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京都大学
Destructible gaps に関する強制概念とその積

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1 Introduction and notation

1.1 Introduction

This note is a part of the paper [23].

In this paper, we deal with destructible gaps. A destructible gap is an ($\omega_1, \omega_1$)-gap which can be destroyed by a forcing extension preserving cardinals. A destructible gap has a characterization similar to a Suslin tree ([2]). A Suslin tree is an $\omega_1$-tree having no uncountable chains and antichains. On the other hand, for an ($\omega_1, \omega_1$)-pregap $(A, B) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ with the set $a_\alpha \cap b_\alpha$ empty for every $\alpha \in \omega_1$, we say here that $\alpha$ and $\beta$ in $\omega_1$ are compatible if

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset.$$

Then by the characterization due to Kunen and Todorčević, we notice that an ($\omega_1, \omega_1$)-pregap is a destructible gap iff it has no uncountable pairwise compatible and incompatible subsets of $\omega_1$. (We must notice that from results of Farah and Hirschorn [8, 9], the existence of a destructible gap is independent with the existence of a Suslin tree.)

One of differences from an $\omega_1$-tree is that any ($\omega_1, \omega_1$)-pregap have never had an uncountable chain and antichain at the same time. We have forcing notions related to an ($\omega_1, \omega_1$)-pregap.

Definition 1.1 (E.g. [5, 11, 18, 19]). Let $(A, B) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ be an ($\omega_1, \omega_1$)-pregap with $a_\alpha \cap b_\alpha = \emptyset$ for every $\alpha \in \omega_1$.

1. $F(A, B) := \{\sigma \in [\omega_1]^{<\omega}; \forall \alpha \neq \beta \in \sigma, (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset\}$, ordered by reverse inclusion.

2. $S(A, B) := \{\sigma \in [\omega_1]^{<\omega}; \bigcup_{\alpha \in \sigma} a_\alpha \cap \bigcup_{\alpha \in \sigma} b_\alpha = \emptyset\}$, ordered by reverse inclusion.

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We note that $\mathcal{F}(A, B)$ forces $(A, B)$ to be indestructible and $S(A, B)$ forces $(A, B)$ to be separated. Using these forcing notions, we can express characterizations of being a gap and destructibility.

**Theorem 1.2** (E.g. [5, 11, 18, 19]). Let $(A, B)$ be an $(\omega_1, \omega_1)$-pregap. Then:

1. $(A, B)$ forms a gap iff $\mathcal{F}(A, B)$ has the countable chain condition.
2. $(A, B)$ is destructible (may not be a gap) iff $S(A, B)$ has the countable chain condition.

Therefore we say that $(A, B)$ is a destructible gap if both $\mathcal{F}(A, B)$ and $S(A, B)$ have the ccc. As in the case of a Suslin tree, by the product lemma for forcings, we note that $\mathcal{F}(A, B) \times S(A, B)$ does not have the ccc, and we will see that e.g., we may have two destructible gaps $(A, B)$ and $(C, D)$ so that all variations $X_0(A, B) \times X_1(A, B)$ have the ccc.

In [10], it is proved that for any family $\{(A_i, B_i); i \in I\}$ of $(\omega_1, \omega_1)$-gaps, the finite support product $\prod_{i \in I} \mathcal{F}(A_i, B_i)$ has the countable chain condition. It means that generically making gaps indestructible cannot separate any $(\omega_1, \omega_1)$-gap. So we arise a question whether or not the above statement is also true for adding interpolations. We prove that this question cannot be decided from ZFC, i.e.

**Theorem 1.** It is consistent with ZFC that for any family $\{(A_i, B_i); i \in I\}$ of destructible gaps, the product forcing notion $\prod_{i \in I} S(A_i, B_i)$ has the countable chain condition.

**Theorem 2.** It is consistent with ZFC that there are two destructible gaps $(A, B)$ and $(C, D)$ such that the product forcing notion $S(A, B) \times S(C, D)$ does not have the countable chain condition.

(We note that the statement in Theorem 1 (and the next theorem) is trivially true if there are no destructible gaps. For example, if Martin's Axiom holds, then all $(\omega_1, \omega_1)$ gaps are indestructible. But it is really consistent with ZFC that the statement in Theorem 1 plus there are many destructible gaps. see the proof of Theorem 1.)

Moreover, we prove the following theorem which is a version of Larson's theorem [14, Theorem 4.6] for a destructible gap.

**Theorem 3.** It is consistent with ZFC that there exists a destructible gap $(A, B)$ such that $S(A, B)$ forces that all $(\omega_1, \omega_1)$-gaps are indestructible.

### 1.2 Notation

A pregap in $\mathcal{P}(\omega)/\text{fin}$ is a pair $(A, B)$ of subsets of $\mathcal{P}(\omega)$ such that for all $a \in A$ and $b \in B$, the set $a \cap b$ is finite. For subsets $a$ and $b$ of $\omega$, we say that $a$ is almost contained in $b$ (and denote $a \subseteq^* b$) if $a \setminus l$ is a subset of $b$ for some $l \in \omega$. For a pregap $(A, B)$, both ordered sets $(A, \subseteq^*)$ and $(B, \subseteq^*)$ are well ordered and
these order type are $\kappa$ and $\lambda$ respectively, then we say that a pregap $(A, B)$ has the type $(\kappa, \lambda)$ or a $(\kappa, \lambda)$-pregap. Moreover if $\kappa = \lambda$, we say that the pregap is symmetric. For a pregap $(A, B)$, we say that $(A, B)$ is separated if for some $c \in \mathcal{P}(\omega)$, $a \subseteq^* c$ and the set $c \cap b$ is finite for every $a \in A$ and $b \in B$. If a pregap is not separated, we say that it is a gap. Moreover if a gap has the type $(\kappa, \lambda)$, it is called a $(\kappa, \lambda)$-gap.

For an ordinal $\alpha$, if we say that $(a_\xi, b_\xi; \xi \in \alpha)$ is a pregap, we always assume that

- if $\xi < \eta$ in $\alpha$, $a_\xi \subseteq^* a_\eta$ and $b_\xi \subseteq^* b_\eta$, and
- for every $\xi \in \alpha$, the set $a_\xi \cap b_\xi$ is empty.

Our other notation is quite standard in set theory. (See [4, 12].)

## 2 Products of forcing notions adding interpolations

The referee of the paper [10] has proved the following theorem. (For the proof of the following theorem, see the proof of Claim 2.11 in the proof of Lemma 2.10.)

**Theorem 2.1 ([10, Theorem 4]).** Let $n \in \omega$ and $(A_i, B_i)$ be $(\omega_1, \omega_1)$-gaps for $i < n$. Then $\prod_{i<n} \mathcal{F}(A_i, B_i)$ has the countable chain condition.

This theorem says that the forcing a gap to be indestructible cannot force any $(\omega_1, \omega_1)$-gap to be separated. But as seen below, we cannot prove from ZFC that the forcing gaps to be separated does not force a gap to be indestructible. The point of the proofs in this section is the homogeneity of the forcing notion $\mathcal{S}(A, B)$ for a destructible gap $(A, B)$ with some property below. For a homogeneity, we give some definitions.

**Definition 2.2 ([18, Definition 2]).** We say that pregaps $(A, B)$ and $(C, D)$ are equivalent if $(A, B)$ and $(C, D)$ are cofinal each others.

We notice that if pregaps $(A, B)$ and $(C, D)$ are equivalent, then $(A, B)$ is a gap iff so is $(C, D)$ and $(A, B)$ is destructible iff so is $(C, D)$. We note that any $(\omega_1, \omega_1)$-pregap has an equivalent pregap $(A, B)$ such that $\mathcal{S}(A, B)$ is homogeneous. The similar property of the following one is appeared in the proof of [6, Proposition 2.5].

**Definition 2.3 ([22]).** We say that a pregap $(A, B) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ admits finite changes if for all $\alpha < \omega_1$, $a_\alpha \cap b_\alpha$ is empty and the set $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite, and for any $\beta < \alpha$ with $\beta = \eta + k$ for some $\eta \in \text{Lim} \cap \alpha$ and $k \in \omega$, $H, J \in [\omega]^{<\omega}$ with $H \cup J = \emptyset$ and $i > \max(H \cup J)$ there exists $n \in \omega$ so that

$$a_{\eta+n} \cap i = H, \ a_{\eta+n} \setminus i = a_\beta \setminus i, \ b_{\eta+n} \cap i = J, \ \text{and} \ b_{\eta+n} \setminus i = b_\beta \setminus i.$$
For a homogeneity, we need a little strong property of the admission of finite changes.

**Definition 2.4.** We say that a pregap $(A, B) = (a_\alpha, b_\alpha; \alpha \in \omega_1)$ strictly admits finite changes if it admits finite changes and for all $\alpha \neq \beta$ in $\omega_1$, $(a_\alpha, b_\alpha) \neq (a_\beta, b_\beta)$.

We note that any symmetric gap has an equivalent gap which strictly admits finite changes. So the rest of this paper, we consider only $(\omega_1, \omega_1)$-gaps which strictly admits finite changes because of the following propositions.

**Proposition 2.5.** Let $((A_i, B_i); i < n)$ be a finite collection of destructible gaps and $(C_i, D_i)$ a gap equivalent to $(A_i, B_i)$ for each $i < n$. Then for any combination $(X_i; i < n)$, where $X_i$ is either $F$ or $S$, the finite support product $\prod_{i<n} X_i(A_i, B_i)$ has the countable chain condition iff $\prod_{i<n} X_i(C_i, D_i)$ also has the countable chain condition.

**Proof.** Let $(A_i, B_i) = \{a_\xi^i, b_\xi^i; \xi \in \omega_1\}$ and $(C_i, D_i) = \{c_\xi^i, d_\xi^i; \xi \in \omega_1\}$. It suffices to show that if $\prod_{i<n} X_i(A_i, B_i)$ has the countable chain condition, then $\prod_{i<n} X_i(C_i, D_i)$ also has the countable chain condition.

Let $\{p_\alpha; \alpha \in \omega_1\}$ be a family of conditions in $\prod_{i<n} X_i(C_i, D_i)$. Without loss of generality, we may assume that

- the set $\{p_\alpha(i); \alpha \in \omega_1\}$ forms a $\Delta$-system with a root $\sigma_i$ for each $i < n$,
- all $p_\alpha(i) \setminus \sigma_i$ have the same size $k_i$ for each $i < n$ and
- for any $\alpha < \beta$ in $\omega_1$ and $i < n$, 
  $$\max(\sigma_i) < \min(p_\alpha(i) \setminus \sigma_i)$$ \quad and \quad $$\max(p_\alpha(i) \setminus \sigma_i) < \min(p_\beta(i) \setminus \sigma_i).$$

Moreover, we may assume that there exists a family $\{q_\alpha; \alpha \in \omega_1\}$ of conditions in $\prod_{i<n} X_i(A_i, B_i)$ and a natural numbers $m_i$ for each $i < n$ such that

- for any $\alpha < \beta$ in $\omega_1$ and $i < n$, 
  $$\max(p_\alpha(i) \setminus \sigma_i) < \min(q_\alpha(i)) \leq \max(q_\alpha(i)) < \min(p_\beta(i) \setminus \sigma_i),$$

- for each $i < n$,
  - if $X_i = F$, then for any $\alpha \in \omega_1$, $q_\alpha(i)$ has the size $k_i$ and for each $\xi \in p_\alpha(i) \setminus \sigma_i$, there is $\eta \in q_\alpha(i)$ such that 
    $$a_\eta^i \setminus m_i \subseteq c_\xi^i \quad \text{and} \quad b_\eta^i \setminus m_i \subseteq d_\xi^i,$$
  - if $X_i = S$, then for any $\alpha \in \omega_1$, $q_\alpha(i) = \{\gamma_\alpha^i\}$ and 
    $$\bigcup_{\xi \in p(\alpha)} c_\xi^i \setminus m_i \subseteq a_{\gamma_\alpha^i} \quad \text{and} \quad \bigcup_{\xi \in p(\alpha)} d_\xi^i \setminus m_i \subseteq b_{\gamma_\alpha^i}.$$
and

- for any $\alpha, \beta \in \omega_1$,

$$\bigcup_{\xi \in p(\alpha)} c_\xi \cap m_i = \bigcup_{\xi \in p(\beta)} c_\xi \cap m_i \quad \text{and} \quad \bigcup_{\xi \in p(\alpha)} d_\xi \cap m_i = \bigcup_{\xi \in p(\beta)} d_\xi \cap m_i.$$  

By the ccc-ness of $\prod_{i<n} \mathcal{X}_i(A_i, B_i)$, we can find different ordinals $\alpha$ and $\beta$ in $\omega_1$ such that $q_\alpha$ and $q_\beta$ are compatible in $\prod_{i<n} \mathcal{X}_i(A_i, B_i)$. Then we notice that $p_\alpha$ and $p_\beta$ are compatible in $\prod_{i<n} \mathcal{X}_i(C_i, D_i)$. \hfill \Box

**Lemma 2.6.** If $(A, B)$ strictly admits finite changes, then $S(A, B)$ is homogeneous as a forcing notion, i.e. for every $\sigma, \tau \in S(A, B)$ there are extensions $\sigma'$ and $\tau'$ of $\sigma$ and $\tau$ respectively such that $S(A, B)[\sigma']$ and $S(A, B)[\tau']$ are isomorphic.

**Proof.** Now we fix $\sigma, \tau \in S(A, B)$. By strict admission of finite changes of $(A, B)$, we can find extensions $\sigma'$ and $\tau'$ of $\sigma$ and $\tau$ respectively such that

(i) $\max\{\alpha \in \omega_1 \cap \operatorname{Lim}; \exists k \in \omega (\alpha + k \in \sigma')\} = \max\{\alpha \in \omega_1 \cap \operatorname{Lim}; \exists k \in \omega (\alpha + k \in \tau')\}$ and

(ii) there exists $N \in \omega$ such that

- for any $\alpha < \beta \in \sigma'$, $a_\alpha \setminus N \subseteq a_\beta \setminus N$ and $b_\alpha \setminus N \subseteq b_\beta \setminus N$,

- for any $\alpha < \beta \in \tau'$, $a_\alpha \setminus N \subseteq a_\beta \setminus N$ and $b_\alpha \setminus N \subseteq b_\beta \setminus N$, and

$$\bigcup_{\alpha \in \sigma'} (a_\alpha \cap N) \cup \bigcup_{\alpha \in \sigma'} (b_\alpha \cap N) = \bigcup_{\alpha \in \tau'} (a_\alpha \cap N) \cup \bigcup_{\alpha \in \tau'} (b_\alpha \cap N) = N.$$  

Then we note that

$$\bigcup_{\alpha \in \sigma'} (a_\alpha \setminus N) = \bigcup_{\alpha \in \tau'} (a_\alpha \setminus N) \quad \text{and} \quad \bigcup_{\alpha \in \sigma'} (b_\alpha \setminus N) = \bigcup_{\alpha \in \tau'} (b_\alpha \setminus N).$$

We note that if $\gamma \in \omega_1$ is such that $\sigma' \cup \{\gamma\}$ is also a condition in $S(A, B)$, then

$$a_\gamma \cap n \subseteq \bigcup_{\alpha \in \sigma'} (a_\alpha \cap n), \quad b_\gamma \cap n \subseteq \bigcup_{\alpha \in \sigma'} (b_\alpha \cap n)$$

and

$$\left( (a_\gamma \setminus n) \cap \left( \bigcup_{\alpha \in \sigma'} (b_\alpha \setminus n) \right) \right) \cup \left( (b_\gamma \setminus n) \cap \left( \bigcup_{\alpha \in \sigma'} (a_\alpha \setminus n) \right) \right) = \emptyset.$$  

We pick any bijection $\pi$ from

$$\mathcal{P} \left( \bigcup_{\alpha \in \sigma'} a_\alpha \cap n \right) \times \mathcal{P} \left( \bigcup_{\alpha \in \sigma'} b_\alpha \cap n \right).$$
onto
\[ \mathcal{P} \left( \bigcup_{\alpha \in \tau'} a_{\alpha} \cap n \right) \times \mathcal{P} \left( \bigcup_{\alpha \in \tau'} b_{\alpha} \cap n \right) \]
and let \( \pi_1 \) and \( \pi_2 \) represent the first and second coordinates of the value of \( \pi \) respectively. We define an isomorphism \( \psi \) from \( S(A, B) \mid \sigma' \) onto \( S(A, B) \mid \tau' \) as follow. Let \( \rho \) be an extension of \( \sigma' \) and \( \beta \in \rho \setminus \sigma' \), say \( \beta = \alpha + k \) for \( \alpha \in \omega_1 \cap \text{Lim} \) and \( k \in \omega, a_\beta = H \cup (a_\alpha \setminus N) \) and \( b_\beta = K \cup (b_\alpha \setminus N) \), where \( H \) and \( K \) are subsets of \( N \). Then we let \( k^0 \) be the unique number such that
\[ a_{\alpha+k^0} = \pi_1 (H, K) \cup (a_\beta \setminus N) \]
and
\[ b_{\alpha+k^0} = \pi_2 (H, K) \cup (b_\beta \setminus N) \].
Then we define \( \beta^0 := \alpha + k^0 \) and
\[ \psi(\rho) := \tau' \cup \{ \beta^0; \beta \in \rho \setminus \sigma' \} \]
By the above note, this is well defined and certainly an isomorphism.

Lemma 2.6 says that the theory in the extension with \( S(A, B) \) can calculate in the ground model when \( (A, B) \) strictly admits finite changes, that is, if some condition in \( S(A, B) \) can force the statement about elements of the ground model, then the statement holds in any extension with \( S(A, B) \).

Assume that \( (A, B) \) is a destructible gap and strictly admits finite changes and that \( \sigma \) and \( \tau \) are conditions in \( S(A, B) \). By strengthening \( \sigma \) and \( \tau \) if need, we may assume that \( \sigma \) and \( \tau \) satisfy the conditions (i) and (ii). When \( \sigma, \tau \) and \( N \) satisfies above conditions, we say that \( \langle \sigma, \tau, N \rangle \) is a good sequence. If \( \langle \sigma, \tau, N \rangle \) is a good sequence, as seen in above lemma, \( S(A, B) \mid \sigma \) and \( S(A, B) \mid \tau \) are isomorphic and a finite bijection \( \pi \) from
\[ \mathcal{P} \left( \bigcup_{\xi \in \sigma} a_\xi \cap N \right) \times \mathcal{P} \left( \bigcup_{\xi \in \sigma} b_\xi \cap N \right) \]
on to
\[ \mathcal{P} \left( \bigcup_{\xi \in \tau} a_\xi \cap N \right) \times \mathcal{P} \left( \bigcup_{\xi \in \tau} b_\xi \cap N \right) \]
induces an isomorphism \( \psi \) from \( S(A, B) \mid \sigma \) onto \( S(A, B) \mid \tau \). We say that \( \psi \) is an isomorphism induced by \( \pi \).

Let \( \{ \langle A_i, B_i \rangle; i \in I \} \) be a family of destructible gaps which strictly admits finite changes and \( p = \langle \sigma_i; i \in I \rangle \) and \( p' = \langle \sigma'_i; i \in I \rangle \) are conditions in the finite support product \( \prod_{i \in I} S(A_i, B_i) \). Then by strengthening conditions, we can find a sequence \( \langle N_i : i \in I \rangle \) of natural numbers with the property that the supports of two conditions are same and for any \( i \in I \cap \text{supp}(p) \), \( \langle \sigma_i, \sigma'_i, N_i \rangle \) is a good
sequence, then we have an isomorphism between \(\prod_{i \in I} S(A_i, B_i) \upharpoonright \langle \sigma_i ; i \in I \rangle\) and \(\prod_{i \in I} S(A_i, B_i) \upharpoonright \langle \sigma'_i ; i \in I \rangle\) induced by finitely many finite bijections. That is, we have

**Lemma 2.7.** Let \(\{(A_i, B_i) ; i \in I\}\) be a family of destructible gaps which strictly admits finite changes. Then the product forcing \(\prod_{i \in I} S(A_i, B_i)\) with a finite support is homogeneous.

Moreover assume all \((A_i, B_i)\) are the same gap \((A, B)\). By strengthening each \(\sigma_i\), we have \(N \in \omega\) such that for any \(i \neq j\) in \(I \cap \mathrm{supp}(P)\), \(\langle \sigma_i, \sigma_j, N \rangle\) is a good sequence. Then we have the collection of isomorphisms \(\psi_{i,j}\) for each \(i, j \in I \cap \mathrm{supp}(P)\) from \(S(A, B)\) onto \(S(A, B)\) which are commutative, by taking finite bijections suitably.

The following lemma is to show Theorem 1.

**Lemma 2.8.** Let \(P\) is a homogeneous forcing notion with the countable chain condition and \((C, D)\) an \((\omega_1, \omega_1)\)-pregap. Then the following statements hold.

1. If the product forcing \(P \times S(C, D)\) does not have the countable chain condition, then the product \(P \times F(C, D)\) has the countable chain condition.

2. If the product forcing \(P \times F(C, D)\) does not have the countable chain condition, then the product \(P \times S(C, D)\) has the countable chain condition.

**Proof.** Both statements follow from the ccc-ness and the homogeneity of \(P\) and the fact that

1. if \(S(C, D)\) does not have the ccc, then \(F(C, F)\) has the ccc, and

2. if \(F(C, D)\) does not have the ccc, then \(S(C, F)\) has the ccc

respectively. \(\square\)

**Proof of Theorem 1.** This theorem is true in the model where there are no destructible gaps. We will build a model for the theorem containing a destructible gap by an iteration with a finite support as follows.

Assume that there is a destructible gap, \(2^{\aleph_1} = \lambda\) and \(\lambda^{\lambda} = \lambda\). At first we take any family \(\Gamma_0\) of destructible gaps which strictly admits finite changes with the property that the finite support product \(\prod_{(A, B) \in \Gamma_0} S(A, B)\) has the ccc (which is a weak property of the independence). By recursion on \(\alpha \in \omega_2\), we construct \(\Gamma_\alpha\) in the \(\alpha\)-th stage of the iteration as follows:

In stage \(\alpha + 1 \in \omega_2\), for a destructible gap \((C, D)\) which strictly admits finite changes (given by a book-keeping map), if \(\prod_{(A, B) \in \Gamma_\alpha} S(A, B) \times S(C, D)\) has the ccc, then let \(\Gamma_{\alpha + 1} := \Gamma \cup \{(C, D)\}\) and does not force in this itand, otherwise, i.e. \(\prod_{(A, B) \in \Gamma_\alpha} S(A, B) \times S(C, D)\) does not have the ccc, then let \(\Gamma_{\alpha + 1} := \Gamma_\alpha\) and force \(F(C, D)\). By Lemma 2.8, \(\prod_{(A, B) \in \Gamma_{\alpha + 1}} S(A, B)\) still has the ccc and by Theorem 2.1, in the extension with \(F(C, D)\), \(F(A, B)\) is still ccc for every
Lemma 2.9 ([10, Lemma B.1]). Let \((a_\alpha, b_\alpha; \alpha \in \omega_1)\) be an \((\omega_1, \omega_1)\)-gap. Then for any uncountable subsets \(I\) and \(J\) of \(\omega_1\), there exist uncountable \(I' \subseteq I\) and \(J' \subseteq J\) such that for every \(\alpha \in I'\) and \(\beta \in J'\), \(a_\alpha \cap b_\beta \neq \emptyset\).

Proof. For each \(\alpha \in \omega_1\), there is a natural number \(n_\alpha\) such that both sets 
\[ \{\xi \in \omega_1; a_\alpha \cap n_\alpha \subseteq a_\xi\} \quad \text{and} \quad \{\eta \in \omega_1; b_\alpha \cap n_\alpha \subseteq b_\eta\} \]
are uncountable. We note that the set
\[ \bigcup_{\xi \in I}(a_\xi \cap n_\xi) \cap \bigcup_{\eta \in J}(b_\eta \cap n_\eta) \]
is not empty because the pregap
\[ (a_\xi \cap n_\xi, b_\eta \cap n_\eta; \xi \in I, \eta \in J) \]
is equivalent to the original one and so is a gap. We take \(\alpha \in I, \beta \in J\) and \(k \in \omega\) such that \(k\) is in the set \((a_\alpha \cap n_\alpha) \cap (b_\beta \cap n_\beta)\). Let \(I' := \{\xi \in I; a_\alpha \cap n_\alpha \subseteq a_\xi\}\) and \(J' := \{\eta \in J; b_\beta \cap n_\beta \subseteq b_\eta\}\) which are as desired. \(\square\)

The next lemma is a variation of [14, Corollary 4.3] for a destructible gap which is the key lemma for proofs of Theorems 2 and 3.

Lemma 2.10. Let \((A, B)\) be a destructible gap and strictly admits finite changes, and \((\mathcal{C}, D)\) be an \(S(A, B)\)-name for an \((\omega_1, \omega_1)\)-gap. Then there exists a ccc forcing notion \(\mathbb{P}\) (which is possibly trivial) such that in the extension with \(\mathbb{P}\), \((A, B)\) is still a destructible gap and \(S(A, B)\) forces \((\mathcal{C}, D)\) to be indestructible.

Proof. At first we define a forcing notion \(\mathbb{Q}\) as follow.

\[ \mathbb{Q} := \left\{ p \in ([\omega_1]^{<\omega})^2 ; p(0) \in S(A, B) \& p(0) \Vdash_{S(A, B)} " p(1) \in S(\mathcal{C}, \mathcal{D}) " \right\}, \]

ordered by

\[ p \leq_{\mathbb{Q}} q \iff p(0) \supseteq q(0) \& p(1) \supseteq q(1). \]

If we have an uncountable antichain in \(\mathbb{Q}\), we have nothing to do, i.e. what we have to do is that we let \(\mathbb{P}\) be the trivial forcing notion.
Assume that $\mathbb{Q}$ has an uncountable antichain $\{q_\alpha; \alpha \in \omega_1\}$. Without loss of generality, we may assume that the set $\{q_\alpha(1); \alpha \in \omega_1\}$ forms a $\Delta$-system with a root $\sigma$ and for all $\alpha < \beta$ in $\omega_1$,

$$\max(\sigma) < \min(q_\alpha(1) \setminus \sigma) \text{ and } \max(q_\alpha(1) \setminus \sigma) < \min(q_\beta(1) \setminus \sigma).$$

Let $\langle c_\alpha, d_\alpha; \alpha \in \omega_1\rangle$ the interpretation of $(\mathcal{C}, \mathcal{D})$ in this extension with $S(A, B)$. Then we can find an uncountable subset $X$ of $\omega_1$ such that the set $\{q_\alpha(0); \alpha \in X\}$ is pairwise compatible in $S(A, B)$ using an interpolation of $(A, B)$. Since $\{q_\alpha; \alpha \in \omega_1\}$ is pairwise incompatible in $\mathbb{Q}$, for all $\alpha \neq \beta$ in $X$,

$$\bigg(\bigcup_{\xi \in q_\alpha(1) \setminus \sigma} c_\xi \cap \bigcup_{\xi \in q_\beta(1) \setminus \sigma} d_\xi\bigg) \cup \bigg(\bigcup_{\xi \in q_\beta(1) \setminus \sigma} c_\xi \cap \bigcup_{\xi \in q_\alpha(1) \setminus \sigma} d_\xi\bigg) \neq \emptyset.$$

Then by our assumption, the following sequence

$$\langle \bigcup_{\xi \in q_\alpha(1) \setminus \sigma} c_\xi, \bigcup_{\xi \in q_\alpha(1) \setminus \sigma} d_\xi; \alpha \in \omega_1 \rangle$$

forms a pregap and is an equivalent gap of $\langle c_\alpha, d_\alpha; \alpha \in \omega_1\rangle$ and so is indestructible. Therefore $S(A, B)$ forces $(\mathcal{C}, \mathcal{D})$ to be indestructible.

Even if $\mathbb{Q}$ has the countable chain condition, we can find a forcing notion $\mathbb{P}$ which adds an uncountable antichain in $\mathbb{Q}$ and preserves the ccc-ness of both $\mathcal{F}(A, B)$ and $S(A, B)$. Let

$$\mathbb{P} := \{P \in [\mathbb{Q}]^{<\omega}; P \text{ is an antichain in } \mathbb{Q}\},$$

ordered by reverse inclusion. Since $(A, B)$ forms a gap, it can be proved that $\mathbb{P}$ has the countable chain condition. Moreover we can show more stronger results. To show them, we use Lemma 2.9. The proof of the following claim is very similar to a proof of Theorem 4 in [10]. And this proof let us know the ccc-ness of $\mathbb{P}$.

**Claim 2.11.** $\mathbb{P} \times \mathcal{F}(A, B)$ has the countable chain condition.

**Proof of Claim 2.11.** Assume that $\{\langle P_\alpha, \sigma_\alpha \rangle; \alpha \in \omega_1\}$ is an uncountable collection of conditions in $\mathbb{P} \times \mathcal{F}(A, B)$. Without loss of generality, we may assume that

- $\{P_\alpha; \alpha \in \omega_1\}$ forms a $\Delta$-system with a root $P$,
- $\{\sigma_\alpha; \alpha \in \omega_1\}$ forms a $\Delta$-system with a root $\sigma$,
- for all $\alpha \in \omega_1$, $P_\alpha \setminus P$ has the same size $k$, and
- for all $\alpha \in \omega_1$, $\sigma_\alpha \setminus \sigma$ has the same size $l$. 
For $\alpha \in \omega_1$, we let $P_\alpha^0 := \{p(0); p \in P_\alpha \setminus P\}$ and denote the $i$-th member of $P_\alpha^0$ and $\sigma_\alpha \setminus \sigma$ by $P_\alpha^0(i)$ and $\sigma_\alpha(j)$ for all $i < k$ and $j < l$ respectively. Using Lemma 2.9 of $\frac{k(k+1)}{2} + \frac{l(l+1)}{2}$ times, we can find uncountable subsets $I_0$ and $I_1$ of $\omega_1$ such that

- for all $\alpha \in I_0$ and $\beta \in I_1$ and $i, j < k$,
  \[ \bigcup_{\xi \in P_\alpha^0(i)} a_\xi \cap \bigcup_{\xi \in P_\beta^0(j)} b_\xi \neq \emptyset, \]

  and

- for all $\alpha \in I_0$ and $\beta \in I_1$ and $i, j < l$,
  \[ a_{\sigma_\alpha(i)} \cap b_{\sigma_\alpha(j)} \neq \emptyset. \]

Then for any $\alpha \in I_0$ and $\beta \in I_1$, $\langle P_\alpha, \sigma_\alpha \rangle$ and $\langle P_\alpha, \sigma_\alpha \rangle$ are compatible in $\mathbb{P} \times \mathcal{F}(A, B)$.

By the fact that $(\dot{C}, \dot{D})$ is an $S(A, B)$-name for a gap and the homogeneity of $S(A, B)$, we can moreover prove the following claim and this completes the proof.

**Claim 2.12.** $\mathbb{P} \times S(A, B)$ has the countable chain condition.

**Proof of Claim 2.12.** Let $\{\langle P_\alpha, \sigma_\alpha \rangle; \alpha \in \omega_1\}$ be in $\mathbb{P} \times S(A, B)$ for all $\alpha \in \omega$. Without loss of generality, we may assume that

- $\{P_\alpha; \alpha \in \omega_1\}$ forms a $\Delta$-system with a root $P$,
- for all $\alpha \in \omega_1$, $P_\alpha \setminus P$ has the same size $m$, and
- for any $\alpha < \beta \in \omega_1$,
  \[ \max \left( \bigcup_{p \in P} p(1) \right) \ < \min \left( \bigcup_{p \in P_\alpha \setminus P} p(1) \right). \]

Let $\{\langle \tau^i_\alpha, v^i_\alpha \rangle; i < m\}$ enumerate the set $P_\alpha \setminus P$ and we denote $\sigma_\alpha$ by $\tau^m_\alpha$ to simplify the notation for all $\alpha \in \omega_1$. Since $(A, B)$ strictly admits finite changes, for every $\alpha \in \omega_1$ and $i \leq m$, there exists $\delta_\alpha^i \in \omega_1$ such that

\[ \bigcup_{\xi \in \tau^i_\alpha} a_\xi = a_{\delta_\alpha^i} \quad \text{and} \quad \bigcup_{\xi \in \tau^i_\alpha} b_\xi = b_{\delta_\alpha^i}. \]
Since $S(A,B)$ has the ccc, for each $i \leq m$, there exists $\rho^i \in S(A,B)$ such that $$\rho^i \Vdash_{S(A,B)} \check{\bar{I}}^i := \left\{ \alpha \in \check{\omega}_1; \delta_\alpha^i \in \check{G} \right\}$$ is uncountable\(^{\ast}\).

We note that $$\rho^i \Vdash_{S(A,B)} \check{I}^i = \left\{ \alpha \in \check{\omega}_1; \delta_\alpha^i \in \check{G} \right\}$$ for all $i \leq m$. By strengthening $\rho^i$'s if need, we may assume that there exists $N \in \omega$ such that for all $i \neq j \leq m$, $\langle \rho^i, \rho^j, N \rangle$ is a good sequence. Then without loss of generality again, we may moreover assume that for all $\alpha, \beta \in \omega_1$ and $i \leq m$, $$a_\delta^i \cap N = a_\delta^i \cap N \quad \text{and} \quad b_\delta^i \cap N = b_\delta^i \cap N.$$ We let $\pi_{i,m}$ be a finite bijection for an isomorphism so that $$\pi_{i,m}(a_\delta \cap N, b_\delta \cap N) = \langle a_\delta \cap N, b_\delta \cap N \rangle$$ for each $i < m$ (and some (any) $\alpha \in \omega_1$) and let $\psi_{i,m}$ be the isomorphism from $S(A,B)|\rho^i$ onto $S(A,B)|\rho^m$ induced by $\pi_{i,m}$. We note that for every $i < m$, the calculations of $\psi_{i,m}$ are absolute and if $\delta_\alpha^i \cup \rho^i \in S(A,B)$, then $$\psi_{i,m}(\delta_\alpha^i \cup \rho^i) = \delta_\alpha^m \cup \rho^m$$ for all $\alpha \in \omega_1$. For each $i \neq j \leq m$, we define $\psi_{i,j} := (\psi_{j,m})^{-1} \circ \psi_{i,m}$. We note that for every $i \neq j \leq m$, $\psi_{i,j}(S(A,B)|\rho^j)$ is an isomorphism onto $S(A,B)|\rho^j$, and if $\delta_\alpha^i \cup \rho^i \in S(A,B)$, then $$\psi_{i,j}(\delta_\alpha^i \cup \rho^i) = \delta_\alpha^j \cup \rho^j$$ for all $\alpha \in \omega_1$. Using Lemma 2.9, since $(\check{C}, \check{D})$ is a name for a gap, we can define $S(A,B)$-names $\check{I}_0^i$ and $\check{I}_1^i$ for $i < m$, such that for each $i < m$,

- $\rho^i \Vdash_{S(A,B)} \check{I}_0^i \cup \check{I}_1^i = \text{uncountable subsets of } \check{I}^i$ \(^{\ast}\),

- $\rho^i \Vdash_{S(A,B)} \check{\bar{I}}^i = \text{for all } \alpha \in \check{I}_0^i \text{ and all } \beta \in \check{I}_1^i, \bigcup \check{\epsilon}_\xi \cap \bigcup \check{d}_\xi \neq 0$ \(^{\ast}\),

- $\rho^0 \Vdash_{S(A,B)} \check{\bar{I}}^0 \subseteq \check{\epsilon}_{m,0}(\check{I}^m)$ and $\check{\bar{I}}^0 \subseteq \psi_{m,0}(\check{I}^m)$ \(^{\ast}\),

and

- $\rho^{i+1} \Vdash_{S(A,B)} \check{\bar{I}}^{i+1} \subseteq \psi_{i+1} \check{\epsilon}_{m}(\check{I}^0)$ and $\check{\bar{I}}^{i+1} \subseteq \psi_{i+1} \check{\epsilon}_{m}(\check{I}^1)$ \(^{\ast}\).

This can be done because for every $i \neq j \leq m$, if $\mu \leq \rho^i$ and $\tau \in [\omega_1]<\omega$ such that $$\mu \Vdash_{S(A,B)} \check{\tau} \in \check{G}$$ then $\psi_{i,j}(\mu) \leq \rho^j$ and $$\psi_{i,j}(\mu) \Vdash_{S(A,B)} \psi_{i,j}(\check{\tau}) \in \check{G}$$
and because of the property of \( \psi_{i,j} \)'s. (We note that \( S(A, B) \) is not separative.)

We take any \( \rho \leq \rho^{m-1} \) and \( \alpha, \beta \in \omega_1 \) such that

\[
\rho \Vdash_{S(A, B)} \alpha \in I_0^{m-1} \text{ and } \beta \in I_1^{m-1}.
\]

Then by the conditions of \( \hat{I}_0^i \) and \( \hat{I}_1^i \), we note that for each \( i < m - 1 \),

\[
\psi_{m-1,i}(\rho) \Vdash_{S(A, B)} \alpha \in I_0^i \text{ and } \beta \in I_1^i.
\]

This means that for every \( i \leq m \), \( \rho \cup \tau^i_\alpha \cup \tau^i_\beta \) is a condition in \( S(A, B) \) and for every \( i < m \),

\[
\rho \cup \tau^i_\alpha \cup \tau^i_\beta \Vdash_{S(A, B)} v^i_\alpha \text{ and } v^i_\beta \text{ are incompatible in } S(C, D).
\]

This implies that \( P_\alpha \cup P_\beta \) is pairwise incompatible in \( Q \) and \( \sigma_\alpha \) and \( \sigma_\beta \) are compatible in \( S(A, B) \), hence \( \langle P_\alpha, \sigma_\alpha \rangle \) and \( \langle P_\beta, \sigma_\beta \rangle \) are compatible in \( P \times S(A, B) \), which completes the proof of the claim.

Proof of Theorem 2. Without loss of generality, we may assume that there are two independent destructible gaps \( (A, B) \) and \( (C, D) \) both of which strictly admit finite changes. Since \( S(A, B) \times \mathcal{F}(C, D) \) is ccc and \( S(A, B) \) is homogeneous, we can consider \( (C, D) \) as an \( S(A, B) \)-name for a gap. As in the proof of Lemma 2.10, let \( P \) be a forcing notion adding an uncountable antichain in \( S(A, B) \times S(C, D) \) by finite approximations. Then not only \( P \times \mathcal{F}(A, B) \) and \( P \times S(A, B) \), but also \( P \times \mathcal{F}(C, D) \) and \( P \times S(C, D) \) have the ccc. So in the extension with \( P \), both \( (A, B) \) and \( (C, D) \) are still destructible gaps and \( S(A, B) \times S(C, D) \) does not have the countable chain condition.

Proof of Theorem 3. This is just a corollary of Lemma 2.10. We fix one destructible gap which strictly admits finite changes, and then by an iteration with a finite support, we can force the desired statement. We note it is upward closed that the forcing notion \( Q \) as in Lemma 2.10 has an uncountable antichain. We notice that the continuum can be large.

References


[22] T. Yorioka. _The diamond principle for the uniformity of the meager ideal implies the existence of a destructible gap_, to appear in AML.

