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# $L^1$ Estimates for Dissipative Wave Equations

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## 1 Introduction and Results

We consider the $L^1$ estimates of the solution $u=u(x,t)$ to the Cauchy problem for the dissipative wave equation :

$$\begin{cases}(\Box+\partial_t)u=0 \quad \text{in} \quad \mathbb{R}^N\times(0,\infty)\\ (u,\partial_t u)|_{t=0}=(u_0,u_1)\end{cases} \tag{1.1}$$

where $\Box+\partial_t=\partial_t^2+\partial_t-\Delta_x$ is the dissipative wave operator with Laplacian $\Delta_x=\sum_{j=1}^N \partial_{x_j}^2$. This equation (1.1) often called the telegraph equation or the damped wave equation.

Matsumura [12] has shown the $L^2$ estimates and the $L^\infty$ estimates of the solution $u(t)$ of (1.1) by using the Fourier transform method. e.g.,

$$\|u(t)\|_{L^2}\leq C(1+t)^{-N/4}(\|u_0\|_{L^2}+\|u_1\|_{H^{-1}}+\|u_0\|_{L^1}+\|u_1\|_{L^1}), \quad t\geq 0,$$

$$\|u(t)\|_{L^\infty}\leq C(1+t)^{-N/2}(\|u_0\|_{H^{[N/2+1]}}+\|u_1\|_{H^{[N/2]}}+\|u_0\|_{L^1}+\|u_1\|_{L^1}), \quad t\geq 0$$

(cf. Kawashima–Nakao–Ono [8] and Hayashi–Kaikina–Naumkin [4]). Then, in this paper we pay attention to the $L^1$ estimates of the solution $u(t)$ of (1.1).

By applying the Fourier transformation in the space variable together with the $L^\infty$ decay estimates of the solution $u(t)$ as in [12], Milani and Han [13] derived the following $L^1$ type estimates of the solution $u(t)$ for large time $t$ :

$$\|\partial_t^k D_x^\beta u(t)\|_{L^1}\leq C\tilde{d}_* t^{-k-|\beta|/2} \qquad \text{for} \quad t\gg 1,$$

where $\tilde{d}_*=\|u_0\|_{H^{k+|\beta|+N/2+1}}+\|u_1\|_{H^{k+|\beta|+N/2}}+\|(1+|\cdot|)^{s_0}u_0\|_{L^1}+\|(1+|\cdot|)^{s_1}u_1\|_{L^1}$ with integers $s_0>(N+k+|\beta|+1)(N+1)-1$ and $s_1>(N+k+|\beta|)(N+1)-1$. The decay rates for large time $t$ seems to be sharp (cf. Ponce [25] for heat equation). However, their estimates should be relaxed the regularity conditions on the initial data and also should be estimated near the origin in time .

On the other hand, concerning the $L^1$ estimates of the solution $u(t)$ for $t\geq 0$ in lower dimensions, there are a few results. Those were given by Marcati and Nishihara [11] for $N=1$, Nishihara [17] for $N=3$, and Ono [19], [20] for $N\leq 3$ (also see Ono [21] for exterior domains), e.g.,

$$\|u(t)\|_{L^1}\leq\begin{cases}C(\|u_0\|_{L^1}+\|u_1\|_{L^1}) & \text{if} \quad N=1\\ C(\|u_0\|_{W^{1,1}}+\|u_1\|_{L^1}) & \text{if} \quad N=2,3\end{cases}$$

for $t \geq 0$, by using the exact solution $S(t)g$ of the dissipative wave equation (1.1) with $(u_0, u_1) = (0, g)$ : For $N = 1$,

$$S(t)g = e^{-t/2} \int_0^t I_0(a\sqrt{t^2 - \rho^2}) G_1(\cdot, \rho)\, d\rho$$

with $G_1(x, \rho) = \frac{1}{2}(g(x + \rho) + g(x - \rho))$. For $N = 2$,

$$S(t)g = e^{-t/2} \int_0^t \cosh(a\sqrt{t^2 - \rho^2}) \frac{\rho}{\sqrt{t^2 - \rho^2}} G_2(\cdot, \rho)\, d\rho$$

with $G_2(x, \rho) = \frac{1}{2\pi} \int_{S^1} g(x + \rho\omega)\, d\omega$ and $S^1 = \{\omega \in \mathbb{R}^2 \,|\, |\omega| = 1\}$. For $N = 3$,

$$S(t)g = e^{-t/2} \frac{1}{t} \partial_t \int_0^t I_0(a\sqrt{t^2 - \rho^2}) \rho^2 G_3(\cdot, \rho)\, d\rho$$

with $G_3(x, \rho) = \frac{1}{4\pi} \iint_{S^2} g(x + \rho\omega)\, d\omega$ and $S^2 = \{\omega \in \mathbb{R}^3 \,|\, |\omega| = 1\}$. Here, $I_0(\cdot)$ is the modified Bessel function of order 0. So, in higher dimensional cases, we will give similar results for the $L^1$ estimates of the solution $u(t)$ of (1.1).

Our main results are as follows.

**Theorem 1.1** *Let $N = 2n$ be even or $N = 2n + 1$ be odd for $n = 1, 2, \cdots$ . Suppose that the initial data*

$$u_0 \in W^{n,1} \quad \textit{and} \quad u_1 \in W^{n-1,1}.$$

*Then, the solution $u(t)$ satisfies*

$$\|u(t)\|_{L^1} \leq C(\|u_0\|_{W^{n,1}} + \|u_1\|_{W^{n-1,1}}), \quad t \geq 0.$$

Here, we set

$$W^{\ell,1} = \{\phi \in L^1 \,|\, D^\beta \phi \in L^1, |\beta| \leq \ell\}.$$

Theorem 1.1 is proved by estimating directly the representation formulas of the solution $u(t)$ as in Section 2 and we will give the outline of the proof of Theorem 1.1 in the following section (see Ono [23] and [24] for details).

As a corollary of Theorem 1.1 together with the $L^2$ estimates of the solution $u(t)$ as in [12], we immediately have the following.

**Corollary 1.2** *For $1 \leq p < 2$,*

$$\|u(t)\|_{L^p} \leq C d_{0,n} (1 + t)^{-(N/2)(1-1/p)}, \quad t \geq 0$$

*with $d_{0,n} = \|u_0\|_{L^2} + \|u_1\|_{H^{-1}} + \|u_0\|_{W^{n,1}} + \|u_1\|_{W^{n-1,1}}$.*

By induction, we have

**Theorem 1.3** *Let $m \geq 1$. Suppose that the initial data $(u_0, u_1)$ belong to $(H^{m+1} \cap W^{n,1}) \times (H^m \cap W^{n-1,1})$. Then, the solution $u(t)$ satisfies that for $0 \leq k + |\beta| \leq m$ and $k \neq m$,*

$$\|\partial_t^k D_x^\beta u(t)\|_{L^1} \leq C d_{m+1,n} (1 + t)^{-k-|\beta|/2}, \quad t \geq 0$$

*with* $d_{m+1,n} = \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} + \|u_0\|_{W^{n,1}} + \|u_1\|_{W^{n-1,1}}$.

*Moreover, for* $1 \le p < 2$ *and for* $0 \le k + |\beta| \le m$ *and* $k \ne m$,

$$\|\partial_t^k D_x^\beta u(t)\|_{L^p} \le C d_{m+1,n}(1+t)^{-k-|\beta|/2-(N/2)(1-1/p)}, \quad t \ge 0.$$

We note that Marcati and Nishihara [11] for $N = 1$ and Nishihara [17] for $N = 3$ derived the $L^p$–$L^q$ type estimates with $1 \le p \le q \le \infty$ of the solution $u(t)$ (cf. Hosono–Ogawa [5] and Narazaki [15] for $L^p$–$L^q$ type estimates with some $p \ne 1$).

## 2 Representation formulas

By Courant and Hilbert's book [1], we know the representation formula of the solution $w(t)$ to the Cauchy problem for the following wave equation (cf. [26], [27]) :

$$\Box w = a^2 w \quad \text{with} \quad (w, \partial_t w)|_{t=0} = (0, g). \tag{2.1}$$

Then, we observe the relation

$$S(t)g = e^{-at} w(t), \tag{2.2}$$

where $S(t)g$ is the solution of

$$(\Box + 2a\partial_t)v = 0 \quad \text{with} \quad (v, \partial_t v)|_{t=0} = (0, g).$$

Therefore, by the Duhamel principle (e.g. [2]), the solution $u(t)$ of (1.1) is expressed as

$$u(t) = \partial_t S(t) u_0 + S(t)(u_0 + u_1) \quad \text{with} \quad a = 1/2. \tag{2.3}$$

Thus, in order to get the $L^1$ estimates of the solution $u(t)$ of (1.1), we need to estimate the $L^1$ estimates of the function $S(t)g$ and its derivatives $\partial_t S(t)g$.

### 2.1 Even dimension $N = 2n$

We first consider the even dimensional cases (i.e. $N = 2n$).

Define a new function $\Phi(y)$ by

$$\Phi(y) = (e^y + e^{-y})\frac{1}{y},$$

then we obtain from (2.2) and the representation formula of $w(t)$ as in [1] that

$$S(t)g = e^{-at} t^{-2n} (t^3 \partial_t)^{n-1} (t^{2(2-n)} R(t)),$$

where

$$R(t) = \int_0^t \Phi(a\sqrt{t^2 - \rho^2})\, \rho^{2n-1} G(\rho)\, d\rho$$

and

$$G(\rho) = \frac{a}{2}\left(\frac{1}{2\pi}\right)^n \int_{S^{2n-1}} g(x+\rho\omega)\, d\omega$$

with $S^{2n-1} = \{\omega \in \mathbb{R}^{2n} \mid |\omega| = 1\}$.

By an elementary calculation, we observe that

$$S(t)g = e^{-at}t^{-n+1}\left(ct^{-n+2}\partial_t R(t) + ct^{-n+3}\partial_t^2 R(t) + \cdots + ct^{-1}\partial_t^{n-2}R(t) + \partial_t^{n-1}R(t)\right). \tag{2.4}$$

Dividing the integration in time $t$ in $R(t)$ into two parts, we have

$$R(t) = \left(\int_0^{t^{3/4}} + \int_{t^{3/4}}^t\right) \Phi(a\sqrt{t^2-\rho^2})\, \rho^{2n-1}G(\rho)\, d\rho \equiv R_1(t) + R_2(t)$$

and we denote (2.4) with $R_1(t)$ (resp. $R_2(t)$) instead of $R(t)$ by $S_1(t)g$ (resp. $S_2(t)g$), that is,

$$S(t)g = S_1(t)g + S_2(t)g\,.$$

For $t \geq 2$, we see that

$$\partial_t^k R_1(t) = \sum_{j=1}^k \partial_t^{k-j} f_j(t) + \int_0^{t^{3/4}} \partial_t^k \Phi(a\sqrt{t^2-\rho^2})\, \rho^{2n-1}G(\rho)\, d\rho$$

with $f_j(t) = \partial_t^{j-1}\left(\Phi(a\sqrt{t^2-\rho^2})\right) \rho^{2n-1}G(\rho)|_{\rho=t^{3/4}} \cdot \partial_t(t^{3/4})$.

Inductively, we define $\Phi_k(y)$ by

$$\Phi_0(y) = e^y - e^{-y} \quad \text{and} \quad \Phi_k(y) = \Phi_{k-1}'(y)\frac{1}{y}\,.$$

Then, we observe that

$$\Phi_k(y) = \frac{e^y}{y^k}\left(1 + \frac{c}{y} + \frac{c}{y^2} + \cdots + \frac{c}{y^{k-1}}\right) + \frac{e^{-k}}{y^k}\left(\pm 1 + \frac{c}{y} + \frac{c}{y^2} + \cdots + \frac{c}{y^{k-1}}\right).$$

By an elementary calculation together with the fact $e^{-at}e^{-a\sqrt{t^2-\rho^2}} \leq e^{-a\rho^2/(2t)}$ for $0 < \rho < t$, we obtain

**Lemma 2.1** (i) *For $t \geq 2$ and $0 < \rho < t^{3/4}$,*

$$e^{-at}\Phi_k(a\sqrt{t^2-\rho^2}) \leq Ct^{-k}e^{-a\rho^2/(2t)}\,.$$

(ii) *For $0 < \rho < t$,*

$$e^{-at}\Phi_0(a\sqrt{t^2-\rho^2}) \leq Ce^{-a\rho^2/(2t)}\,,$$
$$e^{-at}\Phi_1(a\sqrt{t^2-\rho^2}) \leq Ct^{-1/2}e^{-a\rho^2/(2t)}\frac{1}{\sqrt{t-\rho}}\,.$$

(iii) *For $t \geq 2$ and $t^{3/4} < \rho < t$,*

$$e^{-at}\Phi_0(a\sqrt{t^2-\rho^2}) \leq Ce^{-a\sqrt{t}/2},$$
$$e^{-at}\Phi_1(a\sqrt{t^2-\rho^2}) \leq Ct^{-1/2}e^{-a\sqrt{t}/2}\frac{1}{\sqrt{t-\rho}}.$$

Since $\Phi_1(y) = \Phi(y)$ and $\partial_t(a\sqrt{t^2-\rho^2}) = a^2t/(a\sqrt{t^2-\rho^2})$, it follows that

$$\begin{aligned}\partial_t^{2\ell}\Phi(a\sqrt{t^2-\rho^2}) =& (a^2t)^{2\ell}\Phi_{2\ell+1}(a\sqrt{t^2-\rho^2}) + ct^{2\ell-2}\Phi_{2\ell}(a\sqrt{t^2-\rho^2})\\ &+ \cdots + ct^2\Phi_{\ell+2}(a\sqrt{t^2-\rho^2}) + c\,\Phi_{\ell+1}(a\sqrt{t^2-\rho^2})\end{aligned}$$

and

$$\begin{aligned}\partial_t^{2\ell+1}\Phi(a\sqrt{t^2-\rho^2}) =& (a^2t)^{2\ell+1}\Phi_{2\ell+2}(a\sqrt{t^2-\rho^2}) + ct^{2\ell-1}\Phi_{2\ell+1}(a\sqrt{t^2-\rho^2})\\ &+ \cdots + ct^3\Phi_{\ell+3}(a\sqrt{t^2-\rho^2}) + ct\,\Phi_{\ell+2}(a\sqrt{t^2-\rho^2}).\end{aligned}$$

Using the above identities and lemma, we have that

$$\|S_1(t)g\|_{L^1} \leq Ce^{-at/2}\|g\|_{W^{n-2,1}} + C\|g\|_{L^1}, \quad t \geq 2.$$

Moreover, we observe the following estimates :

$$\|S_2(t)g\|_{L^1} \leq Ce^{-a\sqrt{t}/3}\|g\|_{W^{n-1,1}}, \quad t \geq 2$$

and

$$\|S(t)g\|_{L^1} \leq C\|g\|_{W^{n-1,1}}, \quad t \leq 2,$$

and hence, we obtain the $L^1$ estimate of $S(t)g$ :

$$\|S(t)g\|_{L^1} \leq Ce^{-a\sqrt{t}/3}\|g\|_{W^{n-1,1}} + C\|g\|_{L^1}, \quad t \geq 0.$$

From (2.4), we see that

$$\begin{aligned}\partial_t S(t)g + aS(t)g = e^{-at}\Big(&ct^{-2n+2}\partial_t R(t) + ct^{-2n+3}\partial_t^2 R(t) + \cdots\\ &+ ct^{-n}\partial_t^{n-1}R(t) + t^{-n+1}\partial_t^n R(t)\Big)\end{aligned}$$

and

$$\begin{aligned}\partial_t S(t)g = e^{-at}\Big(&\sum_{k=1}^{n-1} ct^{-2n+1+k}\partial_t^k R(t) - a\sum_{k=1}^{n-2} ct^{-2n+2+k}\partial_t^k R(t)\\ &+ t^{-n+1}\big(\partial_t^n R(t) - a\partial_t^{n-1}R(t)\big)\Big) \equiv T_1(t)g + T_2(t)g,\end{aligned}$$

where $T_i(t)g$ has $R_i(t)$ instead of $R(t)$ for $i = 1, 2$.

Then, we observe the following estimates :

$$\|T_1(t)g\|_{L^1} \leq Ce^{-at/2}\|g\|_{W^{n-1,1}} + Ct^{-1}\|g\|_{L^1}, \quad t \geq 2$$

and

$$\|T_2(t)g\|_{L^1} \leq Ce^{-a\sqrt{t}/3}\|g\|_{W^{n,1}}, \quad t \geq 2$$

and

$$\|\partial_t S(t)g\|_{L^1} \leq C\|g\|_{W^{n,1}}, \quad t \leq 2,$$

and hence, we obtain the $L^1$ estimate of $\partial_t S(t)g$ :

$$\|\partial_t S(t)g\|_{L^1} \leq Ce^{-a\sqrt{t}/3}\|g\|_{W^{n,1}} + C(1+t)^{-1}\|g\|_{L^1}, \quad t \geq 0.$$

Therefore, by (2.3), we immediately obtain that

$$\begin{aligned}\|u(t)\|_{L^1} &\leq \|\partial_t S(t)u_0\|_{L^1} + \|S(t)(u_0+u_1)\|_{L^1} \\ &\leq e^{-a\sqrt{t}/3}(\|u_0\|_{W^{n,1}} + \|u_1\|_{W^{n-1,1}}) + C(\|u_0\|_{L^1} + \|u_1\|_{L^1})\end{aligned}$$

for $t \geq 0$, which implies Theorem 1.1 in even dimensions.

### 2.2 Odd dimension $N = 2n+1$

Next we consider the odd dimensional cases (i.e. $N = 2n+1$).
From (2.2) and the representation formula of $w(t)$ of (2.1) as in [1], we have that

$$S(t)g = e^{-at}t^{-2n-2}(t^3\partial_t)^n(t^{2(1-n)}R(t)),$$

where

$$R(t) = \int_0^t I_0(a\sqrt{t^2-\rho^2})\,\rho^{2n}G(\rho)\,d\rho$$

and

$$G(\rho) = \frac{1}{2}\big(\frac{1}{2\pi}\big)^n \int_{S^{2n}} g(x+\rho\omega)\,d\omega$$

with $S^{2n} = \{\omega \in \mathbb{R}^{2n+1} \,|\, |\omega| = 1\}$. Here, $I_\nu(y)$ is the modified Bessel function of order $\nu$ and is given by

$$I_\nu(y) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+1+\nu)}\big(\frac{y}{2}\big)^{2m+\nu}$$

with the Gamma function $\Gamma(\cdot)$ and satisfies the following properties (see [16]) :

$$\begin{aligned}I_{\nu+1}(y) &= I_\nu'(y) - \frac{\nu}{y}I_\nu(y), \\ I_\nu(y) &= \frac{e^y}{\sqrt{2\pi y}}\big(1+O(y^{-1})\big) \quad \text{as} \quad y \to \infty, \\ I_\nu(y) &= \frac{1}{\Gamma(\nu+1)}\big(\frac{y}{2}\big)^\nu + O(y^{\nu+2}) \quad \text{as} \quad y \to 0.\end{aligned}$$

By an elementary calculation, we have

$$S(t)g = e^{-at}t^{-n}\left(ct^{-n+1}\partial_t R(t) + ct^{-n+2}\partial_t^2 R(t) + \cdots + ct^{-1}\partial_t^{n-1}R(t) + \partial_t^n R(t)\right)$$

Differentiating $R(t)$ in time $t$, we have that

$$\partial_t^k R(t) = \sum_{j=1}^{k} \partial_t^{k-j} f_j(t) + \int_0^t \partial_t^k I_0(a\sqrt{t^2-\rho^2})\, \rho^{2n} G(\rho)\, d\rho$$

with $f_j(t) = \left(\partial_t^{j-1} I_0(a\sqrt{t^2-\rho^2})\right)\big|_{\rho=t} \cdot t^{2n}G(t)$ for $1 \le j \le k$.

Inductively, we define $\Lambda_k(y)$ by

$$\Lambda_0(y) = I_0(y) \quad \text{and} \quad \Lambda_k(y) = \Lambda'_{k-1}(y)\frac{1}{y}\,. \tag{2.5}$$

Then, noting $\partial_t \Lambda_k(a\sqrt{t^2-\rho^2}) = a^2 t\, \Lambda_{k+1}(a\sqrt{t^2-\rho^2})$, we observe that

$$\begin{aligned}\partial_t^{2\ell} I_0(a\sqrt{t^2-\rho^2}) &= (a^2t)^{2\ell}\Lambda_{2\ell}(a\sqrt{t^2-\rho^2}) + ct^{2\ell-2}\Lambda_{2\ell-1}(a\sqrt{t^2-\rho^2}) \\ &\quad + \cdots + ct^2\Lambda_{\ell+1}(a\sqrt{t^2-\rho^2}) + c\Lambda_\ell(a\sqrt{t^2-\rho^2})\end{aligned}$$

and

$$\begin{aligned}\partial_t^{2\ell+1} I_0(a\sqrt{t^2-\rho^2}) &= (a^2t)^{2\ell+1}\Lambda_{2\ell+1}(a\sqrt{t^2-\rho^2}) + ct^{2\ell-1}\Lambda_{2\ell}(a\sqrt{t^2-\rho^2}) \\ &\quad + \cdots + ct^3\Lambda_{\ell+2}(a\sqrt{t^2-\rho^2}) + ct\,\Lambda_{\ell+1}(a\sqrt{t^2-\rho^2})\,.\end{aligned}$$

In order to estimate the function $\Lambda_k(y)$ defined by (2.5), we use the following lemma.

**Lemma 2.2** *The function $\Lambda_k(y)$ $(k = 0, 1, 2, \cdots)$ satisfies that*

$$\Lambda_k(y) = I_k(y)\frac{1}{y^k} \quad \textit{and} \quad \Lambda_k(0) = \frac{1}{2^k\, k!}\,.$$

The following estimates of the function $\Lambda_k(a\sqrt{t^2-\rho^2})$ are crucial for the $L^1$ estimates of $S(t)g$ and $\partial_t S(t)g$.

**Lemma 2.3** *For $t \ge 2$, it holds that*

$$\begin{aligned} e^{-at}\Lambda_k(a\sqrt{t^2-\rho^2}) &\le Ct^{-k-1/2}e^{-a\rho^2/(2t)} \quad \textit{if} \quad 0 \le \rho < t^{3/4}\,, \\ e^{-at}\Lambda_k(a\sqrt{t^2-\rho^2}) &\le Ct^{-1/2}e^{-a\sqrt{t}/2} \quad \textit{if} \quad t^{3/4} \le \rho < \sqrt{t^2-1}\,, \\ e^{-at}\Lambda_k(a\sqrt{t^2-\rho^2}) &\le Ce^{-at} \quad \textit{if} \quad \sqrt{t^2-1} \le \rho \le t \end{aligned}$$

*with some constant $C$.*

Then, using the above identities and lemma, we observe the following estimates :

$$\|S(t)g\|_{L^1} \le Ce^{-at/2}\|g\|_{W^{n-1,1}} + C\|g\|_{L^1}\,, \quad t \ge 0$$

and
$$\|\partial_t S(t)g\|_{L^1} \le Ce^{-at/2}\|g\|_{W^{n,1}} + C\|g\|_{L^1}, \quad t \ge 0.$$
Therefore, by (2.3), we immediately obtain that
$$\begin{aligned}\|u(t)\|_{L^1} &\le \|\partial_t S(t)u_0\|_{L^1} + \|S(t)(u_0+u_1)\|_{L^1}\\ &\le Ce^{-at/2}(\|u_0\|_{W^{n,1}} + \|u_1\|_{W^{n-1,1}}) + C(\|u_0\|_{L^1} + \|u_1\|_{L^1})\end{aligned}$$
for $t \ge 0$, which implies Theorem 1.1 in odd dimensions.

## 3 Application

We consider the global existence, uniqueness, and asymptotic behavior of solutions to the Cauchy problem for the semilinear dissipative wave equations :
$$\begin{cases}(\Box + \partial_t)\, u = f(u) & \text{in} \quad \mathbb{R}^N \times (0,\infty)\\ (u, \partial_t u)|_{t=0} = (\varepsilon u_0, \varepsilon u_1)\end{cases} \tag{3.1}$$
with $f(u) = |u|^{\alpha+1}$, $|u|^\alpha u$ for $\alpha > 0$, and a small parameter $\varepsilon > 0$.

The critical exponent $\alpha_c(N)$ for global and non-global existence problems in the $L^1 \cap L^2$-framework is $\alpha_c(N) = 2/N$ and this number is often called Fujita's exponent. Indeed, Fujita [3] proved that the related semilinear heat equations have no non-trivial, global solutions if $\alpha \le 2/N$ and have global, small data solutions if $\alpha > 2/N$.

Todorova and Yordanov [28] have shown that when $2/N < \alpha \le 2/[N-2]^+$, there exist global solutions of the dissipative wave equation (3.1) with small initial data $(u_0, u_1) \in H^1 \times L^2$ satisfying compactly support conditions (cf. Matsumura [12] for $\alpha \ge 1$ and $\alpha > 2/N$). On the other hand, when $\alpha < 2/N$, (3.1) with the nonlinearity $f(u) = |u|^{\alpha+1}$ has no non-trivial global solutions (cf. Ikehata and Ohta [7] for $f(u) = |u|^\alpha u$). Later, in the case of $\alpha = 2/N$, Zhang [29] and Kirane and Qafsaoui [9] derived non-global existence theorems (cf. Li and Zhou [10] for $N = 1, 2$).

The global solvability problem under non-compactly support conditions on the initial data is a difficult and an interesting problem, because we can not use Poincar 's inequality and its related structure of the solutions in the a-priori estimate. In particular, when $N \ge 3$ and $\alpha < 1$, the estimate of $L^1$ norm of the nonlinear term $f(u)$ and thus the $L^p$ type, $1 \le p < 2$, estimates of the solution $u(t)$ will be requested in the analysis, and hence, the problem will become difficult and attracts us. (Cf. Nakao and Ono [14] for $\alpha \ge N/4$ in the $L^2$-framework. See [8], [18] for $f(u) = -|u|^\alpha u$ and for large data.)

Recently, when $N = 3$, Nishihara [17] has proved a global existence theorem for the initial data $(u_0, u_1) \in (W^{1,\infty} \cap W^{1,1}) \times (L^\infty \cap L^1)$ and $\alpha > 2/N$ together with $L^p$ decay estimates for $p \ge 1$. On the other side, in [20] we have solved this problem when $N \le 3$ and $2/N < \alpha \le 2/(N-2)$ for the initial data $(u_0, u_1) \in (H^1 \cap W^{1,1}) \times (L^2 \cap L^1)$ (or $(H^1 \cap L^1) \times (L^2 \cap L^1)$ if $N = 1$), and moreover, we have derived the sharp decay estimates on $L^p$-norm with $p \ge 1$ of the solutions. (See Ikehata and Ohta [7] and Ikehata, Miyaoka and Nakatake [6] for $N = 1, 2$ and $\alpha > 2/N$.)

Quite recently, Narazaki [15] has shown global existence theorems for $N \le 5$ and $2/N < \alpha \le 2/(N-2)$ under the assumptions on the initial data $(u_0, u_1) \in (H^2 \cap$

$W^{1,1+1/\alpha} \cap W^{1,1+\alpha} \cap L^1) \times (H^1 \cap L^{1+1/\alpha} \cap L^{1+\alpha} \cap L^1)$ and derived $L^p$ decay estimates for $p \geq 1+\alpha$ of the solutions. Also, Hayashi, Kaikina, and Naumkin [4] have obtained the global solutions in any dimensions for initial data on suitable weighted Soblev spaces.

Our aims in this section are to prove the global existence theorem by the method used $L^1$ estimates as in [17] and [20] which is different from [15], and to derive the sharp decay estimates on $L^p$ norm with $p \geq 1$ of the solutions (see Ono [22] for details).

**Theorem 3.1** *Let $N = 4, 5$. Suppose that the initial data $(u_0, u_1)$ belong to $(H^1 \cap W^{2,1}) \times (L^2 \cap W^{1,1})$ and*

$$2/N < \alpha \leq 2/(N-2) \quad \text{and} \quad \alpha \geq 1/2\,.$$

*Then, there exists $\varepsilon_0 > 0$ such that the problem* (3.1) *admits a unique global solution $u(t)$ belonging to $C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ for each $\varepsilon \leq \varepsilon_0$ and this solution satisfies*

$$\|\nabla_x u(t)\|_{L^2} \leq C d_{1,2}(1+t)^{-1/2-N/4}\,, \tag{3.2}$$

$$\|\partial_t u(t)\|_{L^2} \leq C d_{1,2}(1+t)^{-1-N/4}\,, \tag{3.3}$$

*and for $1 \leq p \leq 2N/(N-2)$,*

$$\|u(t)\|_{L^p} \leq d_{1,2}(1+t)^{-(N/2)(1-1/p)}\,, \tag{3.4}$$

*where $d_{1,2} = \|u_0\|_{H^1} + \|u_1\|_{L^2} + \|u_0\|_{W^{2,1}} + \|u_1\|_{W^{1,1}}$.*

**Theorem 3.2** *Let $N = 4, 5$. Suppose that the initial data $(u_0, u_1)$ belong to $(H^2 \cap W^{2,1}) \times (H^1 \cap W^{1,1})$ and*

$$2/N < \alpha \leq 2/[N-4]^+\,.$$

*Then, there exists $\varepsilon_0 > 0$ such that the problem* (3.1) *admits a unique global solution $u(t)$ belonging to $C([0,\infty); H^2) \cap C^1([0,\infty); H^1)$ for each $\varepsilon \leq \varepsilon_0$ and this solution satisfies* (3.2)–(3.4) *with $d_2$ instead of $d_1$ and*

$$\|\nabla_x^2 u(t)\|_{L^2} \leq C d_{2,2}(1+t)^{-1-N/4}\,,$$

$$\|\partial_t \nabla_x u(t)\|_{L^2} \leq C d_{2,2}(1+t)^{-3/2-N/4}\,,$$

*where $d_{2,2} = \|u_0\|_{H^2} + \|u_1\|_{H^1} + \|u_0\|_{W^{2,1}} + \|u_1\|_{W^{1,1}}$.*

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