then we see from Theorem 1.1 that for any initial data ϕ_0 belonging to the range of the modified wave operator $\widetilde{W}_+$, there exists a unique global solution $u \in C(\mathbb{R}^+; L^2)$ of the Cauchy problem for equation (1.1) which has a modified free profile u_p. More precisely, u satisfies the asymptotic formula of Theorem 1.1. However it is not clear how to describe the initial data beloging to the range of the operator $\widetilde{W}_+$.

Remark 1.2. If $\phi \in H^{0,2}$ and $\widehat{\phi}(0) = 0$, then $\phi \in H^{0,2} \cap \dot{H}^{-\alpha}$ for $0 \leq \alpha < 1 + \frac{n}{2}$ with $n = 1, 2$. This follows from the fact that $\dot{H}^0 = L^2 \supset H^{0,2}$ and the following inequalities:

(a) $\||\cdot|^{-\alpha}f\|_{L^2} \leq C\||\cdot|^{-\alpha+1}\nabla f\|_{L^2}$ for $\alpha > \frac{n+1}{2}$, provided that $f(0) = 0$,

(b) $\||\cdot|^{-\alpha+1}f\|_{L^2} \leq C\|f\|_{H^{1,0}}$ for $1 < \alpha < 1 + \frac{n}{2}$ with $n = 1, 2$.

Note that this implies that $\int \phi(x)dx = 0$ and $\phi \in H^{0,2}$, then $\phi \in H^{0,2} \cap \dot{H}^{-\alpha}$.

Remark 1.3. In the previous paper [1], we considered the Cauchy problem for the cubic nonlinear Schrödinger equation

$$iu_t + \frac{1}{2}u_{xx} = N(u), \quad x \in \mathbb{R},\ t > 1$$
$$u(1,x) = u_1(x), \quad x \in \mathbb{R},$$

where $N(u) = \lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3$. $\lambda_j \in \mathbb{C}$. $j = 1, 2, 3$. It was shown that there exists a global small solution $u \in C([1,\infty), L^\infty)$, if the initial data u_1 belong to some analytic function space and are sufficiently small. For the coefficients λ_j it was assumed that there exists $\theta_0 > 0$ such that

$$\mathrm{Re}\,\Big(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir}\Big) \geq C > 0,$$
$$\mathrm{Im}\,\Big(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir}\Big) r \geq C r^2,$$

for all $|r| < \theta_0$. and also it was assumed that the initial data $u_1(x)$ are such that

$$\big|\arg e^{-\frac{i}{2}\xi^2} \widehat{u_1}(\xi)\big| < \theta_0, \quad \inf_{|\xi| \leq 1} |\widehat{u_1}(\xi)| \geq C\varepsilon,$$

where ε is a small positive constant depending on the size of the initial data in a suitable norm. Moreover it was shown that there exist unique final states $W_+, r_+ \in L^\infty$ and $0 < \gamma < 1/20$ such that the asymptotic statement

$$u(t,x) = \frac{(it)^{-\frac{1}{2}} W_+(\frac{x}{t}) e^{\frac{ix^2}{2t}}}{\sqrt{1 + \chi(\frac{x}{t}) |W_+(\frac{x}{t})|^2 \log \frac{t^2}{t+x^2}}} + O\Big(t^{-\frac{1}{2}} \big(1 + \log \frac{t^2}{t+x^2}\big)^{-\frac{1}{2}-\gamma}\Big)$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$, where $\gamma > 0$ and $\chi(\xi)$ is given by

$$\chi(\xi) = \mathrm{Re}\,\Big(\frac{\lambda_1}{\sqrt{3}} \exp(2ir_+(\xi)) - i\lambda_2 \exp(-2ir_+(\xi)) + \frac{\lambda_3}{\sqrt{3}} \exp(-4ir_+(\xi))\Big).$$

This asymptotic formula shows that, in the short range region $|x| < \sqrt{t}$. the solution has an additional logarithmic time decay comparing with the corresponding linear case. Thus we can see that the vanishing condition at the origin on the Fourier transform of the final data seems to be essential for our result in the present article.

For the convenience of the reader we now state the strategy of the proof. We consider the linearized version of equation (1.1)

$$\mathcal{L}u = N_n(v) + G_n(v), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

We take

$$u_0(t,x) = \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}(\frac{x}{t}) \exp\big(- i\lambda_0 |\widehat{\phi}(\frac{x}{t})|^{\frac{2}{n}} \log t\big)$$

as the first approximation for solutions to (1.1). By a direct calculation we get

$$\mathcal{L}u_0 = G_n(u_0) + R_1(t),$$

where $R_1(t)$ is a remainder term. Hence

$$\mathcal{L}(u - u_0) = N_n(v) + G_n(v) - G_n(u_0) + R_1.$$

We define the second approximation u_1 for solutions of (1.1) as

$$u_1(t) = -i \int_{\infty}^{t} U(t-\tau) N_n(u_0)\, d\tau$$

which implies that

$$\mathcal{L}u_1 = N_n(u_0)$$

and

$$\begin{aligned} u(t) - u_0(t) = &-i \int_{\infty}^{t} U(t-\tau)(N_n(v) - N_n(u_0) + G_n(v) - G_n(u_0))\, d\tau \\ &- i \int_{\infty}^{t} U(t-\tau) R_1(\tau)\, d\tau + u_1(t). \end{aligned}$$

We define the function space

$$X = \{f \in C([T,\infty); L^2); \|f\|_X < \infty\}$$

$$\|f\|_X = \sup_{t\in[T,\infty)} t^b \|f(t) - u_0(t)\|_{L^2} + \sup_{t\in[T,\infty)} t^b \Big(\int_t^{\infty} \|f(t) - u_0(t)\|_{X_n}^4\, dt\Big)^{1/4},$$

where

$$X_1 = L^{\infty},\ X_2 = L^4,\ b > \frac{n}{4}.$$

In order to get the result we need to prove the following estimate for $u_1(t)$,

$$\|u_1(t)\| + \Big(\int_t^{\infty} \|u_1(\tau)\|_{X_n}^4\, d\tau\Big)^{1/4} \le C(\||\cdot|^{-\widetilde{\delta}}\widehat{\phi}\| + \|\phi\|_{H^{0,2}})^{1+\frac{2}{n}} t^{-\widetilde{\delta}/2},$$

for $n/2 < \widetilde{\delta} < 2$. which is the main estimate of the present article. Note that the choice of u_1 differs from that used in the previous papers.

2. Preliminaries

Lemma 2.1. *We have for $\omega \neq 1$. $f, g \in L^1 \cap L^2$ and $h \in C^2$,*

$$\int_\infty^t h(i\tau)U(t-\tau)\Delta(e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau} f(\frac{x}{\tau}))\,d\tau$$
$$= -\frac{2i\omega}{1-\omega}h(it)e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f(\frac{x}{t})$$
$$-\frac{2\omega}{(1-\omega)^2}\int_\infty^t \Big(\sum_{(F,k)} F(i\tau)e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau} k(\frac{x}{\tau})$$
$$- i\omega U(t-\tau)\int_\infty^\tau \sum_{(F,k)} F'(is)e^{\frac{i\omega x^2}{2s}} e^{ig(\frac{x}{s})\log s} k(\frac{x}{s})\,ds$$
$$- i\omega U(t-\tau)\int_\infty^\tau \sum_{(F,k)} F(is)e^{\frac{i\omega x^2}{2s}} e^{ig(\frac{x}{s})\log s}\frac{1}{s}k(g-\frac{in}{2})(\frac{x}{s})\,ds\Big)\,d\tau + R(t),$$

where the summation is taken over $(F,k) = (h', f), (h\tau^{-1}, f(g - in/2))$,

$$R(t) = -\frac{i\omega}{(1-\omega)^2}\int_\infty^t U(t-\tau)\int_\infty^\tau \sum_{(F,k)} F(is)R_{0,k}(s)\,ds\,d\tau$$
$$+\frac{1}{1-\omega}\int_\infty^t h(i\tau)U(t-\tau)R_{0,f}(\tau)\,d\tau,$$

and

$$R_{0,k}(t) = e^{\frac{i\omega x^2}{2t}} k(\frac{x}{t})\Delta e^{ig\left(\frac{x}{t}\right)\log t} + 2i\frac{1}{t^2}\sum \partial_j g(\frac{x}{t})\partial_j k(\frac{x}{t})e^{\frac{i\omega x^2}{2t}} e^{ig(\frac{x}{t})\log t}\log t$$
$$+\frac{1}{t^2}(\Delta k)(\frac{x}{t})e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t}.$$

Lemma 2.1 is proved in Lemma 2.1 in [2].

Denote

$$\widetilde{R}_1(t) = \int_\infty^t U(t-\tau)\int_\infty^\tau F(is)R_{0,k}(s)\,ds\,d\tau$$
$$\widetilde{R}_2(t) = \int_\infty^t U(t-\tau)h(i\tau)R_{0,k}(\tau)\,d\tau,$$

where

$$R_{0,k}(t) = e^{\frac{i\omega x^2}{2t}} k(\frac{x}{t})\Delta e^{ig\left(\frac{x}{t}\right)\log t} + 2i\frac{1}{t^2}\sum \partial_j g(\frac{x}{t})\partial_j k(\frac{x}{t})e^{\frac{i\omega x^2}{2t}} e^{ig(\frac{x}{t})\log t}\log t$$
$$+\frac{1}{t^2}(\Delta k)(\frac{x}{t})e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t}.$$

Lemma 2.2. *Let*

$$|F(it)| \le C|t|^{-2-\frac{n}{2}}, \quad |h(it)| \le C|t|^{-1-\frac{n}{2}}.$$

Then

$$\|\widetilde{R}_j(t)\|_{L^2} + (\int_t^\infty \|\widetilde{R}_j(t)\|_{X_n}^4\, dt)^{1/4}$$
$$\le Ct^{-2}(\|\Delta k\|_{L^2} + \|\nabla k\cdot\nabla g\|_{L^2}\log t + \|k\Delta g\|_{L^2}\log t + \|k\nabla g\cdot\nabla g\|_{L^2}(\log t)^2),$$

where $X_1 = L^\infty, X_2 = L^4$.

Lemma 2.2 is shown in Lemma 2.3 in [2].

Lemma 2.3. *Assume that* $|G(it)| + |t||G'(it)| \le C|t|^{-q-\frac{n}{2}}$, *then*

$$\|\int_\infty^t G(i\tau)e^{\frac{i\omega x^2}{2\tau}}e^{ig(\frac{x}{\tau})\log s}k(\frac{x}{\tau})\,d\tau\|_{L^p}$$
$$\le \begin{cases} Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}(1-\frac{2}{p})}\||\cdot|^{-\delta}k\|_{L^p} \\ +Ct^{-\frac{\tilde\delta}{2}-q+1-\frac{n}{2}(1-\frac{2}{p})}(\||\cdot|^{1-\tilde\delta}\nabla k\|_{L^p} + \||\cdot|^{1-\tilde\delta}k\nabla g\|_{L^p}\log t), \\ \qquad \textit{for } 0<\delta,\tilde\delta<2,\ 1\le p<\infty, \\ Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}(1-\frac{1}{p})}\||\cdot|^{-\delta}k\|_{L^\infty} \\ +Ct^{-\frac{\tilde\delta}{2}-q+1-\frac{n}{2}(1-\frac{1}{p})}(\||\cdot|^{1-\tilde\delta}\nabla k\|_{L^\infty} + \||\cdot|^{1-\tilde\delta}k\nabla g\|_{L^\infty}\log t), \\ \qquad \textit{for } 0<\delta,\tilde\delta<2-\frac{n}{p},\ 1\le p<\infty. \end{cases}$$

Proof. Using the identity

$$\frac{1}{1-\frac{i\omega x^2}{2\tau}}\partial_t\tau e^{\frac{i\omega x^2}{2\tau}} = e^{\frac{i\omega x^2}{2\tau}}$$

we have

$$\int_\infty^t G(i\tau)e^{\frac{i\omega x^2}{2\tau}}e^{ig(\frac{x}{\tau})\log\tau}k(\frac{x}{\tau})\,d\tau$$
$$= \int_\infty^t G(i\tau)e^{ig(\frac{x}{\tau})\log\tau}k(\frac{x}{\tau})\Big(\frac{1}{1-\frac{i\omega x^2}{2\tau}}\partial_\tau\tau e^{\frac{i\omega x^2}{2\tau}}\Big)\,d\tau$$
$$= G(it)k(\frac{x}{t})e^{ig(\frac{x}{t})\log t}\Big(\frac{1}{1-\frac{i\omega x^2}{2t}}te^{\frac{i\omega x^2}{2t}}\Big)$$
$$-\int_\infty^t \tau e^{\frac{i\omega x^2}{2\tau}}\partial_\tau\Big(G(i\tau)k(\frac{x}{\tau})\frac{1}{1-\frac{i\omega x^2}{2\tau}}e^{ig(\frac{x}{\tau})\log\tau}\Big)\,d\tau.$$

We also obtain

$$\Big\|G(it)k(\frac{x}{t})e^{ig(\frac{x}{t})\log t}\big(\frac{1}{1-\frac{i\omega x^2}{2t}}te^{\frac{i\omega x^2}{2t}}\big)\Big\|_{L^p}$$
$$\le Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}}\Big(\int\Big(\frac{\left|\frac{x}{t^{1/2}}\right|^\delta}{1+\left|\frac{x}{t^{1/2}}\right|^2}\left|\frac{x}{t}\right|^{-\delta}k(\frac{x}{t})\Big)^p dx\Big)^{1/p}$$
$$\le \begin{cases} Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}(1-\frac{2}{p})}\||\cdot|^{-\delta}k\|_{L^p}, & 0<\delta<2, 1\le p<\infty \\ Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}(1-\frac{1}{p})}\||\cdot|^{-\delta}k\|_{L^\infty}, & 0<\delta<2-\frac{n}{p}, 1\le p<\infty \end{cases}$$

and in the same way we get

$$\Big\| te^{\frac{i\omega x^2}{2t}} \partial_t \Big(G(it) k(\frac{x}{t}) \frac{1}{1-\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} \Big) \Big\|_{L^p}$$

$$\leq \begin{cases} Ct^{-\frac{\delta}{2}-q-\frac{n}{2}(1-\frac{2}{p})} \Big\| |\cdot|^{-\delta} k \Big\|_{L^p} \\ +Ct^{-\frac{\widetilde{\delta}}{2}-q-\frac{n}{2}(1-\frac{2}{p})} (\Big\| |\cdot|^{1-\widetilde{\delta}} \nabla k \Big\|_{L^p} + \Big\| |\cdot|^{1-\widetilde{\delta}} k \nabla g \Big\|_{L^p} \log t), \\ \quad \text{for } 0 < \delta, \widetilde{\delta} < 2,\ 1 \leq p < \infty, \\ Ct^{-\frac{\delta}{2}-q-\frac{n}{2}(1-\frac{1}{p})} \Big\| |\cdot|^{-\delta} k \Big\|_{L^\infty} \\ +Ct^{-\frac{\widetilde{\delta}}{2}-q-\frac{n}{2}(1-\frac{1}{p})} (\Big\| |\cdot|^{1-\widetilde{\delta}} \nabla k \Big\|_{L^\infty} + \Big\| |\cdot|^{1-\widetilde{\delta}} k \nabla g \Big\|_{L^\infty} \log t), \\ \quad \text{for } 0 < \delta, \widetilde{\delta} < 2 - \frac{n}{2},\ 1 \leq p < \infty. \end{cases}$$

Hence we have the result of the lemma. □

Finally we state the Strichartz estimate for $\int_s^t \mathcal{U}(t-\tau) f(\tau)\, d\tau$ obtained by Yajima [6].

Lemma 2.4. *For any pairs (q, r) and (q', r') such that $0 \leq \frac{2}{q} = \frac{n}{2} - \frac{n}{r} < 1$ and $0 \leq \frac{2}{q'} = \frac{n}{2} - \frac{n}{r'} < 1$. for any (possibly unbounded) interval I and for any $s \in \overline{I}$ the Strichartz estimate*

$$\Big(\int_I \| \int_s^t U(t-\tau) f(\tau)\, d\tau \|_{L^r}^q \, dt \Big)^{\frac{1}{q}} \leq C \Big(\int_I \| f(t) \|_{L^{\overline{r}'}}^{\overline{q}'} \, dt \Big)^{\frac{1}{\overline{q}'}},$$

is true with a constant C independent of I and s, where $\frac{1}{r} + \frac{1}{\overline{r}} = 1$ and $\frac{1}{q} + \frac{1}{\overline{q}} = 1$.

3. Proof of Theorem 1.1

In this section, following [2], we prove Theorem 1.1.

We consider the linearized version of equation (1.1)

$$\mathcal{L}u = N_n(v) + G_n(v), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.1}$$

We take

$$u_0(t, x) = \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}(\frac{x}{t}) \exp\Big(-i\lambda_0 |\widehat{\phi}(\frac{x}{t})|^{\frac{2}{n}} \log t \Big)$$

as the first approximation for solutions of (3.1). By a direct calculation we get

$$\mathcal{L}u_0 = G_n(u_0) + R_1,$$

where

$$R_1(t) = \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}(\frac{x}{t}) \frac{1}{2} \Delta \exp(-i\lambda_0 |\widehat{\phi}(\frac{x}{t})|^{\frac{2}{n}} \log t)$$
$$- \frac{2}{n} \lambda_0 \frac{1}{t^2} \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \nabla \widehat{\phi}(\frac{x}{t}) \exp(-i\lambda_0 |\widehat{\phi}(\frac{x}{t})|^{\frac{2}{n}} \log t)$$

$$\times 2\,\mathrm{Re}\,\nabla\widehat{\phi}(\frac{x}{t})\overline{\widehat{\phi}(\frac{x}{t})}|\widehat{\phi}(\frac{x}{t})|^{\frac{2}{n}-2}\log t$$
$$+\frac{1}{2}\frac{1}{(it)^{\frac{n}{2}}}e^{\frac{ix^2}{2t}}t^{-2}\Delta\widehat{\phi}(\frac{x}{t})\exp(-i\lambda_0|\widehat{\phi}(\frac{x}{t})|^{\frac{2}{n}}\log t).$$

Hence

$$\mathcal{L}(u-u_0)=N_n(v)+G_n(v)-G_n(u_0)+R_1.$$

By Lemma 2.4 we obtain

$$\begin{aligned}&\|\int_t^\infty U(t-\tau)R_1(\tau)\,d\tau\|_{L^2}\\&\quad+\Big(\int_t^\infty\|\int_t^\infty U(t-\tau)R_1(\tau)\,d\tau\|_{X_n}^4\,dt\Big)^{1/4}\\&\quad\le C\int_t^\infty\|R_1(\tau)\|_{L^2}\,d\tau\le Ct^{-1}(\log t)^2\|\phi\|_{H^{0,2}}^{1+\frac{2}{n}}\end{aligned}\tag{3.2}$$

since by the Hölder inequality we have

$$\begin{aligned}&\|R_1(t)\|_{L^2}\\&\le Ct^{-2}\|\Delta\widehat{\phi}\|_{L^2}+Ct^{-2}(\log t)^2\|\widehat{\phi}\|_{L^\infty}^{\frac{2}{n}-1}\|\nabla\widehat{\phi}\|_{L^4}^2+Ct^{-2}(\log t)\|\widehat{\phi}\|_{L^\infty}^{\frac{2}{n}}\|\Delta\widehat{\phi}\|_{L^2}\\&\le Ct^{-2}(\log t)^2\|\phi\|_{H^{0,2}}^{1+\frac{2}{n}}.\end{aligned}$$

We now define u_1 as

$$u_1(t)=-i\int_\infty^t U(t-\tau)N_n(u_0)\,d\tau$$

which implies $\mathcal{L}u_1=N_n(u_0)$ and

$$\begin{aligned}&u(t)-u_0(t)\\&=-i\int_\infty^t U(t-\tau)(N_n(v)-N_n(u_0)+G_n(v)-G_n(u_0))\,d\tau\\&\quad-i\int_\infty^t U(t-\tau)R_1(\tau)\,d\tau+u_1(t).\end{aligned}\tag{3.3}$$

Note that

$$i\partial_t u_1(t)=N_n(u_0)+\frac{i}{2}\int_\infty^t U(t-\tau)\Delta N_n(u_0)\,d\tau.\tag{3.4}$$

Now, we define the function space

$$X=\left\{f\in C([T,\infty);L^2);\|f\|_X<\infty\right\},\ \text{where}$$

$$\|f\|_X=\sup_{t\in[T,\infty)}t^b\|f(t)-u_0(t)\|_{L^2}+\sup_{t\in[T,\infty)}t^b\Big(\int_t^\infty\|f(t)-u_0(t)\|_{X_n}^4\,dt\Big)^{1/4},$$

and

$$X_1=L^\infty,\quad X_2=L^4,\quad b>\frac{n}{4}.$$

Let X_ρ be a closed ball in X with a radius ρ and a center u_0. Let $v \in X_\rho$. From (3.4) and Lemma 2.1 it follows that

$$
\begin{aligned}
i\partial_t u_1(t) = N_n(u_0) + \frac{i}{2} \sum_{(\omega,h,g,f)} \Big(-\frac{2i\omega}{1-\omega} h(it) e^{\frac{i\omega x^2}{2t}} e^{ig(\frac{x}{t})\log t} f(\frac{x}{t})\\
- \frac{2\omega}{(1-\omega)^2} \int_\infty^t \Big(\sum_{(F,k)} F(i\tau) e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau} k(\frac{x}{\tau})\\
- i\omega U(t-\tau) \int_\infty^\tau \sum_{(F,k)} F'(is) e^{\frac{i\omega x^2}{2s}} e^{(\frac{x}{s})\log s} k(\frac{x}{s})\, ds\\
- i\omega U(t-\tau) \int_\infty^\tau \sum_{(F,k)} F(is) e^{\frac{i\omega x^2}{2s}} e^{ig(\frac{x}{s})\log s} \frac{1}{s} k(g - \frac{in}{2})(\frac{x}{s})\, ds\Big)\, d\tau + R(t),
\end{aligned}
$$

where the summation with respect to (ω, h, g, f) is taken over

$$
\begin{aligned}
(\omega, h, g, f) = &\Big(3, (it)^{-3/2}, \lambda_0|\hat{\phi}(\frac{x}{t})|^2, \lambda_1\hat{\phi}(\frac{x}{t})^3\Big),\\
&\Big(-1, (-i)^{-1/2}t^{-3/2}, \lambda_0|\hat{\phi}(\frac{x}{t})|^2, \lambda_2\hat{\phi}(\frac{x}{t})\overline{\hat{\phi}(\frac{x}{t})}^2\Big),\\
&\Big(-3, (-it)^{-3/2}, \lambda_0|\hat{\phi}(\frac{x}{t})|^2, \lambda_3\overline{\hat{\phi}(\frac{x}{t})}^3\Big),
\end{aligned}
$$

when $n = 1$, and

$$
(\omega, h, g, f) = \Big(2, (it)^{-1}, \lambda_0|\hat{\phi}(\frac{x}{t})|, \lambda_1\hat{\phi}(\frac{x}{t})^2\Big), \Big(-2, (-it)^{-1}, \lambda_0|\hat{\phi}(\frac{x}{t})|, \lambda_2\overline{\hat{\phi}(\frac{x}{t})}^2\Big),
$$

when $n = 2$, and the summation with respect to (F, k) is taken over $(F, k) = (h', f), (h\tau^{-1}, f(g - in/2))$. We have

$$
\begin{aligned}
&G_n(v) - G_n(u_0)\\
&= \lambda_0|v|^{\frac{2}{n}}v - \lambda_0|u_0|^{\frac{2}{n}}u_0\\
&= \lambda_0(|v|^{\frac{2}{n}} - |u_0|^{\frac{2}{n}})(v - u_0) + \lambda_0(|v|^{\frac{2}{n}} - |u_0|^{\frac{2}{n}})u_0 + \lambda_0|u_0|^{\frac{2}{n}}(v - u_0).
\end{aligned}
$$

Therefore, by the Strichartz estimate we get

$$
\begin{aligned}
&\|\int_t^\infty U(t-\tau)(G_n(v) - G_n(u_0))\, d\tau\|_{L^2}\\
&+ \Big(\int_t^\infty \|\int_t^\infty U(t-\tau)(G_n(v) - G_n(u_0))\, d\tau\|_{X_2}^4\, dt\Big)^{1/4}\\
\le& C\Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2}^2\, d\tau\Big)^{\frac{1}{2}} \Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{X_2}^4\, d\tau\Big)^{1/4} \qquad (3.5)\\
&+ C\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2}\|u_0(\tau)\|_{L^\infty}\, d\tau\\
\le& C\rho^{1+\frac{2}{n}}t^{-2b+\frac{1}{2}} + Ct^{-b}\rho\|\phi\|_{L^1},
\end{aligned}
$$

for $n = 2$. Also

$$
\begin{aligned}
&\| \int_t^\infty U(t-\tau)(\mathcal{G}_n(v) - G_n(u_0))\, d\tau \|_{L^2} \\
&\quad + \Big(\int_t^\infty \| \int_t^\infty U(t-\tau)(G_n(v) - G_n(u_0))\, d\tau \|_{X_1}^4 \, dt \Big)^{1/4} \\
&\le C\Big(\int_t^\infty \| |v(\tau) - u_0(\tau)|^3 \|_{L^1}^{\frac{4}{3}} \, d\tau \Big)^{3/4} \\
&\quad + C \int_t^\infty \| |v(\tau) - u_0(\tau)| |u_0(\tau)|^2 \|_{L^2} \, d\tau \\
&\le C\Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^\infty}^{\frac{4}{3}} \|v(\tau) - u_0(\tau)\|_{L^2}^{\frac{8}{3}} d\tau \Big)^{3/4} \\
&\quad + C \int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2} \|u_0(\tau)\|_{L^\infty}^2 \, d\tau \\
&\le C\Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^\infty}^4 \, d\tau)^{\frac{1}{4}} (\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2}^4 d\tau \Big)^{1/2} \\
&\quad + C \int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2} \|u_0(\tau)\|_{L^\infty}^2 \, d\tau \\
&\le C\rho \Big(\int_t^\infty \rho^4 \tau^{-4b} d\tau \Big)^{1/2} + C\rho \|\phi\|_{L^1}^2 \int_t^\infty \tau^{-b-1} d\tau \\
&\le C\rho^3 t^{-3b+\frac{1}{2}} + Ct^{-b} \rho \|\phi\|_{L^1}^2,
\end{aligned} \tag{3.6}
$$

for $n = 1$, where we have used the facts that $b > n/4$ and

$$
|G_n(v) - G_n(u_0)| \le C(|v - u_0|^{\frac{2}{n}} + |u_0|^{\frac{2}{n}})|v - u_0|.
$$

Similarly, we see that the above estimate holds valid with G_n replaced by N_n. Thus by (3.2), (3.3), (3.5) and (3.6)

$$
\begin{aligned}
&\|u(t) - u_0(t)\|_{L^2} + \Big(\int_t^\infty \|u(\tau) - u_0(\tau)\|_{X_n}^4 \, d\tau \Big)^{1/4} \\
&\quad \le C\rho^{1+\frac{2}{n}} t^{-(1+\frac{2}{n})b+\frac{1}{2}} + Ct^{-b} \rho \|\phi\|_{L^1}^{\frac{2}{n}} + Ct^{-1} (\log t)^2 \|\phi\|_{H^{0,2}}^{1+\frac{2}{n}} \\
&\qquad + \|u_1(t)\|_{L^2} + \Big(\int_t^\infty \|u_1(\tau)\|_{X_n}^4 \, d\tau \Big)^{1/4}.
\end{aligned} \tag{3.7}
$$

To get the result we now estimate $u_1(t)$. By Lemma 2.1, Lemma 2.2 and Lemma 2.3 we get

$$
\begin{aligned}
&\|u_1(t)\|_{L^2} + \Big(\int_t^\infty \|u_1(\tau)\|_{X_n}^4 d\tau \Big)^{1/4} \\
&\quad \le C(\| | \cdot |^{-\tilde{\delta}} \widehat{\phi} \|_{L^2} + \|\phi\|_{H^{0,2}})^{1+\frac{2}{n}} t^{-\frac{\delta}{2}},
\end{aligned} \tag{3.8}
$$

for $\frac{n}{2} < \widetilde{\delta} < 2$, where we have used the fact that

$$\begin{aligned}
&\| \int_t^\infty \int_s^\infty U(s-\tau) f(\tau)\, d\tau\, ds \|_{X_n} \\
&\le C \int_t^\infty s^{-\alpha} s^{\alpha} \| \int_s^\infty U(s-\tau) f(\tau)\, d\tau \|_{X_n}\, ds \\
&\le C \Big(\int_t^\infty s^{-\frac{4}{3}\alpha}\, ds \Big)^{3/4} \Big(\int_t^\infty s^{4\alpha} \| \int_s^\infty \mathcal{U}(s-\tau) f(\tau)\, d\tau \|_{X_n}^4\, ds \Big)^{1/4} \\
&\le C t^{-\alpha+\frac{3}{4}} \Big(\int_t^\infty s^{4\alpha} \| \int_s^\infty U(s-\tau) f(\tau)\, d\tau \|_{X_n}^4\, ds \Big)^{1/4}
\end{aligned}$$

with $\alpha \ge 1$, from which it follows that

$$\begin{aligned}
&\Big(\int_{\widetilde{t}}^\infty \| \int_t^\infty \int_s^\infty U(s-\tau) f(\tau)\, d\tau\, ds \|_{X_n}^4\, dt \Big)^{1/4} \\
&\le C \Big(\int_{\widetilde{t}}^\infty t^{-4\alpha+3} \Big(\int_t^\infty \| \int_s^\infty U(s-\tau) \tau^{\alpha} f(\tau)\, d\tau \|_{X_n}^4\, ds \Big) dt \Big)^{1/4} \\
&\le C \Big(\int_{\widetilde{t}}^\infty t^{-4\alpha+3} \Big(\int_t^\infty \| \tau^{\alpha} f(\tau) \|_{L^2}\, d\tau \Big)^4 dt \Big)^{1/4} \\
&\le C t^{-\alpha+1-\beta} \sup_t t^{\beta} \int_t^\infty \| \tau^{\alpha} f(\tau) \|_{L^2}\, d\tau \\
&\le C t^{-\beta} \sup_t t^{\beta} \int_t^\infty \| \tau^{\alpha} f(\tau) \|_{L^2}\, d\tau.
\end{aligned}$$

By virtue of (3.7) and (3.8), taking $\frac{n}{2} < \widetilde{\delta} < 2, b = \frac{\widetilde{\delta}}{2}$, we get

$$\begin{aligned}
&\| u(t) - u_0(t) \|_{L^2} + \Big(\int_t^\infty \| u(\tau) - u_0(\tau) \|_{X_n}^4 d\tau \Big)^{1/4} \\
&\le C(\| | \cdot |^{-\widetilde{\delta}} \widehat{\phi} \| + \| \phi \|_{H^{0,2}})^{1+\frac{2}{n}} t^{-b}.
\end{aligned} \tag{3.9}$$

Since the norm of the final state $\|\phi\|_{H^{0,2}} + \|\phi\|_{\dot{H}^{-\delta}}$ is sufficiently small, estimate (3.9) implies that there exists a sufficiently small radius $\rho > 0$ such that the mapping $\mathcal{M} v = u$. defined by equation (3.1), transforms the set X_ρ into itself. In the same way as in the proof of estimate (3.9) we find that $\mathcal{M}$ is a contraction mapping in X_ρ. This completes the proof of the theorem.

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