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Kyoto University
Geometry of finite-dimensional maps (Pasynkov の定理の精密化)

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Abstract. In [2 and 3], Pasynkov proved the following theorem: If $f : X \to Y$ is a map of compacta such that $f$ is a $k$-dimensional map and $\dim Y = p < \infty$, then the set of maps $g$ in the space $C(X, I^{p+2k+1})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1}$ is an embedding is a $G_\delta$-dense subset of $C(X, I^{p+2k+1})$. In this paper, furthermore we investigate the geometric properties of finite-dimensional maps and finite-to-one maps. We prove that if $f : X \to Y$ is a map as above, then for each $0 \leq i \leq p + k$, the set of maps $g$ in the space $C(X, I^{p+2k+1-i})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1-i}$ is an $(i+1)$-to-1 map is a $G_\delta$-dense subset of $C(X, I^{p+2k+1-i})$. The case $i = 0$ implies the result of Pasynkov. Also, if $Y$ is a one point set, our result implies the following Hurewicz's theorem: If $\dim X = n < \infty$ and $0 \leq i \leq n$, then the set of maps $g$ in the space $C(X, I^{2n+1-i})$ such that $g : X \to I^{2n+1-i}$ is an $(i+1)$-to-1 map is a $G_\delta$-dense subset of $C(X, I^{2n+1-i})$. As a corollary, we have the following representation theorem of finite-dimensional maps: For a map $f : X \to Y$ of compacta such that $0 \leq k < \infty$ and $\dim Y = p < \infty$, $f$ is a $k$-dimensional map if and only if $f$ can be represented as the composition $f = g_{p+2k+1} \circ \ldots \circ g_{p+k+2} \circ g_{p+k+1} \circ g_{p+k} \circ \ldots \circ g_{1}$ of maps $g_{i}$ $(i = 1, 2, \ldots, p + k + 1)$ parallel to the unit interval $I$ such that $g_{i}$ is an $(i+1)$-to-1 map for each $i = 1, 2, \ldots, p + k$ and $g_{p+k+1}$ is a zero-dimensional map.

\[ X = X_{0} \xrightarrow{g_{1}} X_{1} \xrightarrow{g_{p+k}} X_{p+k} \xrightarrow{g_{p+k+1}} X_{p+k+1} \]
\[ \xrightarrow{g_{p+k+2}} X_{p+k+2} \xrightarrow{g_{p+2k+1}} X_{p+2k} = Y \]

1 Introduction.

All spaces considered in this paper are assumed to be separable metric spaces. Maps are continuous functions. Let $I = [0, 1]$ be the unit interval. By a compactum we mean a nonempty compact metric space. Let $X$ and $Y$ be compacta. Then $C(X, Y)$ denotes the space of all maps $g : X \to Y$ with the usual sup-metric. Note that $C(X, Y)$ is a complete metric space.

A map $f : X \to Y$ is a $k$-dimensional map ($0 \leq k < \infty$) if for each $y \in Y$ $\dim f^{-1}(y) \leq k$, where $\dim Z$ denotes the topological dimension of a space $Z$. If a map $f : X \to Y$ is a $k$-dimensional map, we write $\dim f \leq k$. A map $f : X \to Y$ is a $k$-to-1 map if for each $y \in Y$, the cardinal number $|f^{-1}(y)|$ of $f^{-1}(y)$ is equal to or less than $k$. 
In [2 and 3], Pasynkov proved that if \( f : X \rightarrow Y \) is a \( k \)-dimensional map from a compactum \( X \) to a finite dimensional compactum \( Y \), then there is a map \( g : X \rightarrow I^k \) such that \( \dim (f \times g) = 0 \). Also, he proved that if \( f : X \rightarrow Y \) is a map of compacta such that \( f \) is a \( k \)-dimensional map and \( \dim Y = p < \infty \), then the set of maps \( g \) in the space \( C(X, I^{p+2k+1}) \) such that the diagonal product \( f \times g : X \rightarrow Y \times I^{p+2k+1} \) is an embedding is a \( G_\delta \)-dense subset of \( C(X, I^{p+2k+1}) \).

In this paper, furthermore we investigate the geometric properties of finite-dimensional maps and finite-to-one maps. We prove that if \( f : X \rightarrow Y \) is a map of compacta such that \( f \) is a \( k \)-dimensional map and \( \dim Y = p < \infty \), then for each \( 0 \leq i \leq p + k \), the set of maps \( g \) in the space \( C(X, I^{p+2k+1-i}) \) such that the diagonal product \( f \times g : X \rightarrow Y \times I^{p+2k+1-i} \) is an \((i+1)\)-to-1 map is a \( G_\delta \)-dense subset of \( C(X, I^{p+2k+1-i}) \). Note that the restriction \( g|f^{-1}(y) : f^{-1}(y) \rightarrow I^{p+2k+1-i} \) is an \((i+1)\)-to-1 map for each \( y \in Y \). Also, note that the case \( i = 0 \) implies the result of Pasynkov, and our proof in this paper is different from the proof of Pasynkov (see [3]). Also, if \( Y \) is a one point set, our result implies that if \( \dim X = n < \infty \) and \( 0 \leq i \leq n \), then the set of maps \( g \) in the space \( C(X, I^{2n+1-i}) \) such that \( g : X \rightarrow I^{2n+1-i} \) is an \((i+1)\)-to-1 map is a \( G_\delta \)-dense subset of \( C(X, I^{2n+1-i}) \). As a corollary, we have the following representation theorem of finite-dimensional maps: For a map \( f : X \rightarrow Y \) of compacta such that \( 0 \leq k < \infty \) and \( \dim Y = p < \infty \), \( f \) is a \( k \)-dimensional map if and only if \( f \) can be represented as the composition \( f = g_p \circ g_{p+k+1} \circ \cdots \circ g_{p+k+2} \circ \cdots \circ g_1 \) of maps \( g_i \) \((i = 1, 2, \ldots, p + 2k + 1)\) parallel to the unit interval \( I \) (for the definition, see section 3) such that \( g_i \) is an \((i+1)\)-to-1 map for each \( i = 1, 2, \ldots, p + k \) and \( g_{p+k+1} \) is a zero-dimensional map.

\[
\begin{array}{cccccccccc}
X = X_0 & \xrightarrow{g_1} & X_1 & \xrightarrow{\ldots} & \xrightarrow{g_{p+k}} & X_{p+k} & \xrightarrow{g_{p+k+1}} & X_{p+k+1} \\
& \xrightarrow{g_{p+k+2}} & X_{p+k+2} & \xrightarrow{\ldots} & X_{p+2k} & \xrightarrow{g_{p+2k+1}} & X_{p+2k+1} = Y
\end{array}
\]

Note that the maps \( g_i \) \((p + k + 2 \leq i \leq p + 2k + 1)\) are 1-dimensional maps.

2 Main theorem.

A map \( h : X \rightarrow Y \) is a \((p, \epsilon)\)-map \((\epsilon > 0)\) if for each \( y \in Y \), there are subsets \( A_1, A_2, \ldots, A_p \) of \( h^{-1}(y) \) such that \( h^{-1}(y) = \bigcup_{i=1}^{p} A_i \) and \( \text{diam} \ A_i < \epsilon \) for each \( i \). Let \( f : X \rightarrow Y \) be a map and \( A \subset X \). Then \( f|A : A \rightarrow Y \) is a strict embedding for \( f \) if \( f|A \) is an embedding and \( f^{-1}(f(A)) = A \). Note that \( f|A : A \rightarrow Y \) is a strict embedding for \( f \) if and only if \( A \subset \{ x \in X | f^{-1}(f(x)) = \{x\} \} \).

In this paper, we need the following key lemma of Toruńczyk [4, Lemma 2].

**Lemma 2.1.** Let \( \epsilon > 0 \). Suppose that \( f : X \rightarrow Y \) is a map of compacta with \( \dim f = 0 \) and \( \dim Y = p < \infty \). For each \( i = 1, 2, \ldots, l \), let \( K_i \) and \( L_i \) be closed
disjoint subsets of $X$. Then there are open subsets $E_i$ of $X$ separating $X$ between $K_i$ and $L_i$ such that $f|(Cl(E_1) \cup \ldots \cup Cl(E_i))$ is a $(p, \varepsilon)$-map.

The next proposition was proved by Pasynkov in [2] (see also [4, Corollary 1] and [1, p. 48]).

**Proposition 2.2.** If $f : X \to Y$ is a $k$-dimensional map from a compactum $X$ to a finite dimensional compactum $Y$, then the set of maps $g$ in $C(X, I^k)$ such that $\dim (f \times g) = 0$ is a $G_{\delta}$-dense subset of $C(X, I^k)$.

The following lemma is easily proved.

**Lemma 2.3.** Let $X$ and $Y$ be compacta and $A$ a closed subset of $X$. Let $C(X, Y; A, p)$ be the set of all maps $g : X \to Y$ such that $g|A$ is a $p$-to-1 map. Then $C(X, Y; A, p)$ is $G_{\delta}$ in $C(X, Y)$.

**Theorem 2.4.** If $f : X \to Y$ is a map of compacta such that $f$ is a $k$-dimensional map and $\dim Y = p < \infty$, then for each $0 \leq i \leq p + k$, the set of maps $g$ in the space $C(X, I^{p+2k+1-i})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1-i}$ is an $(i+1)$-to-1 map is a $G_{\delta}$-dense subset of $C(X, I^{p+2k+1-i})$. Hence the restriction $g|f^{-1}(y) : f^{-1}(y) \to I^{p+2k+1-i}$ is an $(i+1)$-to-1 map for each $y \in Y$.

3 Finite-dimensional maps and compositions of maps parallel to the unit interval.

A map $f : X \to Y$ is said to be **embedded in a map** $f_0 : X_0 \to Y_0$ (see [2 and 3]) if there exists embeddings $g : X \to X_0$ and $h : Y \to Y_0$ such that $h \circ f = f_0 \circ g$. A map $f : X \to Y$ is **parallel** to the unit interval $I$ (see [2 and 3]) if $f$ can be embedded in the natural projection $p : Y \times I \to Y$. In [2 and 3], Pasynkov proved the following theorem: If $f : X \to Y$ is a map such that $\dim f = k$ and $\dim Y < \infty$, then $f$ can be represented as the composition $f = h_k \circ \ldots h_1 \circ g$ of a zero-dimensional map $g$ and maps $h_i$ ($i = 1, 2, \ldots, k$) parallel to the unit interval $I$ (see Proposition 2.2).

In this section, furthermore we study the properties of finite-dimensional maps and compositions of maps parallel to the unit interval. In fact, we show that the zero-dimensional map $g$ as in the above theorem of Pasynkov can be represented as a composition of some special maps parallel to $I$.

First, we prove the following proposition (Proposition 3.2) which is related to results of Uspenskij [6], Tuncali and Valov [5]. Our proof is similar to the proof of Theorem 2.4. We give the proof which is different from the proofs of Uspenskij, Tuncali and Valov (see [6] and [5]).

**Lemma 3.1.** Let $X, Y$ and $Z$ be compacta and $0 \leq k < \infty$. Let $T$ be the set of maps $g = u \times v : X \to Y \times Z$ in $C(X, Y \times Z)$ such that $\dim v(u^{-1}(y)) \leq k$ for each $y \in Y$. Then $T$ is a $G_{\delta}$-set of $C(X, Y \times Z)$. 
Proposition 3.2. Let \( f : X \to Y \) be a map of compacta such that \( f \) is a \( k \)-dimensional map and \( \dim Y = p < \infty \). Let \( T \) be the set of all maps \( h = q \times u : X \to I^k \times I \) in \( C(X, I^{k+1}) \) such that \( \dim h(f^{-1}(y)) \leq k \), \( \dim u((f \times g)^{-1}(y, t)) = 0 \) for each \( y \in Y \), \( t \in I^k \), \( \dim (f \times g) = 0 \) and \( f \times h \) is an \((p + k + 1)\)-to-1 map. Then \( T \) is a \( G_\delta \)-dense subset of \( C(X, I^{k+1}) \).

Corollary 3.3. Let \( f : X \to Y \) be a map of compacta such that \( f \) is a \( k \)-dimensional map and \( \dim Y = p < \infty \). Let \( \tilde{E}(X, I^{p+2k+1}) \) be the set of maps \( g \) in the space \( C(X, I^{p+2k+1}) \) such that (1) \( f \times g \) is an embedding, (2) for each \( 1 \leq i \leq p + k \), \( f \times (p_i \circ g) : X \to Y \times I^{p+i} \) is an \((i+1)\)-to-1 map, and (3) for \( h = p_{p+k} \circ g = g' \times u : X \to I^k \times I \), \( \dim h(f^{-1}(y)) \leq k \), \( \dim u((f \times g')^{-1}(y, t)) = 0 \) for each \( y \in Y \) and \( t \in I^k \), and \( \dim (f \times g') = 0 \), where \( p_i : I^{p+2k+1} \to I^{p+i} \) is the natural projection. Then \( \tilde{E}(X, I^{p+2k+1}) \) is a \( G_\delta \)-dense subset of \( C(X, I^{p+2k+1}) \).

Now, we have the following representation theorem of finite-dimensional maps.

Theorem 3.4. Let \( f : X \to Y \) be a map of compacta such that \( 0 \leq k < \infty \) and \( \dim Y = p < \infty \). Then \( f \) is a \( k \)-dimensional map if and only if \( f \) can be represented as the composition

\[
\begin{align*}
Y \times I^{p+2k+1} & \overset{f \times g}{\longrightarrow} X \\
\downarrow P_r & \quad \downarrow P_r \\
Y \times I^{p+2k+1-i} & \overset{p_{r-i}}{\longrightarrow} Y
\end{align*}
\]

Remark. In the proof of Theorem 3.4, the maps \( g_i \) \((i = 1, 2, \ldots, p + k + 1)\) parallel to \( I \) such that \( g_i \) is an \((i+1)\)-to-1 map for each \( i = 1, 2, \ldots, p + k \) and \( g_{p+k+1} \) is a zero-dimensional map.

\[
X = X_0 \overset{g_1}{\longrightarrow} X_1 \ldots \overset{g_{p+k}}{\longrightarrow} X_{p+k} \overset{g_{p+k+1}}{\longrightarrow} X_{p+k+1} = Y
\]

Remark. In the proof of Theorem 3.4, the maps \( g_i \) \((i = 1, 2, \ldots, p + k)\) satisfy the condition that \( g_i \circ \ldots \circ g_1 \) \((i \leq p + k)\) is an \((i+1)\)-to-1 map. In particular, \( g_i \) \((i \leq p + k)\) is an \((i+1)\)-to-1 map.
References


