Point vortex dynamics in background fields on surfaces

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Contents

1 Introduction **2**

2 Vector calculus on surfaces **4**

3 Euler-Arnold flows on surfaces **9**

4 Point vortex dynamics **15**

5 de Rham current **18**
5.1 Basic concepts and properties 18
5.2 Weak formulation of vector fields 23

6 Weak Euler-Arnold flows **25**

7 Main results **29**

8 Applications **34**

Abstract

The point vortex dynamics in background fields on surfaces is justified as an Euler-Arnold flow in the sense of de Rham currents. We formulate a current-valued solution of the Euler-Arnold equation with a regular-singular decomposition. For the solution, we first prove that, if the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$ for $n = 1, \ldots, N$, $q_n(t)$ is a solution of the point vortex equation. Conversely, we next prove that, if $q_n(t)$ is a solution of the point vortex equation for $n = 1, \ldots, N$, there exists a current-valued solution of the Euler-Arnold equation with a regular-singular decomposition such that the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$. From the viewpoint of applications, the mathematical justification is of a significance since the point vortex dynamics in the rotational vector field on the unit sphere is adapted as a mathematical model of geophysical flows in order to take the effect of the Coriolis force on inviscid flows into considerations.

1 Introduction

The motion of incompressible and inviscid fluids in the Euclidean plane is governed by the Euler equation and its solution is called an Euler flow. Since the fluid velocity and the pressure can be recovered from the vorticity, an Euler flow is determined from a solution of the vorticity equation. Namely, assuming ω_t is a solution of the vorticity equation,

$$\partial_t \omega_t + (-\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle \cdot \nabla)\omega_t = 0,$$

where $\mathcal{J}$ is the symplectic matrix and G_H is the Green function for the Laplacian, we obtain an Euler flow (v_t, p_t) which is defined by

$$v_t = -\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle, \quad p_t = \langle G_H, \mathrm{div}(v_t \cdot \nabla)v_t\rangle, \tag{1}$$

On the other hand, the formulae (1) still make sense in the sense of distributions when we give a time-dependent distribution Ω_t by a linear combination of delta functions centered at $q_n(t)$ for $n = 1, \dots, N$. Then, replacing ω_t by Ω_t in (1), we formally obtain a fluid velocity V_t and a pressure P_t. However, we can not define the dynamics of $q_n(t)$ from the vorticity equation. Instead, to determine the evolution of $q_n(t)$ by V_t, Helmholtz considered the following regularized equation for $q_n(t)$ [21].

$$\begin{aligned} \dot{q}_n &= \lim_{q \to q_n} \left[V_t(q) + \mathcal{J}\mathrm{grad}\langle G_H, \Gamma_n \delta_{q_n(t)}\rangle(q) \right] \qquad (2) \\ &= -\mathcal{J}\mathrm{grad} \sum_{\substack{m=1 \\ m \neq n}}^{N} \Gamma_m G_H(q_n, q_m) \equiv v_n(q_n). \end{aligned}$$

It is called the *point vortex equation*, and the solution of (2) is called the *point vortex dynamics*. Then, a natural question arises; How can we interpret (V_t, P_t) as an Euler flow in an appropriate mathematical sense? In other words, we need to determine a space of solutions of the Euler equation to which (V_t, P_t) belongs. Since L^p space does not contains Ω_t, a more sophisticated space is to be considered. This is one of problems in the analysis of the 2D Euler equation as discussed in [16, 17, 18].

From the viewpoint of applications, the point vortex dynamics is sometimes considered in the fluid velocity $X_t \in \mathfrak{X}^r(\mathbb{R}^2)$ of an Euler flow, called the point vortex dynamics in the background field X_t. Then, the evolution of $q_n(t)$ is governed by the following equation.

$$\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega v_n(q_n(t)), \quad n = 1, \dots N \tag{3}$$

for a given parameter $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. An experimental study confirms the importance of background fields in two-dimensional turbulence [28].

The purpose of this paper is justifying the point vortex dynamics in background fields as an Euler flow mathematically. To this end, we establish a weak formulation of the Euler equation in the space of currents, which is developed in the theory of geometric analysis and geometric measure theory. Since the notion of currents is defined not only for the Euclidean plane but also general curved surface, the formulation established here can be naturally generalized for surfaces. The Euler equation is generalized for the case of surface by Arnold, whose generalization is called the Euler-Arnold equation. From the viewpoint of applications, it is of a significance to justify the point vortex dynamics in a background field on curved surfaces as an Euler-Arnold flow. Since the point vortex dynamics in the rotational vector field on the unit sphere is adapted as a mathematical model of geophysical flows in order to take the effect of the Coriolis force on inviscid flows into considerations [20] for instance.

This paper is organized as follows. In Section 2, we reformulate vector calculus in the plane from the viewpoint of differential forms. In Section 3, we derive the Euler-Arnold equation and investigate steady solutions. In Section 4, we introduce the point vortex dynamics and its generalizations as another dynamical model of incompressible and inviscid fluids. In Section 5, we review basic concepts of the theory of de Rham currents. Some notions in vector calculus are reformulated for the application to fluid dynamics. In Section 6, we examine the Euler-Arnold equation for more details from the viewpoint of currents. We formulate a current-valued solution of the Euler-Arnold equation on surfaces with a regular-singular decomposition, which we call a *C^r-decomposable weak Euler-Arnold flow*. In Section 7, our main results are stated and proved. In the first theorem, we prove that, for a given C^r-decomposable weak Euler-Arnold flow, if the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$ for $n = 1, \ldots, N$, $q_n(t)$ is the solution of the point vortex equation. Conversely, we next prove that, if $q_n(t)$ is a solution of the point vortex equation for $n = 1, \ldots, N$, there exists a C^r-decomposable weak Euler-Arnold flow such that the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$. In Section 8, we apply the main results to some problems in fluid dynamics. As a consequence, the point vortex dynamics in a background field X_t (3) on a surface (M, g) is justified as C^r-decomposable weak Euler-Arnold flow and the pressure p_t satisfies

$$p_t = P_t + (2\beta_X - 1)g(X_t, V_t - X_t) + (2\beta_\omega - 1)|V_t - X_t|^2/2,$$

for some a time-dependent function $P_t \in C^r(M)$ as a C^r-decomposable weak Euler-Arnold flow. The present paper is written as the author's Ph.D thesis to make the preprint [25] self-contained.

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2 Vector calculus on surfaces

We review vector calculus in the plane from the viewpoint of calculus of differential forms. Let us remember that the curl operator curl : $X = {}^{\mathsf{T}}(X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2) \to \operatorname{curl} X \in C^{r-1}(\mathbb{R}^2)$ and the divergence operator div : $X = {}^{\mathsf{T}}(X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2) \to \operatorname{div} X \in C^{r-1}(\mathbb{R}^2)$ are defined by

$$\operatorname{curl} X = \partial_1 X^2 - \partial_2 X^1, \quad \operatorname{div} X = \partial_1 X^1 + \partial_2 X^2.$$

In particular, a vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$ is said to be irrotational (or curl-free), if $\operatorname{curl} X = 0$. And a vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$ is said to be incompressible (or divergence-free), if $\operatorname{div} X = 0$. For each $X \in \mathfrak{X}^r(\mathbb{R}^2)$, the function $\omega = \operatorname{curl} X \in C^{r-1}(\mathbb{R}^2)$ is called the vorticity. A vector field $X = {}^{\mathsf{T}}(X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2)$ is called the gradient vector field, if there exists a function $\phi \in C^{r+1}(\mathbb{R}^2)$ such that $X = \operatorname{grad} \phi = {}^{\mathsf{T}}(\partial_1 \phi, \partial_2 \phi)$. Then, the function ϕ is called the potential. A vector field $X = {}^{\mathsf{T}}(X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2)$ is called the Hamiltonian vector field, if there exists a function $\psi \in C^{r+1}(\mathbb{R}^2)$ such that $X = -\mathcal{J} \operatorname{grad} \psi = {}^{\mathsf{T}}(\partial_2 \psi, -\partial_1 \psi)$, where $\mathcal{J}$ is the following matrix.

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The function ψ is called the Hamiltonian, or the stream-function. Let us remember that every irrotational vector field in the plane is a gradient vector field and every incompressible vector field in the plane is a Hamiltonian vector field. In addition, the following formula is useful in 2D fluid dynamics.

$$\operatorname{curl} \mathcal{J} \operatorname{grad} \psi = \triangle \psi, \tag{4}$$

where $\triangle = \partial_1^2 + \partial_2^2$ is the Laplacian. Hence, every incompressible vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$ is written as $X = -\mathcal{J} \operatorname{grad} \psi$ for some $\psi \in C^{r+1}(\mathbb{R}^2)$, and the vorticity ω satisfies

$$\omega = \operatorname{curl}(-\mathcal{J} \operatorname{grad} \psi) = -\triangle \psi. \tag{5}$$

Let G_H denote the Green's function for the Laplacian $-\triangle$. Since the solution ψ of the Poisson equation (5) is given as $\psi = \langle G_H, \omega \rangle$, the vector field can be presented by the vorticity as

$$X = -\mathcal{J} \operatorname{grad} \langle G_H, \omega \rangle, \tag{6}$$

where $\langle G_H, \cdot \rangle$ is the convolution with the Green function G_H. The equality (6) is called the Biot-Savart law. In particular, since $G_H(x, y) = -(4\pi)^{-1} \log(x^2 + y^2)$, the equation (6) is written as

$$X(x_0, y_0) = \int_{\mathbb{R}^2} \frac{{}^{\mathsf{T}}(-y, x)}{2\pi(x^2 + y^2)} \omega(x - x_0, y - y_0) \, \mathrm{d}\, x \, \mathrm{d}\, y.$$

Let us reformulate the fact stated the above in terms of vector calculus through differential forms. Note that every vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$ is conventionally written as $X = {}^{\mathsf{T}}(X^1, X^2)$. However, in the theory of manifolds, it is rewritten as $X = X^1\partial_1 + X^2\partial_2$ by using an orthogonal basis (∂_1, ∂_2) of a tangent space of $\mathbb{R}^2$. Let us remember that the orthogonal basis is identified with the partial derivative acting on functions. Every vector field $X = X^1\partial_1 + X^2\partial_2 \in \mathfrak{X}^r(\mathbb{R}^2)$ is converted to a 1-form $X^\flat = X^1 \,\mathrm{d}\, x^1 + X^2 \,\mathrm{d}\, x^2 \in \Omega^1_{[r]}(\mathbb{R}^2)$. The operator $\flat : X \in \mathfrak{X}^r(\mathbb{R}^2) \to X^\flat \in \Omega^1_{[r]}(\mathbb{R}^2)$ is called the flat operator and the corresponding 1-form $X^\flat$ is called the velocity form. Conversely, every 1-form $\alpha = \alpha_1 \,\mathrm{d}\, x^1 + \alpha_2 \,\mathrm{d}\, x^2 \in \Omega^1_{[r]}(\mathbb{R}^2)$ is also converted to a vector field $\alpha_\sharp = \alpha_1\partial_1 + \alpha_2\partial_2 \in \mathfrak{X}^r(\mathbb{R}^2)$. The operator $\sharp : \alpha \in \Omega^1_{[r]}(\mathbb{R}^2) \to \alpha_\sharp \in \mathfrak{X}^r(\mathbb{R}^2)$ is called the sharp operator and the corresponding vector field $\alpha_\sharp$ is called the dual vector field. It is clear that $X = (X^\flat)_\sharp$ and $\alpha = (\alpha_\sharp)^\flat$. We see that the curl operator and the divergence operator is written by the differential operator d and the Hodge-$*$ operator as

$$\operatorname{curl} X = *\,\mathrm{d}\, X^\flat, \quad \operatorname{div} X = *\,\mathrm{d}\,{*X^\flat}. \tag{7}$$

To see this, let us remember the following property of the differential operator d and the Hodge-$*$ operator in the plane. For each p-form $\alpha \in \Omega^p_{[r]}(\mathbb{R}^2)$, $p = 0, 1, 2$, $\mathrm{d}\,\alpha \in \Omega^{p+1}_{[r]}(\mathbb{R}^2)$ and $*\alpha \in \Omega^{2-p}_{[r]}(\mathbb{R}^2)$ satisfy

$$\mathrm{d}\,\alpha = \begin{cases} \partial_1\alpha \,\mathrm{d}\, x^1 + \partial_2\alpha \,\mathrm{d}\, x^2, & \text{if } p = 0, \\ (\partial_1\alpha_2 - \partial_2\alpha_1) \,\mathrm{d}\, x^1 \wedge \mathrm{d}\, x^2, & \text{if } p = 1, \\ 0, & \text{if } p = 2, \end{cases}$$

$$*\alpha = \begin{cases} \alpha \,\mathrm{d}\, x^1 \wedge \mathrm{d}\, x^2, & \text{if } p = 0, \\ -\alpha_2 \,\mathrm{d}\, x^1 + \alpha_1 \,\mathrm{d}\, x^2, & \text{if } p = 1, \\ \alpha_{12}, & \text{if } p = 2. \end{cases}$$

Hence, it follows that

$$\begin{aligned} *\,\mathrm{d}\, X^\flat &= *\,\mathrm{d}(X^1 \,\mathrm{d}\, x^1 + X^2 \,\mathrm{d}\, x^2) = *(\partial_1 X^2 - \partial_2 X^1) \,\mathrm{d}\, x^1 \wedge \mathrm{d}\, x^2 \\ &= \partial_1 X^2 - \partial_2 X^1, \\ *\,\mathrm{d}\,{*X^\flat} &= *\,\mathrm{d}\,{*(X^1 \,\mathrm{d}\, x^1 + X^2 \,\mathrm{d}\, x^2)} = *\,\mathrm{d}(-X^2 \,\mathrm{d}\, x^1 + X^1 \,\mathrm{d}\, x^2) \\ &= *(\partial_1 X^1 + \partial_2 X^2) \,\mathrm{d}\, x^1 \wedge \mathrm{d}\, x^2 = \partial_1 X^1 + \partial_2 X^2. \end{aligned}$$

In the same manner, we obtain

$$\operatorname{grad} \phi = (\mathrm{d}\,\phi)_\sharp, \quad \mathcal{J}\operatorname{grad} \psi = (*\,\mathrm{d}\,\psi)_\sharp, \tag{8}$$

since

$$\mathrm{d}\,\phi = \partial_1\phi\,\mathrm{d}\,x^1 + \partial_2\phi\,\mathrm{d}\,x^2, \quad *\,\mathrm{d}\,\psi = -\partial_2\psi\,\mathrm{d}\,x^1 + \partial_1\psi\,\mathrm{d}\,x^2.$$

Owing to $*\,\mathrm{d} * \mathrm{d}\,\psi = \triangle\psi$, the equality (4) is obtained.

Based on the reformulation of vector calculus, we naturally generalize the above notions to those for the case of curved surfaces. Let (M, g) be a connected orientable 2-dimensional Riemannian manifold, called a surface, with or without boundary, which can be compact or non-compact. We extend the Riemannian metric g on the tangent bundle TM to the metric on a vector bundle E over M. Namely, g is defined as a section of $(E \otimes E)^*$ such that for all $p \in M$, g_p is an inner product on the vector space E_p. Hodge-$*$ operator associated with the metric g and the volume form dVol_g is defined by $\alpha \wedge *\beta = g(\alpha, \beta)\mathrm{dVol}_g$ for all $\alpha, \beta \in \Omega^p_{[r]}(M)$. In particular, $*$ rotates vectors and covectors by the degree $+\pi/2$ owing to $\dim M = 2$. We assume the existence of the Green function for the Hodge Laplacian $\triangle = \delta\,\mathrm{d} + \mathrm{d}\,\delta$ on (M, g), where $\delta = *\,\mathrm{d}\,*$ is the codifferential operator. Note that the existence of a Green function on (M, g) is discussed in [3, 24]. If M is a surface with boundary, we assume every vector field X satisfies the slip boundary condition on ∂M, that is $X|_{\partial M} \in \mathfrak{X}(\partial M)$. We also assume every vector field has compact support.

We first define the flat operator $\flat : X \to \mathfrak{X}^r(M) \to X^\flat \in \Omega^1_{[r]}(M)$ by $X^\flat = g(X, \cdot\,)$. The sharp operator $\sharp : \alpha_\sharp \in \Omega^1_{[r]}(M) \to \alpha_\sharp \in \mathfrak{X}^r(M)$ is defined as the inverse of the flat operator. The flat operator and the sharp operator become isomorphisms between $\mathfrak{X}^r(M)$ and $\Omega^1_{[r]}(M)$. From this reason, the flat operator and the sharp operator are also called the musical isomorphisms. Based on the equality (7), for the case of surfaces, we next define the curl operator $\mathrm{curl} : \mathfrak{X}^r(M) \to C^{r-1}(M)$ and the divergence operator $\mathrm{div} : \mathfrak{X}^r(M) \to C^{r-1}(M)$ by

$$\mathrm{curl}\, X = *\,\mathrm{d}\, X^\flat, \quad \mathrm{div}\, X = \delta X^\flat.$$

A vector field $X \in \mathfrak{X}^r(M)$ is said to be irrotational, if $\mathrm{curl}\, X = 0$. A vector field $X \in \mathfrak{X}^r(M)$ is said to be incompressible, if $\mathrm{div}\, X = 0$. For each $X \in \mathfrak{X}^r(M)$, the function $\omega = \mathrm{curl}\, X \in C^{r+1}(M)$ is called the vorticity. Owing to the equality (8), we define the gradient operator $\mathrm{grad} : \phi \in C^r(M) \to \mathrm{grad}\,\phi \in \mathfrak{X}^{r-1}(M)$ by $\mathrm{grad}\,\phi = (\mathrm{d}\,\phi)_\sharp$. The symplectic matrix $\mathcal{J}$ is interpreted as the complex structure $\mathcal{J} : X \in TM \to (*X^\flat)_\sharp \in TM$. A vector field $X \in \mathfrak{X}^r(M)$ is called the gradient vector field, if there exists a function

$\phi \in C^{r+1}(M)$ such that $X = \operatorname{grad}\phi$. Then, the function ϕ is called the potential. A vector field $X \in \mathfrak{X}^r(M)$ is called the Hamiltonian vector field on the symplectic manifold (M, dVol_g), if there exists a function $\psi \in C^{r+1}(\mathbb{R}^2)$ such that $X = -\mathcal{J}\operatorname{grad}\psi$. Then, the equality (4) for the case of surfaces holds true, since

$$\operatorname{curl}\mathcal{J}\operatorname{grad}\psi = *\,\mathrm{d}(*\,\mathrm{d}\,\psi) = \delta\,\mathrm{d}\,\psi = \triangle\psi.$$

On the other hand, there is an incompressible vector field which is not Hamiltonian vector field on the surface as long as M is simply connected whereas, in the plane, every incompressible vector field is a Hamiltonian vector field. Indeed, for every incompressible vector field $X \in \mathfrak{X}^r(M)$, $X^\flat$ is closed 1-form but it should be exact 1-form when X is a Hamiltonian vector field. In the same reason, there is a irrotational vector field which is not a gradient vector field. Hence, in order to recover incompressible vector fields from the vorticity, we need to restrict the incompressible vector fields to certain appropriate vector fields. As a trivial restriction, the incompressible vector field X is taken as a Hamiltonian vector field $-\mathcal{J}\operatorname{grad}\psi$. Then, the vorticity ω of $X = -\mathcal{J}\operatorname{grad}\psi$ satisfies

$$\omega = *\,\mathrm{d}(-*\,\mathrm{d}\,\psi) = -\triangle\psi, \tag{9}$$

which yields that ψ is determined by a solution of the Poisson equation (9). Owing to the slip boundary condition for X, ψ obeys the Dirichlet boundary condition with a constant boundary value, since

$$\mathrm{d}\,\psi = *X^\flat = 0 \quad \text{on } \partial M.$$

We define $G_H \in C^\infty(M \times M \setminus \Delta)$ as the fundamental solution of the above Poisson problem and call it the *hydrodynamic Green function*.

Definition 2.1. *A function $G_H \in C^\infty(M \times M \setminus \Delta)$ is called a hydrodynamic Green function, if the function G_H is a solution of the following boundary value problem in $\mathcal{D}_0'(M)$.*

$$\begin{aligned}
-\triangle G_H(x, x_0)[\phi] &= \begin{cases} *\phi(x_0) - \dfrac{1}{\mathrm{Area}(M)}\displaystyle\int_M \phi, & \textit{if } M \textit{ is closed}, \\ *\phi(x_0), & \textit{otherwise}, \end{cases} \\
G_H(x, x_0) &= G_H(x_0, x), \\
\mathrm{d}\,G_H &= 0 \quad \textit{on } \partial M.
\end{aligned}$$

Let us recall the existence of G_H is assumed throughout this paper. We also note that the boundary value of G_H is arbitrarily fixed as long as the summation of the flux on each boundary component is equals to 1 and that the difference of this choice is at most harmonic functions, say ψ_0. There are several definitions of the hydrodynamic Green's function according to the boundary value and asymptotic behavior near ends of surfaces [9, 11]. In what follows, let us choose a hydrodynamic Green function G_H. Hence, by solving the Poisson problem (30), we obtain $\psi = \psi^0 + \langle G_H, \omega \rangle$ for some harmonic function ψ^0, which gives the Biot-Savart law on the surface.

$$X = -\mathcal{J} \operatorname{grad} \psi = -\mathcal{J} \operatorname{grad} \langle G_H, \omega \rangle. \tag{10}$$

As another restriction, for an arbitrarily fixed incompressible vector field $X \in \mathfrak{X}^r(M)$, the incompressible vector field $Y \in \mathfrak{X}^r(M)$ is given by $Y = X - \mathcal{J} \operatorname{grad} \psi$ for some $\psi \in C^{r+1}(M)$. The case stated the above corresponds to the case of $X = 0$. Then, we can also recover the vector fields from the vorticity, since the relative vorticity $\omega = * \mathrm{d}(Y - X)^\flat \in \Omega^0_{[r-1]}(M)$ satisfies

$$\omega = * \mathrm{d}(-\mathcal{J} \operatorname{grad} \psi)^\flat = -\triangle \psi,$$

which gives $Y = X - \mathcal{J} \operatorname{grad}(\psi^0 + \langle G_H, \omega \rangle)$. In this paper, we will use this restriction to recover an incompressible vector field from a singular vorticity in a weak sense.

3 Euler-Arnold flows on surfaces

The motion of incompressible and inviscid fluids on a simply connected domain U in the Euclidean plane is governed by the Euler equation,

$$\partial_t v_t + (v_t \cdot \nabla) v_t = -\operatorname{grad} p_t, \quad \operatorname{div} v_t = 0, \tag{11}$$

where $(v_t \cdot \nabla) = v_t^1 \partial_1 + v_t^2 \partial_2$. We consider the slip boundary condition $v_t \cdot n = 0$ on the boundary of U. The fluid velocity field $v_t \in \mathfrak{X}(U)$ and the scalar pressure $p_t \in C(U)$ are the unknowns at time $t \in (0, T]$. A solution of the Euler equation (11) is called an Euler flow. Applying the curl operator to the Euler equation, we have the following equation for the vorticity $\omega_t = \operatorname{curl} v_t = \partial_1 v_t^2 - \partial_2 v_t^1$.

$$\partial_t \omega_t + (v_t \cdot \nabla) \omega_t = 0,$$

which is called the vorticity equation. By the Biot-Savart law, we have

$$v_t = -\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle,$$

which yields the vorticity equation is rewritten with ω_t only.

$$\partial_t\omega_t + -(\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle \cdot \nabla)\omega_t = 0. \tag{12}$$

Then, the pressure p_t is also presented by ω_t, since, applying the divergence operator to (11), we obtain

$$\mathrm{div}(v_t \cdot \nabla)v_t = -\,\mathrm{div}\,\mathrm{grad}\, p_t = -\triangle p_t,$$

which implies that p_t is determined as a solution of this Poisson equation. Hence, if ω_t is a solution of (12), the Euler flow (v_t, p_t) is given by

$$v_t = -\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle, \quad p_t = \langle G_H, \mathrm{div}(\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle \cdot \nabla)\,\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle\rangle. \tag{13}$$

Obviously, when $\omega_t = 0$ for each time t, it is a steady solution of (12). Then, the velocity field v_t is irrotational and the pressure satisfies the steady Bernoulli law.

$$p = -|v|^2/2.$$

When the flow field is a curved surface, we derive the Euler equation as the variational equation for the kinematic energy of incompressible flows. We denote the space of all incompressible vector fields on a surface (M, g) with compact support by $\mathrm{SVect}(M)$ and the space of all area-preserving diffeomorphisms on the surface with compact support by $\mathrm{SDiff}(M)$. Then, every incompressible vector field $v \in \mathrm{SVect}(M)$ generates an area-preserving diffeomorphism $v_t \in \mathrm{SDiff}(M)$. Conversely, the infinitesimal generator of $\Phi : t \in [0, T] \to \Phi_t \in \mathrm{SDiff}(M)$ is an incompressible vector field $\partial_t\Phi_t \in \mathrm{SVect}(M)$. For each path $c : t \in [0, T] \to c_t \in \mathrm{SDiff}(M)$, we define the L^2-kinematic energy $E(c)$ of the path c by

$$E(c) = \frac{1}{2}\int_0^T \int_M g(\partial_t c_t, \partial_t c_t)\mathrm{dVol}_g\, \mathrm{d}\, t.$$

Let us denote the variation of the path c by $\Phi : (-\varepsilon, \varepsilon) \times [0, T] \to \Phi_t^s \in \mathrm{SDiff}(M)$, in which $c = \Phi^0$. It follows from the chain rule that the time-dependent vector field $\partial_t\Phi_t^0 \in \mathrm{SVect}(M)$, denoted by v_t, satisfies

$$\partial_t^2\Phi_t^0(x)|_{t=\tau} = (\partial_t v_t + \nabla_{v_t} v_t)(\Phi_\tau(x))$$

for each $x \in M$ and each $\tau \in [0, T]$ where ∇ is the Levi-Civita connection on (M, g). We write the time-dependent vector field $\partial_s \Phi_t^s|_{s=0} \in \mathrm{SVect}(M)$ by u_t, which satisfies $u_0 = u_T = 0$. For the first variation of E, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\, s} E(\Phi^s)\bigg|_{s=0} &= \int_0^T \int_M (\partial_t \Phi_t^0, \partial_s \partial_t \Phi_t^s|_{s=0}) \mathrm{dVol}_g \,\mathrm{d}\, t \\
&= \int_0^T \int_M (\partial_t \Phi_t^0, \partial_t u_t) \mathrm{dVol}_g \,\mathrm{d}\, t \\
&= \left[\int_M (\partial_t \Phi_t^0, u_t) \mathrm{dVol}_g \,\mathrm{d}\, t \right]_0^T - \int_0^T \int_M (\partial_t^2 \Phi_t^0, u_t) \mathrm{dVol}_g \,\mathrm{d}\, t \\
&= - \int_0^T \int_M (\partial_t v_t + \nabla_{v_t} v_t, u_t) \mathrm{dVol}_g \,\mathrm{d}\, t.
\end{aligned}$$

Hence, if the path c is a critical point of E, then for every $t \in [0, T]$ and every $u_t \in \mathrm{SVect}(M)$,

$$\int_M (\partial_t v_t + \nabla_{v_t} v_t, u_t) \mathrm{dVol}_g = 0.$$

By Hodge-Kodaira decomposition, there exists a time-dependent function $p_t \in C^{r+1}(M)$ such that

$$\partial_t v_t + \nabla_{v_t} v_t = -\operatorname{grad} p_t. \tag{14}$$

The evolution equation (14) is called the Euler-Arnold equation [1, 2]. The unknowns, v_t and p_t are called the fluid velocity and the pressure. The solution (v_t, p_t) of (14) is called a (classical) Euler-Arnold flow.

In what follows, we fix an Euler-Arnold flow $(v_t, p_t) \in \mathfrak{X}^r(M) \times C^r(M)$ at a time $t \in (0, T]$. For convenience, we omit the subscript t unless otherwise stated in what follows. Let us introduce some equivalent formulations of the Euler-Arnold equation. They are derived from the following dual formulation.

$$\partial_t v^\flat + \nabla_v v^\flat = -\,\mathrm{d}\, p. \tag{15}$$

Based on the fact that

$$\nabla_v v^\flat = \mathcal{L}_v\, v^\flat - \mathrm{d}\, |v|^2/2, \tag{16}$$

$$\nabla_v v^\flat = i_v \,\mathrm{d}\, v^\flat + \mathrm{d}\, |v|^2/2, \tag{17}$$

we obtain two equivalent formulations of (15),

$$\partial_t v^\flat + \mathcal{L}_v\, v^\flat = -\,\mathrm{d}(p - |v|^2/2), \tag{18}$$

$$\partial_t v^\flat + i_v\,\mathrm{d}\, v^\flat = -\,\mathrm{d}(p + |v|^2/2), \tag{19}$$

where $\mathcal{L}$ and i are the Lie derivative and the interior multiplication respectively. Applying the operator $*\,\mathrm{d}$ to (18), we have the vorticity equation.

$$\partial_t \omega + \mathcal{L}_v\, \omega = 0. \tag{20}$$

Hence, the vorticity of the Euler-Arnold flow is also a Lagrange invariance. If v is a solution of (20), the pressure is determined by v, since, applying the codifferential operator $\delta = *\,\mathrm{d}\,*$ to (15), we have

$$\mathrm{div}\,\nabla_v v = \delta \nabla_v v^\flat = -\triangle p,$$

which yields

$$p = \langle G_H, \mathrm{div}\,\nabla_v v\rangle.$$

Moreover, if v is a Hamiltonian vector field, owing to the Biot-Savart law (10), the vorticity equation is written with ω only.

$$\partial_t \omega + \mathcal{L}_{-\,\mathcal{J}\mathrm{grad}\langle G_H, \omega\rangle}\, \omega = 0. \tag{21}$$

Hence, if ω_t is a solution of (21), the Euler-Arnold flow (v_t, p_t) is given by

$$v_t = -\,\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle, \quad p_t = \langle G_H, \mathrm{div}\,\nabla_{-\,\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle} -\,\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle\rangle. \tag{22}$$

We notice that the equality (18) contains both v and $v^\flat$. On the other hand, the equality (19) gives a formulation of the Euler-Arnold equation with $v^\flat$ only. As a matter of fact, let us recall the Riemannian metric can be extended to the metric on the cotangent bundle, which yields $|v^\flat|$ makes sense and $|v^\flat| = |v|$ holds. Owing to $\dim M = 2$, we have

$$i_v\,\mathrm{d}\, v^\flat = \omega i_v \mathrm{dVol}_g = (*\,\mathrm{d}\, v^\flat) * v^\flat,$$

which yields that the equality (19) is deduced to

$$\partial_t v^\flat + (*\,\mathrm{d}\, v^\flat) * v^\flat + \mathrm{d}\,|v^\flat|^2/2 = -\,\mathrm{d}\, p.$$

This formulation plays a key role in deriving a weak formulation of the Euler-Arnold equation.

Let us examine these expressions of the advection term, (16) and (17) in more detail. We will see that each of them leads to a steady solution of the Euler-Arnold equation. They are derived from the covariant derivative operator $\nabla v : X \in \mathfrak{X}^r(M) \to \nabla_X v \in \mathfrak{X}^r(M)$ and its adjoint operator $\nabla^* v$ for each $v \in \mathfrak{X}^r(M)$, which is defined by

$$g(\nabla^* v(X), Y) = g(X, \nabla v(Y))$$

for every $X, Y \in \mathfrak{X}^r(M)$. In what follows, we fix vector fields $v, X, Y \in \mathfrak{X}^r(M)$ arbitrarily. It follows from the compatibility of the metric that

$$g(\nabla^* v(v), X) = g(v, \nabla v(X)) = g(v, \nabla_X v) = X|v|^2/2 = \mathrm{d}\,|v|^2/2(X),$$

which gives

$$\nabla^* v(v)^\flat = \mathrm{d}\,|v|^2/2. \tag{23}$$

Let us decompose ∇v into the symmetric part $\nabla^{\mathrm{sym}} v$ and skew-symmetric part $\nabla^{\mathrm{sk}} v$, that is, for each $X \in \mathfrak{X}^r(M)$,

$$\begin{aligned} \nabla^{\mathrm{sym}} v(X) &= \nabla v(X) + \nabla^* v(X), \\ \nabla^{\mathrm{sk}} v(X) &= \nabla v(X) - \nabla^* v(X). \end{aligned}$$

In particular, by (23), we obtain

$$\begin{aligned} \nabla^{\mathrm{sym}} v(v)^\flat &= \nabla v(v)^\flat + \nabla^* v(v)^\flat = \nabla_v v^\flat + \mathrm{d}\,|v|^2/2, \\ \nabla^{\mathrm{sk}} v(v)^\flat &= \nabla v(v)^\flat - \nabla^* v(v)^\flat = \nabla_v v^\flat - \mathrm{d}\,|v|^2/2. \end{aligned}$$

By the definition of the Lie derivative, we have

$$\begin{aligned} (\mathcal{L}_v\, g)(X, Y) &= v g(X, Y) - g([v, X], Y) - g(X, [v, Y]) \\ &= g(\nabla_v X, Y) + g(X, \nabla_v Y) \\ &\quad - g(\nabla_v X - \nabla_X v, Y) - g(X, \nabla_v Y - \nabla_Y v) \\ &= g(\nabla_X v, Y) + g(X, \nabla_Y v) \\ &= g(\nabla v(X), Y) + g(X, \nabla v(Y)) \\ &= g(\nabla^{\mathrm{sym}} v(X), Y). \end{aligned}$$

Hence, the equality (16) holds true, since

$$\begin{aligned}(\mathcal{L}_v\, v^\flat)(X) &= v(v^\flat(X)) - v^\flat([v,X]) = vg(v,X) - g(v,[v,X])\\ &= (\mathcal{L}_v\, g)(v,X) = g(\nabla^{\mathrm{sym}} v(v), X) = (\nabla_v v^\flat + \mathrm{d}\,|v|^2/2)(X).\end{aligned}$$

Let us compute $\mathrm{d}\, v^\flat$ in the similar manner.

$$\begin{aligned}\mathrm{d}\, v^\flat(X,Y) &= X(v^\flat(Y)) - Y(v^\flat(X)) - v^\flat([X,Y])\\ &= Xg(v,Y) - Yg(v,X) - g(v,[X,Y])\\ &= g(\nabla_X v, Y) + g(v, \nabla_X Y) - g(\nabla_Y v, X)\\ &- g(v, \nabla_Y X) - g(v, \nabla_X Y - \nabla_Y X)\\ &= g(\nabla_X v, Y) - g(\nabla_Y v, X)\\ &= g(\nabla v(X), Y) - g(\nabla v(Y), X)\\ &= g(\nabla^{\mathrm{sk}} v(X), Y).\end{aligned}$$

From this, we see that

$$i_v\, \mathrm{d}\, v^\flat(X) = \mathrm{d}\, v^\flat(v,X) = g(\nabla^{\mathrm{sk}} v(v), X) = (\nabla_v v^\flat - \mathrm{d}\,|v|^2/2)(X),$$

which establishes the equality (17). For example, when the flow field is the Euclidean plane, we have

$$\begin{aligned}\mathrm{d}\,|v|^2/2 &= (v^1\partial_1 v^1 + v^2\partial_1 v^2)\,\mathrm{d}\,x^1 + (v^1\partial_2 v^1 + v^2\partial_2 v^2)\,\mathrm{d}\,x^2,\\ \nabla_v v^\flat &= (v^1\partial_1 v^1 + v^2\partial_2 v^1)\,\mathrm{d}\,x^1 + (v^1\partial_1 v^2 + v^2\partial_2 v^2)\,\mathrm{d}\,x^2,\\ i_v\,\mathrm{d}\,v^\flat &= (-v^2\partial_1 v^2 + v^2\partial_2 v^1)\,\mathrm{d}\,x^1 + (v^1\partial_1 v^2 - v^1\partial_2 v^1)\,\mathrm{d}\,x^2,\\ \mathcal{L}_v\, v^\flat &= (2v^1\partial_1 v^1 + v^2\partial_2 v^1 + v^2\partial_1 v^2)\,\mathrm{d}\,x^1 + (2v^2\partial_2 v^2 + v^1\partial_1 v^2 + v^1\partial_2 v^1)\,\mathrm{d}\,x^2.\end{aligned}$$

Let us derive steady solutions of the Euler-Arnold equation from the equalities, (16) and (17). We first examine a vector field $v \in \mathfrak{X}^r(M)$ such that $\mathcal{L}_v\, g = 0$, which is called the Killing vector field. Then, the Euler-Arnold equation is deduced from (18) to

$$\mathrm{d}(p - |v|^2/2) = 0.$$

Hence, every Killing vector field $v \in \mathfrak{X}^r(M)$ is the fluid velocity of a steady Euler-Arnold flow (v,p) and the pressure satisfies $p = |v|^2/2$ up to constant.

For instance, when the flow field is the Euclidean plane, the Killing vector field has a constant vorticity. Our next concern is a vector field $v \in \mathfrak{X}^r(M)$ such that $\mathrm{d}\, v^\flat = 0$. Owing to $\omega = *\,\mathrm{d}\, v^\flat$, the vector field is irrotational. In the same manner as the case of the plane, we conclude every irrotational vector field $v \in \mathfrak{X}^r(M)$ is the fluid velocity of a steady Euler-Arnold flow (v, p) and the pressure p satisfies $p = -|v|^2/2$ up to constant owing to (19).

4 Point vortex dynamics

The point vortex dynamics is another dynamical model of incompressible and inviscid fluids, which is formally derived from the Euler flow. As we see in Section 3, if a time-dependent function ω_t is a solution of the vorticity equation (12), then an Euler flow (v_t, p_t) is determined by ω_t as

$$v_t = -\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle, \quad p_t = \langle G_H, \mathrm{div}(\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle \cdot \nabla)\, \mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle\rangle. \tag{24}$$

These formula (12) and (24) are still valid in the sense of distributions when we give a time-dependent vorticity distribution $\Omega_t = \sum_{n=1}^{N} \Gamma_n \delta_{q_n(t)}$ in which $(\Gamma_n)_{n=1}^{N} \in (\mathbb{R} \setminus \{0\})^N$, $q_n(t)$ is a given 1-parameter family, and $\delta_{q_n(t)}$ denotes the delta function centered at $q_n(t)$ for $n = 1, \ldots, N$. Then, a time-dependent vector field $V_t \in \mathfrak{X}(U \setminus \{q_n(t)\}_{n=1}^{N})$ and a time-dependent function $P_t \in C(U \setminus \{q_n(t)\}_{n=1}^{N})$ are given by

$$V_t(q) = -\mathcal{J}\mathrm{grad} \sum_{n=1}^{N} \Gamma_n G_H(q, q_n(t)), \quad P_t = \langle G_H, \mathrm{div}(V_t \cdot \nabla) V_t\rangle. \tag{25}$$

Since (V_t, P_t) is no longer an Euler flow in a regular sense, we can not define the dynamics of $q_n(t)$ from the Euler equation. Instead, to determine the evolution of $q_n(t)$ by V_t, Helmholtz considered the following regularized equation for $q_n(t)$ [21].

$$\begin{aligned} \dot{q}_n &= \lim_{q \to q_n} \left[V_t(q) + \mathcal{J}\mathrm{grad}\langle G_H, \Gamma_n \delta_{q_n(t)}\rangle(q) \right] \qquad (26) \\ &= -\mathcal{J}\mathrm{grad} \sum_{\substack{m=1 \\ m \neq n}}^{N} \Gamma_m G_H(q_n, q_m) \equiv v_n(q_n). \end{aligned}$$

It is called the *point vortex equation*, and the solution of (26) is called the *point vortex dynamics*. Since the evolution equation for the vorticity Ω_t are determined in a different way from the vorticity equation, the problem arises as to whether the point vortex dynamics is an Euler flow in a mathematically appropriate sense. Although the problem is not well-established mathematically, the equation (26) plays a significant role in the application of fluid dynamics as mathematical models of many fluid phenomena with localized vortex structures, in which the fluid velocity and the pressure are obtained by (25). On the other hand, until this problem is solved, it is still unclear whether qualitative understanding of the fluid phenomena based on the point vortex dynamics is valid for Euler flows. With this reason, it is of a significance to justify the point vortex dynamics as an Euler flow in an appropriate mathematical sense.

The point vortex dynamics is sometimes considered in the presence of the velocity $X_t \in \mathfrak{X}^r(\mathbb{R}^2)$ of an Euler flow, in which the evolution of $q_n(t)$ is governed by the following equation.

$$\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega v_n(q_n(t)), \quad n = 1, \dots N, \tag{27}$$

for a given $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. Some experimental studies confirm the importance of background fields in two-dimensional turbulence [28]. For the solution $q_n(t)$ of (27), a time-dependent vector field V_t and a time-dependent function P_t are thus defined by

$$V_t(q) = X_t(q) - \mathcal{J}\mathrm{grad} \sum_{n=1}^{N} \Gamma_n G_H(q, q_n(t)), \quad P_t = \langle G_H, \mathrm{div}(V_t \cdot \nabla)V_t \rangle. \tag{28}$$

As is the case without background field, we are concerned with whether (V_t, P_t) which is defined by (28) becomes an Euler flow. However, for general background fields, even if the background field comes from a classical solution of the Euler equation, any space of solutions containing (V_t, P_t) has not yet been established.

When the flow field is a curved surface, in the same manner of the case of the Euclidean plane, the point vortex dynamics is derived from the fact that, if ω_t is a solution of the vorticity equation (21), we obtain an Euler-Arnold flow (v_t, p_t) which is defined by

$$v_t = -\mathcal{J}\mathrm{grad}\langle G_H, \omega_t \rangle, \quad p_t = \langle G_H, \mathrm{div}\, \nabla_{\mathcal{J}\mathrm{grad}\langle G_H, \omega_t \rangle} \mathcal{J}\mathrm{grad}\langle G_H, \omega_t \rangle \rangle.$$

Taking a linear combination of delta functions $\Omega_t = \sum_{n=1}^{N} \Gamma_n \delta_{q_n(t)}$ for $(\Gamma_n)_{n=1}^{N}$ and $q_n(t) \in M$ as the time-dependent distribution, we obtain a time-dependent vector field $V_t \in \mathfrak{X}(M \setminus \{q_n(t)\}_{n=1}^N)$ and a time-dependent function $P_t \in C(M \setminus \{q_n(t)\}_{n=1}^N)$.

$$V_t(q) = -\mathcal{J}\mathrm{grad}\sum_{n=1}^{N}\Gamma_n G_H(q, q_n(t)), \quad P_t = \langle G_H, \mathrm{div}\,\nabla_{V_t}V_t\rangle.$$

As an analogy of the Helmholtz principle, let us consider the following regularized equation for $q_n(t)$.

$$\begin{aligned} \dot{q}_n &= \lim_{q\to q_n}\left[V_t(q) + \mathcal{J}\mathrm{grad}\,\Gamma_n \log d(q, q_n(t))\right] \\ &= -\mathcal{J}\mathrm{grad}_{q_n}\sum_{\substack{m=1\\ m\neq n}}^{N}\Gamma_m G_H(q_n, q_m) + \Gamma_n R(q_n) \equiv v_n(q_n), \end{aligned} \tag{29}$$

where $d \in C^\infty(M \times M)$ is the geodesic distance on (M, g). The equation (29) is also called the point vortex equation, and the solution is called the point vortex dynamics on the surface. The function R is referred to as the Robin function [9, 11], which is defined by

$$R(q) = \lim_{q_0\to q} G_H(q, q_0) + \frac{1}{2\pi}\log d(q, q_0).$$

Point vortex dynamics on surfaces is originally motivated by the applications to geophysical fluids [4]. Later, the point vortex dynamics is investigated in the many surfaces: a sphere [15], a hyperbolic disc [14], multiply connected domains [22], a cylinder [19], a flat torus [27], the Bolza surface [10], surfaces of revolution diffeomorphic to the plane [13], the sphere [7] and the torus [23]. There are some derivations of the point vortex dynamics on surfaces. The derivation stated above is based on the analogy of Helmholtz principle. Another derivation of point vortex dynamics from the vorticity equation (20) and the generalized Newton law has been recently developed in [11]. Note that point vortex dynamics on surfaces is formally derived independently from the Euler-Arnold flows. As is the case of the Euclidean plane, we can ask whether (V_t, P_t) is an Euler-Arnold flow. However, there is no result on the problem for curved surfaces as well as the plane. Besides the problem as we see in the case of the Euclidean plane, due to the generalization of point vortex dynamics to the case of curved surfaces, the other problems arise in justifying the point vortex dynamics as an Euler-Arnold flow. First, in the analogy of the Helmholtz principle, the fluid velocity is regularized by the geodesic distance. There is no particular reason why the geodesic distance should be taken to regularize the fluid velocity. Another choice of the regularizing function has been recently discussed in [12]. Second, there exists an Euler-Arnold flow such that it is not Hamiltonian owing to the Hodge decomposition, whereas the point vortex dynamics is formulated as a Hamiltonian dynamics. Hence, there is a gap between non-Hamiltonian Euler-Arnold flows and the point vortex dynamics.

For the case of a curved surface, the point vortex dynamics in a background field is formulated as well as for the cased of the plane by taking an Euler-Arnold flow as the background field instead of an Euler flow. Namely, the point vortex dynamics on the surface with a background field $X_t \in \mathfrak{X}^r(M)$ is defined as a solution of

$$\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega v_n(q_n(t)), \quad n = 1, \dots N,$$

where $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ is a given parameter and the vector field $X_t \in \mathfrak{X}^r(M)$ is the fluid velocity of a given Euler-Arnold flow. For example, in the application to geophysical flows, the point vortex dynamics in a background field on the unit sphere is adopted as a mathematical model of incompressible and inviscid fluid flows on the unit sphere with Coriolis force [20]. On the other hand, as is the case with no background field, there is no theoretical result on the problem whether (V_t, P_t) is an Euler-Arnold flow.

5 de Rham current

5.1 Basic concepts and properties

We review some basic notions in the theory of de Rham currents, which are alternatives of the Schwartz distributions in the Euclidean space and which are required to give a weak formulation of the Euler-Arnold equation on curved surfaces. Roughly speaking, currents are differential forms with distribution coefficients in local charts. A good reference of the theory of de Rham currents is given in [6].

A p-current is defined as a continuous linear functional over $\mathbb{R}$ on $\mathcal{D}^{2-p}(M)$, which is the space of all $(2-p)$-forms on M with compact support. Let $T[\phi]$ denote the coupling of the p-current T and $\phi \in \mathcal{D}^{2-p}(M)$. The elements of $\mathcal{D}^p(M)$ are often called test forms. The space of all p-currents on M is denoted by $\mathcal{D}'_p(M)$. The calculus of differential forms such as the differential operator d, the Hodge-$*$ operator, the codifferential operator $\delta = *\,\mathrm{d}\,*$, Hodge Laplacian $\triangle = \mathrm{d}\,\delta + \delta\,\mathrm{d}$ can be extended to currents via test forms. In fact, for a given $T \in \mathcal{D}'_p(M)$, d and $*$ are defined by $\mathrm{d}\,T[\phi] = (-1)^{p+1}T[\mathrm{d}\,\phi]$ for each $\phi \in \mathcal{D}^{1-p}(M)$ and $*T[\varphi] = (-1)^{p(2-p)}T[*\varphi]$ for each $\varphi \in \mathcal{D}^p(M)$. Thus the notions for differential forms such as (co)closedness, (co)exactness and harmonicity can also be defined with respect to currents.

For instance, the space $\mathcal{D}'_0(M)$ corresponds to the space of distributions on M. For any p-form $\alpha \in \Omega^p(M)$, it is naturally identified with a p-current when we define a functional $I(\alpha)$ on $\mathcal{D}^{2-p}(M)$ as $I(\alpha)[\phi] = \int_M \alpha \wedge \phi$. For $p \in M$, we define *the delta current*, say $\delta_p \in \mathcal{D}'_0(M)$, by $\delta_p[\phi] = *\phi(p)$. This is the counterpart of the delta function in the theory of distributions.

We now introduce $\chi_p : v \in \mathfrak{X}^r(U) \to \chi_p v \in \mathcal{D}'_1(M)$ for a given open subset $U \subset M$ and $p \in U$ by $\chi_p v[\phi] = \phi_p(v_p)$ for each $\phi \in \mathcal{D}^1(M)$. It is characterized as a limit of the mean value for the vector field around a geodesic circle as follows.

Proposition 5.1. *Fix an open subset $U \subset M$, $p \in U$ and $v \in \mathfrak{X}^r(U)$. Then for any $\phi \in \mathcal{D}^1(M)$, we have*

$$\chi_p v[\phi] = \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(p)} \frac{1}{\pi} g(v, \mathcal{J}\,\mathrm{grad}\log d(p,q))\phi_q.$$

Proof. Let us take a complex coordinate (z) centered at p with $z(p) = 0$. Since g is presented as $g = \lambda^2|\,\mathrm{d}\,z|^2$ for some $\lambda \in C^\infty(U)$, the distance $d(p,q)$ is written as $d(p,q) = \lambda(0)|z| + O(\varepsilon)$. Writing $v = v^z\partial_z + v^{\bar z}\partial_{\bar z}$, we deduce from $*\,\mathrm{d}\,z = -\mathrm{i}\,\mathrm{d}\,z$ and $*\,\mathrm{d}\,\bar z = \mathrm{i}\,\mathrm{d}\,\bar z$ that

$$\begin{aligned} g(v, \mathcal{J}\,\mathrm{grad}\log d(p,q)) &= *\,\mathrm{d}\log d(p,q)[v] = *\,\mathrm{d}\log|z|[v] + O(\varepsilon) \\ &= \frac{\mathrm{i}}{2}\left(-\frac{\mathrm{d}\,z}{z} + \frac{\mathrm{d}\,\bar z}{\bar z}\right)[v^z\partial_z + v^{\bar z}\partial_{\bar z}] + O(\varepsilon) \\ &= \frac{\mathrm{i}}{2}\left(-\frac{v^z}{z} + \frac{v^{\bar z}}{\bar z}\right) + O(\varepsilon). \end{aligned}$$

By the residue theorem, we conclude that

$$\int_{\partial B_\varepsilon(p)} \frac{1}{\pi} g(v, \mathcal{J} \operatorname{grad} \log d(p,q))\phi = \int_{\partial B_\varepsilon(p)} \frac{\mathrm{i}}{2\pi} \left(-\frac{v^z}{z} + \frac{v^{\bar{z}}}{\bar{z}} \right) (\phi_z \,\mathrm{d}\, z + \phi_{\bar{z}} \,\mathrm{d}\, \bar{z}) + O(\varepsilon)$$

$$= v^z(0)\phi_z(0) + v^{\bar{z}}(0)\phi_{\bar{z}}(0) + O(\varepsilon)$$
$$\to \chi_p v[\phi].$$

□

Let $\Omega^p_{[r]\mathrm{loc}}(M)$ denote the space of all p-forms on nonempty open subset $U \subset M$, which are called local p-form. In the same manner as $L^p_{\mathrm{loc}}(M)$, the topology of $\Omega^p_{[r]\mathrm{loc}}(M)$ can be defined. For each local p-form $\alpha \in \Omega^p_{[r]\mathrm{loc}}(M)$, there exists a maximal open subset U such that $\alpha \in \Omega^p_{[r]}(U)$. Then, the closed subset $\mathsf{S}^r(\alpha) = M \setminus U$ is called the singular support of α. A p-current $T \in \mathcal{D}'_p(M)$ is said to be C^r, if there exists a local p-form $\alpha_T \in \Omega^p_{[r]\mathrm{loc}}(M)$ such that for every $\phi \in \mathcal{D}^{2-p}(M \setminus \mathsf{S}^r(\alpha_T))$, $T[\phi] = I(\alpha_T)[\phi]$. Let $\mathcal{D}'^r_p(M)$ denote the space of all C^r p-currents. For each $T \in \mathcal{D}'^r_p(M)$, the subset $\mathsf{S}^r(T) = \mathsf{S}^r(\alpha_T)$ and the local p-form α_T is called the singular support of T and the density of T. Owing to the fundamental lemma of calculus of variation, the density is uniquely determined. Thus, the map $K : T \in \mathcal{D}'^r_p(M) \to K(T) = \alpha_T \in \Omega^p_{[r]\mathrm{loc}}(M)$ is well-defined and called the derivative. For example, for each $p \in M$, the delta current δ_p is C^∞ since $\delta_p = I(0)$ in $M \setminus \{p\}$. We thus obtain $\mathsf{S}^\infty(\delta_p) = \{p\}$ and $K(\delta_p) = 0$. In this paper, all currents are C^r and the singular support consists of a finite set of points.

In Definition 2.1, we state the definition of the hydrodynamic Green function in the sense of distribution. Let us restate the definition in the sense of currents. For each $x_0 \in M$, we define $G_{x_0} \in \mathcal{D}'^\infty_0(M)$ as $K(G_{x_0})(x) = G_H(x, x_0)$. Then, the definition of the hydrodynamic Green function $G_H \in C^\infty(M \times M \setminus \Delta)$ is rewritten in terms of the current G_{x_0} as follows. For each $(x, x_0) \in M \times M \setminus \Delta$ and each $\phi \in \mathcal{D}^2(M)$,

$$\begin{aligned} -\triangle G_{x_0}[\phi] &= \begin{cases} *\phi(x_0) - \dfrac{1}{\mathrm{Area}(M)} \displaystyle\int_M \phi, & \text{if } M \text{ is closed,} \\ *\phi(x_0), & \text{otherwise,} \end{cases} \\ G_{x_0} &= G_x, \\ \mathrm{d}\, K(G_{x_0}) &= 0 \quad \text{on } \partial M. \end{aligned}$$

Let us remember that $\Omega^1_{[r]}(M)$ is identified with $\mathfrak{X}^r(M)$ by the musical isomorphism $\sharp : \alpha \in \Omega^1_{[r]}(M) \to \alpha_\sharp \in \mathfrak{X}^r(M)$, satisfying $\alpha[X] = g(\alpha_\sharp, X)$ for all $X \in \mathfrak{X}^r(M)$. Every C^r 1-current T generates a vector field allowing singularities in $\mathsf{S}(T)$. For each $T \in \mathcal{D}'^r_1(M)$, we define $T_\sharp \in \mathfrak{X}^r(M \setminus \mathsf{S}^r(T))$ by $T_\sharp = K(T)_\sharp$. As an example, for each $\psi \in \mathcal{D}'^r_0(M)$, we define $\mathcal{J}\operatorname{grad}\psi \in \mathfrak{X}^{r-1}(M \setminus \mathsf{S}^{r-1}(\mathrm{d}\,\psi))$ by $\mathcal{J}\operatorname{grad}\psi = K(*\,\mathrm{d}\,\psi)_\sharp$. The vector field $\mathcal{J}\operatorname{grad}\psi$ stands for the Hamiltonian vector field induced from the Hamiltonian ψ with singularities in $\mathsf{S}(\psi)$. We will use this notion when we take a vector field generated by point vortices. In what follows, for a given $T \in \mathcal{D}'^r_p(M)$, we abbreviate $\mathsf{S}^r(T)$ to $\mathsf{S}(T)$. Moreover, we denote $K(T)$ briefly by T as long as no confusion arises. In particular, for a given $T \in \mathcal{D}'^r_1(M)$, when we write $|T|^2$, it stands for not the multiplication of currents but the multiplication of the local 1-form $|\alpha_T|^2$. Similarly, $(*\,\mathrm{d}\,T) * T$ means the multiplication of the local 0-form $*\,\mathrm{d}\,\alpha_T$ and the local 1-form $*\alpha_T$. This treatment is sensitive when we formulate the nonlinear term in the sense of currents.

We define the principal value p.v. : $T \in \Omega^p_{[r]\mathrm{loc}}(M) \to \mathrm{p.v.}\,T \in \mathcal{D}'_p(M)$ by

$$\mathrm{p.v.}\,T[\phi] = \lim_{\varepsilon\to 0} \int_{M\setminus B_\varepsilon(\mathsf{S}(T))} T \wedge \phi$$

for each $\phi \in \mathcal{D}^{2-p}(M)$ if the limit exists. The domain of p.v., say Dom(p.v.), is defined as the space of p-currents in which the limit exists for every $\phi \in \mathcal{D}^{2-p}(M)$.

We apply the calculus of currents to the derivation of the Biot-Savart kernel in the Euclidean plane $(\mathbb{C}, |\,\mathrm{d}\,z|^2)$. First, let us consider a 0-current $\psi = \langle G_H, \delta_0\rangle \in \mathcal{D}'_0(\mathbb{C})$. Since $\psi = I(G_H(z,0))$ on $\mathbb{C}\setminus\{0\}$, we deduce that ψ is C^∞ and that $\mathsf{S}(\psi) = \{0\}$. We next compute $u = -*\,\mathrm{d}\,\psi \in \mathcal{D}'^\infty_1(\mathbb{C})$. For each $\phi \in \mathcal{D}^1(\mathbb{C})$,

$$\begin{aligned}
u[\phi] = \mathrm{d}\,\psi[*\phi] = -\psi[\mathrm{d}*\phi] &= -\int_{\mathbb{C}\setminus\{0\}} G_H(z,0)\,\mathrm{d}*\phi \\
&= \int_{\mathbb{C}\setminus\{0\}} -\mathrm{d}(G_H(z,0)*\phi) + \mathrm{d}\,G_H(z,0)\wedge *\phi \\
&= \int_{\mathbb{C}\setminus\{0\}} -*\,\mathrm{d}\,G_H(z,0)\wedge\phi = I(-*\,\mathrm{d}\,G_H(z,0))[\phi],
\end{aligned}$$

which yields $u = I(-*\,\mathrm{d}\,G_H(z,0))$ in $\mathbb{C}\setminus\{0\}$. It follows from $*\,\mathrm{d}\,z = -\mathrm{i}\,\mathrm{d}\,z$ and $*\,\mathrm{d}\,\bar z = \mathrm{i}\,\mathrm{d}\,\bar z$ that $-*\,\mathrm{d}\,G_H(z,0) = \mathrm{i}(\partial_z G_H(z,0)\,\mathrm{d}\,z - \partial_{\bar z} G_H(z,0)\,\mathrm{d}\,\bar z)$.

Therefore we obtain the Biot-Savart kernel as follows.

$$\begin{aligned}
-\mathcal{J}\operatorname{grad}\psi &= (-*\mathrm{d}\,G_H(z,0))_\sharp = \mathrm{i}(2\partial_z G_H(z,0)\partial_{\bar{z}} - 2\partial_{\bar{z}} G_H(z,0)\partial_z) \\
&= \mathrm{i}\left(-\frac{1}{2\pi z}\partial_{\bar{z}} + \frac{1}{2\pi\bar{z}}\partial_z\right) = \frac{\mathrm{i}}{2\pi|z|^2}(z\partial_z - \bar{z}\partial_{\bar{z}}) \\
&= \frac{1}{2\pi(x^2+y^2)}(-y\partial_x + x\partial_y).
\end{aligned}$$

Remark 1. We compute the vorticity of $-\mathcal{J}\operatorname{grad}\psi$, in which p.v. plays a key role in the computation by the density of ψ. Indeed, since $u = I(-*\mathrm{d}\,G_H(z,0))$ in $\mathbb{C}\setminus\{0\}$, $*\mathrm{d}\,u = I(-*\mathrm{d}*\mathrm{d}\,G_H(z,0)) = I(-\triangle G_H(z,0)) = I(0)$ in $\mathbb{C}\setminus\{0\}$. On the other hand, we have $*\mathrm{d}\,u = -*\mathrm{d}*\mathrm{d}\,\psi = -\triangle\psi = \omega = \delta_0$. This indicates that we need to take the singular behavior into account in order to calculate the vorticity by the density. For each $\phi \in \mathcal{D}^2(\mathbb{C})$, owing to $-*\mathrm{d}\,G_H(z,0) \in \Omega^1_{[r]\mathrm{loc}}(\mathbb{C})$, we see that

$$\begin{aligned}
\mathrm{d}\,\mathrm{p.v.}\,u[\phi] &= \mathrm{p.v.}\,u[\mathrm{d}\phi] = \lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus B_\varepsilon(0)} -*\mathrm{d}\,G_H(z,0)\wedge\mathrm{d}*\phi \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus B_\varepsilon(0)} \mathrm{d}(*\phi*\mathrm{d}\,G_H(z,0)) - *\phi\,\mathrm{d}*\mathrm{d}\,G_H(z,0) \\
&= \lim_{\varepsilon\to 0} -\int_{\partial B_\varepsilon(0)} *\phi*\mathrm{d}\,G_H(z,0) \\
&\quad - \lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus B_\varepsilon(0)} *\phi\triangle G_H(z,0)\,\mathrm{d}\,z\wedge\mathrm{d}\,\bar{z} \\
&= \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(0)} *\phi\mathrm{i}(\partial_z G_H(z,0)\,\mathrm{d}\,z - \partial_{\bar{z}} G_H(z,0)\,\mathrm{d}\,\bar{z}) \\
&= *\phi(0),
\end{aligned}$$

which yields $*\mathrm{d}\,\mathrm{p.v.}\,u = \delta_0$ in $\mathcal{D}'_0(\mathbb{C})$. Therefore, in order to recover the vorticity from the density, we need to consider $*\mathrm{d}\,\mathrm{p.v.}\,u$ instead of $*\mathrm{d}\,u$.

Let us introduce the operator $\mathfrak{L}$ by $\mathfrak{L} = \mathrm{d}\,\mathrm{p.v.} : T \in \Omega^p_{[r]\mathrm{loc}}(M) \to \mathfrak{L}T \in \mathcal{D}'_{p+1}(M)$, which is called the *localizing operator*. For each $\phi \in \mathcal{D}^{p+1}(M)$, we have

$$\mathfrak{L}T[\phi] = \mathrm{d}\,\mathrm{p.v.}\,T[\phi] = (-1)^{p+1}\,\mathrm{p.v.}\,T[\mathrm{d}\,\phi].$$

The domain of $\mathfrak{L}$, $\mathrm{Dom}(\mathfrak{L})$, is the space of p-currents T in which $\mathrm{p.v.}\,T[\mathrm{d}\,\phi]$ is well-defined for every $\phi \in \mathcal{D}^{1-p}(M)$. If $T \in \mathrm{Dom}(\mathfrak{L})$ satisfies $\mathrm{d}\,T = 0$ in

$M \setminus \mathsf{S}(T)$, then we obtain

$$\mathfrak{L}\,T[\phi] = (-1)^{p+1}\,\mathrm{p.v.}\,T[\mathrm{d}\,\phi] = (-1)^{p+1} \lim_{\varepsilon\to 0} \int_{M\setminus B_\varepsilon(\mathsf{S}(T))} T \wedge \mathrm{d}\,\phi$$
$$= -\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(\mathsf{S}(T))} T \wedge \phi,$$

since $(-1)^{p+1} T \wedge \mathrm{d}\,\phi = -\,\mathrm{d}(T \wedge \phi) + \mathrm{d}\,T \wedge \phi$. Hence, $\mathfrak{L}\,T$ is determined by the asymptotic behavior of T near the singular support $\mathsf{S}(T)$. The name of *localizing* operator is named after this property.

5.2 Weak formulation of vector fields

Based on the fact that $\mathfrak{X}^r(M)$ is isomorphic to $\Omega^1_{[r]}(M)$ through the musical isomorphism $\flat : v \in \mathfrak{X}^r(M) \to v^\flat = g(v, \cdot\,) \in \Omega^1_{[r]}(M)$, we can extend the notions associated with vector fields such as the divergence, the vorticity and the slip-boundary condition in the sense of currents. We will use these notions to formulate the Euler-Arnold equations in the sense of currents.

As we see in Section 2, the divergence and the vorticity of $v \in \mathfrak{X}^r(M)$ is defined by $\delta v^\flat \in \Omega^0_{[r-1]}(M)$ and $*\,\mathrm{d}\,v^\flat \in \Omega^0_{[r-1]}(M)$. Hence, it is reasonable to define the divergence and the vorticity of a 1-current $\alpha \in \mathcal{D}'_1(M)$ by $\delta\alpha \in \mathcal{D}'_0(M)$ and $*\,\mathrm{d}\,\alpha \in \mathcal{D}'_0(M)$. Owing to $\dim M = 2$, the slip boundary condition $v|_{\partial M} \in \mathfrak{X}(\partial M)$ is written as the condition that $*v^\flat = 0$ on ∂M. By analogy, we define the slip boundary condition for a C^r 1-current $\alpha \in \mathcal{D}'^r_1(M)$ with $\partial M \cap \mathsf{S}(\alpha) = \emptyset$ by $*K(\alpha) = 0$ on ∂M. Next, we reformulate the notion of the Biot-Savart law on surfaces in the sense of currents. As we see in Section 2, for a given incompressible vector field $X \in \mathfrak{X}^r(M)$ and a given $\psi \in C^{r+1}(M)$, the incompressible vector field $Y = X - \mathcal{J}\mathrm{grad}\,\psi \in \mathfrak{X}^r(M)$ can be recovered from the vorticity $Y = X - \mathcal{J}\mathrm{grad}\langle G_H, \omega\rangle$, where $\omega = *\,\mathrm{d}(Y - X)^\flat \in \Omega^0_{[r-1]}(M)$. Let us extend this formulation with respect to vector fields to currents by replacing $Y^\flat \in \Omega^1_{[r]}(M)$ with $\alpha \in \mathcal{D}'_1(M)$. That is to say, for a given incompressible vector field $X \in \mathfrak{X}^r(M)$, we consider a 1-current $\alpha \in \mathcal{D}'_1(M)$ such that $\alpha - X^\flat$ is coexact, or equivalently there exists $\psi \in \mathcal{D}'_0(M)$ such that $\alpha = X^\flat - *\,\mathrm{d}\,\psi$. Defining the relative vorticity $\omega \in \mathcal{D}'_0(M)$ to $X \in \mathfrak{X}^r(M)$ by $\omega = *\,\mathrm{d}(\alpha - X^\flat) \in \mathcal{D}'_0(M)$, we obtain

$$-\triangle\psi = -\delta\,\mathrm{d}\,\psi = *\,\mathrm{d}(\alpha - X^\flat) = \omega, \tag{30}$$

which yields $\psi = \psi^0 + \langle G_H, \omega\rangle$ up to harmonic function ψ^0 since the kernel of the Laplacian $-\triangle$ consists of harmonic functions. In the present paper, we consider a special form of a singular vorticity as follows.

Definition 5.2. *Fix $N \in \mathbb{Z}$, $(\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N$ and $(q_n)_{n=1}^N \in Q_N = (\mathrm{Int}M)^N \setminus \{(q_n)_{n=1}^N \in (\mathrm{Int}M)^N \,|\, \exists i, j,\, q_i = q_j\}$. A 0-current $\omega \in \mathcal{D}_0'(M)$ is called a singular vorticity of point vortices placed on $\{q_n\}_{n=1}^N \subset M$, if for each $\phi \in \mathcal{D}^2(M)$,*

$$\omega[\phi] = \sum_{n=1}^N \Gamma_n * \phi(q_n) + c\int_M \phi,$$

where

$$c = \begin{cases} -\dfrac{1}{\mathrm{Area}(M)}\displaystyle\sum_{n=1}^N \Gamma_n, & \textit{if } M \textit{ is closed,} \\ 0, & \textit{otherwise.} \end{cases}$$

Then every solution $\psi \in \mathcal{D}_0'^{\infty}(M)$ of the Poisson problem $-\triangle\psi = \omega$ is presented by

$$\psi = \psi^0 + \sum_{n=1}^N \Gamma_n G_{q_n}$$

up to a harmonic function ψ^0, since for each $\phi \in \mathcal{D}^2(M)$,

$$\begin{aligned} -\triangle\psi[\phi] &= -\triangle\psi_0[\phi] + \sum_{n=1}^N \Gamma_n(-\triangle G_{q_n}) \\ &= \sum_{n=1}^N \Gamma_n \left(*\phi(q_n) + \frac{1}{\mathrm{Area}(M)}\int_M \phi \right) \\ &= \omega. \end{aligned}$$

Identifying the delta function with the Dirac measure, we see that

$$K(\omega) = c, \quad K(\psi)(p) = \psi^0(p) + \sum_{n=1}^N \Gamma_n G_H(p, q_n),$$

which yields $\mathsf{S}(\omega) = \mathsf{S}(\psi) = \{q_n\}_{n=1}^N$.

Let us now fix $N \in \mathbb{Z}$, $(\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N$ and $(q_n)_{n=1}^N \in Q_N$. For each $n \in \{1, \ldots, N\}$, a vector field $v_n \in \mathfrak{X}^2(B_r(q_n))$ with sufficiently small $r \in \mathbb{R}_{>0}$ is defined by

$$v_n(q) = -\mathcal{J}\mathrm{grad}_q \left[\sum_{m=1}^N \Gamma_m G_H(q_m, q) + \frac{\Gamma_n}{2\pi} \log d(q_n, q) \right].$$

It follows from the regularity theorem for a linear elliptic operator [3] that v_n is C^1. Note that, in particular,

$$v_n(q_n) = -\mathcal{J}\mathrm{grad}_{q_n} \left[\sum_{m \neq n}^N \Gamma_m G_H(q_m, q_n) + \Gamma_n R(q_n) \right].$$

Since $\sum_{m \neq n}^N \Gamma_m G_H(q_m, q_n) + \Gamma_n R(q_n)$ is smooth on Q_N, $v : (q_n)_{n=1}^N \in Q_N \to (v_n(q_n))_{n=1}^N \in TQ_N$ is a smooth vector field on Q_N. The point vortex dynamics is defined by a solution of the following ordinary differential equation,

$$\dot{q}_n(t) = v_n(q_n(t)), \quad n = 1, \ldots, N.$$

For a given Euler-Arnold flow $(X_t, P_t) \in \mathfrak{X}^r(M) \times C^r(M)$ and $(\beta_X, \beta_\omega) \in \mathbb{R}^2$, the point vortex dynamics in the background field X_t is defined as a solution of the following ordinary differential equation,

$$\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega v_n(q_n(t)), \quad n = 1, \ldots, N, \tag{31}$$

called the point vortex equation.

6 Weak Euler-Arnold flows

As we see in Section 3, the Euler-Arnold equation on surfaces is presented as

$$\partial_t v^\flat + (*\,\mathrm{d}\, v^\flat) * v^\flat + \mathrm{d}\, |v^\flat|^2/2 = -\,\mathrm{d}\, p. \tag{32}$$

Before we replace differential forms in (32) with currents, we need to deal with the nonlinear term carefully in order to avoid multiplication of currents. Based on the fact that multiplication of local p-forms is still valid and a local p-form is converted to a current by taking the principle value, the Euler-Arnold equation is reformulated for $\alpha_t \in \mathcal{D}_1'^r(M)$ and $p_t \in \Omega^0_{[r]\mathrm{loc}}(M)$ as follows.

$$\partial_t\, \mathrm{p.v.}\, \alpha_t + \mathrm{p.v.}\{(*\,\mathrm{d}\, \alpha_t) * \alpha_t + \mathrm{d}\, |\alpha_t|^2/2\} = -\,\mathrm{p.v.}\,\mathrm{d}\, p_t \quad \text{in } \mathcal{D}_1'(M), \tag{33}$$

if each of terms is contained in Dom(p.v.), where we abbreviate $K(\alpha_t)$ to α_t. When we focus on evolution of vorticity, the vorticity equation is useful rather than the Euler-Arnold equation. Let us remember that the vorticity equation is obtained by applying the differential operator d to the Euler-Arnold equation. Hence, applying the differential operator d to (33), we obtain the vorticity equation corresponding to (33) if each density in (33) is contained in $\mathrm{Dom}(\mathrm{d}\,\mathrm{p.v.}) = \mathrm{Dom}(\mathfrak{L})$.

Definition 6.1. *A pair of time-dependent currents $(\alpha_t, p_t) \in \mathcal{D}_1'^r(M) \times \Omega^0_{[r]\mathrm{loc}}(M)$ is called a weak Euler-Arnold flow, if the following conditions are satisfied:*

1. $\alpha_t \in \mathrm{Dom}(\mathfrak{L})$ *and* $(*\,\mathrm{d}\,\alpha_t) * \alpha_t + \mathrm{d}\,|\alpha_t|^2/2 \in \mathrm{Dom}(\mathfrak{L})$;
2. $\mathrm{d}\,p_t \in \mathrm{Dom}(\mathfrak{L})$;
3. $\partial_t\,\mathfrak{L}\,\alpha_t + \mathfrak{L}\,\{(*\,\mathrm{d}\,\alpha_t) * \alpha_t + \mathrm{d}\,|\alpha_t|^2/2\} = -\,\mathfrak{L}\,\mathrm{d}\,p_t$ *in* $\mathcal{D}_2'(M)$;
4. $\delta\alpha_t = 0$ *in* $\mathcal{D}_1'(M)$;
5. $\partial M \cap \mathsf{S}(\alpha_t) = \emptyset$ *and* $*\alpha_t = 0$ *on* ∂M.

In particular, we call the third condition the weak Euler-Arnold equation and α_t the velocity current.

We decompose a weak Euler-Arnold flow into a regular part and a singular part, thereby discussing the decomposition of each term of the weak Euler-Arnold equation. As we see in Section 5.2, for a given $X \in \mathfrak{X}^r(M)$, a 1-current $\alpha \in \mathcal{D}_1'(M)$ is recovered from the relative vorticity $\omega = *\,\mathrm{d}(\alpha - X^\flat)$, if $\alpha - X^\flat$ is coexact. Based on this fact, we assume that, for a given weak Euler-Arnold flow $(\alpha_t, p_t) \in \mathcal{D}_1'^r(M) \times \Omega^0_{[r]\mathrm{loc}}(M)$, there exists a time-dependent vector field $X_t \in \mathfrak{X}^r(M)$ such that $u_t = \alpha_t - X_t^\flat$ is coexact for each time t, that is, there exists $\psi_t \in \mathcal{D}_0'(M)$ such that $\alpha_t - X_t^\flat = u_t = -*\,\mathrm{d}\,\psi_t$. The vorticity of X is denoted by $\omega_X = *\,\mathrm{d}\,X_t^\flat$. In other words, we consider a weak Euler-Arnold flow such that the singular part of α can be described by a Hamiltonian system on the surface in the sense of currents. Then, ψ and u are determined by the relative vorticity $\omega = *\,\mathrm{d}(\alpha - X^\flat)$ and a harmonic function ψ^0 as follows.

$$\psi = \psi^0 + \langle G_H, \omega\rangle, \quad u = -*\,\mathrm{d}(\psi^0 + \langle G_H, \omega\rangle).$$

For the corresponding densities, it follows from

$$\begin{aligned}
\partial_t \mathfrak{L}\,\alpha - \partial_t \mathfrak{L}\,X^\flat &= \partial_t \mathfrak{L}\,u \\
\mathfrak{L}(*\,\mathrm{d}\,\alpha) * \alpha - \mathfrak{L}\,\omega_X * X^\flat &= \mathfrak{L}(\omega + \omega_X) * (u + X^\flat) - \mathfrak{L}\,\omega_X * X^\flat \\
&= \mathfrak{L}\{(\omega_X + \omega) * u + \omega * X^\flat\}, \\
\mathfrak{L}\,\mathrm{d}\,|\alpha|^2 - \mathfrak{L}\,\mathrm{d}\,|X|^2 &= \mathfrak{L}\,\mathrm{d}\,g(X^\flat + u, X^\flat + u) - \mathfrak{L}\,\mathrm{d}\,g(X^\flat, X^\flat) \\
&= \mathfrak{L}\,\mathrm{d}\,g(2X^\flat + u, u)
\end{aligned}$$

that, if $(X, P) \in \mathfrak{X}^r(M) \times C^r(M)$ is a classical Euler-Arnold flow,

$$\partial_t X^\flat + \omega_X * X^\flat + \mathrm{d}\,|X|^2 = -\,\mathrm{d}\,P,$$

the weak Euler-Arnold equation (Definition 6.1-3) is reduced to

$$\partial_t \mathfrak{L}\,u + \mathfrak{L}\{(\omega_X + \omega) * u + \omega * X^\flat + \mathrm{d}\,g(2X^\flat + u, u)/2\} = -\,\mathfrak{L}\,\mathrm{d}(p - P). \quad (34)$$

We shall compute the advection term and the pressure term when ω is a singular vorticity of point vortices. To get insights on this calculation, let us compute these terms in the case where (M, g) is the Euclidean plane $(\mathbb{C}, |\,\mathrm{d}\,z|^2)$ as an example. Let us take a weak Euler flow $(\alpha_t, p_t) \in \mathcal{D}_1'^r(\mathbb{C}) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{C})$. We also suppose that there exists a classical Euler flow $(X_t, P_t) \in \mathfrak{X}^r(M) \times C^r(M)$ such that $u_t = \alpha_t - X_t^\flat \in \mathcal{D}_1'^r(\mathbb{C})$ is coexact for each time t, and its relative vorticity is given by $\sum_{n=1}^N \Gamma_n \delta_{z_n}$ for a given $N \in \mathbb{Z}_{\geq 1}$, $(\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N$ and $\{z_n\}_{n=1}^N \subset \mathbb{R}^2$. Notice that $\mathsf{S}(u) = \mathsf{S}(\omega) = \{z_n\}_{n=1}^N$. Then, let us compute each terms of the weak Euler-Arnold equation (Definition 6.1-3). It follows from $\omega = I(0)$ that

$$\mathfrak{L}\,\omega * (u + X^\flat) = 0(\mathfrak{L}\,{*u} + \mathfrak{L}\,X^\flat) = 0.$$

We can see $\mathfrak{L}\,\omega_X * u = \mathfrak{L}\,{*u} = 0$. Let us fix $\phi \in \mathcal{D}^0(\mathbb{C})$ and sufficiently small $\varepsilon > 0$. Defining $v_n \in \mathfrak{X}^\infty(\mathbb{C} \setminus \{q_m\}_{m \neq n}^N)$ by $v_n^\flat = u - (\Gamma_n/2\pi) * \mathrm{d}\log|z - z_n|$, by the Stokes theorem, we deduce

$$\begin{aligned}
\int_{\mathbb{C}\setminus B_\varepsilon(\mathsf{S}(u))} *u \wedge \mathrm{d}\,\phi &= \int_{\mathbb{C}\setminus B_\varepsilon(\mathsf{S}(u))} -\,\mathrm{d}(\phi * u) + \phi\,\mathrm{d}\,{*u} = \sum_{n=1}^N \int_{\partial B_\varepsilon(z_n)} \phi * u \\
&= \sum_{n=1}^N \int_{\partial B_\varepsilon(z_n)} \phi \left(v_n^\flat - \frac{\Gamma_n}{2\pi}\,\mathrm{d}\log|z - z_n| \right) = O(\varepsilon),
\end{aligned}$$

which yields $\mathfrak{L} *u = 0$. Since

$$\left| \int_{\mathbb{C}\setminus B_\varepsilon(\mathsf{S}(u))} \omega_X * u \wedge \mathrm{d}\,\phi \right| \leq ||\omega_X|_{\mathrm{supp}\,\mathrm{d}\,\phi}||_\infty \left| \int_{\mathbb{C}\setminus B_\varepsilon(\mathsf{S}(u))} *u \wedge \mathrm{d}\,\phi \right|,$$

it follows from $\mathfrak{L} *u = 0$ that $\mathfrak{L}\,\omega_X *u = 0$. In contrast, Proposition 5.1 yields that

$$\mathfrak{L}\,\mathrm{d}\,g(X^\flat, u) = -\sum_{n=1}^N \Gamma_n\,\mathrm{d}\,\chi_{z_n} X, \quad \mathfrak{L}\,\mathrm{d}\,g(u,u)/2 = -\sum_{n=1}^N \Gamma_n\,\mathrm{d}\,\chi_{z_n} v_n,$$

since

$$\begin{aligned}
\int_{\mathbb{C}\setminus B_\varepsilon(\mathsf{S}(u))} \mathrm{d}\,g(X^\flat, u) \wedge \mathrm{d}\,\phi &= -\sum_{n=1}^N \int_{\partial B_\varepsilon(z_n)} g(X^\flat, u)\,\mathrm{d}\,\phi \\
&= -\sum_{n=1}^N \int_{\partial B_\varepsilon(z_n)} \left(g(X, v_n^\flat) + \frac{\Gamma_n}{2\pi} g(X, \mathcal{J}\mathrm{grad}\log|z|) \right) \mathrm{d}\,\phi \\
&\to -\sum_{n=1}^N \Gamma_n\,\mathrm{d}\,\chi_{z_n} X[\phi] \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{C}\setminus B_\varepsilon(\mathsf{S}(u))} \mathrm{d}\,g(u/2, u) \wedge \mathrm{d}\,\phi &= -\sum_{n=1}^N \int_{\partial B_\varepsilon(z_n)} g(u/2, u)\,\mathrm{d}\,\phi \\
&= -\sum_{n=1}^N \int_{\partial B_\varepsilon(z_n)} \left\{ |v_n|^2/2 + \frac{\Gamma_n}{2\pi} g(v_n, \mathcal{J}\mathrm{grad}\log|z|) \right\} \mathrm{d}\,\phi \\
&\quad -\sum_{n=1}^N \int_{\partial B_\varepsilon(z_n)} \frac{1}{2} \left(\frac{\Gamma_n}{2\pi} |*\mathrm{d}\log|z - z_n|| \right)^2 \mathrm{d}\,\phi \\
&\to -\sum_{n=1}^N \Gamma_n\,\mathrm{d}\,\chi_{z_n} v_n[\phi] \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

This illustrates that the leading terms in the advection term consists of $\mathfrak{L}\,\mathrm{d}\,g(X^\flat, u)$ and $\mathfrak{L}\,\mathrm{d}\,|u|^2/2$. This fact will be generally confirmed to hold true for every surfaces in Lemma 7.3. Based on this, we introduce a model for the pressure that $\mathfrak{L}\,p$ is killed out with a linear combination of singular terms $\mathfrak{L}\,\mathrm{d}\,g(X^\flat, u)$ and $\mathfrak{L}\,\mathrm{d}\,|u|^2/2$, that is

$$p = P + (2\beta_X - 1)g(X^\flat, u) + (2\beta_\omega - 1)|u|^2/2$$

for some $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. This mathematical model can be interpreted that the singular behavior of the pressure is balanced with the interaction energy density $g(X^\flat, u)$ and the kinetic energy density $|u|^2/2$ with a growth rate (β_X, β_ω). Summarizing the above, we propose the following regular-singular decomposition of a weak Euler-Arnold flow.

Definition 6.2. *A weak Euler-Arnold flow* $(\alpha_t, p_t) \in \mathcal{D}_1'^r(M) \times \Omega^0_{[r]\mathrm{loc}}(M)$ *is said to be* C^r *decomposable* $(r \geq 1)$*, if there exists a classical Euler-Arnold flow* $(X_t, P_t) \in \mathfrak{X}^r(M) \times C^r(M)$ *and* $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ *such that the following conditions are satisfied for each time* t*.*

1. $\alpha_t - X_t^\flat$ *is coexact;*

2. $p_t = P_t + (2\beta_X - 1)g(X_t^\flat, u_t) + (2\beta_\omega - 1)|u_t|^2/2.$

Then we call X_t *a background field of* α_t*,* $\alpha_t - X_t$ *a relative velocity current and* (β_X, β_ω) *a growth rate of* p_t*.*

Let us note that the C^r decomposability of the weak Euler-Arnold flow guarantees the existence of the decomposition but there is no mention of the uniqueness of the decomposition. Hence, when we study a C^r decomposable weak Euler-Arnold flow $(\alpha_t, p_t) \in \mathcal{D}_1'^r(M) \times \Omega^0_{[r]\mathrm{loc}}(M)$, we need to fix a classical Euler-Arnold flow $(X_t, P_t) \in \mathfrak{X}^r(M) \times C^r(M)$ and a parameter $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ such that the velocity field X_t is a background field of α_t and the parameter (β_X, β_ω) is a growth rate of p_t. If a weak Euler-Arnold flow (α_t, p_t) is C^r decomposable $(r \geq 1)$, the equation (34) is written without the pressure term as follows.

$$\partial_t\,\mathfrak{L}\,u + \mathfrak{L}\{(\omega_X + \omega) * u + \omega * X^\flat + \mathrm{d}\,g(2\beta_X X^\flat + \beta_\omega u, u)\} = 0. \tag{35}$$

7 Main results

Let us fix $N \in \mathbb{Z}_{\geq 1}$, $(\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N$ and a C^r one-parameter family $\Phi : t \in [0, T] \to \Phi_t \in \mathrm{Diff}^r(Q_N)$ in what follows. Let us denote by $(q_n(t))_{n=1}^N =$

$\Phi_t((q_n(0))_{n=1}^N)$ an orbit of Φ. We first prove that for a given C^r-decomposable weak Euler-Arnold flow, if the relative vorticity is given by a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$, $q_n(t)$ is a solution of the point vortex equation (31), which defines the point vortex dynamics in a background field.

Theorem 7.1. *Let $(\alpha_t, p_t) \in \mathcal{D}_1'^r(M) \times \Omega^0_{[r]\mathrm{loc}}(M)$ be a C^r-decomposable weak Euler-Arnold flow ($r \geq 1$). Fix a background field X_t of α_t, a growth rate (β_X, β_ω) of p_t. Suppose the relative vorticity ω_t is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$. Then, $q_n(t)(n = 1, \ldots, N)$ is a solution of the point vortex equation* (31).

Conversely, we next prove that if $q_n(t)$ is a solution of the point vortex equation (31), there exists a C^r-decomposable weak Euler-Arnold flow such that the relative vorticity is given by a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$.

Theorem 7.2. *Fix a classical Euler-Arnold flow $(X_t, P_t) \in \mathfrak{X}^r(M) \times C^r(M)$ and $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. Let $\omega_t \in \mathcal{D}_0'(M)$ be a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$. Define a time-dependent current $u_t \in \mathcal{D}_1'^\infty(M)$ by $u_t = -*\mathrm{d}\langle G_H, \omega_t\rangle$. Suppose $q_n(t)(n = 1, \ldots, N)$ is a solution of the point vortex equation* (31)*. Then, the following pair of time-dependent currents α_t and p_t defines a C^r-decomposable weak Euler-Arnold flow.*

$$\begin{aligned}
&\alpha_t = X_t^\flat + u_t \in \mathcal{D}_1'^r(M), \\
&p_t = P_t + (2\beta_X - 1) g(X_t^\flat, u_t) + (2\beta_\omega - 1)|u_t|^2/2 \in \Omega^0_{[r]\mathrm{loc}}(M).
\end{aligned}$$

The following lemma plays a key role in the proofs of Theorem 7.1 and 7.2.

Lemma 7.3. *Fix a time-dependent vector field $X_t \in \mathfrak{X}^r(M)$, $r \geq 1$ and $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. Let ω_t be a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$. Define a time-dependent current $u_t \in \mathcal{D}_1'^\infty(M)$ by $u_t = -*\mathrm{d}\langle G_H, \omega_t\rangle$. Then, we have*

$$\partial_t \mathfrak{L}\, u = \sum_{n=1}^N \Gamma_n \,\mathrm{d}\,\chi_{q_n} \dot{q}_n, \tag{36}$$

$$\mathfrak{L}\{(\omega_X + \omega) * u + \omega * X^\flat\} = 0, \tag{37}$$

$$\mathfrak{L}\{\mathrm{d}\, g(2\beta_X X^\flat + \beta_\omega u, u)\} = -\sum_{n=1}^N \Gamma_n \,\mathrm{d}\,\chi_{q_n}(\beta_X X + \beta_\omega v_n). \tag{38}$$

Proof. In what follows, since we fix $t \in [0, T]$, we may omit the subscript t unless otherwise stated. We decompose u into the regular part $v_n^\flat$ and the singular part $* \mathrm{d}_q(\Gamma_n/2\pi) \log d(q, q_n)$ as follows.

$$u = v_n^\flat + * \mathrm{d}_q \frac{\Gamma_n}{2\pi} \log d(q, q_n). \tag{39}$$

Without loss of generality, a geodesic polar coordinate $(\rho, \theta) = (d(q, q_n), \theta)$ can be taken in the neighborhood of q_n, satisfying

$$* \mathrm{d} \log \rho = \mathrm{d}\, \theta + O(\rho). \tag{40}$$

Let us fix $\phi \in \mathcal{D}^0(M)$ and sufficiently small $\varepsilon > 0$. We remember the relation $\alpha \wedge \mathrm{d}\, \phi = - \mathrm{d}(\phi\alpha) + \phi \, \mathrm{d}\, \alpha$ for each $\alpha \in \Omega^1(M)$. To show (36), we see that $\mathfrak{L}\, u = \omega$. By the Stokes theorem, we have

$$\begin{aligned} \int_{M \setminus B_\varepsilon(\mathsf{S}(u))} u \wedge \mathrm{d}\, \phi &= \int_{M \setminus B_\varepsilon(\mathsf{S}(u))} - \mathrm{d}(\phi u) + \phi \, \mathrm{d}\, u \\ &= \sum_{n=1}^{N} \int_{\partial B_\varepsilon(q_n)} \phi u + \int_{M \setminus B_\varepsilon(\mathsf{S}(\omega))} \omega * \phi. \end{aligned}$$

It follows from (39) and (40) that the first term becomes

$$\begin{aligned} \int_{\partial B_\varepsilon(q_n)} \phi u &= \int_{\partial B_\varepsilon(q_n)} (\phi(q) - \phi(q_n) + \phi(q_n)) \left(v_n^\flat + * \mathrm{d}_q \frac{\Gamma_n}{2\pi} \log \rho(q) \right) \\ &= \frac{\Gamma_n}{2\pi} \phi(q_n) \int_{\partial B_\varepsilon(q_n)} \mathrm{d}\, \theta + O(\varepsilon) = \Gamma_n \phi(q_n) + O(\varepsilon). \end{aligned}$$

Regarding the second term, we have

$$\int_{M \setminus B_\varepsilon(\mathsf{S}(\omega))} \omega * \phi = c \int_{M \setminus B_\varepsilon(\mathsf{S}(\omega))} * \phi.$$

Thus we obtain

$$\mathfrak{L}\, u[\phi] = \sum_{n=1}^{N} \Gamma_n \phi(q_n) + c \int_M * \phi = \omega[\phi],$$

which yields (36). We compute each term in (37). Owing to $\omega = I(c)$, we have $\mathfrak{L}\, \omega * X^\flat = c\, \mathfrak{L} * X^\flat = 0$. Since $\mathfrak{L}\, \omega * u = c\, \mathfrak{L} * u$ and $|\, \mathfrak{L}\, \omega_X * u[\phi]| \leq$

$||\omega_X|_{\operatorname{supp} \mathrm{d}\,\phi}||_\infty | \mathfrak{L} * u[\phi]|$, if we prove $\mathfrak{L} * u = 0$, the assertion (37) follows. Indeed, we have

$$\begin{aligned}\mathfrak{L} * u[\phi] &= \lim_{\varepsilon\to 0} \int_{M\setminus B_\varepsilon(\mathsf{S}(u))} *u \wedge \mathrm{d}\,\phi = \lim_{\varepsilon\to 0} \int_{M\setminus B_\varepsilon(\mathsf{S}(u))} \{-\,\mathrm{d}(\phi \wedge *u) + \phi\,\mathrm{d} * u\} \\ &= \sum_{n=1}^{N} \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(q_n)} \phi * u = \sum_{n=1}^{N} \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(q_n)} \phi \left(*v_n^\flat - \mathrm{d}_q \frac{\Gamma_n}{2\pi} \log \rho(q) \right) = 0.\end{aligned}$$

It is easy to check (38) by Proposition 5.1. We see from (39) that

$$\begin{aligned}\int_{M\setminus B_\varepsilon(\mathsf{S}(u))} \mathrm{d}\,g(2\beta_X X^\flat, u) \wedge \mathrm{d}\,\phi &= -\sum_{n=1}^{N} \int_{\partial B_\varepsilon(q_n)} g(2\beta_X X^\flat, u)\,\mathrm{d}\,\phi \\ &= -\sum_{n=1}^{N} \frac{\Gamma_n}{2\pi} \int_{\partial B_\varepsilon(q_n)} g(2\beta_X X^\flat, *\,\mathrm{d} \log d(q, q_n))\,\mathrm{d}\,\phi + O(\varepsilon) \\ &= -\sum_{n=1}^{N} \frac{\Gamma_n}{2\pi} \int_{\partial B_\varepsilon(q_n)} g(2\beta_X X, \mathcal{J} \operatorname{grad} \log d(q, q_n))\,\mathrm{d}\,\phi + O(\varepsilon) \\ &\to -\sum_{n=1}^{N} \Gamma_n\,\mathrm{d}\,\chi_{q_n}(\beta_X X)[\phi] \quad \text{as } \varepsilon \to 0,\end{aligned}$$

and

$$
\begin{aligned}
\int_{M\setminus B_\varepsilon(\mathsf{S}(u))} \mathrm{d}\, g(\beta_\omega u, u) \wedge \mathrm{d}\,\phi &= -\sum_{n=1}^{N} \int_{\partial B_\varepsilon(q_n)} g(\beta_\omega u, u)\,\mathrm{d}\,\phi \\
&= -\sum_{n=1}^{N} \frac{\Gamma_n}{2\pi} \int_{\partial B_\varepsilon(q_n)} g(\beta_\omega v_n^\flat, *\,\mathrm{d}\log d(q, q_n))\,\mathrm{d}\,\phi + O(\varepsilon) \\
&\quad - \sum_{n=1}^{N} \frac{\Gamma_n}{2\pi} \int_{\partial B_\varepsilon(q_n)} \frac{\Gamma_n}{2\pi} |*\,\mathrm{d}\log d(q, q_n)|^2\,\mathrm{d}\,\phi + O(\varepsilon) \\
&= -\sum_{n=1}^{N} \frac{\Gamma_n}{2\pi} \int_{\partial B_\varepsilon(q_n)} g(2\beta_\omega v_n, \mathcal{J}\operatorname{grad}\log d(q, q_n))\,\mathrm{d}\,\phi + O(\varepsilon) \\
&\to -\sum_{n=1}^{N} \Gamma_n\,\mathrm{d}\,\chi_{q_n}(\beta_\omega v_n)[\phi] \quad \text{as } \varepsilon \to 0,
\end{aligned}
$$

which completes the proof. □

We now show the two main theorems by using Lemma 7.3.

Proof of Theorem 7.1. Since the background field X and the relative velocity current u satisfy the assumptions of Lemma 7.3, the equalities (36)-(38) hold true. Since X and u come from a C^r-decomposable weak Euler-Arnold flow, they satisfy the equation (35). Substituting (36)-(38) into (35), we obtain

$$
\sum_{n=1}^{N} \Gamma_n\,\mathrm{d}\,\chi_{q_n}\left\{\dot{q}_n - (\beta_X X + \beta_\omega v_n)\right\} = 0,
$$

which is the conclusion as desired. □

Proof of Theorem 7.2. We first prove that (α_t, p_t) is a weak Euler-Arnold flow. Owing to Lemma 7.3, it is easy to check that α_t and p_t satisfy the conditions in Definition 6.1 except for the weak Euler-Arnold equation. In addition, we see that

$$
\partial_t\,\mathfrak{L}\,\alpha = \partial_t\,\mathfrak{L}(X^\flat + u) = \partial_t\,\mathrm{d}\,X^\flat + \sum_{n=1}^{N} \Gamma_n\,\mathrm{d}\,\chi_{q_n}\dot{q}_n,
$$

$$\begin{aligned}\mathfrak{L}\{(*\,\mathrm{d}\,\alpha)*\alpha\} &= \mathfrak{L}\{(*\,\mathrm{d}\,X^\flat)*X^\flat\} + \mathfrak{L}\{(\omega_X+\omega)*u+\omega*X^\flat\}\\ &= \mathrm{d}\{(*\,\mathrm{d}\,X^\flat)*X^\flat\}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{L}\,\mathrm{d}(|\alpha|^2/2+p) &= \mathfrak{L}\,\mathrm{d}(|X^\flat|^2/2+P) + \mathfrak{L}\{\mathrm{d}\,g(2\beta_X X^\flat+\beta_\omega u, u)\}\\ &= -\sum_{n=1}^{N}\Gamma_n\,\mathrm{d}\,\chi_{q_n}(\beta_X X+\beta_\omega v_n).\end{aligned}$$

Since (X,P) is an Euler-Arnold flow and q_n is a solution of the point vortex equation (31), we deduce

$$\partial_t\,\mathfrak{L}\,\alpha_t+\mathfrak{L}\{(*\,\mathrm{d}\,\alpha_t)*\alpha_t\}+\mathfrak{L}\,\mathrm{d}(|\alpha_t|^2/2+p_t) = \sum_{n=1}^{N}\Gamma_n\,\mathrm{d}\,\chi_{q_n}\{\dot{q}_n-(\beta_X X+\beta_\omega v_n)\}=0,$$

which yields (α_t,p_t) is a weak Euler-Arnold flow. By definition, it is obvious that the weak Euler-Arnold flow (α_t,p_t) is C^r-decomposable. $\square$

8 Applications

As applications of these theorems, we now discuss two examples of point vortex dynamics in a background field: two identical point vortices in a linear shear in the Euclidean plane $(\mathbb{C},\mathrm{d}\,z\,\mathrm{d}\,\bar{z})$ and N-point vortices on a surface in an irrotational flow.

Let us first check that the point vortex equation without any background field in the plane is obtained from our results as a special case. Let us set $\beta_X=0$ and $\beta_\omega=1$. Then, the point vortex equation (31) is deduced as follows.

$$\dot{q}_n(t)=v_n(q_n). \tag{41}$$

When the flow field is the plane, it follows from $G_H(z,z_0)=-(1/2\pi)\log|z-z_0|$ that for a given singular vorticity ω of point vortices placed on $\{z_n\}_{n=1}^N\subset\mathbb{C}$,

$$u=-*\,\mathrm{d}\langle G_H,\omega\rangle = I\left(-*\,\mathrm{d}\,\psi_0-*\,\mathrm{d}\sum_{n=1}^{N}\Gamma_n G_H(z,z_n)\right)$$

for some harmonic function ψ_0. We can choose $\psi_0 = 0$ without loss of generality. From $*\,\mathrm{d}\,z = -\mathrm{i}\,\mathrm{d}\,z$ we deduce that

$$\begin{aligned} -*\,\mathrm{d}\,G_H(z, z_n) &= \partial_z G_H(-*\,\mathrm{d}\,z) + \partial_{\bar{z}} G_H(-*\,\mathrm{d}\,\bar{z}) \\ &= \mathrm{i}\partial_z G_H\,\mathrm{d}\,z - \mathrm{i}\partial_{\bar{z}} G_H\,\mathrm{d}\,\bar{z}. \end{aligned}$$

Therefore, the dual vector field of u, denoted by $u_\sharp = u^z\partial_z + u^{\bar{z}}\partial_{\bar{z}} \in \mathfrak{X}^\infty(M \setminus \mathsf{S}(u))$, can be written as

$$u^z(z) = \sum_{n=1}^{N} \Gamma_n(-2\mathrm{i}\partial_{\bar{z}} G_H(z, z_n)) = \frac{\mathrm{i}}{2\pi} \sum_{n=1}^{N} \frac{\Gamma_n}{\bar{z} - \bar{z}_n},$$

since $u_\sharp^\flat = u^z\,\mathrm{d}\,\bar{z} + u^{\bar{z}}\,\mathrm{d}\,z$. In the same manner, denoting $v_n(z)$ by $v_n(z) = v_n^z\partial_z + v_n^{\bar{z}}\partial_{\bar{z}}$, we obtain

$$v_n^z(z_n) = \sum_{\substack{m=1 \\ m\neq n}}^{N} \Gamma_m(-2\mathrm{i}\partial_{\bar{z}_n} G_H(z_n, z_m)) = \frac{\mathrm{i}}{2\pi} \sum_{\substack{m=1 \\ m\neq n}}^{N} \frac{\Gamma_m}{\bar{z}_n - \bar{z}_m}.$$

Hence, we deduce from the point vortex equation (41) to the following equation.

$$\dot{z}_n = \frac{\mathrm{i}}{2\pi} \sum_{\substack{m=1 \\ m\neq n}}^{N} \frac{\Gamma_m}{\bar{z}_n - \bar{z}_m}.$$

Two identical point vortices in a linear shear Two identical point vortices in a linear shear is used as a model of the vortex merger in [28]. The vortex merger is characterized as a fundamental process of the inverse cascade in 2D turbulence. In the process, vortices with similarly small scales are affected by the shear flow induced from the surrounding vortices. As a result, small vortices combine to form a vortex with large scale. As a simple model for the vortex merger, two identical point vortices in a linear shear is adopted. To use the notations of this paper, let us set $N = 2$, $\Gamma_1 = \Gamma_2 = \gamma$ and the linear shear $X = (cy, 0)$ where $y = (z - \bar{z})/2\mathrm{i}$. Based on [28], the evolution equation of two identical point vortices in a linear shear placed on $\{q_n(t)\}_{n=1}^2$ is given by

$$\dot{q}_n(t) = X(q_n) + v_n(q_n).$$

Let us apply the main theorems to this fact. First, Theorem 7.2 shows that there exists a C^∞-decomposable weak Euler flow (α_t, p_t) such that the relative vorticity is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^2$. This guarantees the dynamics in [28] comes from a weak Euler flow even though the evolution equation is formally derived with no relation with Euler flows. Second, as an application of Theorem 7.1, let us take a C^∞-decomposable weak Euler flow (α_t, p_t) such that the background field is the linear shear X and the growth rate is $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ and relative vorticity is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^2$ with $\Gamma_1 = \Gamma_2 = \gamma$. From Theorem 7.1, we can deduce that $q_n(t)$ satisfies the point vortex equation (31). It has been established in [28] that vortex merger is determined by the parameter $\mu = c\xi_0^2/\gamma$ where $\xi_0 \in \mathbb{R}$ is the initial distance between two point vortices and it occurs if $\mu < 0$. In the similar argument, we obtain the parameter μ' of the criterion for our case, which gives $\mu' = \mu\beta_X/\beta_\omega$. Moreover, the same criterion is valid, that is the vortex merger occurs if $\mu' < 0$. The difference between the parameters μ' and μ is whether it contains the growth rate (β_X, β_ω) of the pressure. Whereas the criterion by μ does not states the relation between the vortex merger and the pressure, the criterion by μ' tells us that the vortex merger is determined not only by the vorticity of the background field and the circulation of point vortices but also by the growth rate of the pressure relative to interaction energy density and kinetic energy density. Therefore, Theorem 7.1 provides further physical insight into the 2D turbulence that vortex merger is governed by the sign of the growth rate of the pressure p_t.

N-point vortices on a surface in an irrotational flow. In the last case, let us take a C^∞-decomposable weak Euler-Arnold flow (α_t, p_t) such that the background field is an irrotational field $X \in \mathfrak{X}^\infty(M)$ and the relative vorticity is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$. As we see in Section 3, the irrotational field is the fluid velocity of a steady Euler-Arnold flow $(X, P) \in \mathfrak{X}^\infty(M) \times C^\infty(M)$ and the pressure $P \in C^\infty(M)$ satisfies $P = -|X|^2/2$. Let us fix the growth rate $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ and the parameter $(\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N$. For simplicity, we ignore the interaction between the background field and point vortices, that is we can assume $\beta_X = 0$. Then owing to Theorem 7.1, $q_n(t)$ and p_t satisfy

$$\begin{cases} \dot{q}_n = \beta_\omega v_n(q_n), \\ p_t = -|X_t + u_t|^2 + \beta_\omega |u_t|^2. \end{cases} \tag{42}$$

Let us focus on two cases $\beta_\omega = 0$ and 1. In the case $\beta_\omega = 0$, it follows from $\dot{q}_n(t) = 0$ that each of point vortices does not move. Hence, we deduce that (α_t, p_t) is a steady solution. Moreover, the pressure is a type of the steady Bernoulli law $p = -|X + u|^2/2 = -|\alpha|^2/2$. On the other hand, for $\beta_\omega = 1$ we obtain the conventional point vortex equation and $p = -|\alpha|^2/2 + |u|^2$. This formula of the pressure is equivalent to the Bernoulli law with point vortices for the unsteady solution. As a corollary, a generalization of steady and unsteady Bernoulli law with N-point vortices to the case where the flow field is a curved surface can be obtain as follows.

Corollary 8.1 (Generalized steady Bernoulli law with point vortices)**.** *If a C^∞-decomposable weak Euler-Arnold flow (α_t, p_t) on a surface satisfies that the background field of α_t is an irrotational field $X \in \mathfrak{X}^\infty(M)$ and that the pressure is given by $p_t = -|\alpha_t|^2/2$ and that the relative vorticity is a singular vorticity of point vortices, then (α_t, p_t) is a steady solution of the weak Euler-Arnold equations.*

Corollary 8.2 (Generalized unsteady Bernoulli law with point vortices)**.** *If a C^∞-decomposable weak Euler-Arnold flow (α_t, p_t) on a surface satisfies that the background field of α_t is an irrotational field and that $p_t = -|\alpha_t|^2/2 + |u_t|^2$ and that the relative vorticity is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$, then for every $n \in \{1, \dots, N\}$, $q_n(t)$ is a solution of the point vortex equation:*

$$\dot{q}_n(t) = v_n(q_n(t)).$$

Let us finally discuss the role of the growth rate β_ω in the motion of point vortices and the pressure given in (42). Denoting $Q_n(t)$ by the solution of $\dot{Q}_n = v_n(Q_n)$, the solution $q_n(t)$ in (42) can be written as $q_n(t) = Q_n(\beta_\omega t)$. Letting $\beta_\omega \to 0$, we see that q_n moves very slowly on the orbit of Q_n. In this sense, β_ω stands for the flexibility of the motion of point vortices as well as the growth rate of the pressure relative to the kinetic energy density. We notice that as $\beta_\omega \to 0$ the pressure p_t converges to $-|\alpha_t|^2/2$ in $\Omega^0_{[r]\mathrm{loc}}(M)$, which is consistent with the generalized steady Bernoulli law. From this we can observe that point vortices are frozen if the pressure p_t is sufficiently close to the generalized steady Bernoulli law. As a consequence, we conclude that β_ω describes the the flexibility of the motion of point vortices and the growth rate of the pressure and that point vortices are slower to move as the pressure is sufficiently close to the generalized steady Bernoulli law.

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