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CHARACTERIZATIONS OF NORMAL COVERS ON RECTANGULAR PRODUCTS AND INFINITE PRODUCTS

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1. Introduction

An open cover $\mathcal{O}$ of a topological space $X$ is normal if there is a sequence $\{\mathcal{U}_n\}$ of open covers of $X$ such that each $\mathcal{U}_n$ is a star-refinement of $\mathcal{U}_n$ for each $n \in \omega$, where $\mathcal{U}_0 = \mathcal{O}$.

We may well know the following characterization of normal covers of topological spaces. For example, it is seen in [AS, p.122], [Ho, Theorems 1.2 and 1.4] and [Mo, Theorem 1.2] etc.

**Theorem 1.1** [Stone-Michael-Morita]. Let $X$ be a topological space and $\mathcal{O}$ an open cover of $X$. Then the following are equivalent.

(a) $\mathcal{O}$ is normal.
(b) $\mathcal{O}$ has a $\sigma$-locally finite cozero refinement.
(c) $\mathcal{O}$ has a $\sigma$-discrete cozero refinement.
(d) $\mathcal{O}$ has a locally finite cozero refinement.
(e) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cozero refinement which has a shrinking consisting of zero-sets.

Now, we recall that a product space $X \times Y$ is said to be rectangular if every finite cozero cover of $X \times Y$ has a $\sigma$-locally finite refinement consisting of cozero rectangles. This concept was introduced by Pasynkov [Pa] in dimension theory.

The following is easily seen by the definition (see [HM, Lemma 1]).

**Fact 1.2.** A product space $X \times Y$ is rectangular if and only if every normal cover of $X \times Y$ has a $\sigma$-locally finite refinement consisting of cozero rectangles.

In this report, for normal covers of rectangular products, we give some characterizations analogous to Stone-Michael-Morita's Theorem above in terms of refinements consisting of cozero rectangles. Next, we can apply these characterizations to the strong rectangularity and the base-paracompactness of rectangular products as well as in [Y2]. Finally, we also give the same kind of characterization of normal covers on infinite products of metrizable spaces.

These results for normal covers of rectangular products are included in the paper [Y3] with their complete proofs. But the paper does not refer to normal covers of infinite products stated in the last section.

Throughout this paper, all spaces are topological spaces without any separation axiom. However, paracompact spaces are assumed to be Hausdorff.
Let $X$ be a space and $\mathcal{U}$ a cover of $X$. A cover $\mathcal{V}$ of $X$ is called a refinement of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. A cover $\{W_{U} : U \in \mathcal{U}\}$ of $X$ is called a shrinking of $\mathcal{U}$ if $\overline{W_{U}} \subset U$ for each $U \in \mathcal{U}$.

Let $X \times Y$ be a product space. A subset of the form $A \times B$ in $X \times Y$ is called a rectangle. For a subset $R$ in $X \times Y$, $R'$ and $R''$ denote the projections of $R$ into $X$ and $Y$, respectively. A rectangle $R = R' \times R''$ is called a cozero rectangle (zero rectangle) if $R'$ and $R''$ are cozero-sets (zero-sets) in $X$ and $Y$, respectively.

For the sake of convenience, a cover $\mathcal{G}$ of a product space $X \times Y$ is said to be cozero rectangular (resp., zero rectangular, rectangular) if each member of $\mathcal{G}$ is a cozero rectangle (resp., zero rectangle, rectangle) in $X \times Y$.

2. $X$-rectangular products

A product space $X \times Y$ is said to be $X$-rectangular [Oh] if every finite cozero cover $\mathcal{O}$ of $X \times Y$ has a cozero rectangular refinement $\mathcal{G}$ such that $\pi_{X}(\mathcal{G}) = \{G' : G \in \mathcal{G}\}$ is $\sigma$-locally finite in $X$.

**Lemma 2.1.** A product space $X \times Y$ is $X$-rectangular if and only if every finite cozero cover $\mathcal{O}$ of $X \times Y$ has a cozero rectangular refinement $\mathcal{G}$ such that $\pi_{X}(\mathcal{G}) = \{G' : G \in \mathcal{G}\}$ is $\sigma$-discrete in $X$.

**Remark.** $X$-rectangular products were originally defined for Tychonoff products in [Oh]. In the case of a Tychonoff product $X \times Y$, the proof of Lemma 3.1 can be obtained by a modification of that of [Oh, Theorem 2.2]. However, the assumption that $Y$ is Tychonoff is necessary in his proof, because the Stone-Čech compactification $\beta Y$ of $Y$ has to be used there.

**Theorem 2.2.** Let $X \times Y$ be an $X$-rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.

(a) $\mathcal{O}$ is normal.
(b) $\mathcal{O}$ has a $\sigma$-locally finite cozero rectangular refinement.
(c) $\mathcal{O}$ has a $\sigma$-discrete cozero rectangular refinement.
(d) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
(e) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cozero rectangular refinement which has a zero rectangular shrinking.

It is pointed out in the proof of [Ta, Theorem 1] that if $X$ is a metric space, then the rectangularity of $X \times Y$ implies the $X$-rectangularity. So we have

**Corollary 2.3.** Let $X$ be a metric space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.

(a) $\mathcal{O}$ is normal.
(b) $\mathcal{O}$ has a $\sigma$-locally finite cozero rectangular refinement.
(c) $\mathcal{O}$ has a $\sigma$-discrete cozero rectangular refinement.
(d) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
(e) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cozero rectangular refinement which has a zero rectangular shrinking.
Remark. If a product space $X \times Y$ is not rectangular, there is a normal cover of $X \times Y$ which has no $\sigma$-locally finite cozero rectangular refinement (see Fact 1.2). In fact, there is a non-rectangular product with a metric factor (see [Pr], [Ta]). So we cannot exclude the assumption of rectangularity of $X \times Y$ in Corollary 2.3.

3. Products with a $\sigma$-Space Factor

Recall that a regular $T_1$-space $X$ is a $\sigma$-space if there is a $\sigma$-discrete (closed) net of $X$.

Theorem 3.1. Let $X$ be a paracompact $\sigma$-space and $Y$ a space. Let $\mathcal{O}$ be a normal cover of $X \times Y$. Then the following are equivalent.

(a) $\mathcal{O}$ has a $\sigma$-locally finite cozero rectangular refinement.
(b) $\mathcal{O}$ has a $\sigma$-discrete cozero rectangular refinement.
(c) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
(d) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cozero rectangular refinement which has a zero rectangular shrinking.

Theorem 3.1 immediately yields the following extension of Corollary 2.3.

Corollary 3.2. Let $X$ be a paracompact $\sigma$-space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.

(a) $\mathcal{O}$ is normal.
(b) $\mathcal{O}$ has a $\sigma$-locally finite cozero rectangular refinement.
(c) $\mathcal{O}$ has a $\sigma$-discrete cozero rectangular refinement.
(d) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
(e) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cozero rectangular refinement which has a zero rectangular shrinking.

Theorem 3.3. If $X$ is a paracompact $\Sigma$-space and $Y$ is a paracompact $P$-space, then every open cover of $X \times Y$ has a locally finite, $\sigma$-discrete, cozero rectangular refinement which has a zero rectangular shrinking.

Comparing Theorems 3.1 and 3.3, it is natural to raise the following question.

Question. Can "$\sigma$-space" be replaced by "$\Sigma$-space" in Theorem 3.1?

4. Products with a Factor Defined by Topological Games

Telgársky [Te] introduced the topological game $G(\mathbf{DC}, X)$, where $\mathbf{DC}$ denotes the class of all spaces which have a discrete cover consisting of compact sets.

According to [GT], a function $s$ from the family of all closed sets in $X$ to itself is called a winning strategy for Player I in $G(\mathbf{DC}, X)$ if it satisfies

(a) $s(F) \in \mathbf{DC}$ and $s(F) \subseteq F$ for each closed set $F$ in $X$,
(b) if $\{F_n\}$ is a decreasing sequence of closed sets in $X$ such that $s(F_n) \cap F_{n+1} = \emptyset$ for each $n \in \omega$, then $\bigcap_{n \in \omega} F_n = \emptyset$.

A space $X$ is said to be $\mathbf{DC}$-like if there is a winning strategy for Player I in $G(\mathbf{DC}, X)$. 
**Theorem 4.1.** Let $X$ be a paracompact DC-like space and $Y$ a space. Let $\mathcal{O}$ be a normal cover of $X \times Y$. Then the following are equivalent.

(a) $\mathcal{O}$ has a $\sigma$-locally finite cozero rectangular refinement.
(b) $\mathcal{O}$ has a $\sigma$-discrete cozero rectangular refinement.
(c) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
(d) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cozero rectangular refinement which has a zero rectangular shrinking.

If a Hausdorff space $X$ is subparacompact and $C$-scattered or it has a $\sigma$-closure-preserving cover consisting of compact sets, then Player I has a winning strategy for $G(\text{DC, } X)$, that is, $X$ is DC-like (see [Te, Theorems 9.7 and 14.7]). So the following is an immediate consequences of Theorem 4.2.

**Corollary 4.2.** Suppose that a paracompact space $X$ is $C$-scattered or has a $\sigma$-closure-preserving cover consisting of compact sets and $Y$ is a space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.

(a) $\mathcal{O}$ is normal.
(b) $\mathcal{O}$ has a $\sigma$-locally finite cozero rectangular refinement.
(c) $\mathcal{O}$ has a $\sigma$-discrete cozero rectangular refinement.
(d) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
(e) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cozero rectangular refinement which has a zero rectangular shrinking.

5. **APPLICATIONS TO RECTANGULAR PRODUCTS**

A product space $X \times Y$ is called a strongly rectangular [Y1] if every finite cozero (or normal) cover of $X \times Y$ has a locally finite cozero rectangular refinement.

Theorem 2.2 immediately yields

**Corollary 5.1.** If a product space $X \times Y$ is $X$-rectangular, then it is strongly rectangular.

Theorem 4.1 immediately yields

**Corollary 5.2.** Let $X$ be a paracompact $\sigma$-space. Then $X \times Y$ is rectangular if and only if it is strongly rectangular for any space $Y$.

Moreover, Corollary 4.2 immediately yields

**Corollary 5.3.** Let $X$ be a paracompact space which is $C$-scattered or has a $\sigma$-closure-preserving cover consisting of compact sets. Then $X \times Y$ is rectangular if and only if it is strongly rectangular for any space $Y$.

A Hausdorff space $X$ is said to be base-paracompact [Po] if there is a base $\mathcal{B}$ of $X$ such that $|\mathcal{B}| = w(X)$ and every open cover of $X$ has a locally finite refinement consisting of members of $\mathcal{B}$.
Proposition 5.4. Let $X$ and $Y$ be base-paracompact spaces. Assume that every normal cover of $X \times Y$ has a locally finite cozero rectangular refinement which has a zero rectangular shrinking. If $X \times Y$ is paracompact, then it is base-paracompact.

Theorem 3.1 and Proposition 5.4 immediately yield

Corollary 5.5. Let $X$ be a base-paracompact $\sigma$-space and $Y$ a base-paracompact space. If $X \times Y$ is paracompact and rectangular, then it is base-paracompact.

Zhong [Zh] actually proved that the product $X \times Y$ of a stratifiable space $X$ and a paracompact space $Y$ is rectangular if it is (countably) paracompact. So our Corollaries 5.2 and 5.5 are extensions of [Y2, Corollaries 4.2 and 4.4], respectively.

6. INFINITE PRODUCTS OF METRIZABLE SPACES

Let $X = \prod_{\lambda \in \Lambda} X_{\lambda}$ be a product space. A subset of the form $\prod_{\lambda \in \Lambda} Y_{\lambda}$ in $X$ is called a rectangle if there is a finite subset $\theta$ of $\Lambda$ such that $Y_{\lambda} = X_{\lambda}$ for each $\lambda \in \Lambda \setminus \theta$. Recall that a subset $U$ of $X$ is $R$-distinguished if $\pi_{R}^{-1}\pi_{R}(U) = U$ holds, where $R \subseteq \Lambda$ and $\pi_{R}$ denotes the projection from $X$ onto $\prod_{\lambda \in R} X_{\lambda}$. A subset $V$ of $X$ is called a cylinder (resp., $\omega$-cylinder) if $V$ is $R$-distinguished for some finite (resp., countable) $R \subseteq \Lambda$.

rectangle $\Rightarrow$ cylinder $\Rightarrow$ $\omega$-cylinder

A cover $\mathcal{G}$ of $X$ is said to be rectangular (resp., cylindrical, $\omega$-cylindrical) if each member of $\mathcal{G}$ is a rectangle (resp., cylinder, $\omega$-cylinder) in $X$.

By Theorem 4.3, the following is easily seen by induction.

Lemma 6.1. Let $X = \prod_{i \leq n} X_{i}$ be a finite product of paracompact $\Sigma$-spaces. Then every open cover of $X$ has a locally finite, $\sigma$-discrete, cozero rectangular refinement which has a rectangular shrinking.

For the countable product case, we have

Lemma 6.2. Let $X = \prod_{n \in \omega} X_{n}$ be a countable product of metrizable spaces. Then every binary open cover of $X$ has a locally finite, $\sigma$-discrete, open cylindrical refinement which has a cylindrical shrinking.

For the uncountable product case, we can prove

Lemma 6.3. Let $X = \prod_{\lambda \in \Lambda} X_{\lambda}$ be a product of metrizable spaces. Then every binary cover of $X$ by open $F_{\sigma}$-sets has a locally finite, $\sigma$-discrete, open $\omega$-cylindrical refinement which has a $\omega$-cylindrical shrinking.

A basic idea of the proof of Lemma 6.3 is due to Yamazaki's which is found in that of [Ya, Theorem 1.3].

Remark. Lemma 6.3 is an extension of [Kl, Theorem 1].

Finally, by Lemmas 6.1, 6.2 and 6.3, we can obtain the following result.
Theorem 6.4. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product of metrizable spaces and $\mathcal{G}$ an open cover of $X$. Then the following are equivalent.

(a) $\mathcal{G}$ is normal.
(b) $\mathcal{G}$ has a $\sigma$-locally finite open rectangular refinement.
(c) $\mathcal{G}$ has a $\sigma$-discrete open rectangular refinement.
(d) $\mathcal{G}$ has a locally finite open rectangular refinement.
(e) $\mathcal{G}$ has a locally finite, $\sigma$-discrete, open rectangular refinement which has a rectangular shrinking.

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