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On a decomposition space of a weak self-similar set*)

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1. Introduction

We call the function f_j defined on a metric space (X, d) weak contraction provided that there exists a function $\alpha_j(l)$, $l > 0$ such that $0 < \alpha_j(l) < 1$, $\inf_{l>0} \alpha_j(l) > 0$ and $d(f_j(x), f_j(y)) \leqq \alpha_j(l)d(x,y)$ for $d(x,y) < l$, $j = 1, \ldots, m$ $(2 \leqq m < \infty)$. In a complete metric space, there exists a unique nonempty compact weak self-similar set S generated by the system $\{f_j;\ j = 1, \ldots, m\}$ of weak contractions, namely, there exists a set S satisfying the relation $\bigcup_{j=1}^{m} f_j(S) = S$ [1,2,3].

In the present article, we will search the conditions for the weak contractions f_j, $j = 1, \ldots, m$ defined on a compact metric space X to generate a weak self-similar set S which has the property that every nonempty compact metric space is a continuous image of it, and then, we will construct a decomposition space of S, which is homeomorphic to the compact space X.

*) A part of the contents of this article is found in our paper, On a decomposition space of a weak self-similar set, Chaos, Solitons and Fractals **24**(2005)785.

2. Construction of a decomposition space of the set S

Every nonempty, perfect, zero-dimensional, compact metric space is known [4,5] to have the above mentioned property that any nonempty compact metric space is a continuous image of it. If there exists a nonempty, perfect, zero-dimensional, compact weak self-similar set S in a compact metric space X, then, X must be a continuous image of S.

To attain our main results, we prepare the following lemmata.

Lemma 1. Let X be a nonempty compact metric space with a metric d. Assume that the following conditions hold for the weak contraction $f_j,\ j = 1, \ldots, m$.

i) Each f_j is one to one.

ii) The set $\{x \in X;\ f_j(x) = x,\ j = 1\ldots, m\}$[1)] is not degenerate. That is, there exist at least two points x_0 and x_0' such that $f_{j_0}(x_0) = x_0$ and $f_{j_0'}(x_0') = x_0'$.

iii) There exists a point $l_0 > 0$ such that $\sum_{j=1}^{m} \alpha_j(l_0) < 1$.

Then, there exists in X a perfect, zero-dimensional compact set S such that $\bigcup_{j=1}^{m} f_j(S) = S$.

Proof. Let us define a set S by $\bigcap_n X^n = S$. Here, $X^n = \bigcup_{j_1 \cdots j_n \in W_n} X_{j_1 \cdots j_n}$ where $X_{j_1 \cdots j_n} = f_{j_1} \circ \cdots \circ f_{j_n}(X)$ and W_n denotes the set of all words $j_1 \cdots j_n$ with length n on symbols $\{1, \ldots, m\}$. Since $x_0 = f_{j_0} \circ \cdots \circ f_{j_0}(x_0) \in X_{j_0 \cdots j_0} \subset X^n$ for any n, the compact set S is nonempty. We note that for any n and for any $j_1 \cdots j_n \in W_n$, $f_{j_1} \circ \cdots \circ f_{j_n}(x_0) = f_{j_1} \circ \cdots \circ f_{j_n} \circ f_{j_0} \circ \cdots \circ f_{j_0}(x_0) \in X_{j_1 \cdots j_n j_0 \cdots j_0} \subset X^k$ for any $k \geq n+1$. Since the relation $X^{n+1} \subset \cdots \subset X^1$ is obvious, $f_{j_1} \circ \cdots \circ f_{j_n}(x_0) \in \bigcap_k X^k = S$. In the same way, concerning another fixed point x_0', the relation $f_{j_1} \circ \cdots \circ f_{j_n}(x_0') \in S$ must hold for any $j_1 \cdots j_n \in W_n$.

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1) It is known [1] that any weak contraction f defined on a complete metric space X has a fixed point in X.

Next we note that the diameter of $X_{j_1\cdots j_n} \to 0$ $(n \to \infty)$, that is, for any $\varepsilon > 0$, there exists N such that for any $n \geq N$ and for any $j_1 \cdots j_n \in W_n$, the diameter $X_{j_1\cdots j_n} < \varepsilon$. In fact, since the relation $d(f_j(x), f_j(y)) \leqq \tilde{\alpha}_j(d(x,y))d(x,y)$ holds for $\tilde{\alpha}_j(l) = \inf_{p>l} \alpha_j(p)$, each diameter of $X_{j_1\cdots j_n}$ is dominated from above as follows.

$$\text{diameter of } X_{j_1\cdots j_n} \leqq \left(\max_j\{\tilde{\alpha}_j(\text{diameter of } X)\}\right)^n \text{diameter of } X.$$

Now, let x be a point of $S = \bigcap_n X^n$. For any $\varepsilon > 0$, there exists a number N such that for any $j_1 \cdots j_N \in W_N$, the diameter of $X_{j_1\cdots j_N} < \varepsilon$. Then, there exists $j_1 \cdots j_N \in W_N$ such that $x \in X_{j_1\cdots j_N} \subset$ open sphere $S(x, \varepsilon)$. It is evident that both $f_{j_1} \circ \cdots \circ f_{j_N}(x_0)$ and $f_{j_1} \circ \cdots \circ f_{j_N}(x_0')$ are contained in $S(x, \varepsilon)$. But the condition (i) guarantees that the point $f_{j_1} \circ \cdots \circ f_{j_N}(x_0) \in S$ is different from the point $f_{j_1} \circ \cdots \circ f_{j_N}(x_0') \in S$. Therefore, at least one of the two points $f_{j_1} \circ \cdots \circ f_{j_N}(x_0)$ and $f_{j_1} \circ \cdots \circ f_{j_N}(x_0')$ is different from the point x. This means that the point x is an accumulation point of S, and then, it is convinced that the nonempty compact set S must be perfect. Taking the condition (i) into account, we can easily verify the relation $\bigcup_{j=1}^m f_j(S) = S$. As this relation implies that $\bigcup_{j_1\cdots j_n \in W_n} f_{j_1} \circ \cdots \circ f_{j_n}(S) = S$ for any $n \geq 1$, the estimate [2] concerning the Hausdorff dimension of S, $\dim_H S \leq \inf_{l>0} x(l)$ where $\sum_{j=1}^m \tilde{\alpha}_j(l)^{x(l)} = 1$, holds. Then, since the condition (iii) implies $\dim_H S < 1$, the dimension of $S = 0$, immediately follows from the fact [6] that the dimension[2)] of a set does not exceed its Hausdorff dimension. ∎

Next, let us recall the definition of the decomposition space [8] and that of the quotient map [9].

Let $\mathbf{D}$ be a partition of a topological space (X, τ), *i.e.* $\mathbf{D}$ is a collection of nonempty, mutually disjoint subsets of X such that $\cup\mathbf{D} = X$. The topological space $(\mathbf{D}, \tau(\mathbf{D}))$ defined by the decomposition topology $\tau(\mathbf{D}) = \{U \subset \mathbf{D};\ \cup U \in$

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2) Since S is a separable metric space, $\text{ind}S = \text{Ind}S = \dim S$ [7].

$\tau\}$, is called a decomposition space of (X, τ). An onto map $f : (X, \tau) \to (Y, \tau')$ is said to be a quotient map provided that $\tau' = \tau_f$. Here, $\tau_f = \{u' \subset Y;\ f^{-1}(u') \in \tau\}$. Concerning the relationhships between the decomposition space and the quotient map, we have the following general lemma.

Lemma 2. Let f be a quotient map from a space (X, τ) onto a space (Y, τ'). Then, (Y, τ') is homeomorphic to the decomposition space $(\mathbf{D}_f, \tau(\mathbf{D}_f))$ of (X, τ). Here, $\mathbf{D}_f = \{f^{-1}(y);\ y \in Y\}$.

Proof. Let us show that $h : (Y, \tau') \to (\mathbf{D}_f, \tau(\mathbf{D}_f)),\ y \mapsto f^{-1}(y)$ is an homeomorphism. First, it is evident that the map h is one to one and onto. Then, it is sufficient to show that the map is continuous and open. Let U be a nonempty open set of $\mathbf{D}_f$. There exists a nonempty subset B of Y such that $U = \{f^{-1}(y);\ y \in B\}$. $\cup U = \cup\{f^{-1}(y);\ y \in B\} = f^{-1}(B)$. Since the subset U is $\tau(\mathbf{D}_f)$-open, $f^{-1}(B) \in \tau$. The relation $B = h^{-1} \circ h(B) = h^{-1}(U)$ follows immediately from $h(B) = \{f^{-1}(y);\ y \in B\} = U$. This means the continuity of the map h. Next, let B be a nonempty τ'-open, that is, $f^{-1}(B) \in \tau$. Since $\cup h(B) = \cup\{f^{-1}(y);\ y \in B\} = f^{-1}(B)$, $h(B) \in \tau(\mathbf{D}_f)$. Hence, the map h is an open map.∎

It is an elemental fact that the existence of a continuous map f from a compact space (X, τ) onto a T_2 space (Y, τ') implies that $\tau' = \tau_f$. Therefore, the above lemma can be rewrited as follows.

Lemma 3. If there exists a continuous map f from a compact space (X, τ) onto a T_2 space (Y, τ'), then, (Y, τ') is homeomorphic to the decomposition space $(\mathbf{D}_f, \tau(\mathbf{D}_f))$, $\mathbf{D}_f = \{f^{-1}(y);\ y \in Y\}$.

Under the Lemma 3, we can state our results as the following proposition.

Proposition 1. Let X be a nonempty compact metric space, and let S be a weak self-similar set generated from f_j, $j = 1, \dots, m$ satisfying the three conditions in Lemma 1. Then, there exists a continuous function f from S onto X, and the decomposition space $(\mathbf{D}_f, \tau(\mathbf{D}_f))$, $\mathbf{D}_f = \{f^{-1}(x) \subset S;\ x \in X\}$, of S is homeomorphic to X.

3. Metrizablility of the decomposition space

It is known [10,11] that i) If a T_2 space Y is an image of a continuous function f defined on a compact metric space X, then, the decomposition space $(\mathbf{D}_f, \tau(\mathbf{D}_f))$, $\mathbf{D}_f = \{f^{-1}(y) \subset X;\ y \in Y\}$, of X is upper semi continuous [12], and ii) Any upper semicontinuous decomposition of a compact metric space is metrizable.

Therefore, from the above discussions, our decomposition space of S is easily verified to be metrizable.

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