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Dieudonné Completeness and Continuous Selections

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The purpose of this note is to introduce some results in [6] and to show some additional ones. Let X be a topological space and Y a topological vector space. Symbols 2^Y, $\mathcal{K}(Y)$, and $\mathcal{F}_c(Y)$ stand for the set of all non-empty subsets of Y, the set of all non-empty convex subsets of Y, and the set of all non-empty closed convex subsets of Y, respectively. A mapping $f : X \to Y$ is called a *selection* of a mapping $\varphi : X \to 2^Y$ if $f(x) \in \varphi(x)$ for every $x \in X$. A mapping $\varphi : X \to 2^Y$ is *lower semicontinuous* (l.s.c. for short) if the set $\varphi^{-1}(V) = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. A subset S of X is a *zero-set* (respectively a *cozero-set*) if $S = \{x \in X \mid f(x) = 0\}$ (respectively $S = \{x \in X \mid f(x) \neq 0\}$) for some real-valued continuous function f on X. A Hausdorff space X is *paracompact* if every open cover has a locally finite open refinement. A Tychonoff space X is called *realcompact* if every z-ultrafilter (that is, a maximal filter consisting of zero-sets) on X with the countable intersection property has non-empty intersection. For undefined notations and terminology we refer to [1] or [3].

The following is a well-known selection theorem due to Michael.

Theorem 1 (Michael [4]). *A T_1-space X is paracompact if and only if, for every Banach space Y, every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection.*

This result not only guarantees the existence of a selection but describes paracompactness in terms of continuous selections. In addition to this theorem, some topological properties have been characterized by means of continuous selections. Among these results, Blum and Swaminathan [2] characterized realcompactness for Tychonoff spaces of non-measurable cardinal as in Theorem 2.

Before stating Theorem 2, let us recall some terminology introduced by Blum and Swaminathan [2]. An l.s.c. mapping $\varphi : X \to 2^Y$ is said to be *of infinite character* if there exists a neighborhood V of the origin of Y such that the open cover $\{\varphi^{-1}(y+V) \mid y \in Y\}$ of X has no finite subcover; and otherwise φ is called *of finite character*. For a family $\mathcal{S}$ of subsets of a space X, a mapping $\varphi : X \to 2^Y$ is *$\mathcal{S}$-fixed* if $\bigcap\{\varphi(x) \mid x \in S\} \neq \emptyset$ for every $S \in \mathcal{S}$. For a given Tychonoff space X, let $\mathcal{B}$ be a family of subsets of X defined as follows:

$\mathcal{B} = \{B \subset X \mid B$ is a realcompact cozero-set in X and $X \setminus B$ is not compact$\}$.

A cardinality τ is called *measurable* if the discrete space of cardinal τ admits a non-trivial $\{0,1\}$-valued countably additive measure.

Theorem 2 (Blum-Swaminathan [2]). *For a Tychonoff space X of non-measurable cardinal, the following are equivalent:*

(a) *X is realcompact;*

(b) *for every locally convex topological vector space Y, every $\mathcal{B}$-fixed l.s.c. mapping $\varphi : X \to \mathcal{K}(Y)$ is of finite character;*

(c) *for every locally convex topological vector space Y, every $\mathcal{B}$-fixed l.s.c. mapping $\varphi : X \to \mathcal{K}(Y)$ of infinite character admits a continuous selection.*

Let us recall that a Tychonoff space X is *Dieudonné complete* if there exists a complete uniformity on the space X. For a Tychonoff space X, Blum and Swaminathan defined the collection $\mathcal{C}$ of subsets of X as follows:

$$\mathcal{C} = \{C \subset X \mid C \text{ is a Dieudonné complete cozero-set in } X \text{ and } X \setminus C \text{ is not compact}\}.$$

In [6] the following characterizations of realcompactness and of Dieudonné completeness analogous to Theorem 1 are obtained.

Theorem 3 ([6]). *A Tychonoff space X is realcompact if and only if, for every Banach space Y, every $\mathcal{B}$-fixed l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection f such that $f(X)$ is separable.*

Theorem 4 ([6]). *A Tychonoff space X is Dieudonné complete if and only if, for every Banach space Y, every $\mathcal{C}$-fixed l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection.*

In this note, we give characterizations (Theorems 5 and 9) analogous to Theorem 2.

In the implication $(c) \Rightarrow (a)$ of Theorem 2, the assumption that X is of non-measurable cardinal cannot be dropped. Indeed, a discrete space D of measurable cardinal satisfies the condition (c) of Theorem 2 since every set-valued mapping on D has a continuous selection. But D is not realcompact (see [3, 3.11.D]). It is known that every realcompact space is Dieudonné complete and that Dieudonné complete space of non-measurable cardinal is realcompact (see [3, 8.5.13 (h)]). Thus Theorem 2 is valid with substitution of the phrases "Dieudonné complete" for "realcompact", and "$\mathcal{C}$-fixed" for "$\mathcal{B}$-fixed". In fact, Theorem 2 with this substitution is true for Tychonoff spaces of any cardinal, that is, the following holds.

Theorem 5. *For a Tychonoff space X the following are equivalent:*

(a) *X is Dieudonné complete;*

(b) *for every locally convex topological vector space Y, every C-fixed l.s.c. mapping $\varphi : X \to \mathcal{K}(Y)$ is of finite character;*

(c) *for every locally convex topological vector space Y, every C-fixed l.s.c. mapping $\varphi : X \to \mathcal{K}(Y)$ of infinite character admits a continuous selection;*

(d) *for every Banach space Y, every C-fixed l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ of infinite character admits a continuous selection.*

To prove Theorem 5 we need some preparation. Let X be a topological space. For a subset S of X, $\mathrm{cl}_X(S)$ stands for the closure of S in X. Let us denote $C(X)$ the set of all real-valued continuous functions on X. For $f \in C(X)$, set $Z(f) = \{x \in X \mid f(x) = 0\}$ and $\mathrm{Coz}(f) = \{x \in X \mid f(x) \neq 0\}$. A family $\{p_\lambda \mid \lambda \in \Lambda\}$ of continuous functions $p_\lambda : X \to [0,1]$ is called a *partition of unity* on X if $\sum_{\lambda\in\Lambda} p_\lambda(x) = 1$ for each $x \in X$. A partition of unity $\{p_\lambda \mid \lambda \in \Lambda\}$ on X is said to be *locally finite* if the cover $\{\mathrm{Coz}(p_\lambda) \mid \lambda \in \Lambda\}$ of X is locally finite. For an open cover $\mathcal{U}$ of X, a partition of unity $\{p_\lambda \mid \lambda \in \Lambda\}$ on X is *subordinated to* $\mathcal{U}$ if the cover $\{\mathrm{Coz}(p_\lambda) \mid \lambda \in \Lambda\}$ refines $\mathcal{U}$. Let $\mathbf{R}$ and $\mathbf{N}$ be the set of all real numbers and the set of all natural numbers, respectively. For a set A, $l_1(A)$ means the Banach space of all functions $y : A \to \mathbf{R}$ such that $\sum_{a\in A} |y(a)| < \infty$ with the norm $\|y\| = \sum_{a\in A} |y(a)|$. For $a \in A$, let $\pi_a : l_1(A) \to \mathbf{R}$ be the a-th projection.

Lemma 6 (Michael [4]). *Let $\mathcal{U}$ be an open cover of a topological space X. Let $\varphi : X \to 2^{l_1(\mathcal{U})}$ be a mapping defined by*

$$\varphi(x) = \{y \in l_1(\mathcal{U}) \mid \|y\| = 1,\ y(U) \geq 0 \text{ for every } U \in \mathcal{U}, \\ \text{and } y(U) = 0 \text{ for all } U \in \mathcal{U} \text{ with } x \notin U\},$$

for $x \in X$. Then φ is l.s.c. and closed-and-convex-valued. Furthermore, if φ has a continuous selection, then there exists a locally finite partition of unity on X subordinated to $\mathcal{U}$.

For a Tychonoff space X, βX and μX stand for the Stone-Čech compactification of X and the Dieudonné completion of X, respectively.

Theorem 7 (Tamano [5]). *For a Tychonoff space X and a point $a \in \beta X$, $a \in \beta X \setminus \mu X$ if and only if there exists a (locally finite) partition of unity $\{p_\lambda \mid \lambda \in \Lambda\}$ on X such that $a \in \mathrm{cl}_{\beta X}(Z(p_\lambda))$ for each $\lambda \in \Lambda$.*

Proposition 8 ([6]). *Let X be a Tychonoff space. If X is the union of a compact subspace and a Dieudonné complete subspace, then X is Dieudonné complete.*

Proof of Theorem 5. Proof of the implication $(a) \Rightarrow (b)$ is the same as [2, Theorem 2]. Implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (d)$ are obvious. To see $(d) \Rightarrow (a)$, let X be a Tychonoff space satisfying that, for every Banach space Y, every $\mathcal{C}$-fixed l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ of infinite character admits a continuous selection. Assume that X is not Dieudonné complete and take $a_0 \in \mu X \setminus X$. We will deduce a contradiction. Put $\mathcal{U} = \{\mathrm{Coz}(p) \mid p \in C(X),\ a_0 \in \mathrm{cl}_{\beta X}(Z(p))\}$. Then $\mathcal{U}$ is an open cover of X. Let $Y = l_1(\mathcal{U})$ and define a mapping $\varphi : X \to 2^Y$ as in Lemma 6. Then φ is l.s.c. and $\varphi(x) \in \mathcal{F}_c(Y)$ for each $x \in X$.

The mapping φ is $\mathcal{C}$-fixed. To prove this, let $C \in \mathcal{C}$. Then $C = \mathrm{Coz}(h)$ for some $h \in C(X)$ as C is a cozero-set. Since $\mathrm{Coz}(h)$ is Dieudonné complete and $\mathrm{cl}_{\beta X}(Z(h))$ is compact, by Proposition 8, $\mathrm{Coz}(h) \cup \mathrm{cl}_{\beta X}(Z(h))$ is Dieudonné complete and contains X. Thus we have $\mu X \subset \mathrm{Coz}(h) \cup \mathrm{cl}_{\beta X}(Z(h))$, and hence $a_0 \in \mu X \setminus X \subset \mathrm{cl}_{\beta X}(Z(h))$. Thus $C = \mathrm{Coz}(h) \in \mathcal{U}$. Let $y \in l_1(\mathcal{U})$ be the element defined by

$$y(U) = \begin{cases} 1, & \text{if } U = C, \\ 0, & \text{if } U \neq C, \end{cases}$$

for each $U \in \mathcal{U}$. Then $y \in \bigcap\{\varphi(x) \mid x \in C\}$, so that φ is $\mathcal{C}$-fixed.

The mapping φ is of infinite character. For, let $V = \{y \in l_1(\mathcal{U}) \mid \|y\| < 1\}$ and take $y_1, y_2, \ldots, y_k \in Y$ arbitrarily. It suffices to show the collection $\{\varphi^{-1}(y_i + V) \mid i = 1, 2, \ldots, k\}$ does not cover X. Put $\mathcal{U}' = \{U \in \mathcal{U} \mid y_i(U) \neq 0$ for some $i \in \{1, 2, \ldots, k\}\}$. Then $\mathrm{Card}\,\mathcal{U}'$ is countable, so that we may denote $\mathcal{U}' = \{U_i \mid i \in \mathbf{N}\}$. We show that $\mathcal{U}'$ does not cover X. Suppose that $\bigcup \mathcal{U}' = X$. By the definition of $\mathcal{U}$, for $i \in \mathbf{N}$ there exists a continuous mapping $f_i : X \to [0,1]$ such that $W_i = \mathrm{Coz}(f_i)$ and $a_0 \in \mathrm{cl}_{\beta X} Z(f_i)$. Then the mapping $f : X \to \mathbf{R}$ defined by $f(x) = \Sigma_{i=1}^{\infty} f_i(x)/2^i$ for $x \in X$ is continuous and $f(x) > 0$ for every $x \in X$. Define $p_i : X \to \mathbf{R}$ by $p_i(x) = f_i(x)/(2^i f(x))$ for $x \in X$. Then $\{p_i \mid i \in \mathbf{N}\}$ is a partition of unity on X such that $a_0 \in \mathrm{cl}_{\beta X}(Z(p_i))$ for each $i \in \mathbf{N}$. By virtue of Theorem 7, $a_0 \in \beta X \setminus \mu X$. That contradicts the choice of a_0. Thus $\mathcal{U}'$ does not cover X. Choose $x \in X \setminus \bigcup \mathcal{U}'$ and $y \in \varphi(x)$. Then $y(U) = 0$ for each $U \in \mathcal{U}'$, so that $\|y - y_i\| = \Sigma_{U \in \mathcal{U}} |y(U) - y_i(U)| = \Sigma_{U \in \mathcal{U} \setminus \mathcal{U}'} |y(U)| + \Sigma_{U \in \mathcal{U}'} |y_i(U)| \geq \Sigma_{U \in \mathcal{U} \setminus \mathcal{U}'} |y(U)| = \|y\| = 1$, and hence $y \notin y_i + V$ for each $i \in \{1, 2, \ldots, k\}$. Thus $\varphi(x) \cap (y_i + V) = \emptyset$ for each $i \in \{1, 2, \ldots, k\}$, which implies $x \notin \bigcup\{\varphi^{-1}(y_i + V) \mid i = 1, 2, \ldots, k\}$. Therefore φ is of infinite character.

By hypothesis, φ admits a continuous selection $f : X \to Y$. Put $p_U = \pi_U \circ f$ for $U \in \mathcal{U}$. Then $\{p_U \mid U \in \mathcal{U}\}$ is a partition of unity on X such that $\mathrm{Coz}(p_U) \subset U$, and hence $a_0 \in \mathrm{cl}_{\beta X}(Z(p_U))$ for each $U \in \mathcal{U}$. Thus $a_0 \in \beta X \setminus \mu X$ due to Theorem 7, that contradicts the choice of a_0. Hence X is Dieudonné complete. $\square$

A topological space satisfies *the discrete countable chain condition* (DCCC for short) if every discrete collection of non-empty open sets is countable. Every Lindelöf T_1-space and every separable space satisfy the DCCC. We also note that every metrizable space satisfying the DCCC is second countable.

Theorem 9. *For a Tychonoff space X the following are equivalent:*

(a) *X is realcompact;*

(b) *for every locally convex topological vector space Y, every $\mathcal{B}$-fixed l.s.c. mapping $\varphi : X \to \mathcal{K}(Y)$ of infinite character admits a continuous selection f such that $f(X)$ is DCCC;*

(c) *for every Banach space Y, every $\mathcal{B}$-fixed l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ of infinite character admits a continuous selection f such that $f(X)$ is separable.*

Proof. Due to [2, Theorem 2], the implication $(a) \Rightarrow (b)$ of Theorem 2 is valid without assuming that X is of non-measurable cardinal. Thus $(a) \Rightarrow (b)$ holds. The implication $(b) \Rightarrow (c)$ is clear. For the proof of $(c) \Rightarrow (a)$, see the proof of the "if" part of [6, Theorem 1.3]. □

Remark 10. Note that Theorem 9 holds for Tychonoff spaces X of any cardinal. Due to Theorem 9, the implication $(b) \Rightarrow (a)$ of Theorem 2 also holds for a Tychonoff space X of any cardinal.

References

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