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<td>Hosaka, Tetsuya</td>
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Kyoto University
Reflection groups of geodesic spaces 
and Coxeter groups

宇都宮大学教育学部
保坂 哲也 (Tetsuya Hosaka)

The purpose of this note is to introduce a result of my recent paper [8] about cocompact discrete reflection groups of geodesic spaces.

A metric space \((X, d)\) is called a geodesic space if for each \(x, y \in X\), there exists an isometry \(\xi : [0, d(x, y)] \to X\) such that \(\xi(0) = x\) and \(\xi(d(x, y)) = y\) (such \(\xi\) is called a geodesic). We say that an isometry \(r\) of a geodesic space \(X\) is a reflection of \(X\), if

1. \(r^2\) is the identity of \(X\),
2. \(X \setminus F_r\) has strictly two convex components \(X_r^+\) and \(X_r^-\), and
3. \(\text{Int} F_r = \emptyset\),

where \(F_r\) is the fixed-point set of \(r\) which is called the wall of \(r\). An isometry group \(\Gamma\) of a geodesic space \(X\) is called a reflection group, if some set of reflections of \(X\) generates \(\Gamma\). Let \(\Gamma\) be a reflection group of a geodesic space \(X\) and let \(R\) be the set of all reflections of \(X\) in \(\Gamma\). We note that \(R\) generates \(\Gamma\) by definition. Now we suppose that the action of \(\Gamma\) on \(X\) is proper, that is, \(\{\gamma \in \Gamma \mid \gamma x \in B(x, N)\}\) is finite for each \(x \in X\) and \(N > 0\) (cf. [2, p.131]). Then the set \(\{F_r \mid r \in R\}\) is locally finite. Let \(C\) be a component of \(X \setminus \bigcup_{r \in R} F_r\), which is called a chamber. Here we can show that \(\Gamma C = X \setminus \bigcup_{r \in R} F_r\). Then \(\Gamma \overline{C} = X\) and for each \(\gamma \in \Gamma\), either \(C \cap \gamma C = \emptyset\) or \(C = \gamma C\). We say that \(\Gamma\) is a cocompact discrete reflection group of \(X\), if \(\overline{C}\) is compact and \(\{\gamma \in \Gamma \mid C = \gamma C\} = \{1\}\).
**Definition.** A group $\Gamma$ is called a cocompact discrete reflection group of a geodesic space $X$, if

1. $\Gamma$ is a reflection group of $X$,
2. the action of $\Gamma$ on $X$ is proper,
3. for a chamber $C$, $\overline{C}$ is compact, and
4. $\{\gamma \in \Gamma | C = \gamma C\} = \{1\}$.

For example, every Coxeter group is a cocompact discrete reflection group of some geodesic space.

A Coxeter group is a group $W$ having a presentation

$$\langle S | (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where $S$ is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

1. $m(s, t) = m(t, s)$ for each $s, t \in S$,
2. $m(s, s) = 1$ for each $s \in S$, and
3. $m(s, t) \geq 2$ for each $s, t \in S$ such that $s \neq t$.

The pair $(W, S)$ is called a Coxeter system. H.S.M. Coxeter showed that a group $\Gamma$ is a finite reflection group of some Euclidean space if and only if $\Gamma$ is a finite Coxeter group. Every Coxeter system $(W, S)$ induces the Davis-Moussong complex $\Sigma(W, S)$ which is a CAT(0) space ([6], [7], [10]). Then the Coxeter group $W$ is a cocompact discrete reflection group of the CAT(0) space $\Sigma(W, S)$.

Here we obtained the following theorem in [8].

**Theorem.** A group $\Gamma$ is a cocompact discrete reflection group of some geodesic space if and only if $\Gamma$ is a Coxeter group.

Let $\Gamma$ be a cocompact discrete reflection group of a geodesic space $X$, let $C$ be a chamber and let $S$ be a minimal subset of $R$ such that $C = \bigcap_{s \in S} X_s^+$ (i.e. $C \neq \bigcap_{s \in S \setminus \{s_0\}} X_s^+$ for each $s_0 \in S$). Then we can show that $\langle S \rangle C = X \setminus \bigcup_{r \in R} F_r = \Gamma C$. Since $\{\gamma \in \Gamma | C = \gamma C\} = \{1\}$, $S$ generates $\Gamma$. In [8], we have proved that the pair $(\Gamma, S)$ is a Coxeter system.
REFERENCES


DEPARTMENT OF MATHEMATICS, UTSUNOMIYA UNIVERSITY, UTSUNOMIYA, 321-8505, JAPAN

E-mail address: hosaka@cc.utsunomiya-u.ac.jp