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On a decomposition space of a weak self-similar set

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1. Introduction

We call the function $f_j$ defined on a metric space $(X, d)$ weak contraction provided that there exists a function $\alpha_j(l), \ l > 0$ such that $0 < \alpha_j(l) < 1, \ \inf_{l>0} \alpha_j(l) > 0$ and $d(f_j(x), f_j(y)) \leqq \alpha_j(l)d(x, y)$ for $d(x, y) < l, \ j = 1, \ldots, m \ (2 \leqq m < \infty)$. In a complete metric space, there exists a unique nonempty compact weak self-similar set $S$ generated by the system $\{f_j; \ j = 1, \ldots, m\}$ of weak contractions, namely, there exists a set $S$ satisfying the relation $\bigcup_{j=1}^{m} f_j(S) = S$ [1,2,3].

In the present article, we will search the conditions for the weak contractions $f_j, \ j = 1, \ldots, m$ defined on a compact metric space $X$ to generate a weak self-similar set $S$ which has the property that every nonempty compact metric space is a continuous image of it, and then, we will construct a decomposition space of $S$, which is homeomorphic to the compact space $X$.

*) A part of the contents of this article is found in our paper, On a decomposition space of a weak self-similar set, Chaos, Solitons and Fractals 24(2005)785.
2. Construction of a decomposition space of the set $S$

Every nonempty, perfect, zero-dimensional, compact metric space is known [4,5] to have the above mentioned property that any nonempty compact metric space is a continuous image of it. If there exists a nonempty, perfect, zero-dimensional, compact weak self-similar set $S$ in a compact metric space $X$, then, $X$ must be a continuous image of $S$.

To attain our main results, we prepare the following lemmata.

**Lemma 1.** Let $X$ be a nonempty compact metric space with a metric $d$. Assume that the following conditions hold for the weak contraction $f_j$, $j = 1, \ldots, m$.

i) Each $f_j$ is one to one.

ii) The set $\{x \in X; f_j(x) = x, j = 1, \ldots, m\}$ is not degenerate. That is, there exist at least two points $x_0$ and $x_0'$ such that $f_{j_0}(x_0) = x_0$ and $f_{j_0'}(x_0') = x_0'$.

iii) There exists a point $l_0 > 0$ such that $\sum_{j=1}^{m} \alpha_j(l_0) < 1$.

Then, there exists in $X$ a perfect, zero-dimensional compact set $S$ such that $\bigcup_{j=1}^{m} f_j(S) = S$.

**Proof.** Let us define a set $S$ by $\bigcap_n X^n = S$. Here, $X^n = \bigcup_{j_1 \ldots j_n \in W_n} X_{j_1 \ldots j_n}$ where $X_{j_1 \ldots j_n} = f_{j_1} \circ \cdots \circ f_{j_n}(X)$ and $W_n$ denotes the set of all words $j_1 \ldots j_n$ with length $n$ on symbols $\{1, \ldots, m\}$. Since $x_0 = f_{j_0} \circ \cdots \circ f_{j_0}(x_0) \in X_{j_0 \ldots j_0} \subset X^n$ for any $n$, the compact set $S$ is nonempty. We note that for any $n$ and for any $j_1 \ldots j_n \in W_n$, $f_{j_1} \circ \cdots \circ f_{j_n}(x_0) = f_{j_1} \circ \cdots \circ f_{j_n} \circ f_{j_0} \circ \cdots \circ f_{j_0}(x_0) \in X_{j_1 \ldots j_n j_0 \ldots j_0} \subset X^k$ for any $k \geq n + 1$. Since the relation $X^{n+1} \subset \cdots \subset X^1$ is obvious, $f_{j_1} \circ \cdots \circ f_{j_n}(x_0) \in \bigcap_k X^k = S$. In the same way, concerning another fixed point $x_0'$, the relation $f_{j_1} \circ \cdots \circ f_{j_n}(x_0') \in S$ must hold for any $j_1 \ldots j_n \in W_n$.

1) It is known [1] that any weak contraction $f$ defined on a complete metric space $X$ has a fixed point in $X$. 

Next we note that the diameter of $X_{j_{1} \cdots j_{n}} \to 0$ $(n \to \infty)$, that is, for any $\varepsilon > 0$, there exists $N$ such that for any $n \geq N$ and for any $j_{1} \cdots j_{n} \in W_{n}$, the diameter $X_{j_{1} \cdots j_{n}} < \varepsilon$. In fact, since the relation $d(f_{j_{i}}(x), f_{j_{i}}(y)) = \tilde{\alpha}_{j_{i}}(d(x, y))d(x, y)$ holds for $\tilde{\alpha}_{j_{i}}(l) = \inf_{p > l} \alpha_{j_{i}}(p)$, each diameter of $X_{j_{1} \cdots j_{n}}$ is dominated from above as follows.

$$\text{diameter of } X_{j_{1} \cdots j_{n}} \leq \left( \max_{j} \{ \tilde{\alpha}_{j}(\text{diameter of } X) \} \right)^{n} \text{diameter of } X.$$  

Now, let $x$ be a point of $S = \cap_{n} X^{n}$. For any $\varepsilon > 0$, there exists a number $N$ such that for any $j_{1} \cdots j_{N} \in W_{N}$, the diameter of $X_{j_{1} \cdots j_{N}} < \varepsilon$. Then, there exists $j_{1} \cdots j_{N} \in W_{N}$ such that $x \in X_{j_{1} \cdots j_{N}} \subset$ open sphere $S(x, \varepsilon)$. It is evident that both $f_{j_{1}} \circ \cdots \circ f_{j_{N}}(x_{0})$ and $f_{j_{1}} \circ \cdots \circ f_{j_{N}}(x'_{0})$ are contained in $S(x, \varepsilon)$. But the condition (i) guarantees that the point $f_{j_{1}} \circ \cdots \circ f_{j_{N}}(x_{0}) \in S$ is different from the point $f_{j_{1}} \circ \cdots \circ f_{j_{N}}(x'_{0}) \in S$. Therefore, at least one of the two points $f_{j_{1}} \circ \cdots \circ f_{j_{N}}(x_{0})$ and $f_{j_{1}} \circ \cdots \circ f_{j_{N}}(x'_{0})$ is different from the point $x$. This means that the point $x$ is an accumulation point of $S$, and then, it is convinced that the nonempty compact set $S$ must be perfect. Taking the condition (i) into account, we can easily verify the relation $\bigcup_{j=1}^{m} f_{j}(S) = S$. As this relation implies that $\bigcup_{j_{1} \cdots j_{n} \in W_{n}} f_{j_{1}} \circ \cdots \circ f_{j_{n}}(S) = S$ for any $n \geq 1$, the estimate [2] concerning the Hausdorff dimension of $S$, $\dim_{H} S = \inf_{l > 0} x(l)$ where $\sum_{j=1}^{m} \tilde{\alpha}_{j}(l)x(l) = 1$, holds. Then, since the condition (iii) implies $\dim_{H} S < 1$, the dimension of $S = 0$, immediately follows from the fact [6] that the dimension$^{2}$ of a set does not exceed its Hausdorff dimension.

Next, let us recall the definition of the decomposition space [8] and that of the quotient map [9].

Let $D$ be a partition of a topological space $(X, \tau)$, i.e. $D$ is a collection of nonempty, mutually disjoint subsets of $X$ such that $\cup D = X$. The topological space $(D, \tau(D))$ defined by the decomposition topology $\tau(D) = \{ U \subset D; \cup U \in \}$

$^{2}$ Since $S$ is a separable metric space, $\text{ind} S = \text{Ind} S = \dim S [7]$. 


\$\{u' \subset Y; f^{-1}(u') \in \tau\}\$.

Concerning the relationships between the decomposition space and the quotient map, we have the following general lemma.

**Lemma 2.** Let \( f \) be a quotient map from a space \((X, \tau)\) onto a space \((Y, \tau')\). Then, \((Y, \tau')\) is homeomorphic to the decomposition space \((\mathcal{D}_f, \tau(\mathcal{D}_f))\) of \((X, \tau)\).

\(\mathcal{D}_f = \{f^{-1}(y); y \in Y\}\).

**Proof.** Let us show that \( h : (Y, \tau') \rightarrow (\mathcal{D}_f, \tau(\mathcal{D}_f))\), \( y \mapsto f^{-1}(y) \) is an homeomorphism. First, it is evident that the map \( h \) is one to one and onto. Then, it is sufficient to show that the map is continuous and open. Let \( U \) be a nonempty open set of \( \mathcal{D}_f \). There exists a nonempty subset \( B \) of \( Y \) such that \( U = \{f^{-1}(y); y \in B\} \). \( \cup U = \cup \{f^{-1}(y); y \in B\} = f^{-1}(B) \). Since the subset \( U \) is \( \tau(\mathcal{D}_f) \)-open, \( f^{-1}(B) \in \tau \). The relation \( B = h^{-1} \circ h(B) = h^{-1}(U) \) follows immediately from \( h(B) = \{f^{-1}(y); y \in B\} = U \). This means the continuity of the map \( h \). Next, let \( B \) be a nonempty \( \tau' \)-open, that is, \( f^{-1}(B) \in \tau \). Since \( \cup h(B) = \cup \{f^{-1}(y); y \in B\} = f^{-1}(B), h(B) \in \tau(\mathcal{D}_f) \). Hence, the map \( h \) is an open map.

It is an elemental fact that the existence of a continuous map \( f \) from a compact space \((X, \tau)\) onto a \( T_2 \) space \((Y, \tau')\) implies that \( \tau' = \tau_f \). Therefore, the above lemma can be rewrited as follows.

**Lemma 3.** If there exists a continuous map \( f \) from a compact space \((X, \tau)\) onto a \( T_2 \) space \((Y, \tau')\), then, \((Y, \tau')\) is homeomorphic to the decomposition space \((\mathcal{D}_f, \tau(\mathcal{D}_f))\), \( \mathcal{D}_f = \{f^{-1}(y); y \in Y\} \).

Under the Lemma 3, we can state our results as the following proposition.
Proposition 1. Let $X$ be a nonempty compact metric space, and let $S$ be a weak self-similar set generated from $f_j$, $j = 1, \ldots, m$ satisfying the three conditions in Lemma 1. Then, there exists a continuous function $f$ from $S$ onto $X$, and the decomposition space $(D_f, \tau(D_f))$, $D_f = \{f^{-1}(x) \subset S; \ x \in X\}$, of $S$ is homeomorphic to $X$.

3. Metrizability of the decomposition space

It is known [10,11] that i) If a $T_2$ space $Y$ is an image of a continuous function $f$ defined on a compact metric space $X$, then, the decomposition space $(D_f, \tau(D_f))$, $D_f = \{f^{-1}(y) \subset X; \ y \in Y\}$, of $X$ is upper semi continuous [12], and ii) Any upper semicontinuous decomposition of a compact metric space is metrizable.

Therefore, from the above discussions, our decomposition space of $S$ is easily verified to be metrizable.

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References


