A characterization of coactions which fix Cartan subalgebras

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1 Preparation

In this section, we summarize the basic facts about measured groupoids and von Neumann algebras associated to them. Further details regarding these objects can be found in [3], [8], [9]. We also briefly discuss actions of locally compact quantum groups on von Neumann algebras.

We assume that all von Neumann algebras in this paper have separable preduals, and

\[(X, \mu) : \text{standard Borel space,}\]
\[\mathcal{R} : \text{discrete measured equivalence relation on } (X, \mu),\]
\[\nu : \text{left counting measure on } \mathcal{R},\]
\[\sigma : \text{normalized 2-cocycle on } \mathcal{R},\]
\[\mathcal{R}(x) := \{y \in X : (x, y) \in \mathcal{R}\},\]
\[[\mathcal{R}] := \{\varphi : \text{bimeasurable nonsingular transformations}\]
\[\text{such that } \varphi(x) \text{ is in } \mathcal{R}(x) \text{ for a.e. } x \text{ in } X\},\]
\[\Gamma(\varphi) := \{(x, \varphi(x)) : x \in \text{Dom}(\varphi)\} \quad (\varphi \in [\mathcal{R}]).\]
Definition 1. (1) We define a von Neumann algebra $W^*(\mathcal{R}, \sigma)$ and a von Neumann subalgebra $W^*(X)$ which act on $L^2(\mathcal{R}, \nu)$ by the following:

$$W^*(\mathcal{R}, \sigma) := \{ L^\sigma(f) : f \text{ is a left finite function on } \mathcal{R} \}'',$$

$$W^*(X) := \{ L^\sigma(d) : d \in L^\infty(X, \mu) \},$$

where we regard $L^\infty(X, \mu)$ as functions on the diagonal of $\mathcal{R}$, and $L^\sigma(f)$ is defined by

$$\{ L^\sigma(f)\xi\}(x, z) := \sum_{y:(x,y)\in \mathcal{R}} f(x, y)\xi(y, z)\sigma(x, y, z).$$

(2) Let $A$ be a von Neumann algebra and $D$ be a subalgebra of $A$. We call $D$ is a Cartan subalgebra of $A$ if $D$ satisfies the following:

(i) $D$ is maximal abelian in $A$,

(ii) $D$ is regular in $A$, i.e., the normalizer $N_A(D)$ generates $A$, where

$$N_A(D) := \{ u \in A : u \text{ is unitary and } uDu^* = D \}.$$

(iii) there exists a faithful normal conditional expectation $E_D$ from $A$ onto $D$.

Theorem 2 ([3, Theorem 1]). For each inclusion of a von Neumann algebra $A$ and a Cartan subalgebra $D$ of $A$, there exists a standard Borel space $(X, \mu)$ and a discrete measured equivalence relation $\mathcal{R}$ on $X$ with a normalized 2-cocycle $\sigma$ such that $(D \subseteq A)$ is isomorphic to $(W^*(X) \subseteq W^*(\mathcal{R}, \sigma))$.

Theorem 3 ([1, Corollary 3.5]). Suppose $A$ is a von Neumann algebra with a Cartan subalgebra $D$ of $A$ such that $A = W^*(\mathcal{R}, \sigma)$ and $D = W^*(X)$. Then there exists a bijective correspondence between the set of Borel subrelations $S$ of $\mathcal{R}$ on $(X, \mu)$ and the set of von Neumann subalgebras $B$ of $A$ which contain $D$:

$$B \mapsto S_B \subseteq \mathcal{R}$$

$$S \mapsto W^*(S, \sigma) := \{ L^\sigma(f) \in A : \text{supp}(f) \subseteq S \} \subseteq A.$$
Let $G = (M, \Delta, \varphi, \psi)$ be a locally compact quantum group ($M$ is a von Neumann algebra, $\Delta : M \mapsto M \otimes M$ is a coproduct, $\varphi$ (resp. $\psi$) is a left (resp. right) invariant weight on $M$). A normal unital injective $\ast$-homomorphism $\alpha$ from $A$ onto $M \otimes A$ is called an action of $G$ on $A$ if $\alpha$ satisfies the following:

$$(\Delta \otimes id_A)\alpha = (id_M \otimes \alpha)\alpha.$$ 

In particular, if $G$ is cocommutative, i.e., $M$ is equal to the group von Neumann algebra $W^*(K)$ which is generated by the left regular representation $\lambda_K$ of a locally compact group $K$, and $\Delta$ is equal to $\Delta_K : \lambda_K(k) \mapsto \lambda_K(k) \otimes \lambda_K(k)$, then the action $\alpha$ is called a coaction of $K$.

### 2 A reduction to coaction case

In the discussion that follows, we fix a von Neumann algebra $A$ and a Cartan subalgebra $D$ of $A$ with an equivalence relation $\mathcal{R}$ on $(X, \mu)$ and a normalized 2-cocycle $\sigma$ of $\mathcal{R}$ such that the pair $(D \subseteq A)$ is equal to $(W^*(X) \subseteq W^*(\mathcal{R}, \sigma))$.

We assume that the action $\alpha$ fixes $D$, i.e., $\alpha(d) = 1 \otimes d$ for each $d \in D$. It follows that the fixed-point algebra $A^\alpha := \{a \in A : \alpha(a) = 1 \otimes a\}$ is an intermediate subalgebra for $D \subseteq A$.

We will prove that each such a action should be a coaction.

**Proposition 4.** Under the situation as above, the von Neumann subalgebra $\{(id_M \otimes \omega)(\alpha(a)) : a \in A, \omega \in A_\ast\}'$ of $M$ is contained in $IG(G)'$, where

$$IG(G) := \{u \in M : u is\ unitary\ and\ \Delta(u) = u \otimes u\}$$

is the intrinsic group of $G$.

In particular, if $\alpha$ is faithful, then $\alpha$ is a coaction of some locally compact group.

**Proof.** For each $u \in \mathcal{N}_A(D)$, set $w := \alpha(u)(1 \otimes u^*) \in M \otimes A$. Since $u$ normalizes $D$, for any $d \in D$, we have

$$w(1 \otimes d) = \alpha(u)(1 \otimes u^*)d = \alpha(u)(1 \otimes u^*du)(1 \otimes u^*) = \alpha(u)(1 \otimes u^*) = \alpha(du)(1 \otimes u^*) = (1 \otimes d)w.$$
Hence $w$ belongs to $(M \otimes A) \cap (C \otimes D)' = M \otimes D$. So we may and do assume that $w$ is an $M$-valued function. Moreover, we have

$$(\Delta \otimes id_A)(w) = (\Delta \otimes id_A)(\alpha(u)(1 \otimes u^*))$$

$$= (id_M \otimes \alpha)(\alpha(u))(1 \otimes 1 \otimes u^*)$$

$$= (id_M \otimes \alpha)(\alpha(u)(1 \otimes u^*))(1 \otimes \alpha(u))(1 \otimes 1 \otimes u^*)$$

$$= w_{12}w_{23}$$

Hence $w$ is an $IG(G)$-valued function. So we have that $\alpha(u) = w(1 \otimes u)$ belongs to $IG(G)^\sigma \otimes A$. Since $N_A(D)$ generates $A$, we get the conclusion. $\square$

3 Coactions derived from 1-cocycles

Let $K$ be a locally compact group. A Borel map $c : \mathcal{R} \to K$ is called a 1-cocycle if $c$ satisfies the following:

$$c(x, x) = 1_K \quad \text{for a.e. } x \in X,$$

$$c(x, y)c(y, z) = c(x, z) \quad \text{for a.e. } (x, y, z) \in \mathcal{R}^2.$$  

Each 1-cocycle $c$ into $K$ determines a unitary $U_c$ on $L^2(K) \otimes L^2(\mathcal{R})$ by

$$\{U_c\xi\}(k, x, y) = \xi(c(x, y)^{-1}k, x, y).$$

Since $c$ is a 1-cocycle, the map

$$\alpha_c(a) := U_c(1 \otimes a)U_c^* \quad (a \in A)$$

is a coaction of $K$. In fact, $\alpha_c$ is defined by the following:

$$\{\alpha_c(L^\sigma(f))\xi\}(k, x, z) := \sum_{y : (y, x) \in \mathcal{R}} f(x, y)\xi(c(x, y)^{-1}k, y, z)\sigma(x, y, z).$$

By the definition of $\alpha_c$, we have that the fixed-point algebra $A^{\alpha_c}$ is equal to $W^*(\text{Ker}(c), \sigma)$.

We claim that the converse also holds.

**Theorem 5.** For each coaction $\alpha$ of $K$ on $A$ which satisfies $D \subseteq A^\alpha \subseteq A$, there exists a Borel 1-cocycle $c : \mathcal{R} \to K$ such that $\alpha$ is equal to $\alpha_c$. 
Proof. Suppose $u$ is in $\mathcal{N}_A(D)$. By the definition, $\text{Ad} u$ determines an automorphism $\rho \in [\mathcal{R}]$. Set $w := \alpha(u)(1 \otimes u^*)$. By using the same argument as in the proof of Proposition 4, $w$ is a $W^*(K)$-valued function. Moreover, for almost all $x \in X$, $w(x)$ is equal to $\lambda_K(k(x))$ for some $k(x) \in K$. We note that the map $k$ depends only on $\rho$. Now, we define a map $c$ from the graph $\Gamma(\rho^{-1})$ to $K$ by the following:

$$c(\rho(x), x) := k(x) \quad (x \in \text{Dom}(\rho))$$

By using this construction, we can define a map $c$ from $\mathcal{R}$ to $K$. We note that the map $c$ is well-defined, i.e., if there exists $\rho_1$ and $\rho_2$ in $[\mathcal{R}]$ and a measurable subset $E \subseteq X$ such that $\rho_1(x) = \rho_2(x)$ for all $x \in E$, then there exists null set $F \subseteq X$ such that $c(\rho_1(x), x) = c(\rho_2(x), x)$ for all $x \in E \setminus F$. It is easy to check that $c$ is a 1-cocycle. Moreover, we have that $\alpha(u)$ is equal to $\alpha_c(u)$ for all $u \in \mathcal{N}_A(D)$. Hence we conclude that $\alpha$ is equal to $\alpha_c$. \hfill \Box

By using the above characterization, we will develop a theory of coactions in terms of 1-cocycles.

In the rest of this paper, we fix a coaction $\alpha$ of $K$ on $A$ and a 1-cocycle $c : \mathcal{R} \to K$ which satisfies $\alpha_c = \alpha$. We denote by $\hat{\mathcal{G}}(K)_{\alpha_c} \ltimes W^*(\mathcal{R}, \sigma)$ the crossed product of $A$ by $\alpha$, i.e.,

$$\hat{\mathcal{G}}(K)_{\alpha_c} \ltimes W^*(\mathcal{R}, \sigma) := (L^\infty(K) \otimes K \vee \alpha_c(W^*(\mathcal{R}, \sigma)))^\prime.$$  

We recall that a unitary $V \in W^*(K) \otimes A$ is called an $\alpha$-1-cocycle if $V$ satisfies the following:

$$(\hat{\Delta}_K \otimes id_A)(V) = V_{23}(id_M \otimes \alpha)(V).$$

Another coaction $\alpha'$ of $K$ on $A$ is said to be cocycle conjugate to $\alpha$ if there exists an $\alpha$-1-cocycle $V$ and a $*$-automorphism $\theta$ of $A$ such that

$$(id_M \otimes \theta) \circ \alpha' \circ \theta^{-1} = \text{Ad} V \circ \alpha.$$

For each Borel map $\phi : X \to K$, a unitary $(V_\xi)(k, x, y) := \xi(\phi(x)^{-1} k, x, y)$ is an $\alpha$-1-cocycle. So we get the following

**Proposition 6.** Suppose a Borel 1-cocycle $c : \mathcal{R} \to K$ is cohomologous to another Borel 1-cocycle $c'$, i.e., there exists a Borel map $\phi : X \to K$ such that $c'(x, y) = \phi(x)c(x, y)\phi(y)^{-1}$ for a.e. $(x, y) \in \mathcal{R}$. Then the coaction $\alpha_c$ is cocycle conjugate to $\alpha_{c'}$. Hence the crossed product $\hat{\mathcal{G}}(K)_{\alpha_c} \ltimes A$ is isomorphic to $\hat{\mathcal{G}}(K)_{\alpha_{c'}} \ltimes A$. 


4 Connes spectrum and asymptotic range

Let $c: \mathcal{R} \rightarrow K$ be a Borel 1-cocycle from an equivalence relation $\mathcal{R}$ into a locally compact group $K$. Again we consider the coaction $\alpha_c$ of $K$ on the von Neumann algebra $A := W^*(\mathcal{R}, \sigma)$. We will show that the Connes spectrum of the coaction $\alpha_c$ can be described in terms of the 1-cocycle $c$.

For each such a 1-cocycle $c: \mathcal{R} \rightarrow K$, the essential range $\sigma(c)$ is the smallest closed subset $F$ of $K$ such that $c^{-1}(F)$ has complement of $\nu$ measure zero. It is easy to check that $k \in K$ belongs to $\sigma(c)$ if and only if, for any (compact) neighborhood $U$ of $k$, one has $\nu(c^{-1}(U)) > 0$. The asymptotic range $r^*(c)$ of the 1-cocycle $c$ is by definition $\cap \{\sigma(c_B) : B \subseteq X, \mu(B) > 0\}$, where $c_B$ stands for the restriction of $c$ to the reduction $\mathcal{R}_B$ by $B$.

**Theorem 7.** The Connes spectrum $\Gamma(\alpha_c)$ of $\alpha_c$ is equal to the asymptotic range $r^*(c)$.

To prove this theorem, we use the following

**Lemma 8.** Let $L^\sigma(f) \in A$ and $\omega \in A(K)$, where $A(K)$ is the Fourier algebra $W^*(K)_*$ of $K$. Then $(\alpha_c)_\omega(L^\sigma(f)) := (\omega \otimes id)(\alpha_c(L^\sigma(f)))$ equals $L^\sigma((\omega \circ c)f)$

**Proof.** We may and do assume that $\omega$ has the form $\omega = \omega_{\eta_1, \eta_2}$ for some $\eta_1, \eta_2 \in L^2(K)$. For any $\zeta_1, \zeta_2 \in L^2(\mathcal{R})$, we have

$$((\alpha_c)_\omega(L^\sigma(f)) \zeta_1 \mid \zeta_2) = (\alpha_c(L^\sigma(f)) (\eta_1 \otimes \zeta_1) \mid \eta_2 \otimes \zeta_2)$$

$$= \int \sum_{y, (y, x) \in \mathcal{R}} \eta_1(c(x, y)^{-1}k) \eta_2(k) \cdot f(x, y) \zeta_1(y, z) \sigma(x, y, z) \overline{\zeta_2(x, z)} d\nu(x, z) dk$$

$$= \int \sum_{y, (y, x) \in \mathcal{R}} \omega(c(x, y)) f(x, y) \zeta_1(y, z) \sigma(x, y, z) \overline{\zeta_2(x, z)} d\nu(x, z)$$

$$= (L^\sigma((\omega \circ c)f) \zeta_1 \mid \zeta_2).$$

Thus we are done. \[\square\]

**Proof of Theorem 7.** Since the center $Z(A^\alpha)$ is contained in $D$, we have

$$\Gamma(\alpha_c) = \cap \{\text{Sp}((\alpha_c)^e) : e \text{ : non-zero projection in } D\}.$$

Hence, it suffices to show that $\text{Sp}(\alpha_c) = \sigma(c)$. 
Let $k \in \sigma(c)$. Take any compact neighborhood $U$ of $k$. Since $\nu(c^{-1}(U)) > 0$, there exists a measurable subset $E \subseteq c^{-1}(U)$ such that $\nu(E) > 0$ and $L^\sigma(\chi_E) \in A$. Then define $a := L^\sigma(\chi_E) \in A \setminus \{0\}$. If $\omega \in A(K)$ vanishes on some neighborhood of $U$, then, by Lemma 8, we have $(\alpha_c)_{\omega}(a) = 0$. From [6, Chapter IV, Lemma 1.2 (ii)], it follows that $\text{Sp}_{\alpha_c}(a) \subseteq U$. Hence $a$ belongs to $A^{\alpha_c}(U)$. By [6, Chapter IV, Lemma 1.2 (iv)], $k$ lies in $\text{Sp}(\alpha_c)$.

Conversely suppose that $k \in \text{Sp}(\alpha_c)$. We will show that, for each open neighborhood $V$ of $k$, $c^{-1}(V)$ is not a $\nu$-null set. Indeed, if $\nu(c^{-1}(V))$ is equal to 0 for some $V$, we have $L^\sigma(f) = L^\sigma(f_{\chi_{c^{-1}(V)^c}})$ for each $L^\sigma(f) \in A$. So, for each $\omega \in A(K)$ such that $\text{supp} \omega \subseteq U$, by Lemma 8, we have

$$(\alpha_c)_{\omega}(L^\sigma(f)) = L^\sigma(f_{\chi_{c^{-1}(V)}})(\omega \circ c) = 0.$$ 

So we conclude that $(\alpha_c)_{\omega}(a) = 0$ for each $a \in A$ and $\omega \in A(K)$ such that $\text{supp} \omega \subseteq U$. In the meantime, since $V$ is open, for each $h \in V$, there exists $\omega \in A(K)$ such that $\omega(h) = 1$ and $\text{supp} \omega \subseteq V$. This shows that for each $a \in A$, $h \not\in \text{Sp}_{\alpha_c}(a)$. This contradicts [6, Chapter IV, Lemma 1.2(iv)]. Therefore $k$ belongs to $\sigma(c)$.

By using the above theorem and [4, Lemma 1.13], we get the following

**Corollary 9 (cf. [5]).** Let $A$ be an AFD type II factor. Suppose that $\alpha$ and $\alpha'$ are coactions of a locally compact group $K$ on $A$ such that each of $A^\alpha$ and $A^\alpha'$ contains a Cartan subalgebra of $A$. If $\Gamma(\alpha) = \Gamma(\alpha') = K$, then $\alpha$ is cocycle conjugate to $\alpha'$.

**Proof.** Suppose that $A^\alpha$ (resp. $A^\alpha'$) contains a Cartan subalgebra $D_1$ (resp. $D_2$) of $A$. By [2], there exists a $*$-automorphism $\theta$ of $A$ such that $\theta(D_1) = D_2$. Set $\alpha_\theta := (id_{W^*(K)} \otimes \theta^{-1}) \circ \alpha \circ \theta$. Then we have $A^{\alpha_\theta} = \theta(A^\alpha)$. So $D_2 = \theta(D_1) \subseteq \theta(A^\alpha) = A^{\alpha_\theta}$. Clearly, $\alpha_\theta$ is cocycle conjugate to $\alpha$. Hence it suffices to assume from the outset that $D_1 = D_2 =: D$.

We may assume that the inclusion $(D \subseteq A)$ is of the form $(L^\infty(X) \subseteq W^*(\mathcal{R}))$ for an amenable ergodic type II equivalence relation $\mathcal{R}$ on a standard Borel space $(X, \mu)$ with an invariant measure $\mu$. By Theorem 5 there exist Borel 1-cocycles $c$ and $c'$ from $\mathcal{R}$ to $K$ such that $\alpha = \alpha_c$ and $\alpha' = \alpha_{c'}$. By Theorem 7, we have $r^*(c) = r^*(c') = K$. So we may apply [4, Lemma 1.13], and obtain that there exist cocycles $\overline{c}$ and $\overline{c'}$ cohomologous to $c$ and $c'$ respectively as 1-cocycles on $\mathcal{R}$ such that $\overline{c}$ is equal to $\overline{c'} \circ \rho$ for some $\rho \in N[\mathcal{R}]$, the normalizer of $\mathcal{R}$. By Proposition 6, $\alpha$ (resp. $\alpha'$) is cocycle conjugate
to $\alpha_\overline{c}$ (resp. $\alpha_\overline{c'}$). Furthermore, a direct computation shows that for each $X \in W^*(\mathcal{R})$,

$$\alpha_{\overline{c}\circ \rho}(X) = (1 \otimes \Phi_\rho^{-1})(\alpha_{\overline{c}}(\Phi_\rho(X))),$$

where $\Phi_\rho$ is an automorphism on $W^*(\mathcal{R})$ which is defined by

$$\Phi_\rho(L(f)) := L(f \circ \rho).$$

So we conclude that $(1 \otimes \Phi_\rho)\alpha_{\overline{c}\circ \rho} = \alpha_{\overline{c}} \circ \Phi_\rho$, i.e., $\alpha_{\overline{c}\circ \rho}$ is conjugate to $\alpha_\overline{c}$. Hence $\alpha$ is cocycle conjugate to $\alpha'$. $\Box$

5 Exchangeability for a 1-cocycle with a smaller range within the cohomology class

Suppose that there exists a closed subgroup $H$ of $K$ which cohomologous to $c$ and the range is contained in $H$. By regarding $c'$ as a 1-cocycle into $H$, we obtain the crossed product $\hat{G}(H)_{\alpha_{c'}} \ltimes A$ and the dual action $\overline{\alpha_{c'}}$ of $H$. It follows that the dual action $\overline{\alpha_{c}}$ of $K$ is induced from $\overline{\alpha_{c'}}$. Namely, there exists an isomorphism $\Pi$ from $\hat{G}(K)_{\alpha_c} \ltimes A$ onto $L^\infty(K/H) \otimes (\hat{G}(H)_{\alpha_{c'}} \ltimes A)$ such that $\Pi \circ (\alpha_c)_k = \delta_k \circ \Pi$, where the action $\delta$ of $K$ is the induced action of $\overline{\alpha_{c'}}([7])$.

We will show that the converse also holds.

**Theorem 10 (cf. [9, Theorem 3.5]).** Let $c : \mathcal{R} \to K$ be a Borel 1-cocycle and $H$ be a closed subgroup of $K$. Then the following are equivalent:

1. There exists a Borel 1-cocycle $c_0 : \mathcal{R} \to K$, cohomologous to $c$, such that the range of $c_0$ is contained in $H$.

2. There exists an injective *-homomorphism $\Theta$ from $L^\infty(K/H)$ into the center of the crossed product $\hat{G}(K)_{\alpha_c} \ltimes A$ such that $\Theta \circ \ell_k = (\overline{\alpha_c})_k \circ \Theta$ for all $k \in K$, where $\ell_k$ comes from the left translation by $k$ on $K/H$. Equivalently, if $Y$ is the measure-theoretic spectrum of the center of the crossed product (i.e., the measure space on which the Mackey action (the Poincaré flow) of $K$ is considered), then it is an extension of the $K$-space $K/H$.

3. The covariant system $\{\hat{G}(K)_{\alpha_c} \ltimes A, K, \overline{\alpha_c}\}$ is induced from some system $\{P, H, \beta\}$. 


If one of (1) \sim (3) occurs, then one can take \( \{P, H, \beta\} \) to be \( \widehat{\mathcal{G}}(H)_{\alpha_c} \ltimes A, H, \alpha_c \), where \( \mathcal{C} : H \rightarrow H \) is the 1-cocycle obtained by regarding \( c_0 \) as an \( H \)-valued 1-cocycle.

Proof. It is easy to check hat the condition (2) follows (1). By using the Imprimitivity Theorem of [7], (2) is equivalent to (3). So we will prove (2)\Rightarrow(1).

If such a \(*\)-homomorphism \( \Theta \) exists, then by using [7], the dual action \( (\alpha_c)_k \) is induced from an action \( \beta \) of \( H \) on a von Neumann algebra \( P \). We denote the induced action of \( \beta \) by \( \delta \). By the assumption, there exists a \(*\)-isomorphism \( \Pi \) from \( \widehat{\mathcal{G}}(K)_{\alpha_c} \ltimes A \) onto \( L^\infty(K/H) \otimes P \) such that \( \Pi \circ (\alpha_c)_k = \delta_k \circ \Pi \) for all \( k \in K \).

A direct computation shows that \( \Pi(\alpha_c(A)) \) is equal to \( C \otimes P^\beta \). Moreover, since \( \beta \) is defined by \( \beta_h := \text{Ad}(\lambda_H(h) \otimes 1)|_P \), there exists a dual action \( \beta' \) on \( H \) which is conjugate to \( \beta \). So there exist a von Neumann algebra \( B \) and a coaction \( \tau \) of \( H \) on \( B \) such that the dual action \( (\alpha_c) \) is conjugate to the induced action by \( \tau \). In particular, we have

\[
\widehat{\mathcal{G}}(K)_{\alpha_c} \ltimes A \cong L^\infty(K/H) \otimes \widehat{\mathcal{G}}(H)_{\tau} \ltimes B
\]

Under the above isomorphism, we have that there exists a isomorphism \( \eta \) from \( A \) onto \( B \) such that the fixed-point subalgebra \( B^\tau \) contains a Cartan subalgebra \( \eta(D) \). So \( \tau \) comes from a 1-cocycle \( c_0 : H \rightarrow H \). By the construction, we conclude that \( c_0 \) is cohomologous to \( c \) as a cocycle into \( K \).

Therefore we complete the proof. \( \square \)

References


