# Loewner chains and evolution families on parallel slit half-planes

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**Abstract**

In this thesis, we define and study Loewner chains and evolution families on finitely multiply-connected domains in the complex plane. These chains and families consist of conformal mappings on parallel slit half-planes and have one and two "time" parameters, respectively. By analogy with the case of simply connected domains, we develop a general theory of Loewner chains and evolution families on multiply connected domains and, in particular, prove that they obey the chordal Komatu–Loewner differential equations driven by measure-valued processes. Our method involves Brownian motion with darning, as do some recent studies.

# Contents

# Chapter 1

# Introduction

The aim of this thesis is to study "Loewner chains" and "evolution families" on finitely multiply-connected domains in the complex plane $\mathbb{C}$. These two concepts are originally defined on simply connected domains and connected to the Loewner differential equation, which is of great use in complex analysis and in probability theory. In this thesis, we first provide a review on previous studies. We then develop a general theory of Loewner chains and evolution families in our sense and, in particular, deduce the chordal Komatu–Loewner differential equation.

## 1.1 Purpose of the study

In Chapter 6 of Pommerenke's celebrated book [52], a *Loewner chain* on the unit disk $\mathbb{D}$ of $\mathbb{C}$ is defined as a family of univalent functions $f_t \colon \mathbb{D} \to \mathbb{C}$, $t \in [0, \infty)$, with the following properties:

- $f_t(0) = 0$ and $f_t'(0) = e^t$ for any $t \geq 0$;
- $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for any $0 \leq s \leq t$.

Here, a complex-valued function on a domain in $\mathbb{C}$ is said to be univalent if it is holomorphic and injective. We also use the adjective "conformal" if it is also surjective. For a Loewner chain $(f_t)_{t\geq 0}$, the composites $\phi_{t,s} := f_t^{-1} \circ f_s$, $0 \leq s \leq t$, "measure" the difference between $f_t$ and $f_s$ and enjoy the semigroup property $\phi_{u,s} = \phi_{u,t} \circ \phi_{t,s}$ for any $0 \leq s \leq t \leq u$. The family $(\phi_{t,s})$ is referred to as the evolution family of $(f_t)$. More generally, a family of univalent self-mappings $\phi_{t,s} \colon \mathbb{D} \to \mathbb{D}$, $0 \leq s \leq t$, is called an *evolution family* on $\mathbb{D}$ if the following are satisfied:

- $\phi_{t,t}(z) = z$ for any $t \geq 0$;

- $\phi_{t,s}(0) = 0$ and $\phi'_{t,s}(0) = e^{-(t-s)}$ for any $0 \leq s \leq t$;
- $\phi_{u,s} = \phi_{u,t} \circ \phi_{t,s}$ for any $0 \leq s \leq t \leq u$.

A Loewner chain $(f_t)_{t\geq 0}$ and an evolution family $(\phi_{t,s})_{0\leq s\leq t}$ are absolutely continuous solutions to the *(radial) Loewner differential equations*

$$\frac{\partial f_t(z)}{\partial t} = z f'_t(z) p(z,t) \qquad \text{a.e. } t \geq 0, \tag{1.1}$$

$$\frac{\partial \phi_{t,s}(z)}{\partial t} = -\phi_{t,s}(z) p(\phi_{t,s}(z), t) \qquad \text{a.e. } t \geq s \text{ for each } s \geq 0. \tag{1.2}$$

Here, $p(z,t)$ is a *Herglotz vector field*, whose definition will be given later.

The differential equations (1.1) and (1.2) provide a powerful variational method for studying univalent functions on $\mathbb{D}$. Probably the most famous application is de Branges' proof [24] of Bieberbach's conjecture, which says that any univalent function $f$ on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$ has a Taylor expansion $f(z) = z + \sum_{n\geq 2} a_n z^n$ with $|a_n| \leq n$ for all $n \geq 2$. Proving this conjecture was a fairly difficult problem despite its simple statement. Although the conjecture and Löwner's work [47] on the case $n = 3$ were done in 1916 and in 1923, respectively, a conclusive proof of de Branges was given as late as 1985.

In 2000, another remarkable application of Loewner's method was proposed by Schramm [58] in probability theory. His theory concerns random models related to statistical physics, such as percolation, Ising model, and Gaussian free field. These models typically attach one of two (or more) values to every point of a simply connected domain, say the upper half-plane $\mathbb{H}$, under some probability law. Suppose that such a model on $\mathbb{H}$ has different boundary conditions on the positive and negative real axes. Let $\gamma$ be the interface curve growing between two phases in $\mathbb{H}$ from the origin, which is the border of the given boundary conditions. Schramm applied Loewner's method to a particular choice of Loewner chain. Namely, he considered the random univalent functions $g_t \colon \mathbb{H} \setminus \gamma(0,t] \to \mathbb{H}$ and obtain the (chordal) Loewner differential equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}. \tag{1.3}$$

Here, $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion. The constant $\kappa$ depends on the model. The family $(g_t)_{t\geq 0}$ is called the *stochastic Loewner evolution* or *Schramm–Loewner evolution* with parameter $\kappa$ (SLE$_\kappa$ for short). It is a surprising fact that the information of several different two-dimensional models can be encoded into one-dimensional Brownian motion via Loewner's method. This feature brought a drastic progress to the research of two-dimensional random models related to statistical physics. Several conjectures coming from physics ideas, which had seemed hard to verify, were proved by means of stochastic analysis.

From the viewpoints of geometric function theory and of statistical physics, it is a natural problem to extend the above-mentioned concepts of complex analysis and of probability theory to multiply connected domains. The aim of this thesis, the development of a general theory of Loewner chains and evolution families on multiply connected domains, is a central part of this problem, indeed. With this aim, we go basically by analogy with the theory on simply connected domains, but there are some difficulties in doing so.

The proof of the Loewner equations (1.1) and (1.2) involves simple inequalities of holomorphic functions on $\mathbb{D}$ such as the Schwarz lemma, distortion theorem, and growth theorem. For example, an application of Schwarz's lemma yields the inequality

$$|f_t(z) - f_s(z)| \leq \frac{8|z|}{(1-|z|)^4}(e^t - e^s), \quad 0 \leq s \leq t,$$

which is important to show the $t$-differentiability of $f_t(z)$ (see Chapter 6 of Pommerenke [52]). The nice quantitative properties listed above reflect the simple forms of Möbius transformations, i.e., conformal automorphisms on $\mathbb{D}$. However, conformal transformations on multiply connected domains do not have such simple forms, and holomorphic functions there are difficult to deal with quantitatively. We need some ideas to deduce the ($t$-)differentiability of Loewner chains and evolution families in our extended sense.

As a solution to the problem in the preceding paragraph, we make use of probabilistic concepts and, in particular, develop some potential theory of *Brownian motion with darning* (BMD for short). Its precise definition is postponed until Section 3.1, but for an $(N+1)$-connected domain $D = \mathbb{H} \setminus \bigcup_{j=1}^N A_j$, BMD is a diffusion process on the set $D^* = D \cup \{a_1^*, \dots, a_N^*\}$. This set is the quotient space of $\mathbb{H}$ obtained by regarding each hole $A_j$ as a single point $a_j^*$. It turns out that BMD-harmonic functions on $D^*$ have a close connection to holomorphic functions on $D$. This fact was efficiently used in a series of papers written by Chen, Fukushima and Rohde [17], Chen and Fukushima [16], and Chen, Fukushima and Suzuki [18]. We elaborate part of their results and add some new estimates about BMD in Chapter 3.

Now, in order to specify our goal more, let us return to the definition of $p(z,t)$ in (1.1) and (1.2). A holomorphic function $p(z)$ on $\mathbb{D}$ is called a Herglotz function if $p(0) = 1$ and $\Re p(z) > 0$ for any $z \in \mathbb{D}$. A function $p(z,t)$

of two variables $z \in \mathbb{D}$ and $t \in [0, \infty)$ is called a Herglotz vector field if the following hold:

- $z \mapsto p(z,t)$ is a Herglotz function for each $t$;
- $t \mapsto p(z,t)$ is a Lebesgue measurable function for each $z$.

The triangular diagram in the left of Figure 1.1 shows the one-to-one correspondence given by the relations

$$\phi_{t,s}(z) = (f_t^{-1} \circ f_s)(z) \qquad \text{for any } 0 \leq s \leq t, \tag{1.4}$$

$$f_s(z) = \lim_{t\to\infty} e^t \phi_{t,s}(z) \qquad \text{for any } s \geq 0 \tag{1.5}$$

in addition to (1.1) and (1.2). We note that any Herglotz function $p(z)$ has the Herglotz representation

$$p(z) = \int_{\partial\mathbb{D}} S_{\mathbb{D}}(z,\zeta)\, \mu(d\zeta) \tag{1.6}$$

for some Borel probability measure $\mu$ on $\partial\mathbb{D}$. Here, $S_{\mathbb{D}}(z,\zeta) := (\zeta+z)/(\zeta-z)$ is the Schwarz kernel. Hence, for a Herglotz vector field $p(z,t)$, there exists a *measure-valued* process $t \mapsto \nu_t$ such that

$$p(z,t) = \int_{\partial\mathbb{D}} S_{\mathbb{D}}(z,\zeta)\, \nu_t(d\zeta). \tag{1.7}$$

Substituting this expression into (1.1) and (1.2), we observe that these differential equations are driven by the process $(\nu_t)_{t\geq 0}$ via the kernel $S_{\mathbb{D}}$. This interpretation of Loewner equations, illustrated in the right diagram of Figure 1.1, is quite natural in view of SLE. In the SLE case, the family $(g_t)_{t\geq 0}$ is generated from a multiple of Brownian motion, i.e., from the process of Dirac measures $\delta_{\sqrt{\kappa}B_t}$, $t \geq 0$, on $\mathbb{R}$ via (1.3), and the right-hand side of (1.3) is given in terms of the Cauchy kernel $(z-\zeta)^{-1}$. *It is such a relation among Loewner chains, evolution families, and (measure-valued) driving processes that we shall extend to multiply connected domains.* The concrete way to do this is summarized in the next subsection.

## 1.2 Overview of the results

We formulate our results on *parallel slit half-planes*. Here, a parallel slit half-plane is a domain obtained by removing some line segments parallel to the real axis from $\mathbb{H}$. Any $(N+1)$-connected domain is mapped conformally

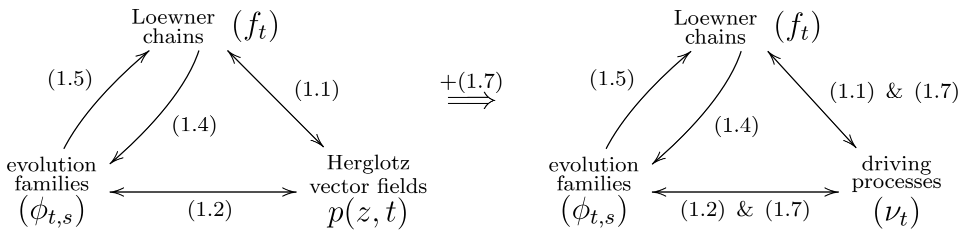


Figure 1.1: The relations studied in this thesis.

onto some parallel slit half-plane of $N$ slits (see Courant [23, Theorem 2.3] for example). In this sense, parallel slit half-planes can be regarded as "canonical" representatives of conformal equivalence classes of multiply connected domains.

Our main results are threefold. As preliminaries, basic assumptions are introduced in Sections 4.1 and 4.2, which are imposed on univalent functions throughout this thesis. Univalent functions treated in this thesis map a parallel slit half-plane $D_1$ into another slit domain $D_2$ and are asymptotic to the identity mapping $z \mapsto z$ as $z \to \infty$. For such a function $f\colon D_1 \to D_2$, we provide the integral representation

$$f(z) = z + \pi \int_{\mathbb{R}} \Psi_{D_1}(z, \xi)\, \mu_f(d\xi), \quad z \in D_1, \tag{1.8}$$

with the measure $\mu_f$ given by $\mu_f(d\xi) = \pi^{-1} \Im f(\xi + i0)\, d\xi$ in Theorem 4.3. Here, the kernel $\Psi_{D_1}$ is exactly the *complex Poisson kernel* of BMD on $D_1^*$. This is our first result.

The representation (1.8) on $D_1$ corresponds to the Herglotz representation (1.6) on $\mathbb{D}$. A typical way to prove the latter is applying Schwarz's formula, a complex version of Poisson's integral formula, on the disk of radius $r \in (0, 1)$ and then taking the limit as $r \to 1$. However, more work is required for proving (1.8) along a similar line of reasoning, because we do not know much about the dependence of the integral kernel $\Psi_{D_1}$ on the parallel slit half-plane $D_1$. If we try to approximate $D_1$ by a sequence $(D^{(n)})_{n\in\mathbb{N}}$ of smaller slit domains, we have to show the convergence $\Psi_{D^{(n)}}(z, \xi) \to \Psi_{D_1}(z, \xi)$ in a strong sense stated in Proposition 3.9. Also, we have to know the behavior of $\Psi_{D_1}(z, \xi)$ as $z \to \infty$ because the normalization condition $f(z) \sim z$ $(z \to \infty)$ is imposed. Since existing results on $\Psi_{D_1}$ in the above-mentioned works [17, 16] are not sufficient, we refine part of them in Chapter 3, combining analytic and probabilistic observations.

After proving the integral representation formula, we define Loewner chains and evolution families on parallel slit half-planes. Let $D_t$, $t \in [0, T]$,

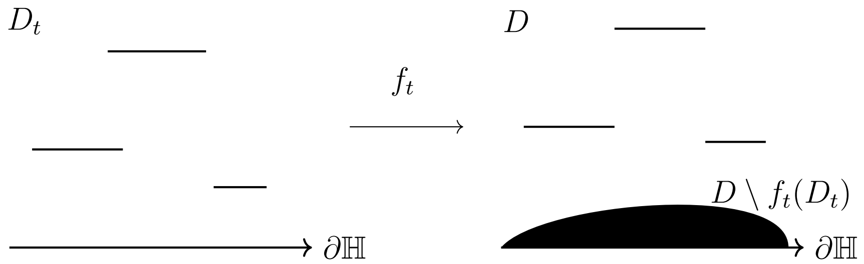


Figure 1.2: A Loewner chain $(f_t)_t$ over parallel slit half-planes $(D_t)_t$.

and $D$ be parallel slit half-planes. In our definition, a *Loewner chain* $(f_t)_{t\in[0,T]}$ over $(D_t)_{t\in[0,T]}$ consists of univalent functions $f_t\colon D_t \to D$ with the property that $f_s(D_s) \subset f_t(D_t)$ for $0 \le s < t \le T$ (see Figure 1.2). This property is analogous to the second one of Loewner chains on $\mathbb{D}$ in Section 1.1. An *evolution family* $(\phi_{t,s})_{0\le s\le t\le T}$ over $(D_t)_{t\in[0,T]}$ consists of univalent functions $\phi_{t,s}\colon D_t \to D_s$ which have the same semigroup property as that of $f_t^{-1} \circ f_s$. We apply (1.8) to $\phi_{t,0}$ and define $\lambda(t) := \mu_{\phi_{t,0}}(\mathbb{R})$. This quantity has two meanings: the (angular) residue of $\phi_{t,0}$ at infinity and the *(BMD) half-plane capacity* of the hull[1] $D_t \setminus \phi_{t,0}(D_0)$. We regard $\lambda(t)$ as a "canonical" parameter and derive the equality

$$\tilde{\partial}_t^{\lambda} \phi_{t,s}(z) := \lim_{\delta\downarrow 0} \frac{\phi_{t+\delta,s}(z) - \phi_{t-\delta,s}(z)}{\lambda(t+\delta) - \lambda(t-\delta)} = \pi \int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,s}(z), \xi)\, \nu_t(d\xi) \qquad (1.9)$$

in Theorem 5.7. Here, $\nu_t$ is a finite Borel measure on $\mathbb{R}$. A differential equation similar to (1.9) holds for a Loewner chain $(f_t)_{t\in[0,T]}$ as well. The derivation of these equations is our second result, which is central to this thesis. In view of the right diagram of Figure 1.1, this part corresponds to the arrows from Loewner chains and from evolution families, respectively, to driving processes.

The equation (1.9) is named the *chordal Komatu–Loewner differential equation* after Komatu [40], who tried to extend Löwner's work [47] to multiply connected domains. Refining his idea, Bauer and Friedrich [7] provided a way for extending the chordal Loewner equation (1.3) and SLE to parallel slit half-planes. Although the previous and our studies are different in some technical points, we emphasize that the scope of our study is much larger than that of the previous in that the hull $D_t \setminus \phi_{t,0}(D_0)$ is allowed to be even unbounded.

[1] A hull is a relatively closed subset of $\mathbb{H}$ whose complement in $\mathbb{H}$ is simply connected. For example, if $\gamma\colon (0,\infty) \to \mathbb{H}$ is a simple curve with $\gamma(0+) = 0$ as in (1.3), then the trace $\gamma(0,t]$ is a hull.

The relationship between Loewner chains and evolution families is relatively easy to show as in Proposition 5.11. Thus, our remaining problem is to establish the correspondence from driving processes to Loewner chains or to evolution families. This will be done by solving the differential equation (1.9) for each $s \geq 0$ and $z \in D_s$ and by showing that the solutions $\phi_{t,s}(z)$ form an evolution family. Here, it is important that the right-hand side of (1.9) depends on $D_t$. Note that $D_s \neq D_t$ generally, which is a big difference from the classical theory on $\mathbb{D}$ or $\mathbb{H}$. For this reason, we have to determine the evolution of $D_t$ as well as that of $\phi_{t,s}(z)$. Such a situation does not appear in the general theory of (ordinary) differential equations.

For reducing the problem in the preceding paragraph to a simple one, it is helpful to consider the evolution equation of $D_t$ as Bauer and Friedrich [7] did. In other words, we describe the evolution of the parallel slits of $D_t$, using a differential equation similar to (1.9). Actually, if $(\phi_{t,s})_{0 \leq s \leq t \leq T}$ is an evolution family over $(D_t)_{t \in [0,T]}$, then the Komatu–Loewner equation for the endpoints $z_j^\ell(t)$ and $z_j^r(t)$ of the slits holds:

$$\tilde{\partial}_t^\lambda z_j^\ell(t) = \pi \int_{\mathbb{R}} \Psi_{D_t}(z_j^\ell(t), \xi)\, \nu_t(d\xi), \tag{1.10}$$

$$\tilde{\partial}_t^\lambda z_j^r(t) = \pi \int_{\mathbb{R}} \Psi_{D_t}(z_j^r(t), \xi)\, \nu_t(d\xi). \tag{1.11}$$

These equations are obtained through extending (1.9) carefully to points on the slits. Here, we note that this extension is not deduced from a formal replacement of $\phi_{t,s}(z)$ in (1.9) by $z_j^\ell(t)$ and $z_j^r(t)$. See Chen and Fukushima [16, Remark 2.4].

Using (1.10) and (1.11), we discuss the two remaining arrows growing from driving processes in the right diagram of Figure 1.1. This is our last result. Let $(\nu_t)_t$ be a measure-valued driving process. The Komatu–Loewner equation (1.10) and (1.11) for the slits has a local solution, which determines the evolution of $D_t$. Solutions to the Komatu–Loewner equation (1.9) for evolution families then form a two-parameter family $(\phi_{t,s})_{0 \leq s \leq t \leq T}$ of univalent functions. We prove in Theorem 6.9 that this family belongs to a slightly larger class than the one of evolution families. These assertions are quite close to the desired correspondence from a driving process to an evolution family, but we lack the following two properties:

- the uniqueness of a solution to (1.10) and (1.11),
- the finiteness of the residue (or BMD half-plane capacity) at infinity of the solution to (1.9).

In Section 6.3.2, we prove that these properties do hold and that $(\phi_{t,s})$ becomes a genuine evolution family under the additional assumption that $\operatorname{supp}\nu_t$ is locally bounded in $t \in [0, \infty)$. Even if we assume this local boundedness, our results include all the known examples of driving processes in the previous studies about Komatu–Loewner equations. We consider only the arrow from driving processes to evolution families in Chapter 6, but it is easily transferred into the arrow to Loewner chains through Proposition 5.11.

In addition to the main results explained above, we investigate some relationship between the previous and present studies. Let $(f_t)_{t\in[0,T]}$ be a Loewner chain over $(D_t)_{t\in[0,T]}$. We are particularly interested in the question of *how the driving process* $(\nu_t)_{t\in[0,T]}$ *reflects the "geometry" and "continuity" of the domains* $f_t(D_t)$, $t \in [0, T]$. From this point of view, we apply our results to two examples which were studied in previous papers.

## 1.3 Organization of this thesis

Chapter 2 presents a summary of related studies. We first explain Loewner's method on $\mathbb{H}$ developed by Goryainov and Ba [34] in Section 2.1. Their argument and results are the model of Sections 4 and 5. We then return to the review of SLE in Section 2.2. In particular, we choose the topic of multiple SLE and its hydrodynamic limit, which is studied by del Monaco and Schleißinger [25], as a probabilistic application of Loewner's method on $\mathbb{H}$ in Section 2.2.3. This topic is an example in which a measure-valued driving process appears naturally in the study of SLE. Section 2.3 is devoted to previous studies on the Komatu–Loewner differential equation.

In Chapter 3, we study the complex Poisson kernel $\Psi_D(z, \xi)$ of BMD. Section 3.1 collects potential-theoretic basics of BMD that are used subsequently. We examine the integral operator defined by the (complex) Poisson kernel in Section 3.2 and study the asymptotic behavior of $\Psi_D(z, \xi)$ as $z \to \infty$ in Section 3.3. Finally, we review the dependence of $\Psi_D(z, \xi)$ on a variable domain $D$ in Section 3.4. The "local Lipschitz continuity" of $\Psi_D(z, \xi)$ with respect to the variation of $D$ was investigated by Chen, Fukushima and Rohde [17], and we give a partial improvement to be applied to our framework.

Chapters 4 through 7 go along the line of Section 1.2.

In Chapter 4, we study conformal mappings on parallel slit half-planes. Section 4.1 provides a brief summary on normalization conditions of univalent functions at infinity. We then state our first main result in Section 4.2: the integral representation formula (1.8) for conformal mappings which are suitably normalized at infinity on parallel slit half-planes. Section 4.3 treats the enlargement of parallel slit half-planes across the slits and the analytic continuation of holomorphic functions defined on such slit domains. We consider such analytic continuation in order to describe the behavior of conformal mappings *on the slits*.

Chapter 5 is devoted to our second main result. We define evolution families over a family of parallel slit half-planes and collect their basic properties in Section 5.1 and derive the Komatu–Loewner equation (1.9) for evolution families in Section 5.2. We then define Loewner chains and transfer (1.9) into the equation for Loewner chains in Section 5.3. Finally, we deduce the Komatu–Loewner equation (1.10) and (1.11) for the slits in Section 5.4.

Chapter 6 presents our third main result. We confirm that the Komatu–Loewner equation (1.10) and (1.11) for the slits enjoys the Carathéodory condition for any driving process $(\nu_t)_t$ in Section 6.1. This implies the existence of a local solution to (1.10) and (1.11). For domains $(D_t)_t$ determined by such a local solution, we study the behavior of the solutions to the Komatu–Loewner equation (1.9) for evolution families in Sections 6.2 and 6.3. We show that the solutions form a two-parameter family of univalent functions, but this family satisfies only a condition weaker than evolution families should do. In Section 6.3.2, we prove that this family is actually a unique evolution family associated to the driving process $(\nu_t)_t$ under the assumption that $\operatorname{supp} \nu_t$ is uniformly bounded.

Chapter 7 is the application of our results. In Sections 7.1 and 7.2, we derive the Komatu–Loewner equation for the mapping-out functions of hulls with local growth. In Section 7.3, we derive the same kind of equation for the mapping-out function of multiple paths. Although these equations have been obtained in previous studies, our argument provides a new understanding on them.

Chapter 8 collects some remarks for future studies.

Appendices A and B collect auxiliary results needed in Chapter 5. We introduce a continuity condition, which we call $(\mathrm{Lip})_\mu$, on a one-parameter family of holomorphic functions in Appendix A. This condition yields some nice differentiable properties of this family with respect to its "time" parameter. Appendix B is a review of hyperbolic and quasi-hyperbolic distances.

# Chapter 2

# Previous studies

## 2.1 Evolution families on the upper half-plane

In Section 1.1, we have summarized a conceptual understanding of Loewner's method on $\mathbb{D}$, following Pommerenke [52]. On the other hand, we shall owe practical techniques mainly to Goryainov and Ba [34], who studied evolution families on $\mathbb{H}$. We review their study briefly.

A holomorphic self-mapping $f\colon \mathbb{H} \to \mathbb{H}$ is called a *Pick function* and has the *Pick–Nevanlinna representation*

$$f(z) = a + bz + \int_{\mathbb{R}} \left( \frac{1}{\xi - z} - \frac{\xi}{1 + \xi^2} \right) \mu(d\xi). \tag{2.1}$$

Here, $a \in \mathbb{R}$, $b \geq 0$, and $\mu$ is a Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} (1 + \xi^2)^{-1} \mu(d\xi) < \infty$. The representation of Pick–Nevanlinna (2.1) is equivalent to that of Herglotz (1.6) through the Cauchy transform[1].

In order to define evolution families on $\mathbb{H}$ appropriately, it turns out to be natural to consider some normalization conditions on Pick functions. We say that a Pick function $f$ belongs to the class $\mathfrak{P}$ if there exists a finite Borel measure $\mu$ such that

$$f(z) = z + \int_{\mathbb{R}} \frac{1}{\xi - z} \mu(d\xi). \tag{2.2}$$

In addition, we say that $h \in \mathfrak{N}$ if $h(z) = \int_{\mathbb{R}} (\xi - z)^{-1} \mu(d\xi)$ for some finite Borel measure $\mu$. We use (2.2) instead of (2.1) in what follows. Indeed, a

[1]We refer the reader to Section 3, Chapter 5 of Rosenblum and Rovnyak [56] or Bondesson [13, Theorem 2.4.1] for details.

Pick function $f$ belongs to $\mathfrak{P}$ if and only if the following two normalization conditions hold [34, Lemma 1]:

- It is *hydrodynamically normalized at infinity* in the sense that

$$\lim_{\substack{z\to\infty \\ \Im z>\eta}} (f(z)-z) = 0 \quad \text{for any } \eta > 0; \tag{2.3}$$

- There exists a constant $c$ such that

$$\lim_{\substack{z\to\infty \\ \theta<\arg z<\pi-\theta}} z(f(z)-z) = c \quad \text{for any } \theta \in (0,\pi/2).$$

The constant $c$ is called the *angular residue* of $f$ at infinity (see Aleksandrov and Sobolev [3] for example) and given by $c = -\mu(\mathbb{R})$ with $\mu$ as in (2.2).

Let $\mathfrak{S} := \{\, f \in \mathfrak{P} \ ; \ f \text{ is univalent}\,\}$. Goryainov and Ba [34] defined an evolution family $(\phi_{t,s})_{0\le s\le t\le T}$ in $\mathfrak{S}$ by the following three conditions:

- $\phi_{t,t}(z) = z$ for any $t \in [0,T]$;
- $\phi_{u,s} = \phi_{u,t} \circ \phi_{t,s}$ for any $0 \le s \le t \le u \le T$;
- $\phi_{t,s}(z) \to z$ locally uniformly on $\mathbb{H}$ as $s, t \to u$.

The measure $\mu$ of (2.2) in the case $f = \phi_{t,s}$ is designated as $\mu_{t,s}$. The function $\lambda(t) := \mu_{t,0}(\mathbb{R})$ is non-decreasing and continuous, and the identity $\mu_{t,s}(\mathbb{R}) = \lambda(t) - \lambda(s)$ holds.

**Theorem 2.1** (Goryainov and Ba [34, Theorem 3])**.** *Suppose that $\lambda(t)$ is absolutely continuous. For each $s \in [0,T)$ and $z \in \mathbb{H}$, the function $w(t) := \phi_{t,s}(z)$ is an absolutely continuous solution to the differential equation*

$$\frac{dw(t)}{dt} = \dot{\lambda}(t) H(w(t),t) \quad \textit{a.e. } t \in [s,T] \tag{2.4}$$

*with initial value $w(s) = z$. Here, $H(z,t)$ is a function of two variables $z \in \mathbb{H}$ and $t \in [0,T]$. Moreover, the function $t \mapsto H(z,t)$ is Lebesgue measurable for each $z$, and $z \mapsto H(z,t)$ belongs to $\mathfrak{N}$ for each $t$.*

**Theorem 2.2** (Goryainov and Ba [34, Theorem 4])**.** *Suppose that a function $H(z,t)$ enjoys the same property as in Theorem 2.1. Then for every $s \in [0,T)$ and $z \in \mathbb{H}$, there exists a unique absolutely continuous solution $w(t) = w(t;s,z)$ to (2.4) with initial value $w(s) = z$. The functions $\phi_{t,s} \colon z \mapsto w(t;s,z)$, $0 \le s \le t \le T$, form an evolution family in $\mathfrak{S}$.*

Since $H(\cdot,t) \in \mathfrak{N}$ in these theorems, there exists a driving process $(\nu_t)_{t\in[0,T]}$ such that

$$H(z,t) = \int_{\mathbb{R}} \frac{1}{\xi - z}\, \nu_t(d\xi). \tag{2.5}$$

We here sketch how this driving process appears and later will go along a similar line of reasoning in Sections 5.1 and 5.2. Let $0 \le s \le u < v \le T$. By (2.2) in the case $f = \phi_{v,u}$, we have

$$\begin{aligned}\phi_{v,s}(z) - \phi_{u,s}(z) = \phi_{v,u}(\phi_{u,s}(z)) - \phi_{u,s}(z) &= \int_{\mathbb{R}} \frac{1}{\xi - \phi_{u,s}(z)}\, \mu_{v,u}(d\xi) \\ &= (\lambda(v) - \lambda(u)) \int_{\mathbb{R}} \frac{1}{\xi - \phi_{u,s}(z)} \frac{\mu_{v,u}(d\xi)}{\lambda(v) - \lambda(u)}.\end{aligned}$$

Since $\mu_{v,u}(\mathbb{R}) = \lambda(v) - \lambda(u)$, the measures $\mu_{v,u}/(\lambda(v) - \lambda(u))$, $s \le u < v \le T$, are probability ones. We can prove that this family of probability measures converges vaguely to some finite measure $\nu_t$ as $u, v \to t$ for a.e. $t \in [s,T]$. Thus, (2.4) holds with $H(z,t)$ given by (2.5).

Goryainov and Ba [34] did not refer to Loewner chains on $\mathbb{H}$, but a natural definition must be unique. A Loewner chain on $\mathbb{H}$ is a family $(f_t)_{t\in[0,T]} \in C([0,T];\mathfrak{S})$ such that $f_s(\mathbb{H}) \subset f_t(\mathbb{H})$ for $s < t$. Using the fact that $\phi_{t,s} := f_t^{-1} \circ f_s$ is an evolution family in $\mathfrak{S}$, we can easily transform (2.4) into the partial differential equation of $f_t(z)$.

**Remark 2.3** (radial vs. chordal). (i) Evolution families on $\mathbb{H}$ in this subsection are similar to evolution families on $\mathbb{D}$ in Section 1.1, but their fixed points are different. An evolution family $(\phi_{t,s})$ on $\mathbb{D}$ enjoys $\phi_{t,s}(0) = 0$, that is, fixes the *interior* point 0 of $\mathbb{D}$. Such a family is said to be *radial*. In contrast, an evolution family $(\phi_{t,s})$ on $\mathbb{H}$ enjoys $\lim_{z\to\infty}(\phi_{t,s}(z) - z) = 0$, that is, fixes the *boundary* point $\infty$ of $\mathbb{H}$ (in the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$). Such a family is said to be *chordal*. Since these two cases have fixed points with different topological properties, we cannot derive one from the other just by operating the Cayley transform. According to this classification, the theory developed in Sections 5 and 6 is a chordal case on finitely multiply-connected domains.

(ii) Using the theory of complex dynamics, Bracci, Contreras and Diaz-Madrigal [14] proposed a unified treatment of the radial and chordal cases. Some authors call it the *modern Loewner theory*. We use part of their idea in Chapter 5 but do not pursue such a great generality as they established.

## 2.2 Stochastic Loewner evolution

### 2.2.1 Mathematical basics of chordal SLE

Recall from Section 1.1 that SLE arises from the study on the interface curve $\gamma$ of two phases in a random model of statistical physics. We have observed only the radial Loewner equations in Section 1.1, whereas the equation (1.3) in the definition of SLE is chordal. Thus, we have given merely an abstract connection between (1.3) and Loewner's method in Section 1.1. We now provide a more concrete connection between (1.3) and (2.4).

Suppose that $\gamma\colon [0,T] \to \overline{\mathbb{H}}$ is a simple curve with $\gamma(0) \in \partial\mathbb{H}$ and $\gamma(0,T] \subset \mathbb{H}$. For the moment, it does not matter whether $\gamma$ is random or deterministic. By a version of Riemann's mapping theorem, there exists a unique conformal mapping $g_t\colon \mathbb{H} \setminus \gamma(0,T] \to \mathbb{H}$ with the hydrodynamic normalization $\lim_{z\to\infty}(g_t(z) - z) = 0$. This mapping is called the *mapping-out function* of $\gamma(0,T]$ by some authors. The aim is to study the behavior of $\gamma(t)$ through the analysis of $g_t$.

We can easily check that $g_t^{-1} \in \mathfrak{S}$ (cf. Proposition 4.1). In this case, the residue at infinity can be given without (2.2) as follows: By Carathéodory's boundary correspondence[2],

$$\lim_{z\to x_0} \Im g_t(z) = 0, \quad x_0 \in \partial\mathbb{H} \setminus \{\gamma(0)\}. \tag{2.6}$$

The function $g_t(z) - z$ thus extends to a holomorphic function on $\hat{\mathbb{C}} \setminus (\gamma[0,t] \cup \bar{\gamma}[0,t])$ by Schwarz's reflection principle. Here, $\bar{\gamma}(t)$ stands for the complex conjugate of $\gamma(t)$. The function $g_t(z)$ is now expanded as

$$g_t(z) = z + \frac{a_t}{z} + o(z^{-1}), \quad z \to \infty.$$

The positive constant $a_t$ is, by definition, the residue of $g_t$ at infinity. We also consider it to be a characteristic quantity of the curve $\gamma$. From this viewpoint, $a_t$ is called the *half-plane capacity* of the trace $\gamma(0,t]$ and denoted by $\operatorname{hcap}(\gamma(0,t])$.

Since $(g_{T-t}^{-1})_{t\in[0,T]}$ is a chordal Loewner chain, Theorem 2.1 applies. In addition, we can obtain further information in this case. Suppose that $\gamma$ is reparametrized so that $\operatorname{hcap}(\gamma(0,t]) = 2t$. Then $g_t(z)$ is $C^1$ in $t$ for each

[2] Chapter 2 of Pommerenke [53] contains a detailed exposition of the boundary correspondence between two simply connected domains through a conformal mapping. This correspondence also holds between two finitely multiply-connected domains. See Courant [23, Theorem 2.4] and references therein.

$z \in \mathbb{H} \setminus \gamma(0,t]$. The chordal Loewner equation for $g_t(z)$ takes a simpler form

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z \in \mathbb{H}. \tag{2.7}$$

Here, the *driving function* $\xi(t) := \lim_{z \to \gamma(t)} g_t(z) \in \partial\mathbb{H}$ is the image of the tip $\gamma(t)$.

We note that the contrary to the preceding paragraph is not true in general. The solutions $g_t(z)$, $z \in \mathbb{H}$, to (2.7) form the mapping-out function of a bounded hull, but this hull is not necessarily a curve. (We revisit this fact in Section 7.1.) With regard to this point, $\mathrm{SLE}_\kappa$ has nice properties as follows: Let $(B_t)_{t \geq 0}$ be the one-dimensional standard Brownian motion. (1.3) is the same equation as (2.7) with $\xi(t) = \sqrt{\kappa}B_t$ ($\kappa > 0$). The $\mathrm{SLE}_\kappa$ *hull*

$$\begin{aligned} F_t &:= \{\, z \in \mathbb{H}\, ; \text{the solution } g_t(z) \text{ (exists and) remains in } \mathbb{H} \text{ at } t \,\} \\ &= \mathbb{H} \setminus g_t^{-1}(\mathbb{H}), \quad t \geq 0, \end{aligned}$$

is a.s. a simple curve if $0 < \kappa \leq 4$ [55, Theorem 6.1] whereas it is a.s. not a simple curve if $\kappa > 4$ [55, Theorem 6.4]. Even in the latter case, the $\mathrm{SLE}_\kappa$ *trace* $\gamma(t) := \lim_{y \downarrow 0} g_t^{-1}(\sqrt{\kappa}B_t + iy)$ is continuous, and $F_t$ is generated by the curve $\gamma$ [55, Theorem 5.1]. Here, the last statement means that $\mathbb{H} \setminus F_t$ is a unique unbounded component of $\mathbb{H} \setminus \gamma(0,t]$.

**Remark 2.4** (Relationship between (2.4) and (2.7))**.** (i) In his original work [47], Löwner studied a radial version of (2.7). Chapter 6 of Ahlfors [1], for instance, gives a neat exposition on it. Our line of argument on the two equations (2.4) and (2.7) is thus reversed, compared with the history of Loewner equations.

(ii) In some literature on SLE, (2.4) is also discussed as a generalization of (2.7). See Section 1, Chapter 4 of Lawler [44]. He examined (2.4) in his own way under the assumption $\operatorname{supp} \nu_t \subset [-a, a]$, $a \in (0, \infty)$. We also impose this assumption in order for a solution to (1.10) and (1.11) to be unique in Section 6.3.2.

### 2.2.2 Link to physics

$\mathrm{SLE}_\kappa$ is proved to be the scaling limit of several stochastic processes on some lattice as its mesh size goes to zero. For example, it is the limit of loop-erased random walk ($\kappa = 2$), critical Ising interface ($\kappa = 3$), contour line of Gaussian free field ($\kappa = 4$), critical percolation exploration process ($\kappa = 6$), and uniform spanning tree ($\kappa = 8$). We briefly mention[3] a physical structure

[3]See the references [29, 30, 41] for further details.

lying behind these models in common. Before doing so, we refer the reader to Schramm [58], Rohde and Schramm [55], and references therein for looking at the correspondence of $\kappa$ to discrete random models without assuming such a particular knowledge of physics.

We consider a two-dimensional random model of statistical physics and its (inverse) temperature. In many cases, the property of the model in low temperature is much different from that in high temperature, and at critical temperature, the *conformal covariance* emerges. Characteristic physical quantities of the critical model have a certain covariance under conformal transformations. In particular, if we consider the interface curve as in Section 1.1, the probability law of this curve should be conformally invariant. This conformal invariance is a reason why Brownian motion appears in (1.3).

In general, the theory describing conformally covariant fields is called the *conformal field theory* (CFT for short) in physics. In CFT, the infinitesimal generators of conformal transformations form the Witt algebra, and the representation theory of its central extension, the Virasoro algebra, describes the property of conformally covariant fields. The central element $c$, which can be regarded as a complex number, is called the *central charge*.

It is a natural question how SLE is related to CFT. There are several approaches to this question, which we avoid listing here. In any approach, the parameter $\kappa$ and central charge $c$ are connected by the relation

$$c = -\frac{(6-\kappa)(8-3\kappa)}{2\kappa}.$$

See Friedrich and Werner [30] for instance. Since CFT is a rather large theory, links between SLE and CFT motivate us to study variants of SLE, such as multiple(-path) SLE in Section 2.2.3 and SLE on multiply connected domains (or on general Riemann surfaces) in Section 2.3.1.

### 2.2.3 Multiple-path SLE

In Sections 1.1, 2.2.1 and 2.2.2, we have considered the scaling limit of a *single* interface curve in a critical statistical model. If we consider *multiple* interface curves, then the *multiple(-path)*[4] *SLE* appears accordingly (see Figure 2.1). Precisely speaking, the $n$-multiple $\mathrm{SLE}_\kappa$ for $n \geq 1$ is the random family

[4] Here, the adjective "multiple" means that we consider several SLE traces which do not collide with each other. This usage of "multiple" is irrelevant to the connectivity of the underlying domain. Thus, there are no duplications of meaning in the phrase"multiple SLE in a multiply connected domain".

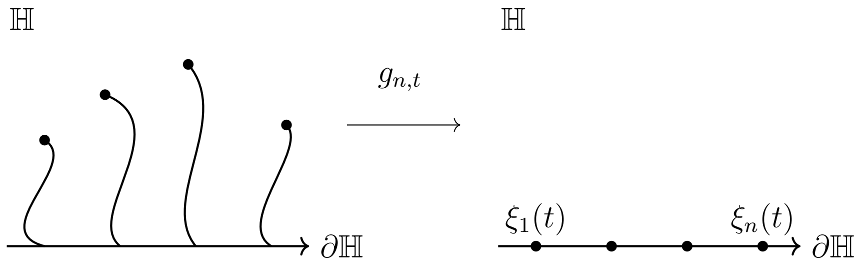


Figure 2.1: The $n$-multiple SLE.

$(g_{n,t})_{t\geq 0}$ determined by the Loewner equation

$$\frac{\partial g_{n,t}(z)}{\partial t} = \frac{1}{n}\sum_{k=1}^{n} \frac{2}{g_{n,t}(z) - \xi_k(t)}, \quad g_{n,0}(z) = z \in \mathbb{H}. \tag{2.8}$$

The driving functions $\xi_k(t)$, $k = 1, \dots, n$, are given by the stochastic differential equation (SDE for short)

$$d\xi_k(t) = \sqrt{\frac{\kappa}{n}}\, dB_k(t) + \frac{1}{n}\sum_{\substack{l=1\\ l\neq k}}^{n} \frac{4}{\xi_k(t) - \xi_l(t)}\, dt. \tag{2.9}$$

Here, $B_k(t)$, $k = 1, \dots, n$, are independent standard Brownian motions. See e.g. Bauer, Bernard and Kytölä [4] and Kozdron and Lawler [42] for the relation to single SLE and to CFT.

In this paper, we focus on a particular aspect of $n$-multiple SLE, apart from the CFT context, that it has the limit as $n \to \infty$. To see this, we recall that (2.9) is exactly a linear time-change of the SDE of *Dyson's non-colliding Brownian motions*. We define the empirical distribution on $\mathbb{R}$ by $\nu_t^{(n)} := n^{-1}\sum_{k=1}^{n} \delta_{\xi_k(t)} \in \mathcal{P}(\mathbb{R})$. Here, $\mathcal{P}(\mathbb{R})$ stands for the set of probability measures on $\mathbb{R}$ endowed with the weak topology.

**Theorem 2.5** (Rogers and Shi [54, Theorem 1]). (i) *Suppose that the sequence of the initial configurations $\nu_0^{(n)}$, $n = 1, 2, \dots$, weakly converges to a probability measure $\nu_0^{(\infty)}$ and that there exists a $C^\infty$ even function $f\colon \mathbb{R} \to [1, \infty)$ diverging to $+\infty$ as $x \to +\infty$ such that $\sup_{n\geq 1}\int_{\mathbb{R}} f(x)\, \nu_0^{(n)}(dx) < \infty$. Then for each fixed $T > 0$, the stochastic process $(\nu_t^{(n)})_{t\geq 0}$ converges in distribution to a (deterministic) process $(\nu_t^{(\infty)})_{t\geq 0}$ in the uniform topology of $C([0,T]; \mathcal{P}(\mathbb{R}))$.*

(ii) *$\nu_t$ weakly converges to a semicircle law as $t \to \infty$.*

Del Monaco and Schleißinger [25] studied the limit of $n$-multiple SLE combining Theorem 2.5 with (2.8). They showed that the sequence of $n$-multiple SLE $(g_{n,t})_t$ converges as $n \to \infty$ to a family $(g_{\infty,t})_t$ of conformal mappings given by the equation

$$\frac{\partial g_{\infty,t}(z)}{\partial t} = \int_{\mathbb{R}} \frac{2}{g_{\infty,t}(z) - \xi} \, \nu_t^{(\infty)}(d\xi) \tag{2.10}$$

in an appropriate sense. See also del Monaco, Hotta and Schleißinger [26] and Hotta and Katori [36] for related studies. (2.10) shows that differential equations like (2.4) rather than (2.7) and measure-valued driving processes appear through a natural procedure also in probability theory.

**Remark 2.6** (Other probabilistic models)**.** In this section, we have seen *chordal* SLE and its multiple version defined via the chordal Loewner equation (2.4). We can also consider *radial* SLE through the radial equation (1.1) or (1.2). Moreover, there are another kind of planar random models, such as (anisotropic) Hastings–Levitov cluster [38] and quantum Loewner evolution [48], defined via the radial Loewner equation. These models are conformally invariant and compared with diffusion-limited aggregation (DLA for brevity) in some contexts. DLA is a cluster which grows randomly as random walkers come from the point at infinity and stick to this cluster one after another. Although we do not know whether the scaling limit of DLA is conformally invariant or not, some researchers expect that DLA and the above two models have some properties in common.

## 2.3 Komatu–Loewner equations

### 2.3.1 Extending Löwner's work and SLE

Komatu [39, 40] tried to extend Löwner's original work [47] to a result on multiply connected domains. Let $D = \mathbb{A}_q \setminus \bigcup_{j=1}^{N} C_j$ be a circularly slit annulus, i.e., an annulus $\mathbb{A}_q = \{\, z \in \mathbb{C} \; ; \; q < |z| < 1 \,\}$ with some concentric arcs $C_j \subset \{\, z \; ; \; |z| = q_j \,\}$ $(q < q_j < 1,\ j = 1, \dots, N)$ removed. Suppose that $\gamma$ is a simple curve in $D$ growing from a point on the outer boundary $\partial\mathbb{D}$. Komatu showed that, if $\gamma$ is parametrized appropriately, the conformal mappings $g_t \colon D \setminus \gamma(0,t] \to D_t$, $t \geq 0$, obey a differential equation, which is called the bilateral[5] Komatu–Loewner equation.

[5] The adjective "bilateral" means that $g_t$ is normalized on the outer and inner boundary components. This case is not the same as radial or chordal one from a topological point of view (see Remark 2.3).

Bauer and Friedrich developed Komatu's idea to obtain the Komatu–Loewner equations on circularly slit disks [5, 6], circularly slit annuli and parallel slit half-planes [7] and discussed a way to define "SLE on multiply connected domains." Below we give a quick review on their result in the chordal setting.

Let $C_j$, $j = 1, \dots, N$ be disjoint line segments in $\mathbb{H}$ parallel to $\partial\mathbb{H}$, and put $D := \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j$. Suppose that $\gamma$ is a simple curve in $D$ growing from a point on $\partial\mathbb{H}$. Then for each $t \in (0, T)$, there exists a unique pair of a parallel slit half-plane $D_t$ and a conformal mapping $g_t \colon D \setminus \gamma(0,t] \to D_t$ with $\lim_{z\to\infty}(g_t(z) - z) = 0$. Assuming that $\gamma$ is parametrized in such a way that $\mathrm{hcap}^D(\gamma(0,t]) := \lim_{z\to\infty} z(g_t(z) - z) = 2t$, Bauer and Friedrich derived[6] the chordal Komatu–Loewner equation [7, Eq. (18)]

$$\frac{\partial g_t(z)}{\partial t} = -2\pi \Psi_{D_t}(g_t(z), \xi(t)), \quad g_0(z) = z \in D, \tag{2.11}$$

with $\xi(t) := \lim_{z\to\gamma(t)} g_t(z)$. We have already mentioned in Section 1.2 that $\Psi_{D_t}(z, \xi)$ in (2.11) is the complex Poisson kernel of BMD[7], but in the original paper [7] this kernel was obtained in a purely complex-analytic manner based on the Green function, harmonic measures, and their periods. This is a classical way to construct a conformal mapping from a given multiply-connected domain onto a slit domain of some standard type (see Section 5, Chapter 6 of Ahlfors [2] for instance).

After deriving (2.11), Bauer and Friedrich considered the evolution equation for the slits of $D_t$. Let $C_{j,t}$ be the slit of $D_t$ associated with $C_j$ by $g_t$ for each $j = 1, \dots, N$. The left and right endpoints of $C_{j,t}$ are designated by $z_j^\ell(t)$ and by $z_j^r(t)$, respectively. Bauer and Friedrich derived the differential equation [7, Eq. (30)]

$$\frac{dz_j^\ell(t)}{dt} = -2\pi \Psi_{D_t}(z_j^\ell(t), \xi(t)), \quad \frac{dz_j^r(t)}{dt} = -2\pi \Psi_{D_t}(z_j^r(t), \xi(t)), \tag{2.12}$$

which was later called the Komatu–Loewner equation for the slits [16, Section 2].

As explained in Section 1.2, the introduction of the equation (2.12) for the slits enables us to solve the equation (2.11) for $g_t(z)$ in the usual manner for ordinary differential equations. Namely, the solution to (2.12) determines

[6] Since their derivation focuses on the left $t$-derivative of $g_t(z)$, we refer the reader to Chen, Fukushima and Rohde [17, Eqs. (1.4) and (9.38)] for a detailed way to obtain the right derivative.

[7] We adopt the notation of Chen, Fukushima and Rohde [17] in this paper. As a result, the kernel $\Psi_{D_t}$ in (2.11) differs from $\Psi_t$ in [7, Eq. (18)] by a multiplicative constant $2\pi$.

the evolution of the slits $C_{j,t}$, $j = 1, \dots, N$, and hence of the domains $D_t$, and then the solution to (2.11) with $D_t$ so obtained determines the value of $g_t(z)$. In particular, Bauer and Friedrich discussed a random evolution $(g_t)_{t \geq 0}$ through these equations combined with a suitable SDE of the driving function $\xi(t)$. In order for the probability law to have conformal invariance (and domain Markov property), this SDE must be of the form ([7, Section 5], [16, Section 3])

$$d\xi(t) = \alpha(\xi(t), D_t)\, dB_t + b(\xi(t), D_t)\, dt. \tag{2.13}$$

Here, the coefficients enjoy the homogeneous properties $\alpha(r\xi, rD) = \alpha(\xi, D)$ and $b(r\xi, rD) = r^{-1}b(\xi, D)$ for any $r > 0$. Later, Chen and Fukushima [16] called such an evolution $(g_t)_{t \geq 0}$ the *stochastic Komatu–Loewner evolution* with coefficients $\alpha$ and $b$ and abbreviated it as $\mathrm{SKLE}_{\alpha,b}$.

**Remark 2.7** (SKLE and moduli diffusion). (i) The slits $C_{j,t}$ determines the conformal equivalence class of $D_t = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_{j,t}$, as is mentioned at the beginning of Section 1.2. For this reason, Bauer and Friedrich regarded (2.12) as a differential equation on the "moduli space" of $(N+1)$-connected planar domains with one marked point on boundary. The system of equations (2.12) and (2.13) determines the "moduli diffusion" $(\xi(t), z_j^{\ell}(t), z_j^{r}(t))$. In fact, Friedrich and Kalkkinen [29] and Kontsevich [41] studied conformally invariant probability measures on the space of paths on Riemann surfaces, which extends $\mathrm{SLE}_{\kappa}$, by means of CFT and differential geometry. Compared with their algebraic and geometric way, the moduli diffusion given by (2.12) and (2.13) expresses the random motion of moduli in an analytic, coordinate-based manner.

(ii) Equations similar to (2.11)–(2.13) also appear in Section 3, Chapter 3 of Zhan's thesis [60]. He defined "harmonic random Loewner chains" on finite Riemann surfaces in a different manner of thinking without solving (2.12) directly.

### 2.3.2 Probabilistic derivations

In his discussion on Laplacian-$b$ motion [45] on multiply connected domains, Lawler noticed that his method can be applied to another derivation of chordal Komatu–Loewner equation (2.11). He gave a probabilistic expression of $g_t$ and showed that $\Psi_{D_t}$ is exactly the complex Poisson kernel of the *excursion reflected Brownian motion* (ERBM for brevity) on parallel slit half-planes [45, Section 5]. Drenning [28] later implemented Lawler's idea in detail.

Up to this point, we have seen two standpoints from which (2.11) is derived. Chen, Fukushima and Rohde [17] performed a systematic study on the chordal Komatu–Loewner equation from both these two standpoints and showed that their hybrid method is quite effective in proving the continuity or differentiablity of the functions at issue. Moreover, they wrote down a more thorough description of the Lipschitz continuity of $\Psi_D$ as a function of variable domain $D$ than the previous one [7, Theorem 4.1]. Such a thorough work helps the analysis of the system (2.12) and (2.13) in the subsequent paper of Chen and Fukushima [16].

Chen, Fukushima and Rohde [17] employed BMD instead of ERBM to elaborate their work. Although ERBM and BMD "essentially" have the same distribution [17, Remark 2.2], the latter has a benefit of the theory of Dirichlet forms and symmetric Markov processes. Chen, Fukushima and Rohde investigated the connection between holomorphy and BMD-harmonicity, identified $\Psi_D$ with the complex Poisson kernel of BMD, and gave a probabilistic expression of mapping-out functions based on BMD. We review some of their results in Section 3.1.

On the basis of the elaborated work [17], Chen and Fukushima [16] closely studied the equation for slits (2.12) and $\mathrm{SKLE}_{\alpha,b}$, which enforces the framework of Bauer and Friedrich [7]. Chen, Fukushima and Suzuki [18] studied the relationship between SKLE and SLE as well. We can see the usefulness of martingale theory in their work.

### 2.3.3 Böhm's result for multiple-paths

Independently of Chen, Fukushima and Rohde [17], Böhm and Lauf [10] established the radial Komatu–Loewner equation for the mapping-out functions of traces of $n$ disjoint simple curves in circularly slit disks. After their paper, Böhm derived the Komatu–Loewner equations on circularly slit disks, circularly slit annuli and parallel slit half-planes in a similar and unified way in his doctor's thesis [9]. Based on his thesis, the chordal equation is stated as follows: Let $D$ be a parallel slit half-plane and $\gamma_k$, $k = 1, \ldots, n$, be $n$ disjoint simple curves in $D$ growing from distinct points on $\partial\mathbb{H}$. There exists a unique conformal mapping $g_t \colon D \setminus \bigcup_{k=1}^{n} \gamma_k(0,t] \to D_t$ hydrodynamically normalized at infinity. Let $a_t := \lim_{z\to\infty} z(g_t(z) - z)$.

**Theorem 2.8** (Böhm [9, Theorem 2.54]). *There exist a Lebesgue null set $N \subset [0,T]$ and functions $c_k(t) \geq 0$, $k = 1, \ldots, n$, such that*

- *$a_t$ and $g_t(z)$ are $t$-differentiable on $[0,T] \setminus N$ for each $z \in D \setminus \bigcup_{k=1}^{n} \gamma_k(0,T]$;*

- $\sum_{k=1}^{n} c_k(t) = \dot{a}_t$ *and*

$$\frac{\partial g_t(z)}{\partial t} = -\frac{\pi}{n}\sum_{k=1}^{n} c_k(t)\Psi_{D_t}(g_t(z), \xi_k(t)) \tag{2.14}$$

*holds for every* $z \in D \setminus \bigcup_{k=1}^{n} \gamma_k(0,T]$ *and* $t \in [0,T] \setminus N$.

*Here,* $\xi_k(t) := \lim_{z\to\gamma_k(t)} g_t(z)$ *for each* $k$.

We note that, in contrast to Section 2.3.1, Theorem 2.8 is formulated without reparametrizing $\gamma_k(t)$'s in a way that $a_t = 2t$. As long as $a_t$ and $g_t(z)$ are absolutely continuous[8] in $t$, the differential equation (2.14) is equivalent to its integrated form

$$g_t(z) = z - \frac{\pi}{n}\sum_{k=1}^{n}\int_0^t c_k(u)\Psi_{D_u}(g_u(z), \xi_k(u))\,du. \tag{2.15}$$

We would like to generalize this connection even if $a_t$ is not absolutely continuous. For example, if $a_t$ is Lebesgue's singular function, then $\dot{a}_t = 0$ for a.e. $t$. Hence the right-hand side of (2.15) equals zero. As this example illustrates, we have to replace the derivative in $t$ by the "derivative in $a_t$" itself in (2.14) for our purpose. We shall in fact do this replacement in Section 7.3, using the general theory constructed in Sections 4 and 5.

**Remark 2.9** (Parametrization of multiple paths)**.** The following assertion is formally similar to Theorem 2.8 but certainly different from it: Given traces $\Gamma_1, \dots, \Gamma_n$ of simple curves, there exist parametrizations $\gamma_k \colon [0,T] \to \Gamma_k$, $k = 1, \dots, n$, such that the conformal mapping $g_t$ associated with $\gamma_k$'s in the above-mentioned way satisfies (2.14) for some *constants* $c_k$, $k = 1, \dots, n$, and *all* $t \in [0,T]$. We do not study this type of problems in the present thesis. We refer the reader to Böhm and Schleißinger [11], Roth and Schleissinger [57] and Starnes [59] for this direction.

[8] Actually, if $a_t$ is absolutely continuous, then so is $g_t(z)$. See Chapter 5.

# Chapter 3

# Complex Poisson kernel of BMD

## 3.1 Basic facts

We collect basic facts on BMD needed later without proof. See Chen, Fukushima and Rohde [17] for the detail.

For a parallel slit half-plane[1] $D = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j$, we put $D^* := D \cup \{c_1^*, \dots, c_N^*\}$. Here, $c_j^*$'s are distinct points not in $D$. To be precise, $D^*$ is the quotient topological space of $\mathbb{H}$ obtained by regarding each of the slits $C_j$ as a single point $c_j^*$. The symbol $m_{D^*}$ denotes the Lebesgue measure on $D^*$ that does not charge $\{c_1^*, \dots, c_N^*\}$. The BMD $Z^* = ((Z_t^*)_{t \geq 0}, (\mathbb{P}_z^*)_{z \in D^*})$ is a unique $m_{D^*}$-symmetric diffusion process on $D^*$ with the following properties:

- The part process of $Z^*$ in $D$ is the absorbing Brownian motion on $D$;
- $Z^*$ admits no killings on $\{c_1^*, \dots, c_N^*\}$.

As absorbing Brownian motion is closely related to harmonic functions, so is BMD to holomorphic functions. We say that a continuous function $v \colon D^* \to \mathbb{R}$ is *BMD-harmonic* on $D^*$ if, for any relatively compact open subset $O$ of $D^*$, it holds that

$$v(z) = \mathbb{E}_z^* \left[ v(Z_{\tau_O}^*) \right], \quad z \in O.$$

Here, $\tau_O$ is the first exit time of $Z^*$ from $O$. Suppose that a continuous function $v$ on $D$ can be extended to a continuous one on $D^*$; that is, for any sequence $(z_n)_{n \in \mathbb{N}}$ with $d^{\mathrm{Eucl}}(z_n, C_j) \to 0$, the sequence $(v(z_n))_{n \in \mathbb{N}}$ converges to a constant $v(c_j^*)$. Here, $d^{\mathrm{Eucl}}$ stands for the Euclidean distance. Such a

[1] We can define BMD on more general spaces. See Chen and Fukushima [15, Chapter 7].

function $v$ is BMD-harmonic if and only if it is harmonic on $D$ in the usual sense and enjoys

$$\int_{\gamma_j} \frac{\partial v}{\partial \boldsymbol{n}_z}(z)\,|dz| = 0$$

for any smooth closed simple curve $\gamma_j$ surrounding $C_j$, $j = 1, \dots, N$. Here, $\boldsymbol{n}_z$ is the unit normal of $\gamma_j$ at $z$ pointing toward $C_j$. This integral is called the *period* of $v$ around $C_j$. The latter condition is also equivalent to the existence of a global harmonic conjugate on $D$. Hence, a continuous function $v$ on $D$ which is extendable to a one on $D^*$ is BMD-harmonic if and only if there exists a holomorphic function $f$ on $D$ such that $\Im f = v$.

Let us now observe that other potential theoretic concepts as well as the harmonicity with respect to BMD can be given by modifying those of absorbing Brownian motions. The absorbing Brownian motion in $\mathbb{H}$ is designated by $Z^{\mathbb{H}} = ((Z_t^{\mathbb{H}})_{t \geq 0}, (\mathbb{P}_z^{\mathbb{H}})_{z \in \mathbb{H}})$. We put, for each $j$,

$$\varphi^{(j)}(z) := \mathbb{P}_z^{\mathbb{H}}\left( Z^{\mathbb{H}}_{\sigma_{\bigcup_{k=1}^N C_k}} \in C_j \right), \quad z \in D.$$

Here, $\sigma_{\bigcup_{k=1}^N C_k}$ is the first hitting time of $Z^{\mathbb{H}}$ to the slits $\bigcup_{k=1}^N C_k$. The $N$-tuple $\Phi_D(z) := (\varphi^{(j)}(z))_{j=1}^N$ is called the *harmonic basis* of $D$. Let $a_{ij}$ be the period of $\varphi^{(j)}$ around $C_i$ for each $i, j \in \{1, \dots, N\}$. The matrix $\boldsymbol{A}_D = (a_{ij})_{i,j=1}^N$ is called the *period matrix* of $D$. By Section 4 of [17], the continuous version $G_D(z, w)$ of the 0-order resolvent kernel of the absorbing Brownian motion $Z^D$ in $D$ coincides with ($\pi^{-1}$ times) the classical Green function of $D$. There also exists a continuous version $G_D^*(z, w)$, $z, w \in D^*$, of the 0-order resolvent kernel of $Z^*$, which is given by the relation

$$G_D^*(z, w) = G_D(z, w) + 2\Phi_D(z)\boldsymbol{A}_D^{-1}\Phi_D(w)^{\mathrm{tr}}, \quad z, w \in D.$$

We also call $G_D^*$ the *BMD Green function* of $D^*$. Using this Green function, we further define the *BMD Poisson kernel*

$$\begin{aligned} K_D^*(z, \xi) &:= -\frac{1}{2}\frac{\partial}{\partial \boldsymbol{n}_\xi} G_D^*(z, \xi) \\ &= K_D(z, \xi) - \Phi_D(z)\boldsymbol{A}_D^{-1}\frac{\partial}{\partial \boldsymbol{n}_\xi}\Phi_D(\xi)^{\mathrm{tr}}. \end{aligned} \tag{3.1}$$

for $z \in D$ and $\xi \in \partial\mathbb{H} = \mathbb{R}$. Here, $\boldsymbol{n}_\xi$ is the unit normal at $\xi$ pointing downward, and $K_D(z, \xi) := -2^{-1}\partial_{\boldsymbol{n}_\xi} G_D(z, \xi)$ is the classical Poisson kernel of $D$. Since $K_D^*(z, \xi)$ is BMD-harmonic in the first variable, there exists a unique function $\Psi_D(z, \xi)$ of two variables $z \in D$ and $\xi \in \partial\mathbb{H}$ such that the following hold:

- $\Im\Psi_D(z,\xi) = K_D^*(z,\xi)$;
- For each $\xi$, the function $z \mapsto \Psi_D(z,\xi)$ is holomorphic;
- For each $\xi$, $\lim_{z\to\infty} \Psi_D(z,\xi) = 0$.

$\Psi_D(z,\xi)$ is, in fact, jointly continuous in $(z,\xi)$ and called the *BMD complex Poisson kernel*. The vector field $\Psi_{D_t}$ in (2.11) is exactly this kernel for $D_t$.

The Poisson kernel of $\mathbb{H}$ is given by

$$K_{\mathbb{H}}(x+iy,\xi) = \frac{1}{\pi}\frac{y}{(x-\xi)^2+y^2}.$$

Hence, by abuse of notation, we write the complex Poisson kernel of $\mathbb{H}$ (i.e., the Cauchy kernel) as

$$\Psi_{\mathbb{H}}(z,\xi) = -\frac{1}{\pi}\frac{1}{z-\xi}.$$

For each $\xi$, this fraction is a conformal mapping from $\mathbb{H}$ onto $\mathbb{H}$ and maps $\xi$ and $\infty$ to $\infty$ and $0$, respectively. For a parallel slit half-plane $D$, the function $\Psi_D(\cdot,\xi)$ is also a conformal mapping from $D$ onto another parallel slit half-plane and maps $\xi$ and $\infty$ to $\infty$ and $0$, respectively. In view of geometric function theory, this property is trivial to experts from the construction. It can be shown by a topological argument [17, Theorem 11.2] as well. Moreover, by Lemma 5.6 (ii) and Eq. (6.1) of [16], the Laurent expansion

$$\Psi_D(z,\xi) = \Psi_{\mathbb{H}}(z,\xi) + \frac{1}{2\pi} b_{\mathrm{BMD}}(\xi;D) + o(1), \quad z\to\xi, \tag{3.2}$$

around $\xi$ holds. Here, $b_{\mathrm{BMD}}(\xi;D)$ is called the *BMD domain constant*.

We state one more analogy to absorbing Brownian motion. Let $v$ be a BMD-harmonic function which is continuous and vanishing at infinity on $\overline{D}$. Then we have

$$v(z) = \mathbb{E}_z^*\left[v(Z_{\zeta^*-}^*)\right] = \int_{\mathbb{R}} K_D^*(z,\xi)v(\xi)\,d\xi. \tag{3.3}$$

Here, $\zeta^*$ is the lifetime of $Z^*$. Moreover,

$$\int_{\mathbb{R}} K_D^*(z,\xi)\,d\xi = \mathbb{P}_z^*\left(Z_{\zeta^*-}^* \in \partial\mathbb{H}\right) = 1 \tag{3.4}$$

for each $z\in D^*$. These two properties follow from the same argument as in Sections 4 and 5 of [17] if $D$ is a bounded domain. Then we can use the conformal invariance of BMD (see Theorem 7.8.1 and Remark 7.8.2 of [15]).

Finally, we give an identity that is the key to the proofs in Sections 3.2 and 3.3. By the strong Markov property of $Z^{\mathbb{H}}$, it holds that

$$G_D(z,\xi) = G_{\mathbb{H}}(z,\xi) - \mathbb{E}_z^{\mathbb{H}}\left[G_{\mathbb{H}}(Z^{\mathbb{H}}_{\sigma_{\bigcup_{j=1}^N C_j}},\xi)\,;\,\sigma_{\bigcup_{j=1}^N C_j} < \infty\right].$$

Taking the normal derivative and substituting it into (3.1), we have the key identity

$$\begin{aligned} K_D^*(z,\xi) - K_{\mathbb{H}}(z,\xi) = &-\mathbb{E}_z^{\mathbb{H}}\left[K_{\mathbb{H}}(Z^{\mathbb{H}}_{\sigma_{\bigcup_{j=1}^N C_j}},\xi)\,;\,\sigma_{\bigcup_{j=1}^N C_j} < \infty\right] \\ &+ \Phi_D(z)\boldsymbol{A}_D^{-1}\frac{\partial}{\partial \boldsymbol{n}_\xi}\Phi_D(\xi)^{\mathrm{tr}}. \end{aligned} \tag{3.5}$$

## 3.2 Complex Poisson integrals

Let $D = \mathbb{H} \setminus \bigcup_{j=1}^N C_j$ be a parallel slit half-plane. The aim of this subsection is to study the basic properties of BMD complex Poisson kernel $\Psi_D(z,\xi)$ as an integral kernel. We begin with the following inversion formula for BMD Poisson kernel $K_D^*(z,\xi)$:

**Proposition 3.1.** *Let $\mu$ be a finite Borel measure on $\mathbb{R}$. Then*

$$\lim_{y\downarrow 0}\int_{\mathbb{R}}\int_a^b K_D^*(x+iy,\xi)\,dx\,\mu(d\xi) = \mu((a,b)) + \frac{\mu(\{a\})+\mu(\{b\})}{2} \tag{3.6}$$

*holds for any $a<b$.*

*Proof.* The formula (3.6) with $K_D^*$ replaced by the classical Poisson kernel $K_{\mathbb{H}}$ is the so-called Stieltjes inversion formula. See Chapter 5, Section 4 of [56] or [13, Theorem 2.4.1] for instance. Therefore, it suffices to prove the convergence

$$\lim_{y\downarrow 0}\int_{\mathbb{R}}\int_a^b |K_D^*(x+iy,\xi) - K_{\mathbb{H}}(x+iy,\xi)|\,dx\,\mu(d\xi) = 0. \tag{3.7}$$

Let $Z^{\mathbb{H}} = ((Z_t^{\mathbb{H}})_{t\geq 0}, (\mathbb{P}_z^{\mathbb{H}})_{z\in\mathbb{H}})$ be the absorbing Brownian motion in $\mathbb{H}$ and $\sigma_A$ be the first hitting time of $Z^{\mathbb{H}}$ to a set $A \subset \mathbb{H}$. We put $\eta_D := \min\{\,\Im z\,;\, z \in \bigcup_{j=1}^N C_j\,\}$. Let $z = x+iy \in D$ with $0<y<\eta_D$. By the gambler's ruin estimate,

$$\mathbb{P}_z^{\mathbb{H}}\left(\sigma_{\bigcup_{j=1}^N C_j} < \infty\right) \leq \mathbb{P}_z^{\mathbb{H}}\left(\sigma_{\{\,z;\Im z=\eta_D\,\}} < \infty\right) \leq \frac{y}{\eta_D}.$$

Hence we have

$$\varphi^{(j)}(z) \leq \mathbb{P}^{\mathbb{H}}_z \left( \sigma_{\bigcup_{j=1}^N C_j} < \infty \right) \leq \frac{y}{\eta_D}, \tag{3.8}$$

$$0 < -\frac{\partial}{\partial \boldsymbol{n}_\xi} \varphi^{(j)}(\xi) \leq \frac{1}{\eta_D} \tag{3.9}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathbb{H}}_z \left[ K_{\mathbb{H}}(Z^{\mathbb{H}}_{\sigma_{\bigcup_{j=1}^N C_j}}, \xi) \,;\, \sigma_{\bigcup_{j=1}^N C_j} < \infty \right] \\
&\leq \mathbb{E}^{\mathbb{H}}_z \left[ (\pi \Im Z^{\mathbb{H}}_{\sigma_{\bigcup_{j=1}^N C_j}})^{-1} \,;\, \sigma_{\bigcup_{j=1}^N C_j} < \infty \right] \\
&\leq \frac{1}{\pi \eta_D} \mathbb{P}^{\mathbb{H}}_z \left( \sigma_{\bigcup_{j=1}^N C_j} < \infty \right) \leq \frac{y}{\pi (\eta_D)^2}.
\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10) with (3.5), we obtain

$$|K^*_D(z, \xi) - K_{\mathbb{H}}(z, \xi)| \leq \left( \frac{1}{\pi} + N \max_{1 \leq i,j \leq N} \left| (\boldsymbol{A}_D^{-1})_{ij} \right| \right) \frac{y}{(\eta_D)^2}, \tag{3.11}$$

which yields (3.7). $\square$

The next lemma gives us a useful estimate on $\Psi_D$. Its proof is almost the same as that of [50, Eq. (3.8)]. Here, $d^{\mathrm{Eucl}}$ stands for the Euclidean distance, and $a \wedge b := \min\{a, b\}$.

**Lemma 3.2.** *The inequality*

$$|\Psi_D(z, \xi)| \leq \frac{4}{\pi} \frac{1}{|z - \xi| \wedge d^{\mathrm{Eucl}}(\xi, \bigcup_{j=1}^N C_j)} \tag{3.12}$$

*holds for $z \in D$ and $\xi \in \partial \mathbb{H}$.*

*Proof.* It suffices to prove (3.12) in the case $\xi = 0$ because $\Psi_D(z, \xi) = \Psi_{D-\xi}(z - \xi, 0)$ holds for any $\xi \in \partial \mathbb{H}$ by [16, Eq. (3.31)]. Here, $D - \xi = \{\, z \,;\, z + \xi \in D \,\}$.

We define $r^{\mathrm{in}}_D := d^{\mathrm{Eucl}}(0, \bigcup_{j=1}^N C_j)$ and $T(z) := -1/z$. For each $r \in (0, r^{\mathrm{in}}_D]$, the function

$$h_r(z) := \frac{1}{\pi r} (T \circ \Psi_D)(rz, 0)$$

is univalent on $\mathbb{D}$. Its Taylor expansion around the origin is given by

$$\begin{aligned}
h_r(z) &= \frac{1}{\pi r} \cdot \frac{-1}{-\frac{1}{\pi r z} + \frac{1}{2\pi} b_{\mathrm{BMD}}(0; D) z + o(z)} \\
&= z + \frac{r b_{\mathrm{BMD}}(0; D)}{2} z^2 + o(z^2).
\end{aligned}$$

Hence we have $B(0,1/4) \subset h_r(\mathbb{D})$ by Koebe's one-quarter theorem (e.g. Chapter 7, Section 7 of [56]). This inclusion is equivalent to $\Psi_D(r\mathbb{D},0) \supset B(0,4(\pi r)^{-1})^c$. Since $\Psi_D(\cdot,0)$ is injective on $D$, we finally obtain

$$\Psi_D(D \setminus (r\mathbb{D}),0) \subset \overline{B(0,4(\pi r)^{-1})} \tag{3.13}$$

for all $r \in (0, r_D^{\mathrm{in}}]$, which yields (3.12) for $\xi = 0$. □

**Remark 3.3.** We would like to correct minor mistakes in [50]. As seen from the proof of Lemma 3.2, the right-hand side of [50, Eq. (3.8)] should be $4/(\pi r)$, not $1/(4\pi r)$. Also, the inequality in [50, Theorem 3.1 (ii)] should be replaced by $\zeta \geq y_0^2/16$.

**Corollary 3.4.** *For each $z \in D$, the function $\xi \mapsto \Psi_D(z,\xi)$ belongs to $C_\infty(\mathbb{R})$, the set of (complex-valued) continuous functions on $\mathbb{R}$ vanishing at infinity.*

By Corollary 3.4, we can consider the integral of BMD complex Poisson kernels with respect to finite Borel measures.

**Lemma 3.5.** (i) *The integral*

$$\Psi_D[\mu](z) := \int_{\mathbb{R}} \Psi_D(z,\xi)\,\mu(d\xi)$$

*defines a holomorphic function on $D$ for any finite Borel measure $\mu$ on $\mathbb{R}$.*

(ii) *Let $U \subset D$ be a set having an accumulation point in $D$. If there exist two finite Borel measures $\mu_1$ and $\mu_2$ on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \Psi_D(z,\xi)\,\mu_1(d\xi) = \int_{\mathbb{R}} \Psi_D(z,\xi)\,\mu_2(d\xi) \tag{3.14}$$

*for all $z \in U$, then $\mu_1 = \mu_2$.*

*Proof.* (i) The integrability of $\Psi_D(z,\xi)$ relative to $\mu(d\xi)$ is clear from Corollary 3.4. Let $U$ be any disk in $D$. Then for a smooth Jordan curve $\gamma$ in $U$,

$$\int_\gamma \Psi_D[\mu](z)\,dz = \int_{\mathbb{R}} \left( \int_\gamma \Psi_D(z,\xi)\,dz \right) \mu(d\xi) = 0$$

Here, Lemma 3.2 enables us to apply Fubuni's theorem. By Morera's theorem, $\Psi_D[\mu]$ is holomorphic on $U$ and hence on $D$.
(ii) By the identity theorem, (3.14) holds for all $z \in D$. Taking the imaginary part of (3.14) and applying Proposition 3.1, we get $\mu_1 = \mu_2$. □

## 3.3 Asymptotic behavior around infinity

We carry over the notations from Section 3.2.

**Proposition 3.6.** *The identity*

$$\lim_{z\to\infty} z\Psi_D(z,\xi) = -\frac{1}{\pi} \tag{3.15}$$

*holds for any* $\xi \in \partial\mathbb{H}$.

*Proof.* The function $z \mapsto \Psi_D(-1/z,\xi)$ is holomorphic around the origin and has a zero at $z=0$. Hence, $z\Psi_D(z,\xi)$ converges as $z\to\infty$, and the limit $\alpha$ is given by

$$\alpha := \lim_{z\to\infty} z\Psi_D(z,\xi) = -\lim_{z\to 0}\frac{1}{z}\Psi_D(-\frac{1}{z},\xi) = -\left.\frac{d}{dz}\Psi_D(-\frac{1}{z},\xi)\right|_{z=0}.$$

In particular, since $\Psi_D(x,\xi)$ is real for $x\in\mathbb{R}\setminus\{\xi\}$, we have

$$\alpha = \lim_{\substack{x\to\infty\\ x\in\mathbb{R}}} x\Psi_D(x,\xi) \in \mathbb{R}.$$

Thus,

$$\alpha = \lim_{\substack{y\uparrow\infty\\ y\in\mathbb{R}}} \Re(iy\Psi_D(iy,\xi)) = -\lim_{\substack{y\uparrow\infty\\ y\in\mathbb{R}}} yK_D^*(iy,\xi) = -\frac{1}{\pi}.$$

Here, the last equality follows from [16, Eq. (A.23)]. □

The rate of convergence in (3.15) may depend on $\xi$, but the next lemma shows that a certain uniform boundedness holds on every sectorial domain $\triangle_\theta := \{\, z\in\mathbb{H}\,;\, \theta < \arg z < \pi-\theta \,\}$, $\theta\in(0,\pi/2)$.

**Lemma 3.7.** *For any* $\theta\in(0,\pi/2)$, *it holds that*

$$\limsup_{\substack{z\to\infty\\ z\in D\cap\triangle_\theta}} \sup_{\xi\in\partial\mathbb{H}} |z\Psi_D(z,\xi)| < \infty. \tag{3.16}$$

*Proof.* (3.16) with $D$ replaced by $\mathbb{H}$ is trivial from the inequality

$$\sup_{\xi\in\partial\mathbb{H}} |z\Psi_{\mathbb{H}}(z,\xi)| = \frac{|z|}{\pi\Im z} < \frac{1}{\pi\sin\theta}, \quad z\in\triangle_\theta.$$

Therefore, it suffices to prove

$$\limsup_{\substack{z\to\infty\\ z\in D\cap\triangle_\theta}} \sup_{\xi\in\partial\mathbb{H}} |z(\Psi_D(z,\xi)-\Psi_{\mathbb{H}}(z,\xi))| < \infty. \tag{3.17}$$

Let us denote the real and imaginary parts of the difference $\Psi_D(z,\xi) - \Psi_{\mathbb{H}}(z,\xi)$ by $u_\xi(z)$ and $v_\xi(z)$, respectively. The imaginary part $v_\xi(z)$ is, by definition, equal to the left-hand side of (3.5). Then (3.17) is equivalent to

$$\limsup_{\substack{z\to\infty \\ z\in D\cap\triangle_\theta}} \sup_{\xi\in\partial\mathbb{H}} |zv_\xi(z)| < \infty \quad \text{and} \tag{3.18}$$

$$\limsup_{\substack{z\to\infty \\ z\in D\cap\triangle_\theta}} \sup_{\xi\in\partial\mathbb{H}} |zu_\xi(z)| < \infty. \tag{3.19}$$

In what follows, we use the notation

$$u_\xi(x,y) = u_\xi(x+iy) \quad \text{and} \quad v_\xi(x,y) = v_\xi(x+iy).$$

for $x, y \in \mathbb{R}$.

Let $r_D^{\mathrm{out}} := \sup\{\, |z| \,;\, z \in \bigcup_{j=1}^N C_j \,\}$. It follows from [44, Eq. (2.12)] that

$$\begin{aligned}\mathbb{P}_z^{\mathbb{H}}(\sigma_{\bigcup_{j=1}^N C_j} < \infty) &\le \mathbb{P}^{\mathbb{H}}_{z/r_D^{\mathrm{out}}}(\sigma_{\overline{\mathbb{D}}\cap\mathbb{H}} < \infty) \\ &= \frac{4r_D^{\mathrm{out}}}{\pi}\frac{\Im z}{|z|^2}\left(1 + O(|z|^{-1})\right) \quad (z\to\infty). \end{aligned}\tag{3.20}$$

We can combine (3.20) with (3.5) as in the proof of Proposition 3.1 to obtain the inequality

$$|v_\xi(z)| \le c_D \frac{\Im z}{|z|^2}\left(1 + O(|z|^{-1})\right) \quad (|z|\to\infty) \tag{3.21}$$

for a constant $c_D$ depending only on $D$. In particular, both the constant $c_D$ and term $O(|z|^{-1})$ are independent of $\xi$. Hence we get (3.18).

We shall derive (3.19) from (3.18) through the Cauchy–Riemann relation. To this end, we give proper estimates on the partial derivatives of $v_\xi$. For $t, x, y \in \mathbb{R}$ with $r_D^{\mathrm{out}} < t < y$, we define

$$V_\xi(x,y;t) := \int_{\mathbb{R}} p(x,y;s,t) v_\xi(s,t)\, ds.$$

Here,

$$p(x,y;s,t) := \frac{1}{\pi}\frac{y-t}{(x-s)^2 + (y-t)^2}$$

is the Poisson kernel for the half-plane $\mathbb{H}_t := \{\, z = x+iy \,;\, y > t \,\}$. Poisson's integral formula (e.g. Exercise 2 in Chapter 4, Section 6.4 of [2]) gives $V_\xi(x,y;t) = v_\xi(x,y)$ because $v_\xi$ is bounded and harmonic on $\mathbb{H}_t$ and continuous on $\overline{\mathbb{H}_t}$. For $y > 2r_D^{\mathrm{out}}$, we have

$$\begin{aligned}\frac{\partial}{\partial x} v_\xi(x,y) &= \frac{\partial}{\partial x} V_\xi\left(x, y; \frac{y}{2}\right) \\ &= \int_{\mathbb{R}} \frac{\partial}{\partial x} p\left(x,y;s,\frac{y}{2}\right) v_\xi\left(s, \frac{y}{2}\right) ds. \end{aligned}\tag{3.22}$$

By the relation between arithmetic and geometric means,

$$\left|\frac{\partial}{\partial x}p(x,y;s,t)\right| \leq \frac{2}{y-t}p(x,y;s,t). \tag{3.23}$$

Using (3.22) and (3.23), we get

$$\left|\frac{\partial}{\partial x}v_\xi(x,y)\right| \leq \frac{8}{y^2}\int_{\mathbb{R}} p\left(x,y;s,\frac{y}{2}\right)\left|\frac{y}{2}v_\xi\left(s,\frac{y}{2}\right)\right| ds \leq \frac{8M}{y^2}. \tag{3.24}$$

Here, the constant

$$M := \sup_{\substack{y>2r_D^{\mathrm{out}} \\ s\in\mathbb{R}}}\left|\frac{y}{2}v_\xi\left(s,\frac{y}{2}\right)\right| = \sup_{\substack{t>r_D^{\mathrm{out}} \\ s\in\mathbb{R}}}|tv_\xi(s,t)|$$

is finite by (3.21).

Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial y}v_\xi(x,y) &= \frac{d}{dy}V_\xi\left(x,y;\frac{y}{2}\right) \\ &= \int_{\mathbb{R}}\left(\frac{d}{dy}p\left(x,y;s,\frac{y}{2}\right)\right)v_\xi\left(s,\frac{y}{2}\right) ds \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} p\left(x,y;s,\frac{y}{2}\right)\left.\frac{\partial}{\partial t}v_\xi(s,t)\right|_{t=\frac{y}{2}} ds\end{aligned} \tag{3.25}$$

Since a derivative of $v_\xi$ is again harmonic, bounded and continuous on $\mathbb{H}_{y/2}$, the second integral in the last line of (3.25) equals $\partial_y v_\xi(x,y)$. Hence

$$\left|\frac{\partial}{\partial y}v_\xi(x,y)\right| \leq \frac{4}{y^2}\int_{\mathbb{R}} p\left(x,y;s,\frac{y}{2}\right)\left|\frac{y}{2}v_\xi\left(s,\frac{y}{2}\right)\right| ds \leq \frac{4M}{y^2}. \tag{3.26}$$

We now derive (3.19) from the above estimates. The Cauchy–Riemann relation implies

$$u_\xi(x,y) = u_\xi(0,y_0) - \int_{y_0}^{y}\partial_x v_\xi(0,t)\,dt + \int_0^x \partial_y v_\xi(s,y)\,ds \tag{3.27}$$

for $x\in\mathbb{R}$ and $y,y_0\in(r_D^{\mathrm{out}},\infty)$. Here,

$$u_\xi(0,y_0) = \Re(\Psi_D(iy_0,\xi) - \Psi_{\mathbb{H}}(iy_0,\xi)) \to 0 \quad \text{as } y_0\to\infty.$$

Hence by (3.24), we can let $y_0\to\infty$ in (3.27) to get

$$u_\xi(x,y) = -\int_\infty^y \partial_x v_\xi(0,t)\,dt + \int_0^x \partial_y v_\xi(s,t)\,ds. \tag{3.28}$$

By (3.24) and (3.26), the two integrals in (3.28) enjoy

$$\left|\int_\infty^y \partial_x v_\xi(0,t)\,dt\right| \le \frac{8M}{y} < \frac{8M}{|z|\sin\theta} \quad \text{and}$$
$$\left|\int_0^x \partial_y v_\xi(s,y)\,ds\right| \le \frac{4M|x|}{y^2} < \frac{4M}{|z|\sin\theta\tan\theta}$$

for $z = x + iy \in \triangle_\theta$ with $|z| > 2r_D^{\mathrm{out}}/\sin\theta$ (i.e., $y > 2r_D^{\mathrm{out}}$). Hence (3.19) follows. □

## 3.4 Dependence on slit domains

A set **Slit** is defined as the totality of vectors

$$\begin{aligned}\boldsymbol{s} &= (\boldsymbol{y}, \boldsymbol{x}^\ell, \boldsymbol{x}^r) \in (0,\infty)^N \times \mathbb{R}^N \times \mathbb{R}^N \\ &= (y_1, \dots, y_N, x_1^\ell, \dots, x_N^\ell, x_1^r, \dots, x_N^r)\end{aligned}$$

such that the following are true:

$$\begin{aligned} &x_j^\ell < x_j^r && \text{for } j = 1, \dots N, \\ &x_j^r < x_k^\ell \text{ or } x_k^r < x_j^\ell && \text{if } y_j = y_k \text{ for } j \neq k. \end{aligned}$$

An element $\boldsymbol{s} \in$ **Slit** represents the endpoints of the slits of a parallel slit half-plane. More precisely, we put

$$\begin{aligned} C_j(\boldsymbol{s}) &:= \{\, z = x + iy_j \,;\, x_j^\ell \le x \le x_j^r \,\}, \\ D(\boldsymbol{s}) &:= \mathbb{H} \setminus \bigcup_{j=1}^N C_j(\boldsymbol{s}). \end{aligned}$$

Then $D(\boldsymbol{s})$ is a parallel slit half-plane of $N$ slits. We also put

$$z_j^\ell := x_j^\ell + iy_j \quad \text{and} \quad z_j^r := x_j^r + iy_j,$$

which are the left and right endpoints of the slit $C_j(\boldsymbol{s})$, respectively. The space **Slit** is endowed with the distance

$$d_{\mathbf{Slit}}(\boldsymbol{s}, \tilde{\boldsymbol{s}}) := \max_{1\le j\le N} \left( |z_j^\ell - \tilde{z}_j^\ell| + |z_j^r - \tilde{z}_j^r| \right).$$

Chen, Fukushima and Rohde [17] established the local Lipschitz continuity of the BMD complex Poisson kernel $\Psi_{\boldsymbol{s}}(z,\xi) := \Psi_{D(\boldsymbol{s})}(z,\xi)$ as a function of $\boldsymbol{s}$. Here, we state part of their result. For a metric space $(X,d)$, we denote the open and closed balls with center $a \in X$ and radius $r > 0$ by $B_X(a,r)$ and $\bar{B}_X(a,r)$, respectively. We drop the subscript $X$ if there is no risk of misunderstanding.

**Proposition 3.8** (cf. [17, Theorem 9.1])**.** *Given a slit vector $\boldsymbol{s}_0 \in \mathbf{Slit}$, let $K$ be a compact subset of $D(\boldsymbol{s}_0)$ and $J$ be a bounded interval. There exist positive constants $\varepsilon_{\boldsymbol{s}_0,K}$ and $L_{\boldsymbol{s}_0,K,J}$ such that the inclusion $K \subset D(\boldsymbol{s})$ and inequality*

$$|\Psi_{\boldsymbol{s}}(z,\xi) - \Psi_{\boldsymbol{s}_0}(z,\xi)| \leq L_{\boldsymbol{s}_0,K,J}\, d_{\mathbf{Slit}}(\boldsymbol{s},\boldsymbol{s}_0)$$

*hold for any $z \in K$, $\xi \in J$ and $\boldsymbol{s} \in B(\boldsymbol{s}_0, \varepsilon_{\boldsymbol{s}_0,K})$. Moreover, $\varepsilon_{\boldsymbol{s}_0,K}$ depends only on $\boldsymbol{s}_0$ and $K$, not on $J$.*

In the proof of the next proposition, it is crucial that $\varepsilon_{\boldsymbol{s}_0,K}$ in Proposition 3.8 is independent of $J$.

**Proposition 3.9.** *Let $\boldsymbol{s}_0 \in \mathbf{Slit}$ and $z_0 \in D(\boldsymbol{s}_0)$ be fixed. Then*

$$\lim_{\substack{\boldsymbol{s}\to\boldsymbol{s}_0 \\ z\to z_0}} \sup_{\xi\in\partial\mathbb{H}} |\Psi_{\boldsymbol{s}}(z,\xi) - \Psi_{\boldsymbol{s}_0}(z_0,\xi)| = 0. \tag{3.29}$$

*Proof.* The function

$$f_{\boldsymbol{s},z}(\xi) := \Psi_{\boldsymbol{s}}(z,\xi), \quad \boldsymbol{s} \in \mathbf{Slit},\ z \in D(\boldsymbol{s})$$

of $\xi$ can be regarded as an element of $C(\mathbb{R} \cup \{\infty\})$ by Corollary 3.4. Note that $\mathbb{R} \cup \{\infty\}$ is homeomorphic to just the circle, which is compact. We take a constant $r > 0$ such that $K := \bar{B}_{\mathbb{C}}(z_0, r) \subset D(\boldsymbol{s}_0)$ and set

$$\varepsilon_0 := \frac{1}{5} \min\{r, \eta_{D(\boldsymbol{s}_0)}, \varepsilon_{\boldsymbol{s}_0,K}\}$$

with $\varepsilon_{\boldsymbol{s}_0,K}$ given as in Proposition 3.8. Here, we recall that $\eta_{D(\boldsymbol{s}_0)} = \min\{\, \Im z\,;\, z \in \bigcup_{j=1}^{N} C_j(\boldsymbol{s}_0)\,\}$. In order to prove (3.29), it suffices to show the relative compactness of

$$\mathcal{F} := \{\, f_{\boldsymbol{s},z}\,;\, \boldsymbol{s} \in B_{\mathbf{Slit}}(\boldsymbol{s}_0,\varepsilon_0),\ z \in B_{\mathbb{C}}(z_0,\varepsilon_0)\,\}$$

in $C(\mathbb{R} \cup \{\infty\})$ equipped with the supremum norm. Indeed, by using Proposition 3.8, we can easily show the pointwise convergence

$$\lim_{\substack{\boldsymbol{s}\to\boldsymbol{s}_0 \\ z\to z_0}} f_{\boldsymbol{s},z}(\xi) = f_{\boldsymbol{s}_0,z_0}(\xi).$$

Thus, the limit of any uniformly convergent subsequence in $\mathcal{F}$ as $(\boldsymbol{s},z) \to (\boldsymbol{s}_0,z_0)$ is unique.

We check the assumptions of the Arzelà–Ascoli theorem. The uniform boundedness of $\mathcal{F}$ is obvious from Lemma 3.2. The equicontinuity of $\mathcal{F}$ at $\infty$ is also trivial from the same lemma. Thus, the proof is complete once we show the equicontinuity at each point $\xi_0 \in \mathbb{R}$. For $\xi \in \mathbb{R}$, we denote by $\widehat{\xi}$ the vector of $\mathbb{R}^{3N}$ whose first $N$ entries are zero and other $2N$ entries are $\xi$. For $\xi \in J := (\xi_0 - \varepsilon_0, \xi_0 + \varepsilon_0)$, we have

$$
\begin{aligned}
|f_{\boldsymbol{s},z}(\xi) - f_{\boldsymbol{s},z}(\xi_0)| &= |\Psi_{\boldsymbol{s}}(z,\xi) - \Psi_{\boldsymbol{s}}(z,\xi_0)| \\
&\le |\Psi_{\boldsymbol{s}-\widehat{\xi}+\widehat{\xi_0}}(z-\xi+\xi_0,\xi_0) - \Psi_{\boldsymbol{s}-\widehat{\xi}+\widehat{\xi_0}}(z,\xi_0)| \\
&\quad + |\Psi_{\boldsymbol{s}-\widehat{\xi}+\widehat{\xi_0}}(z,\xi_0) - \Psi_{\boldsymbol{s}}(z,\xi_0)| \\
&\le \sup_{w \in \bar{B}(z_0,2\varepsilon_0)} |\partial_w \Psi_{\boldsymbol{s}-\widehat{\xi}+\widehat{\xi_0}}(w,\xi_0)||\xi - \xi_0| \\
&\quad + L_{\boldsymbol{s}_0,K,J}\, d_{\mathbf{Slit}}(\boldsymbol{s}, \boldsymbol{s} - \widehat{\xi} + \widehat{\xi_0}).
\end{aligned} \tag{3.30}
$$

Since the family of holomorphic functions

$$
w \mapsto \Psi_{\tilde{\boldsymbol{s}}}(w,\xi_0), \quad \tilde{\boldsymbol{s}} \in B_{\mathbf{Slit}}(\boldsymbol{s}, 2\varepsilon_0),
$$

is locally bounded on the disk $B_{\mathbb{C}}(z, 3\varepsilon_0)$ by Lemma 3.2, so is $\partial_w \Psi_{\tilde{\boldsymbol{s}}}(w,\xi_0)$ by Cauchy's integral formula. In particular,

$$
M := \sup_{\substack{w \in \bar{B}_{\mathbb{C}}(z,2\varepsilon_0) \\ \tilde{\boldsymbol{s}} \in B_{\mathbf{Slit}}(\boldsymbol{s},2\varepsilon_0)}} |\partial_w \Psi_{\tilde{\boldsymbol{s}}}(w,\xi)| < \infty.
$$

Thus, (3.30) implies that

$$
|f_{\boldsymbol{s},z}(\xi) - f_{\boldsymbol{s},z}(\xi_0)| \le (L_{\boldsymbol{s}_0,K,J} + M)|\xi - \xi_0|
$$

for any $\boldsymbol{s} \in B_{\mathbf{Slit}}(\boldsymbol{s}_0, \varepsilon_0)$ and $z \in B_{\mathbb{C}}(z_0, \varepsilon_0)$, which leads us to the equicontinuity of $\mathcal{F}$ at $\xi_0$. $\square$

# Chapter 4

# Conformal mappings on slit domains

## 4.1 Normalization conditions at infinity

We shall treat univalent functions on parallel slit half-planes which are normalized properly at infinity in the argument starting in Section 4.2. We state our normalization conditions and their basic consequences.

Let $f$ be a univalent function defined on a domain which contains some half-plane, i.e., a domain of the form $\mathbb{H}_\eta = \{ z \in \mathbb{C} \,;\, \Im z > \eta \}$, $\eta \geq 0$. The function $f$ is said to be *hydrodynamically normalized at infinity* if

$$\lim_{\substack{z \to \infty \\ z \in \mathbb{H}_\eta}} (f(z) - z) = 0 \quad \text{for some } \eta \geq 0.$$

Whenever we say that $f$ satisfies this property, we implicitly assume that the domain of $f$ contains some half-plane. In addition, if there exists $c \in \mathbb{C}$ such that

$$\lim_{\substack{z \to \infty \\ z \in \triangle_\theta}} z(f(z) - z) = c \quad \text{for any } \theta \in (0, \pi/2),$$

then we call $c$ the *angular residue* of $f$ at infinity. Here, recall that $\triangle_\theta = \{ z \in \mathbb{H} \,;\, \theta < \arg z < \pi - \theta \}$.

Propositions 4.1 and 4.2 below ensure that these normalization conditions are preserved by taking the inverse and composite of functions.

**Proposition 4.1.** *Let $f \colon D \to \mathbb{C}$ be a univalent function which is hydrodynamically normalized at infinity. Then so is the inverse $f^{-1} \colon f(D) \to D$. If, moreover, $f$ has the finite angular residue $c$ at infinity, then that of $f^{-1}$ is $-c$.*

*Proof.* We can take positive constants $\eta$ and $L$ so that

$$|f(z) - z| < 1, \quad z \in \mathbb{H}_\eta \setminus \bar{B}(0, L).$$

Since $\mathbb{H}_{\eta+L} \subset \mathbb{H}_\eta \setminus \bar{B}(0, L)$, it is clear from this inequality that $\mathbb{H}_{\eta+L+1} \subset f(\mathbb{H}_{\eta+L})$. Now, let $\varepsilon > 0$. There exists $L' \geq 0$ such that $|f(z) - z| < \varepsilon$ holds for $z \in \mathbb{H}_\eta \setminus \bar{B}(0, L')$. We have

$$|f^{-1}(w) - w| = |f^{-1}(w) - f(f^{-1}(w))| < \varepsilon$$

for $w \in \mathbb{H}_{\eta+L+1} \setminus \bar{B}(0, L' + 1)$ because $f^{-1}(w) \in \mathbb{H}_{\eta+L} \setminus \bar{B}(0, L')$ for such $w$. Thus, $f^{-1}$ is hydrodynamically normalized at infinity.

Suppose that $f$ has the finite angular residue $c$ at infinity. Let $\theta \in (0, \pi/2)$. For $w \in \triangle_\theta \setminus \bar{B}(0, (\eta + L + 2)/\sin\theta)$, we have $\Im w \geq 2$, $|w| \geq 1$ and $|f^{-1}(w) - w| < 1$ because $w \in \mathbb{H}_{\eta+L+2}$. These inequalities yield

$$\frac{|f^{-1}(w)|}{\Im f^{-1}(w)} < \frac{|w| + 1}{\Im w - 1} < \frac{2|w|}{\Im w/2} < \frac{4}{\sin\theta}.$$

Hence $f^{-1}(\triangle_\theta \setminus \bar{B}(0, (\eta+L+2)/\sin\theta)) \subset \triangle_{\theta'}$ holds with $\theta'$ given by $4\sin\theta' = \sin\theta$. From the identity

$$w(f^{-1}(w) - w) = -f^{-1}(w)(f(f^{-1}(w)) - f^{-1}(w)) - (f^{-1}(w) - w)^2,$$

we get

$$\lim_{\substack{w\to\infty \\ w\in\triangle_\theta}} w(f^{-1}(w) - w) = -\lim_{\substack{z\to\infty \\ z\in\triangle_{\theta'}}} z(f(z) - z) = -c. \qquad \square$$

The proof of the next proposition is quite similar, and we omit it.

**Proposition 4.2.** *Let $f\colon D \to \mathbb{C}$ and $g\colon D' \to \mathbb{C}$ be univalent functions which are hydrodynamically normalized at infinity. Then so is the composite $g|_{D'\cap f(D)} \circ f|_{f^{-1}(D')}$. If, moreover, they have the finite angular residues $c_f$ and $c_g$, respectively, then that of $g \circ f$ is $c_f + c_g$.*

## 4.2 Integral representation

In this section, we establish the integral representation (1.8). Although it is similar to the formula (2.2) coming from the Pick–Nevanlinna representation (2.1), we can hardly expect any analogous proof on parallel slit half-planes. For this reason, we adopt a different approach from that of Goryainov and Ba [34, Lemma 1], using the properties of BMD viewed in Chapter 3.

We say that a set $F \subset \mathbb{H}$ is a (possibly unbounded) $\mathbb{H}$-*hull* if $F$ is relatively closed in $\mathbb{H}$ and if $\mathbb{H} \setminus F$ is a simply connected domain. The term "$\mathbb{H}$-hull" is used only for bounded ones in some cases, but we here use it even if $F$ is unbounded. In the following theorem, the assumption on $f(D_1)$ expresses just the condition that "the holes of $f(D_1)$ coincide with the slits of $D_2$ exactly."

**Theorem 4.3.** *Let $D_1$ and $D_2$ be parallel slit half-planes of $N$ slits and $f\colon D_1 \to D_2$ be a univalent function such that $D_2 \setminus f(D_1)$ is a possibly unbounded $\mathbb{H}$-hull. Then $f$ is hydrodynamically normalized and has a finite angular residue at infinity if and only if there exists a finite Borel measure $\mu$ on $\mathbb{R}$ such that*

$$f(z) = z + \pi \int_{\mathbb{R}} \Psi_{D_1}(z, \xi)\, \mu(d\xi), \quad z \in D_1. \tag{4.1}$$

*In this case, the limit*

$$\Im f(x) := \lim_{y \downarrow 0} \Im f(x + iy)$$

*exists for Lebesgue a.e. $x \in \mathbb{R}$, and $\mu$ is uniquely given by $\mu(d\xi) = \pi^{-1} \Im f(\xi)\, d\xi$. Moreover, the angular residue of $f$ at infinity is $-\mu(\mathbb{R})$.*

*Proof.* We begin with the proof of the "if" part. Suppose that (4.1) holds, and let $\eta$ be sufficiently large. Lemma 3.2 implies that $|\Psi_{D_1}(z, \xi)| \leq 4(\pi\eta_{D_1})^{-1}$ for $z \in \mathbb{H}_\eta$ and $\xi \in \partial\mathbb{H}$. Here, $\eta_{D_1} = \min\{\, \Im z \ ;\ z \in \mathbb{H} \setminus D_1 \,\}$. From the bounded convergence theorem we get

$$\lim_{\substack{z \to \infty \\ z \in \mathbb{H}_\eta}} (f(z) - z) = \lim_{\substack{z \to \infty \\ z \in \mathbb{H}_\eta}} \int_{\mathbb{R}} \Psi_{D_1}(z, \xi)\, \mu(d\xi) = 0. \tag{4.2}$$

Similarly, by (3.15) and (3.16) we have

$$\lim_{\substack{z \to \infty \\ z \in \triangle_\theta}} z(f(z) - z) = \pi \lim_{\substack{z \to \infty \\ z \in \triangle_\theta}} \int_{\mathbb{R}} z \Psi_{D_1}(z, \xi)\, \mu(d\xi) = -\mu(\mathbb{R}). \tag{4.3}$$

We move to the proof of the "only if" part. Suppose that $f$ is hydrodynamically normalized and has a finite angular residue at infinity. The supremum $M := \sup_{\xi \in \mathbb{R}} \Im f(\xi + i\eta_0)$ is finite for some $\eta_0 > 0$. Since $f$ maps $D_1 \setminus \overline{\mathbb{H}_{\eta_0}}$ into $D_2 \setminus \overline{\mathbb{H}_M}$, we see that $\Im f$ is bounded by $M$ on $D_1 \setminus \overline{\mathbb{H}_{\eta_0}}$. Let $Z^* = ((Z^*_t)_{t \geq 0}, (\mathbb{P}^*_z)_{z \in D^*_1})$ be the BMD on $D^*_1$. Given $\eta \in (0, \eta_{D_1})$, the

BMD-harmonic function $\Im f(x+iy)-y$ on $D_1\cap\mathbb{H}_\eta$, which is continuous and vanishing at infinity on $\partial\mathbb{H}_\eta$, enjoys

$$\begin{aligned}\Im f(x+iy)-y &= \mathbb{E}^*_{x+iy}\left[\Im f(Z^*_{\sigma_{\partial\mathbb{H}_\eta}})-\Im Z^*_{\sigma_{\partial\mathbb{H}_\eta}}\ ;\ \sigma_{\partial\mathbb{H}_\eta}<\infty\right]\\ &= \int_{\mathbb{R}} K^*_{D_1\cap\mathbb{H}_\eta}(x+iy,\xi+i\eta)(\Im f(\xi+i\eta)-\eta)\,d\xi\end{aligned}$$

for $z=x+iy\in D_1\cap\mathbb{H}_\eta$ by (3.3). Here, $\sigma_{\partial\mathbb{H}_\eta}$ is the first hitting time of $Z^*$ on $D_1^*$. Letting $\eta\to 0$, we have

$$\Im f(x+iy)-y=\lim_{\eta\downarrow 0}\int_{\mathbb{R}} K^*_{D_1\cap\mathbb{H}_\eta}(x+iy,\xi+i\eta)\Im f(\xi+i\eta)\,d\xi \tag{4.4}$$

for any $z=x+iy\in D_1$.

We can change the order of the limit and integral in (4.4) as follows: Putting $\tilde{D}_{1,\eta}:=\{\,z\in\mathbb{H}\ ;\ z+i\eta\in D_1\cap\mathbb{H}_\eta\,\}$, we get

$$\begin{aligned}\lim_{\eta\downarrow 0} K^*_{D_1\cap\mathbb{H}_\eta}(x+iy,\xi+i\eta) &= \lim_{\eta\downarrow 0} K^*_{\tilde{D}_{1,\eta}}(x+i(y-\eta),\xi)\\ &= K^*_{D_1}(x+iy,\xi)\end{aligned}$$

from Proposition 3.9. Thus, by (3.4) we can use Scheffé's lemma to obtain

$$\lim_{\eta\downarrow 0}\int_{\mathbb{R}}\left|K^*_{D_1\cap\mathbb{H}_\eta}(x+iy,\xi+i\eta)-K^*_{D_1}(x+iy,\xi)\right|\,d\xi=0. \tag{4.5}$$

In addition, the bounded harmonic function $\Im f(\xi+i\eta)$ converges as $\eta\downarrow 0$ for a.e. $\xi\in\mathbb{R}$ by Fatou's theorem (e.g. [32, Corollary 2.5]). Now, we decompose the integral in (4.4) as

$$\begin{aligned}&\int_{\mathbb{R}} K^*_{D_1\cap\mathbb{H}_\eta}(x+iy,\xi+i\eta)\Im f(\xi+i\eta)\,d\xi\\ &=\int_{\mathbb{R}}\Big(K^*_{D_1\cap\mathbb{H}_\eta}(x+iy,\xi+i\eta)-K^*_{D_1}(x+iy,\xi)\Big)\Im f(\xi+i\eta)\,d\xi \\ &\quad+\int_{\mathbb{R}} K^*_{D_1}(x+iy,\xi)\Im f(\xi+i\eta)\,d\xi.\end{aligned} \tag{4.6}$$

Since $\Im f(\xi+i\eta)\le M$, the former integral in the right-hand side of (4.6) converges to zero by (4.5). To the latter integral we can apply the dominated convergence theorem. Thus,

$$\Im f(x+iy)-y=\int_{\mathbb{R}} K^*_{D_1}(x+iy,\xi)\Im f(\xi)\,d\xi. \tag{4.7}$$

Let $\mu(d\xi) := \pi^{-1}\Im f(\xi)\, d\xi$. We have

$$\begin{aligned}\mu(\mathbb{R}) &= \pi \int_{\mathbb{R}} \lim_{y \nearrow \infty} y K^*_{D_1}(iy,\xi)\,\mu(d\xi) \\ &\leq \liminf_{y\nearrow\infty} y \cdot \pi \int_{\mathbb{R}} K^*_{D_1}(iy,\xi) \Im f(\xi)\, d\xi \\ &= \lim_{y\nearrow\infty} y(\Im f(iy) - y) \end{aligned} \tag{4.8}$$

using (3.15), Fatou's lemma and (4.7). The limit in the rightmost side of (4.8) is finite by the assumption of finite angular residue, which implies that $\mu$ is a finite Borel measure on $\mathbb{R}$. The inequality in (4.8) is, in fact, an equality by (3.16) and the bounded convergence theorem. Since the real part of $f$ is uniquely determined by the condition $\lim(f(z) - z) = 0$, we have (4.1).

Finally, the uniqueness of $\mu$ follows from Lemma 3.5 (ii). □

**Remark 4.4** (Limit along BMD paths)**.** (i) In Theorem 4.3, it is assumed that $f\colon D_1 \to D_2$ is univalent, which is the case appearing in this thesis. As is easily seen from the above-mentioned proof, we can replace this assumption with a slightly weaker one that $\Im f$ is BMD-harmonic and is bounded on some strip $\{z\,;\, 0 < \Im z < \eta_0\}$.

(ii) We give a rough sketch of another possible line of proof based on probabilistic potential theory for Theorem 4.3. Let $\mathbb{P}^{*;\xi}_z$ be the Doob transform of $\mathbb{P}^*_z$ by $K^*_{D_1}(\cdot,\xi)$, and suppose that $\hat{v}(\xi) := \lim_{t\nearrow\zeta^*} \Im f(Z^*_t)$ exists $\mathbb{P}^{*;\xi}_z$-a.s. for a.e. $\xi \in \partial\mathbb{H}$. Then by the martingale convergence theorem, we have

$$\begin{aligned}\Im f(z) &= \mathbb{E}^*_z\left[\Im f(Z^*_{\sigma_{\partial\mathbb{H}_\eta}})\right] \\ &= \mathbb{E}^*_z\left[\hat{v}(Z^*_{\zeta^*-})\right] = \int_{\mathbb{R}} K^*_{D_1}(z,\xi)\hat{v}(\xi)\, d\xi.\end{aligned}$$

Here, $\hat{v}(\xi)$ is not the limit along the vertical line in Theorem 4.3 but the one along BMD paths. For absorbing Brownian motion on $\mathbb{H}$, Doob [27] deeply studied the relationship between the limit along vertical lines or within sectors and the one along Brownian paths. In fact, we can also give an alternative proof of Theorem 4.3, simply using the property that the part process of $Z^*$ in $D_1$ is the absorbing Brownian motion on $D_1$.

We introduce a new notation. A unique measure $\mu$ representing $f$ in (4.1) is designated by $\mu_f$ or $\mu(f;\cdot)$. We call the total mass $\mu(f;\mathbb{R})$ the *BMD half-plane capacity* of the $\mathbb{H}$-hull $F = D_2 \setminus f(D_1)$ in $D_2$ and denote it by

$\mathrm{hcap}^{D_2}(F)$. These definitions, which coincide with the classical ones if there are no slits and if $F$ is bounded, go well even in the present case thanks to the following uniqueness result:

**Corollary 4.5.** *Let $D_1$ and $D_2$ be parallel slit half-planes of $N$ slits and $F$ be a possibly unbounded $\mathbb{H}$-hull. A conformal mapping $f\colon D_1 \to D_2 \setminus F$ with the hydrodynamic normalization at infinity is unique if it exists.*

*Proof.* Suppose that two mappings $f$ and $g$ satisfy our assumption. Then $h := g^{-1} \circ f$ is a conformal automorphism on $D_1$ with the hydrodynamic normalization at infinity by Propositions 4.1 and 4.2. The boundary function $\Im h(\xi)$ is zero for all $\xi \in \mathbb{R}$ by the boundary correspondence. In addition, we can see in the same way as in the proof of Theorem 4.3 that $\Im(h(z) - z)$ is BMD-harmonic and bounded. It follows from the maximum principle for BMD-harmonic functions that $\Im(h(z) - z) = 0$ on $D_1$. Thus, there exists a real constant $\alpha$ such that $h(z) - z = \alpha$, but $\alpha$ must be zero by the hydrodynamic normalization. Hence we have $h(z) = z$ and $f = g$. $\square$

Some phrases in the above statements are too lengthy to use them repeatedly. Let us put the following definition for convenience.

**Definition 4.6.** We say that a univalent function $f\colon D_1 \to D_2$ satisfies the *standard assumptions* if the following hold:

- $D_1$ and $D_2$ are parallel slit half-planes;
- $f$ is hydrodynamically normalized and has a finite angular residue at infinity;
- $D_2 \setminus f(D_1)$ is a possibly unbounded $\mathbb{H}$-hull.

The next corollary, which follows immediately from Proposition 4.2, shows the "additivity" of the BMD half-plane capacity:

**Corollary 4.7.** *Let $f\colon D_1 \to D_2$ and $g\colon D_2 \to D_3$ be univalent functions which enjoy the standard assumptions. Then so does $g \circ f$, and*

$$\mu(g \circ f; \mathbb{R}) = \mu(g; \mathbb{R}) + \mu(f; \mathbb{R}). \tag{4.9}$$

*This identity can be rephrased as*

$$\mathrm{hcap}^{D_3}(g(F) \cup G) = \mathrm{hcap}^{D_2}(F) + \mathrm{hcap}^{D_3}(G) \tag{4.10}$$

*in terms of the $\mathbb{H}$-hulls $F = D_2 \setminus f(D_1)$ and $G = D_3 \setminus g(D_2)$.*

## 4.3 Enlargement of slit domains and analytic continuation

### 4.3.1 Enlarged slit domains

Let $E \subset \mathbb{C}$ be a simply connected domain, and $C_j$, $j = 1, \dots, N$, be disjoint horizontal slits in $E$. The left and right endpoints of $C_j$ are denoted by $z_j^\ell = x_j^\ell + iy_j$ and by $z_j^r = x_j^r + iy_j$, respectively. We consider a slit domain $D := E \setminus \bigcup_{j=1}^N C_j$. In this section, we glue $N$ copies of $D$ to the original $D$ along the perimeters of its slits to make a Riemann surface $D^\natural$. Conformal mappings between slit domains are naturally extended to holomorphic mappings on such surfaces.

We shall regard $D^\natural$ as a subset of $(N+1)$ sheets $\mathbb{C} \cup \bigcup_{j=1}^N (\mathbb{C} \times \{j\})$. For each $C_j$, we define its

| | | |
|---|---|---|
| interior | $C_j^\circ := C_j \setminus \{z_j^\ell, z_j^r\}$, | |
| upper edge | $C_j^+ := C_j^\circ$ | $\subset \mathbb{C}$, |
| lower edge | $C_j^- := C_j^\circ \times \{j\}$ | $\subset \mathbb{C} \times \{j\}$ and |
| outer edge | $C_j^\natural := C_j^+ \cup \{z_j^\ell, z_j^r\} \cup C_j^-$ | $\subset \mathbb{C} \cup (\mathbb{C} \times \{j\})$. |

The mirror reflection with respect to the line $\Im z = \eta$ is designated by $\Pi_\eta$. Namely, $\Pi_\eta z = \overline{z} + 2i\eta$. The enlarged surface $D^\natural$ is given by

$$D^\natural := D \cup \bigcup_{j=1}^N \left( C_j^\natural \cup (\Pi_{y_j} D \times \{j\}) \right) \subset \mathbb{C} \cup \bigcup_{j=1}^N (\mathbb{C} \times \{j\}).$$

The set $\{\, z \in D \;;\; \Im z > y_j \,\}$, which lies above $C_j^+$, is glued to the set $\{\, (z,j) \;;\; z \in \Pi_{y_j} D, \Im z < y_j \,\}$, which lies below $C_j^-$. Similarly, $\{\, z \in D \;;\; \Im z < y_j \,\}$ is glued to $\{\, (z,j) \;;\; z \in \Pi_{y_j} D, \Im z > y_j \,\}$. More precisely, a fundamental neighborhoods system $\mathcal{V}(p)$ of $p \in C_j^\natural$ is given as follows: We put $\mathbb{H}_\eta^- := \{\, z \in \mathbb{C} \;;\; \Im z < \eta \,\}$, $\eta \in \mathbb{R}$, and

$$l_D := 2^{-1} \inf\{\, d^{\mathrm{Eucl}}(C_j, \partial \mathbb{H} \cup \textstyle\bigcup_{k \neq j} C_k) \wedge (x_j^r - x_j^\ell) \;;\; j = 1, \dots, N \,\}.$$

If $p = z \in C_j^+$, then $\mathcal{V}(p)$ consists of the sets

$$\left(B(z,\delta) \cap \overline{\mathbb{H}_{y_j}}\right) \cup \left( (B(z,\delta) \cap \mathbb{H}_{y_j}^-) \times \{j\} \right), \quad 0 < \delta \le l_D.$$

If $p = (z,j) \in C_j^-$, then $\mathcal{V}(p)$ consists of the sets

$$\left(B(z,\delta) \cap \mathbb{H}_{y_j}^-\right) \cup \left( (B(z,\delta) \cap \overline{\mathbb{H}_{y_j}}) \times \{j\} \right), \quad 0 < \delta \le l_D.$$

Finally, if $p = z_j^\ell$ or $z_j^r$, then $\mathcal{V}(p)$ consists of the sets

$$U_p(\delta) := (B(p,\delta) \setminus C_j^\circ) \cup ((B(p,\delta) \setminus C_j) \times \{j\}), \quad 0 < \delta \le l_D.$$

The closure of a subset $A$ of $D^\natural$ with respect to the topology defined in this way is designated by $\mathrm{cl}^\natural(A)$.

The complex structure of $D^\natural$ is specified by introducing a local coordinate $\psi_p$ around $p \in D^\natural$ as follows: Let $\mathrm{pr}\colon \mathbb{C} \cup \bigcup_{j=1}^N (\mathbb{C} \times \{j\}) \to \mathbb{C}$ be the projection, i.e., $\mathrm{pr}(z) = \mathrm{pr}((z,j)) = z$ for $z \in \mathbb{C}$ and $j = 1, \dots, N$. A local coordinate around $p \notin \bigcup_{j=1}^N \{z_j^\ell, z_j^r\}$ is given just by pr restricted on a neighborhood $U_p \subset D^\natural \setminus \bigcup_{j=1}^N \{z_j^\ell, z_j^r\}$ of $p$ such that $\mathrm{pr}|_{U_p} : U_p \to \mathbb{C}$ is injective. In particular, we put, for each $j$,

$$\begin{aligned}
R_j^+ &:= \{\, z = x+iy \;;\, x_j^\ell < x < x_j^r, y_j \le y < y_j + l_D \,\} \\
&\quad \cup \big(\{\, z = x+iy \;;\, x_j^\ell < x < x_j^r, y_j - l_D < y < y_j \,\} \times \{j\}\big) \supset C_j^+, \\
R_j^- &:= \{\, z = x+iy \;;\, x_j^\ell < x < x_j^r, y_j - l_D < y < y_j \,\} \\
&\quad \cup \big(\{\, z = x+iy \;;\, x_j^\ell < x < x_j^r, y_j \le y < y_j + l_D \,\} \times \{j\}\big) \supset C_j^-, \\
R_j &:= \{\, z = x+iy \;;\, x_j^\ell < x < x_j^r, |y - y_j| < l_D \,\}.
\end{aligned}$$

If $p \in C_j^+$, we can take $\psi_p$ as $\mathrm{pr}|_{R^+} : R_j^+ \to R_j$. If $p \in C_j^-$, then we can take $\psi_p$ as the same mapping with $+$ replaced by $-$. In the case $p = z_j^\ell$, we introduce the function $\vartheta_j^\ell\colon U_{z_j^\ell}(l_D) \setminus \{z_j^\ell\} \to [-2\pi, 2\pi)$ that satisfies the relation $\exp(i\vartheta_j^\ell(q)) = \mathrm{pr}(q) - z_j^\ell$ for all $q \in U_{z_j^\ell}(l_D) \setminus \{z_j^\ell\}$ and the additional conditions

$$\begin{aligned}
&0 < \vartheta_j^\ell(q) < 2\pi && \text{for} \quad q \in B(z_j^\ell, l_D) \setminus C_j, \\
&\vartheta_j^\ell(q) = 0 && \text{for} \quad q \in C_j^+, \\
&-2\pi < \vartheta_j^\ell(q) < 0 && \text{for} \quad q \in (B(z_j^\ell, l_D) \setminus C_j) \times \{j\} \quad \text{and} \\
&\vartheta_j^\ell(q) = -2\pi && \text{for} \quad q \in C_j^-.
\end{aligned}$$

Using this function, we define

$$\begin{aligned}
\psi_{z_j^\ell}(q) &= \mathrm{sq}_j^\ell(q) \\
&:= \begin{cases} 0 & \text{if } q = z_j^\ell \\ \exp\left(\dfrac{\log|\mathrm{pr}(q) - z_j^\ell| + i\vartheta_j^\ell(q)}{2}\right) & \text{if } q \in U_{z_j^\ell}(l_D) \setminus \{z_j^\ell\} \end{cases}
\end{aligned}$$

Roughly speaking, $\mathrm{sq}_j^\ell(q)$ is the square root of $\mathrm{pr}(q) - z_j^\ell$. Indeed, $\mathrm{sq}_j^\ell$ is a homeomorphism from $U_{z_j^\ell}(l_D)$ onto $B(0, \sqrt{l_D})$, and we can easily see that, for any $q_0 \in U_{z_j^\ell}(l_D) \setminus \{z_j^\ell\}$, there exists a neighborhood $U_{q_0}$ of $q_0$ such that

$$\mathrm{sq}_j^\ell(q) = \sqrt{\mathrm{pr}(q) - z_j^\ell}, \quad q \in U_{q_0},$$

holds with a single branch of the square root chosen appropriately. For $p = z_j^r$, we define $\mathrm{sq}_j^r$ in a similar way.

### 4.3.2 Analytic continuation

Let $D_1 = E_1 \setminus \bigcup_{j=1}^N C_{1,j}$ and $D_2 = E_2 \setminus \bigcup_{j=1}^N C_{2,j}$ be slit domains as in Section 4.3.1 and $f\colon D_1 \to D_2$ be a conformal mapping which associates the slit $C_{1,j}$ with $C_{2,j}$, $j = 1, \ldots, N$, respectively.

By the Schwarz reflection principle, there exists a unique analytic continuation of the function $f$ on the Riemann surface $D_1^\natural$, which we denote by the same symbol $f$ again, with the relation

$$f(p) = \Pi_{y_{2,j}} f(\Pi_{y_{1,j}} \mathrm{pr}(p))$$

for $p \in \Pi_{y_j} D \times \{j\}$. By a similar reasoning, we can define a conformal mapping $f^\natural\colon D_1^\natural \to D_2^\natural$ by

$$\begin{aligned} f^\natural(z) &:= f(z) && \text{for} \quad z \in D_1, \\ f^\natural((z,j)) &:= (f((z,j)), j) && \text{for} \quad z \in \Pi_{y_{1,j}} D_1, \\ f^\natural(p) &:= \lim_{\substack{q \to p \\ q \in D_1^\natural \setminus \{p\}}} f^\natural(q) && \text{for} \quad p \in C_{1,j}^\natural. \end{aligned}$$

Here, the last limit is with respect to the topology of $D_2^\natural$. The relation $\mathrm{pr}(f^\natural(p)) = f(p)$ holds for all $p$ by definition.

**Remark 4.8** (Analytic continuation of preceding equalities)**.** Through analytic continuation, some properties of conformal mappings which we have so far studied are suitably generalized. For example, let $f$ be a univalent function on a parallel slit half-plane $D_1$ with the standard assumptions. Then (4.1) gives

$$f(p) = \mathrm{pr}(p) + \pi \int_{\mathbb{R}} \Psi_{D_1}(f^\natural(p), \xi)\, \mu(f; d\xi).$$

**Remark 4.9** (Double of Riemann surfaces)**.** We have introduced the enlargement operations above in order to treat points on the slits as if they were interior points. This is helpful to analyze the motion of the slits in Chapter 5. Here, the reader may have noticed that we can obtain a similar effect by taking the double of the slit domain. Although doubling is a standard method in the study of finite Riemann surfaces, we have chosen another way of enlargement in order to keep simple relations among $f$, $f^\natural$, and pr such as $\mathrm{pr} \circ f^\natural = f$. Since we shall write down the Komatu–Loewner differential equations in terms of the Euclidean coordinate, our choice goes well with such a coordinate-based way.

For later use, we count the order of zeros of conformal mappings extended as above.

**Lemma 4.10.** *Let $D_1$, $D_2$ and $f$ be as above. For $p_2 \in D_2^\natural$, consider the preimage $p_1 := (f^\natural)^{-1}(p_2)$ and a local coordinate*[1] *$\psi\colon D_1^\natural \supset U_{p_1} \to V_{p_1} \subset \mathbb{C}$ around $p_1$. Then the function $h := f \circ \psi^{-1}\colon V_{p_1} \to \mathrm{pr}(D_2^\natural)$ satisfies the following:*

(i) *If $p_2 \notin \bigcup_{j=1}^N \{z_{2,j}^\ell, z_{2,j}^r\}$, then $h$ has a zero of the first order at $\psi(p_1)$.*

(ii) *If $p_2 \in \bigcup_{j=1}^N \{z_{2,j}^\ell, z_{2,j}^r\}$, then $h$ has a zero of the second order at $\psi(p_1)$.*

*Proof.* We assume that $h$ has a zero of order $m \geq 1$ at $\psi(p_1)$. By Theorem 11 in Chapter 3, Section 3.3 of Ahlfors [2], there are a neighborhood $W_{\mathrm{pr}(p_2)} \subset \mathrm{pr}(D_2^\natural)$ of $\mathrm{pr}(p_2)$ and a neighborhood $\tilde{V}_{p_1} \subset V_{p_1}$ of $\psi(p_1)$ such that $h(z) - w = 0$ has exactly $m$ distinct roots in $\tilde{V}_{p_1}$ for any $w \in W_{\mathrm{pr}(p_2)}$.
(i) Suppose $p_2 \notin \bigcup_{j=1}^N \{z_{2,j}^\ell, z_{2,j}^r\}$. Then $h$ is univalent near $\psi(p_1)$ by definition. Hence $m = 1$.
(ii) Suppose $p_2 \in \{z_{2,j}^\ell, z_{2,j}^r\}$ for $j = 1, \dots, N$. Let $w \in C_{2,j}^\circ \cap W_{\mathrm{pr}(p_2)}$. Then the equation $h(z) - w = 0$ has exactly two roots $\tilde{z}^+$ and $\tilde{z}^-$ that satisfy

$$f^\natural(\psi^{-1}(\tilde{z}^+)) = w \in C_{2,j}^+ \quad \text{and} \quad f^\natural(\psi^{-1}(\tilde{z}^-)) = (w, j) \in C_{2,j}^-.$$

Hence $m = 2$. $\square$

[1] This local coordinate does not necessarily coincide with those in Section 4.3.1.

# Chapter 5

# Loewner chains and evolution families

Let $T \in (0, \infty)$. The symbol $I$ stands for the interval $[0, T)$ or $[0, T]$ throughout this section. We use the following notation:

$$\begin{aligned} I^2_{\leq} &:= \{ (s,t) \in I^2 \,;\, s \leq t \}, \\ I^3_{\leq} &:= \{ (s,t,u) \in I^3 \,;\, s \leq t \leq u \}, \end{aligned}$$

and the same symbols with $\leq$ replaced by $<$ in an obvious manner. We also use symbols similar to those in Section 4.3 concerning the enlargement of slit domains and the analytic continuation. The meaning of such symbols must be clear from the context.

## 5.1 Basic properties of evolution families

**Definition 5.1.** Let $D_t$ be a parallel slit half-plane for each $t \in I$. We say that a two-parameter family of univalent functions $\phi_{t,s} \colon D_s \to D_t$, $(s,t) \in I^2_{\leq}$, with the standard assumptions (Definition 4.6) is a *(chordal) evolution family over* $(D_t)_{t \in I}$ if the following properties hold:

(EF.1) $\phi_{t,t}$ is the identity mapping for each $t \in I$;

(EF.2) $\phi_{u,s} = \phi_{u,t} \circ \phi_{t,s}$ holds on $D_s$ for each $(s,t,u) \in I^3_{\leq}$;

(EF.3) The function $\lambda(t) := \mu(\phi_{t,0}; \mathbb{R})$ is continuous on $I$.

Here, $\mu(\phi_{t,0}; \cdot)$ is the measure defined in Theorem 4.3 and the paragraph after Remark 4.4.

Given an evolution family $(\phi_{t,s})_{(s,t)\in I^2_\leq}$ over $(D_t)_{t\in I}$, we associate a family of slit vectors $\boldsymbol{s}(t) \in \mathbf{Slit}$, $t \in I$, so that $D_t = D(\boldsymbol{s}(t))$ and $C^\natural_j(\boldsymbol{s}(t)) = \phi^\natural_{t,s}(C^\natural_j(\boldsymbol{s}(s)))$ hold for every $(s,t) \in I^2_\leq$. For a fixed time $t_0 \in I$, there are $N!$ vectors $\boldsymbol{s}$ such that $D(\boldsymbol{s}) = D_{t_0}$. If we choose one of such vectors as $\boldsymbol{s}(t_0)$, then the family $(\boldsymbol{s}(t))_{t\in I}$ is determined uniquely by the boundary correspondence.

The following two properties will be used repeatedly in the sequel:

**Lemma 5.2.** *Let $(\phi_{t,s})_{(s,t)\in I^2_\leq}$ be an evolution family over $(D_t)_{t\in I}$ and $(s,t) \in I^2_\leq$.*

(i) *It follows that $\mu(\phi_{t,s}; \mathbb{R}) = \lambda(t) - \lambda(s)$. In particular, the function $\lambda$ is non-decreasing on $I$.*

(ii) *The inequality $\Im\phi_{t,s}(z) \geq \Im z$ holds for any $z \in D_s$.*

*Proof.* (i) The relation $\phi_{t,0} = \phi_{t,s} \circ \phi_{s,0}$ follows from (EF.2). Hence Corollary 4.7 gives

$$\lambda(t) = \mu(\phi_{t,s}; \mathbb{R}) + \lambda(s).$$

(ii) From the integral representation (4.1), we obtain

$$\Im\phi_{t,s}(z) = \Im z + \pi \int_{\mathbb{R}} K^*_{D_s}(z,\xi)\, \mu(\phi_{t,s}; d\xi) \geq \Im z. \qquad \square$$

**Remark 5.3.** As in Remark 4.8, some properties of evolution families are suitably extended through analytic continuation. Let $(s,t,u) \in I^3_\leq$. Then (EF.2) implies that $\phi^\natural_{u,s} = \phi^\natural_{u,t} \circ \phi^\natural_{t,s}$ holds on $D^\natural_s$. Similarly, for $(s,t) \in I^2_\leq$ and $p \in \mathrm{cl}^\natural(D_s)$, the inequality $\Im\phi_{t,s}(p) \leq \Im\,\mathrm{pr}(p)$ follows from Lemma 5.2 (ii). In particular, we have $\eta_{D_t} \geq \eta_{D_s}$.

Owing to (EF.3) and Lemma 5.2 (i), there exists a unique non-atomic Radon measure $m_\lambda$ on $I$ that satisfies

$$m_\lambda((s,t]) = \mu(\phi_{t,s}; \mathbb{R}) = \lambda(t) - \lambda(s).$$

This measure plays an essential role in controlling $\phi_{t,s}$ with respect to $(s,t)$. To clarify the roles of $\lambda$ and $m_\lambda$, let us introduce a new terminology under general assumptions. Let $F\colon I \to \mathbb{R}$ be a non-decreasing continuous function and $f_t$ be a holomorphic function on a Riemann surface $X$ for each $t \in I$. The symbol $m_F$ denotes a unique non-atomic Radon measure on $I$ that satisfies $m_F((s,t]) = F(t) - F(s)$. We consider the following property:

$(\mathrm{Lip})_F$ For any compact subset $K$ of $X$, there exists a constant $L_K$ such that

$$\sup_{p\in K}|f_t(p)-f_s(p)|\le L_K(F(t)-F(s))\quad\text{for } (s,t)\in I^2_{\le}.$$

Suppose that this condition holds. Then clearly $t\mapsto f_t(p)$ is of finite variation on $I$ for each $p\in X$. Hence it induces a complex measure $\kappa_p$ on every compact subinterval of $I$. By $(\mathrm{Lip})_F$, it is absolutely continuous with respect to $m_F$. Moreover, the limit

$$\tilde{\partial}^F_t f_t(p):=\lim_{\delta\downarrow 0}\frac{f_{t+\delta}(p)-f_{t-\delta}(p)}{F(t+\delta)-F(t-\delta)}=\lim_{\delta\downarrow 0}\frac{\kappa_p((t-\delta,t+\delta))}{m_F((t-\delta,t+\delta))}$$

exists for $m_F$-a.e. $t\in I$ and is a version of the Radon–Nikodym derivative $d\kappa_p/dm_F$ by the generalized Lebesgue differentiation theorem (see, e.g., Bogachev [8, Theorem 5.8.8]). We can say more about such a family $(f_t)_t$ by using the normality argument and the Cauchy integral formula. The results so obtained are summarized in Appendix A.

We return to the study of an evolution family $(\phi_{t,s})_{(s,t)\in I^2_{\le}}$.

**Lemma 5.4.** *Let $(\phi_{t,s})_{(s,t)\in I^2_{\le}}$ be an evolution family over $(D_t)_{t\in I}$ and $t_0\in[0,T)$ be fixed. For every $\eta\in(0,\eta_{D_{t_0}})$, $p\in(D_{t_0}\cap\mathbb{H}_\eta)^\natural\subset D^\natural_{t_0}$ and $(s,u)\in(I\cap[t_0,T])^2_{\le}$, the inequality*

$$|\phi_{u,t_0}(p)-\phi_{s,t_0}(p)|\le\frac{12}{\eta}(\lambda(u)-\lambda(s)) \tag{5.1}$$

*holds. In particular, the one-parameter family $(\phi_{t,t_0})_{t\in I\cap[t_0,T]}$ satisfies Condition $(\mathrm{Lip})_\lambda$ on $D^\natural_{t_0}$, and the slit motion $\boldsymbol{s}(t)$ is continuous in $t$.*

*Proof.* Let $(s,u)\in(I\cap[t_0,T])^2_{\le}$. By Remarks 4.8 and 5.3, we have

$$\begin{aligned}\phi_{u,t_0}(p)&=\phi_{u,s}(\phi^\natural_{s,t_0}(p))\\&=\phi_{s,t_0}(p)+\pi\int_{\mathbb{R}}\Psi_{D_s}(\phi^\natural_{s,t_0}(p),\xi)\,\mu(\phi_{u,s};d\xi).\end{aligned}$$

Applying Lemma 5.2 (i) gives

$$\phi_{u,t_0}(p)-\phi_{s,t_0}(p)=\pi(\lambda(u)-\lambda(s))\int_{\mathbb{R}}\Psi_{D_s}(\phi^\natural_{s,t_0}(p),\xi)\,\frac{\mu(\phi_{u,s};d\xi)}{\mu(\phi_{u,s};\mathbb{R})}. \tag{5.2}$$

We note that, if $p\in(D_{t_0}\cap\mathbb{H}_\eta)^\natural\cap(\mathbb{C}\times\{j\})$ for some $\eta\le\eta_{D_{t_0}}$ and $j$, then by the definition of analytic continuation, we have

$$\begin{aligned}\Psi_{D_s}(\phi^\natural_{s,t_0}(p),\xi)&=\Pi_{\Im\Psi_{D_s}(z^\ell_j(s),\xi)}\Psi_{D_s}(\Pi_{y_j(s)}\phi_{s,t_0}(p),\xi)\\&=\overline{\Psi_{D_s}(\phi_{s,t_0}(\Pi_{y_j(t_0)}\operatorname{pr}(p)),\xi)}+2i\Im\Psi_{D_s}(z^\ell_j(s),\xi).\end{aligned}$$

Since $\Pi_{y_j(t_0)}\,\mathrm{pr}(p) \in D_{t_0} \cap \mathbb{H}_\eta$ by the definition of $(D_{t_0} \cap \mathbb{H}_\eta)^\natural$, it follows from Lemmas 3.2 and 5.2 (ii) that

$$\sup_{\xi\in\mathbb{R}} \left|\Psi_{D_s}(\phi^\natural_{s,t_0}(p),\xi)\right| \le \frac{12}{\pi\eta}. \tag{5.3}$$

This inequality is trivially true for $p \in (D_{t_0} \cap \mathbb{H}_\eta)^\natural \cap \mathbb{C}$. Thus, by (5.2) we have

$$\begin{aligned}|\phi_{u,t_0}(p) - \phi_{s,t_0}(p)| &\le \pi(\lambda(u)-\lambda(s)) \sup_{\xi\in\mathbb{R}} \left|\Psi_{D_s}(\phi^\natural_{s,t_0}(p),\xi)\right| \\ &\le \frac{12}{\eta}(\lambda(u)-\lambda(s))\end{aligned}$$

for $p \in (D_{t_0} \cap \mathbb{H}_\eta)^\natural$. The remaining assertions are now obvious. □

**Proposition 5.5.** *Given $t_0 \in I$, let $G$ be a bounded open set with $\overline{G} \subset D_{t_0}$ and $\delta$ be a positive constant such that $\overline{G} \subset D_t$ for all $t \in \bar{B}_I(t_0,\delta)$. The trapezoid $\{\,(s,t)\ ;\ s \in \bar{B}_I(t_0,\delta),\ t \in I \cap [s,T]\,\}$ is denoted by $\mathcal{T}_{t_0,\delta}$. Then the mapping*

$$\mathcal{T}_{t_0,\delta} \ni (s,t) \mapsto \phi_{t,s} \in \mathrm{Hol}(G;\mathbb{C})$$

*is continuous. Here,* $\mathrm{Hol}(G;\mathbb{C})$ *is the set of holomorphic functions on $G$ endowed with the topology of locally uniform convergence.*

*Proof.* Our proof goes in a similar way to Bracci, Contreras and Diaz-Madrigal [14, Proposition 3.5]. However, because our domains are moving, we have to modify their proof with the aid of quasi-hyperbolic distances. Here, without loss of generality, we may and do assume that $G$ is a convex set (say, a small disk).

Let $(s_n,t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}_{t_0,\delta}$ convergent to $(s,t)$. The goal is to show

$$\phi_{t_n,s_n} \to \phi_{t,s} \quad \text{locally uniformly on } G. \tag{5.4}$$

We note that, by (5.1),

$$|\phi_{t_n,s_n}(z) - z| = |\phi_{t_n,s_n}(z) - \phi_{s_n,s_n}(z)| \le \frac{12(\lambda(t_n)-\lambda(s_n))}{\tilde{\eta}_G \wedge \eta_{D_0}} \tag{5.5}$$

for all $z \in G$. Here, $\tilde{\eta}_G := \min_{z\in\overline{G}} \Im z > 0$. Since $\lambda(t_n)-\lambda(s_n) \to \lambda(t)-\lambda(s)$ as $n \to \infty$, the sequence $(\phi_{t_n,s_n})_{n\in\mathbb{N}}$ is locally bounded on $G$ by (5.5). Thus, (5.4) follows from Vitali's theorem[1] once we prove the pointwise convergence

$$\phi_{t_n,s_n}(z) \to \phi_{t,s}(z) \quad \text{for each } z \in G. \tag{5.6}$$

[1] See, e.g., Chapter 7, Section 2 of [56].

Therefore, we prove (5.6) below. Moreover, we can regard the convergence in (5.6) as that relative to the hyperbolic distance $d_{\mathbb{H}}^{\mathrm{Hyp}}$ on $\mathbb{H}$ instead of $d^{\mathrm{Eucl}}$ because they induce the same topology on $\mathbb{H}$.

Let $z \in G$. In order to show that $\phi_{t_n,s_n}(z) \to \phi_{t,s}(z)$, it suffices to prove that any subsequence $(t'_n, s'_n)_{n\in\mathbb{N}}$ of $(t_n, s_n)_{n\in\mathbb{N}}$ has a further subsequence $(t''_n, s''_n)_{n\in\mathbb{N}}$ such that $\phi_{t''_n,s''_n}(z) \to \phi_{t,s}(z)$. Here, we note that the sequence $(t'_n, s'_n)_{n\in\mathbb{N}}$ has a subsequence $(t''_n, s''_n)_{n\in\mathbb{N}}$ with one of the following properties:

(I) $s''_n \le t''_n \le s$ for all $n$;

(II) $s \le s''_n$ for all $n$;

(III) $s''_n \le s \le t''_n$ for all $n$.

Thus, we shall prove that $\phi_{t''_n,s''_n}(z) \to \phi_{t,s}(z)$ in these three cases.

In what follows, we drop the superscript $''$ for the simplicity of notation and assume that $(s_n, t_n)_{n\in\mathbb{N}}$ satisfies one of the three properties above.

Firstly, assume that (I) holds. Since $t \ge s \ge t_n \to t$ in this case, we have $s = t$. Hence we have

$$|\phi_{t_n,s_n}(z) - \phi_{t,s}(z)| = |\phi_{t_n,s_n}(z) - z| \le \frac{12(\lambda(t_n) - \lambda(s_n))}{\pi(\tilde{\eta}_G \wedge \eta_{D_0})} \to 0$$

as $n \to \infty$ by (5.5).

Secondly, assume that (II) holds. By Proposition B.1,

$$\begin{aligned}
&d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s_n}(z), \phi_{t,s}(z)) \\
&\le d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s_n}(z), \phi_{t_n,s}(z)) + d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s}(z), \phi_{t,s}(z)) \\
&\le d_{D_{t_n}}^{\mathrm{Hyp}}(\phi_{t_n,s_n}(z), \phi_{t_n,s_n}(\phi_{s_n,s}(z))) + d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s}(z), \phi_{t,s}(z)) \\
&\le d_{D_{s_n}}^{\mathrm{Hyp}}(z, \phi_{s_n,s}(z)) + d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s}(z), \phi_{t,s}(z)).
\end{aligned} \tag{5.7}$$

By Proposition B.2 and Lemma B.3, we have

$$\begin{aligned}
d_{D_{s_n}}^{\mathrm{Hyp}}(z, \phi_{s_n,s}(z)) &\le d_{D_{s_n}}^{\mathrm{QH}}(z, \phi_{s_n,s}(z)) \\
&\le \frac{2|z - \phi_{s_n,s}(z)|}{d^{\mathrm{Eucl}}(\overline{G}, \partial\mathbb{H} \cup \bigcup_{j=1}^{N} C_{j,s_n})}.
\end{aligned} \tag{5.8}$$

Here, $\phi_{s_n,s}(z) \in G$ if $n$ is large enough because $z \in G$. By Lemma 5.4, the rightmost side of (5.8) converges to zero as $n \to \infty$, and $d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s}(z), \phi_{t,s}(z)) \to 0$. Thus, $d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s_n}(z), \phi_{t,s}(z)) \to 0$ by (5.7).

Finally, assume that (III) holds. By a computation similar to that in (II),

$$\begin{aligned}
&d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s_n}(z), \phi_{t,s}(z)) \\
&\le d_{D_s}^{\mathrm{Hyp}}(\phi_{s,s_n}(z), z) + d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s}(z), \phi_{t,s}(z)).
\end{aligned} \tag{5.9}$$

We also have

$$
\begin{aligned}
d_{D_s}^{\mathrm{Hyp}}(\phi_{s,s_n}(z),z) \le d_{D_s}^{\mathrm{QH}}(\phi_{s,s_n}(z),z) &\le \frac{2|\phi_{s,s_n}(z)-z|}{d^{\mathrm{Eucl}}(\overline{G},\partial\mathbb{H}\cup\bigcup_{j=1}^{N}C_{j,s})} \\
&\le \frac{24(\lambda(t_n)-\lambda(s_n))}{(\tilde{\eta}_G\wedge\eta_{D_0})d^{\mathrm{Eucl}}(\overline{G},\partial\mathbb{H}\cup\bigcup_{j=1}^{N}C_{j,s})} \to 0. \qquad (5.10)
\end{aligned}
$$

Thus, (5.9) and (5.10) yield $d_{\mathbb{H}}^{\mathrm{Hyp}}(\phi_{t_n,s_n}(z),\phi_{t,s}(z)) \to 0$. □

## 5.2 Komatu–Loewner equation for evolution families

Let $(\phi_{t,s})_{(s,t)\in I^2_{\le}}$ be an evolution family over $(D_t)_{t\in I}$. For each $t_0 \in [0,T)$, define

$$
N_{t_0} := \bigcup_{p\in D_{t_0}^{\natural}} \{\, t\in(t_0,T)\ ;\ \tilde{\partial}_t^{\lambda}\phi_{t,t_0}(p) \text{ does not exist}\,\}, \qquad (5.11)
$$

which is a $m_\lambda$-null subset of $I$ by Lemma 5.4 and Proposition A.1.

**Lemma 5.6.** *The identity*

$$
N_{t_0} = N_0 \cap (t_0,T)
$$

*holds for every $t_0 \in (0,T)$.*

*Proof.* Let $t_0 \in (0,T)$ and $t \in N_0 \cap (t_0,T)$. For $p \in D_0^{\natural}$ such that $\tilde{\partial}_t^{\lambda}\phi_{t,0}(p)$ does not exist, we have $\phi_{t,0}(p) = \phi_{t,t_0}(\phi_{t_0,0}^{\natural}(p))$ by (EF.2). $t \in N_{t_0}$ is clear from this equality.

Conversely, assume that $t \in N_{t_0} \setminus N_0$. This assumption implies that $\tilde{\partial}_t^{\lambda}\phi_{t,0}(p)$ exists for every $p \in D_0^{\natural}$. Then using (EF.2) as above, we see that $\tilde{\partial}_t^{\lambda}\phi_{t,t_0}(p)$ exists for $p \in \phi_{t_0,0}^{\natural}(D_0^{\natural}) \subset D_{t_0}^{\natural}$. In fact, this derivative exists for any $p \in D_{t_0}^{\natural}$ because we can take any countable subset of $\phi_{t_0,0}^{\natural}(D_0^{\natural})$ having an accumulation point in $D_{t_0}^{\natural}$ as the set $A$ in Proposition A.1 (ii). However, this implies $t \notin N_{t_0}$, a contradiction. □

Now, we derive the Komatu–Loewner equation for evolution families, a central result of this paper. We use the following notation: For a topological space $X$, the Borel $\sigma$-algebra of $X$ is designated by $\mathcal{B}(X)$. Its completion with respect to a Borel measure $m$ is denoted by $\mathcal{B}^m(X)$. We denote by $\mathcal{M}_{\le 1}(\mathbb{R})$ the set of Borel measures on $\mathbb{R}$ with total mass not greater than one. $\mathcal{M}_{\le 1}(\mathbb{R})$ is compact with respect to the vague topology.

**Theorem 5.7.** *Let $(\phi_{t,s})_{(s,t)\in I^2_{\leq}}$ be an evolution family over $(D_t)_{t\in I}$.*

(i) *For any $t \in (0,T) \setminus N_0$, the limit*

$$\nu_t := \lim_{\delta\downarrow 0} \frac{\mu(\phi_{t+\delta,t-\delta};\cdot)}{\mu(\phi_{t+\delta,t-\delta};\mathbb{R})}$$

*exists in the sense of vague convergence.*

(ii) *$t \mapsto \nu_t$ is a measurable mapping from $(I, \mathcal{B}^{m_\lambda}(I))$ to $(\mathcal{M}_{\leq 1}(\mathbb{R}), \mathcal{B}(\mathcal{M}_{\leq 1}(\mathbb{R})))$. Here, $\nu_t$ is defined suitably (say, as zero) on $N_0$.*

(iii) *For each fixed $t_0 \in [0,T)$, the Komatu–Loewner equation*

$$\tilde{\partial}^\lambda_t \phi_{t,t_0}(p) = \pi \int_{\mathbb{R}} \Psi_{D_t}(\phi^\natural_{t,t_0}(p), \xi)\, \nu_t(d\xi) \tag{5.12}$$

*holds for any $t \in (t_0,T) \setminus N_{t_0}$ and $p \in D^\natural_{t_0}$.*

*Proof.* We fix $t_0 \in [0,T)$ and $t \in (t_0,T) \setminus N_0$ throughout this proof. For any sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero, there exists a subsequence $(\delta'_n)_n$ such that the sequence of probability measures

$$\mu^\sharp_n := \frac{\mu(\phi_{t+\delta'_n,t-\delta'_n};\cdot)}{\mu(\phi_{t+\delta'_n,t-\delta'_n};\mathbb{R})}$$

converges vaguely to $\mu^\sharp_\infty$ as $n \to \infty$. We show that

$$\tilde{\partial}^\lambda_t \phi_{t,t_0}(p) = \pi \int_{\mathbb{R}} \Psi_{D_t}(\phi^\natural_{t,t_0}(p), \xi)\, \mu^\sharp_\infty(d\xi). \tag{5.13}$$

Let $z \in D_{t_0}$ and $n$ be large enough. From (5.2) we get

$$\begin{aligned}
&\left| \frac{\phi_{t+\delta'_n,t_0}(z) - \phi_{t-\delta'_n,t_0}(z)}{\pi(\lambda(t+\delta'_n) - \lambda(t-\delta'_n))} - \int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,t_0}(z), \xi)\, \mu^\sharp_\infty(d\xi) \right| \\
&= \left| \int_{\mathbb{R}} \Psi_{D_{t-\delta'_n}}(\phi_{t-\delta'_n,t_0}(z), \xi)\, \mu^\sharp_n(d\xi) \right. \\
&\qquad \left. - \int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,t_0}(z), \xi)\, \mu^\sharp_\infty(d\xi) \right| \\
&\leq \int_{\mathbb{R}} |\Psi_{D_{t-\delta'_n}}(\phi_{t-\delta'_n,t_0}(z), \xi) - \Psi_{D_t}(\phi_{t,t_0}(z), \xi)|\, \mu^\sharp_n(d\xi) \qquad (5.14) \\
&\quad + \left| \int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,t_0}(z), \xi)\, \mu^\sharp_n(d\xi) - \int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,t_0}(z), \xi)\, \mu^\sharp_\infty(d\xi) \right|.
\end{aligned}$$

In the rightmost side of (5.14), the former integral vanishes as $n \to \infty$ by Proposition 3.9 and the continuity of $\phi_{v,u}$ in $v$ in Lemma 5.4. The remaining term in the rightmost side of (5.14) also tends to zero by Corollary 3.4 and the vague convergence of $(\mu_n^\sharp)_n$. Here, we note that $C_\infty(\mathbb{R})$ is the completion of $C_c(\mathbb{R})$ with respect to the supremum norm. Thus, (5.13) holds for $p = z \in D_{t_0}$. The analytic continuation yields the same equation for all $p \in D_{t_0}^\natural$.

$\mu_\infty^\sharp$ is, in fact, independent of the choice of the above subsequence $(\delta_n')_n$. To prove this, assume that we have another subsequence $(\delta_n'')_n$ of $(\delta_n)_n$ such that

$$\mu_n^\flat := \frac{\mu(\phi_{t+\delta_n'',t-\delta_n''};\cdot)}{\mu(\phi_{t+\delta_n'',t-\delta_n''};\mathbb{R})}$$

converges vaguely to $\mu_\infty^\flat$ as $n \to \infty$. Then (5.13) with $\mu_\infty^\sharp$ replaced by $\mu_\infty^\flat$ holds by the same reasoning. As a result,

$$\int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,t_0}(z),\xi)\,\mu_\infty^\sharp(d\xi) = \int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,t_0}(z),\xi)\,\mu_\infty^\flat(d\xi), \quad z \in D_{t_0}.$$

In particular, we have

$$\int_{\mathbb{R}} \Psi_{D_t}(z,\xi)\,\mu_\infty^\sharp(d\xi) = \int_{\mathbb{R}} \Psi_{D_t}(z,\xi)\,\mu_\infty^\flat(d\xi)$$

for all $z \in \phi_{t,t_0}(D_{t_0})$. Lemma 3.5 (ii) now yields $\mu_\infty^\sharp = \mu_\infty^\flat$, which proves both (i) and (iii).

As for (ii), the measurability of $t \mapsto \nu_t$ follows from Proposition 5.5 and the relation

$$\begin{aligned}\nu_t(B) = \frac{1}{\pi}\lim_{n\to\infty}\Bigg(&\frac{1}{\lambda(t+1/n)-\lambda(t-1/n)} \\ &\times \lim_{m\to\infty}\int_B \Im\phi_{t+1/n,\,t}\left(\xi+\frac{i}{m}\right)\,d\xi\Bigg)\end{aligned}$$

for all bounded $B \in \mathcal{B}(\mathbb{R})$. □

The essence of our proof of Theorem 5.7 can be summarized in the following manner:

**Corollary 5.8.** *Suppose that $(\phi_{t,s})_{(s,t)\in I_\leq^2}$ is an evolution family. Let $t_0 \in [0,T)$ and $t \in [t_0,T)$. For a sequence $(s_n,u_n)_{n\in\mathbb{N}}$ in $[t_0,T)_<^2$ with $s_n \leq t \leq u_n$ and $s_n, u_n \to t$, the following are equivalent:*

(i) $\dfrac{\phi_{u_n,t_0}(p)-\phi_{s_n,t_0}(p)}{\lambda(u_n)-\lambda(s_n)}$ *converges as $n \to \infty$ for every $p \in D_{t_0}^\natural$;*

(ii) $\dfrac{\mu(\phi_{u_n,s_n};\cdot)}{\mu(\phi_{u_n,s_n};\mathbb{R})}$ *converges vaguely as* $n \to \infty$.

*If either of the two is true, then* (5.12) *holds at* $t$ *with* $\tilde{\partial}_t^{\lambda}\phi_{t,t_0}(p)$ *and* $\nu_t$ *replaced by these limits.*

We have considered only the differentiation with respect to $\lambda(t)$ up to this point. It is also reasonable to consider the differentiation with respect to $t$. To this end, a natural manner of thinking is to assume the absolute continuity of $\lambda(t)$ in $t$ or to perform time-change.

In the former standpoint, we suppose that $\lambda(t)$ is absolutely continuous in $t \in I$. Then by $(\mathrm{Lip})_\lambda$, the function $t \mapsto \phi_{t,t_0}(p)$ is also absolutely continuous and hence differentiable in a.e. $t$ in the usual sense. Thus, (5.12) reduces to

$$\frac{\partial \phi_{t,t_0}(p)}{\partial t} = \pi\dot{\lambda}(t)\int_{\mathbb{R}} \Psi_{D_t}(\phi^{\natural}_{t,t_0}(p),\xi)\,\nu_t(d\xi) \tag{5.15}$$

for Lebesgue a.e. $t \in [t_0, T)$.

In the latter standpoint, we suppose that $\lambda(t)$ is (strictly) increasing and that the condition (i) or (ii) in Corollary 5.8 holds for every $t \in I$ and every choice of $(s_n, u_n)_{n\in\mathbb{N}}$. For any increasing continuous function $\theta$ on $I$, we perform time-change as $\tilde{\phi}_{t,s} := \phi_{\theta^{-1}(t),\theta^{-1}(s)}$, $\tilde{D}_t := D_{\theta^{-1}(t)}$, $\tilde{\lambda}(t) := \lambda(\theta^{-1}(t))$, and $\tilde{\nu}_t := \nu_{\theta^{-1}(t)}$. Then

$$\frac{\partial \tilde{\phi}_{t,s}(p)}{\partial \tilde{\lambda}(t)} := \lim_{h\to 0}\frac{\tilde{\phi}_{t+h,s}(p) - \tilde{\phi}_{t,s}(p)}{\tilde{\lambda}(t+h) - \tilde{\lambda}(t)} = \pi\int_{\mathbb{R}} \Psi_{\tilde{D}_t}(\tilde{\phi}^{\natural}_{t,s}(p),\xi)\,\tilde{\nu}_t(d\xi). \tag{5.16}$$

In particular, choosing $\theta = \lambda/2$ gives $\tilde{\lambda}(t) = 2t$ and

$$\frac{\partial \tilde{\phi}_{t,s}(p)}{\partial t} = 2\pi\int_{\mathbb{R}} \Psi_{\tilde{D}_t}(\tilde{\phi}^{\natural}_{t,s}(p),\xi)\,\tilde{\nu}_t(d\xi). \tag{5.17}$$

In this case, $(\tilde{\phi}_{t,s})$ is said to be *parametrized by half-plane capacity*[2] in the SLE context. (5.17) as well as (5.12) provides a natural way to regard $\lambda(t)$ as a canonical parameter.

We now return to the study of evolution families in the general setting. As our first application of the Komatu–Loewner equation (5.12), we look at the case in which an evolution family defined on $[0,T)^2_{\le}$ is extended to one defined on $[0,T]^2_{\le}$.

[2] We have chosen the homeomorphism $\theta = \lambda/2$ so that $\tilde{\lambda}(t) = 2t$, not $\tilde{\lambda}(t) = t$. This coefficient two is just conventional.

**Proposition 5.9.** *Let $(\phi_{t,s})_{(s,t)\in[0,T)^2_\leq}$ be an evolution family with $\lambda(t) = \mu(\phi_{t,0};\mathbb{R})$. There exists a unique evolution family $(\tilde{\phi}_{t,s})_{(s,t)\in[0,T]^2_\leq}$ such that $\tilde{\phi}_{t,s} = \phi_{t,s}$ for all $(s,t) \in [0,T)^2_\leq$ if and only if* $\sup_{0\leq t<T}\lambda(t) < \infty$.

*Proof.* The "only if" part is trivial from $\sup_{0\leq t<T}\lambda(t) = \tilde{\lambda}(T)$. Here, $\tilde{\lambda}(t) := \mu(\tilde{\phi}_{t,0};\mathbb{R})$.

To establish the "if" part, suppose that $\sup_{0\leq t<T}\lambda(t) < \infty$. This is equivalent to $m_\lambda([0,T)) < \infty$. The proof is divided into three steps.

**Step 1.** Fix $t_0 \in [0,T)$. We write (5.12) in the integral form

$$\phi_{t,t_0}(z) = z + \pi\int_I\int_{\mathbb{R}}\Psi_{D_s}(\phi_{s,t_0}(z),\xi)\,\nu_s(d\xi)\mathbf{1}_{[t_0,t)}(s)\,m_\lambda(ds). \tag{5.18}$$

Since

$$\sup_{t_0\leq s<T}|\Psi_{D_s}(\phi_{s,t_0}(z),\xi)| \leq \frac{4}{\pi}\frac{1}{\Im z\wedge\eta_{D_0}}$$

holds for each $z \in D_{t_0}$ by Lemmas 3.2 and 5.2 (ii), the dominated convergence theorem applies to (5.18). Hence $(\phi_{t,t_0}(z))_{t\in[t_0,T)}$ converges as $t \uparrow T$ for each $z \in D_{t_0}$.

**Step 2.** By (4.1) we have

$$\phi_{t,t_0}(z) = z + \pi\int_{\mathbb{R}}\Psi_{D_{t_0}}(z,\xi)\,\mu(\phi_{t,t_0};d\xi). \tag{5.19}$$

Since $\mu(\phi_{t,t_0};\mathbb{R}) \leq m_\lambda([0,T))$, the family $(\phi_{t,t_0})_{t\in[t_0,T)}$ is locally bounded on $D^\natural_{t_0}$ by Remarks 4.8 and 5.3. Hence, Vitali's theorem converts the pointwise convergence of $(\phi_{t,t_0})_{t\in[t_0,T)}$ on $D_{t_0}$, which has been shown in Step 1, into the locally uniform convergence on $D^\natural_{t_0}$. In addition, Hurwitz's theorem guarantees that $\tilde{\phi}_{T,t_0} := \lim_{t\to T}\phi_{t,t_0}$ is univalent on $D_{t_0}$ (see, e.g., Theorem A in Chapter 7, Section 2 of [56]). Moreover, since $(\mu(\phi_{t,t_0};\cdot))_{t\in[t_0,T)}$ is sequentially compact with respect to the vague topology, there is a sequence $(t_n)_{n=1}^\infty$ converging to $T$ such that the limit $\mu_{T,t_0} := \lim_{n\to\infty}\mu(\phi_{t_n,t_0};\cdot)$ exists. Letting $t \to T$ in (5.19) yields

$$\tilde{\phi}_{T,t_0}(z) = z + \pi\int_{\mathbb{R}}\Psi_{D_t}(z,\xi)\,\mu_{T,t_0}(d\xi), \tag{5.20}$$

which is exactly (4.1) with $f = \tilde{\phi}_{T,t_0}$. Thus, $\phi_{T,t_0}$ enjoys the standard assuptions by Theorem 4.3.

**Step 3.** Through Steps 1 and 2, we have seen that $\tilde{\phi}_{T,t_0} = \lim_{u\to T} \phi_{u,t_0}$ is a well-defined univalent function for every $t_0 \in [0, T)$. Since this convergence is uniform on the compact set $C_j^{\natural}(\boldsymbol{s}(t_0))$, the slit motion $\boldsymbol{s}(u)$ converges in the closure of **Slit** in $\mathbb{R}^{3N}$. We put $\boldsymbol{s}(T) := \lim_{u\to T} \boldsymbol{s}(u)$ and $D_T := D(\boldsymbol{s}(T))$. In fact, the limit $\boldsymbol{s}(T)$ belongs to **Slit** because $\tilde{\phi}_{T,t_0}(D_{t_0}) \subset D_T$.

Now, we set $\tilde{\phi}_{t,s} := \phi_{t,s}$ for all $(s,t) \in [0,T)^2_{\leq}$ and define $\tilde{\phi}_{T,T}$ as the identity mapping on $D_T$. The family $(\tilde{\phi}_{t,s})_{(s,t)\in[0,T]^2_{\leq}}$ automatically satisfies (EF.1). For $(s,t) \in [0,T)^2_{\leq}$, we have

$$\tilde{\phi}_{T,s}(z) = \lim_{u\to T} \phi_{u,s}(z) = \lim_{u\to T} \phi_{u,t}(\phi_{t,s}(z)) = \tilde{\phi}_{T,t}(\tilde{\phi}_{t,s}(z)),$$

which implies (EF.2). Moreover,

$$\liminf_{t\nearrow T} \lambda(t) \leq \tilde{\lambda}(T) := \mu_{T,0}(\mathbb{R}) \leq \sup_{0\leq t<T} \lambda(t)$$

holds, which means $\tilde{\lambda}(T) = \lim_{t\to T} \lambda(t)$. Here, the first inequality is shown in the same way as in the proof of Lemma 5.2 (i), and the second one follows from the property of vague convergence. Now, (EF.3) for $(\tilde{\phi}_{t,s})$ is clear, and its uniqueness follows trivially from Proposition 5.5. □

## 5.3 Loewner chains

**Definition 5.10.** Let $D$ and $D_t$, $t \in I$, be parallel slit half-planes. We say that a family of univalent functions $f_t \colon D_t \to D$, $t \in I$, with the standard assumptions is a *(chordal) Loewner chain over* $(D_t)_{t\in I}$ *with codomain* $D$ if the following hold:

(LC.1) $f_s(D_s) \subset f_t(D_t)$ holds for each $(s,t) \in I^2_{\leq}$.

(LC.2) The function $\ell(t) := \mu(f_t; \mathbb{R})$ is continuous on $I$.

The relationship between Loewner chains and evolution families is formulated as follows:

**Proposition 5.11.** (i) *Let* $(f_t)_{t\in I}$ *be a Loewner chain over* $(D_t)_{t\in I}$ *with any codomain. The two-parameter family*

$$\phi_{t,s} := f_t^{-1} \circ f_s, \quad (s,t) \in I^2_{\leq},$$

*is an evolution family, and* $\lambda(t) = \mu(\phi_{t,0}; \mathbb{R})$ *is bounded on* $I$.

(ii) *Let* $(\phi_{t,s})_{(s,t)\in I}$ *be an evolution family over* $(D_t)_{t\in I}$ *with* $\lambda$ *bounded. Its prolongation to* $[0,T]^2_{\le}$*, which is guaranteed by Proposition 5.9, is designated by the same symbol. Then the family*

$$f_t := \phi_{T,t}, \quad t \in [0,T],$$

*is a Loewner chain over* $(D_t)_{t\in I}$ *with codomain* $D_T$*.*

*Proof.* (i) By Proposition 4.1, both $f_t^{-1}$ and $f_s$ obey the hydrodynamic normalization and has a finite angular residue at infinity. Hence, $\phi_{t,s} = f_t^{-1} \circ f_s$ is a univalent function from $D_s$ into $D_t$ that satisfies the standard assumptions by Proposition 4.2. The properties (EF.1)–(EF.3) are trivial.

Because $f_0 = f_t \circ \phi_{t,0}$, Corollary 4.7 yields

$$\ell(0) = \ell(t) + \lambda(t) \ge \lambda(t),$$

which implies that $\lambda$ is bounded.
(ii) For $(s,t) \in [0,T]^2_{\le}$, we have

$$f_s(z) = \phi_{T,s}(z) = \phi_{T,t}(\phi_{t,s}(z)) = f_t(\phi_{t,s}(z)).$$

Hence $f_s(D_s) = f_t(\phi_{t,s}(D_s)) \subset f_t(D_t)$, which implies (LC.1). Moreover, Corollary 4.7 yields

$$\mu(f_s;\mathbb{R}) = \mu(f_t;\mathbb{R}) + \lambda(t) - \lambda(s),$$

which implies (LC.2). $\square$

**Remark 5.12** (Terminal condition on Loewner chains)**.** As Proposition 5.11 shows, in our definition, Loewner chains associated with a fixed evolution family are not unique in general. The uniqueness holds if we add the terminal condition

$$\bigcup_{t\in I} f_t(D_t) = f_T(D_T) = D$$

on a Loewner chain $(f_t)_{t\in I}$.

Thanks to Proposition 5.11, we can analyze a Loewner chain $(f_t)_{t\in I}$ using the associated evolution family $(\phi_{t,s})_{(s,t)\in I^2_{\le}}$. In particular, we can write down the Komatu–Loewner equation for Loewner chains as follows:

**Corollary 5.13.** *Let* $(f_t)_{t\in I}$ *be a Loewner chain, and fix* $t_0 \in I$*.*

(i) $(f_t^{-1})_{t\in I\cap[t_0,T]}$ *satisfies* $(\mathrm{Lip})_\ell$ *on* $f_{t_0}(D_{t_0})^\natural$*.*

(ii) *Let $N_0$ and $\nu_t$ be defined as in Lemma 5.6 and Theorem 5.7 for the evolution family $(\phi_{t,s})$ associated to $(f_t)$ by Proposition 5.11. Then for any $t \in [t_0, T) \setminus N$ and $p \in f_{t_0}(D_{t_0})^\natural$, the following Komatu–Loewner equation holds:*

$$\tilde{\partial}_t^\ell (f_t^{-1})(p) = -\pi \int_{\mathbb{R}} \Psi_{D_t}((f_t^\natural)^{-1}(p), \xi)\, \nu_t(d\xi). \tag{5.21}$$

The differential equation for $f_t(z)$ can be obtained by differentiating the identity $f_t(f_t^{-1}(z)) = z$ in $t$ as well. We omit the detail.

## 5.4 Induced slit motion

The definition of evolution families in this paper involves the evolution of parallel slit half-planes. We derive the Komatu–Loewner equation for the slits accompanied by this evolution, following Bauer and Friedrich [7, Section 4.1] and Chen and Fukushima [16, Section 2].

Let $(\phi_{t,s})_{(s,t)\in I^2_\le}$ be an evolution family over $(D_t)_{t\in I}$. By the paragraph just after Definition 5.1, the vectors $\boldsymbol{s}(t) \in \mathbf{Slit}$, $t \in I$, with $D(\boldsymbol{s}(t)) = D_t$ are determined uniquely, provided that the order of the initial slits $C_j(\boldsymbol{s}(0))$, $j = 1, \ldots, N$, is given. The left and right endpoints of $C_j(t) := C_j(\boldsymbol{s}(t))$ are denoted by $z_j^\ell(t) = x_j^\ell(t) + iy_j(t)$ and by $z_j^r(t) = x_j^r(t) + iy_j(t)$, respectively. These endpoints are continuous in $t$ by Lemma 5.4. We put $p_j^\ell(t) := (\phi_{t,0}^\natural)^{-1}(z_j^\ell(t))$ and $p_j^r(t) := (\phi_{t,0}^\natural)^{-1}(z_j^r(t))$, both of which are points on $C_j^\natural(0) \subset D_0^\natural$.

**Lemma 5.14.** *Fix $t_0 \in [0, T)$. A local coordinate of $p_j^\ell(t_0)$ is denoted by $\psi\colon U_{p_j^\ell(t_0)} \to V_{p_j^\ell(t_0)}$. Then for some neighborhood $J$ of $t$ and constant $L > 0$,*

- *$p_j^\ell(t) \in U_{p_j^\ell(t_0)}$ for $t \in J$, and*
- *$\tilde{z}_j^\ell(t) := \psi(p_j^\ell(t))$ satisfies*

$$|\tilde{z}_j^\ell(t) - \tilde{z}_j^\ell(s)| \le L(\lambda(t) - \lambda(s)), \quad (s,t) \in J^2_\le. \tag{5.22}$$

*The same statement with the superscript $\ell$ replaced by $r$ also holds.*

*Proof.* We put $h_t := \phi_{t,0} \circ \psi^{-1}\colon V_{p_j^\ell(t_0)} \to \mathrm{pr}(D_t^\natural)$. By Lemma 4.10 (ii), we have

$$h_{t_0}'(\tilde{z}_j(t_0)) = 0 \quad \text{and} \quad h_{t_0}''(\tilde{z}_j(t_0)) \neq 0.$$

In addition, $(h_t)_{t\in I}$ satisfies $(\mathrm{Lip})_\lambda$ on $V_{p_j^\ell(t_0)}$. By Proposition A.6, there exist some neighborhood $J$ of $t_0$ and neighborhood $\tilde{V} \subset V_{p_j^\ell(t_0)}$ of $\psi(p_j^\ell(t_0))$, function $\hat{z}\colon J \to \tilde{V}$ and constant $L > 0$ such that

$$h_t'(\hat{z}(t)) = 0, \quad h_t''(\hat{z}(t)) \neq 0 \tag{5.23}$$

for $t \in J$ and

$$|\hat{z}(t) - \hat{z}(s)| \leq L(\lambda(t) - \lambda(s)).$$

for $(s,t) \in J_<^2$ are satisfied. (5.23) combined with Lemma 4.10 implies that $h_t(\hat{z}(t)) \in \bigcup_{k=1}^N \{z_k^\ell(t), z_k^r(t)\}$. By the continuity with respect to $t$, we see that $h_t(\hat{z}(t))$ must coincide with $z_j^\ell(t)$. In other words, $\hat{z}(t) = \tilde{z}_j^\ell(t)$. The proof is now complete. □

**Theorem 5.15.** *For each $j = 1, \ldots, N$, the endpoints $z_j^\ell(t)$ and $z_j^r(t)$ of $C_j(t)$ enjoy the Komatu–Loewner equation for the slits*

$$\tilde{\partial}_t^\lambda z_j^\ell(t) = \pi \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t)}(z_j^\ell(t), \xi)\, \nu_t(d\xi), \tag{5.24}$$

$$\tilde{\partial}_t^\lambda z_j^r(t) = \pi \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t)}(z_j^r(t), \xi)\, \nu_t(d\xi) \tag{5.25}$$

*for $m_\lambda$-a.e. $t \in I$. Here, $(\nu_t)_{t\in I}$ is the process of sub-probability measures defined in Theorem 5.7.*

*Proof.* We prove only (5.24). (5.25) is then obtained just by replacing the superscript $\ell$ with $r$ in the proof of (5.24). Let $N_0 \subset [0,T)$ be the exceptional set defined by (5.11) with $t_0 = 0$.

We fix an arbitrary $t_0 \in [0,T)$ and apply Lemma 5.14 to this $t_0$. We denote a neighborhood of $t_0$ that satisfies the conclusion of Lemma 5.14 by $J$. By (5.22), there is a Lebesgue null set $\tilde{N} \subset J$ such that $\tilde{\partial}_t^\lambda \tilde{z}_j(t)$ exists for every $t \in J \setminus \tilde{N}$. For $t \in J \setminus (N_0 \cup \tilde{N})$, we have

$$\begin{aligned}
\tilde{\partial}_t^\lambda z_j^\ell(t) &= \tilde{\partial}_t^\lambda \left(\phi_{t,0}(p_j^\ell(t))\right) \\
&= \lim_{h\to+0} \frac{\phi_{t+h,0}(p_j^\ell(t+h)) - \phi_{t-h,0}(p_j^\ell(t-h))}{\lambda(t+h) - \lambda(t-h)} \\
&= \lim_{h\to+0} \frac{\phi_{t+h,0}(p_j^\ell(t+h)) - \phi_{t-h,0}(p_j^\ell(t+h))}{\lambda(t+h) - \lambda(t-h)} \\
&\quad + \lim_{h\to+0} \frac{(\phi_{t-h,0}\circ\psi^{-1})(\tilde{z}_j^\ell(t+h)) - (\phi_{t-h,0}\circ\psi^{-1})(\tilde{z}_j^\ell(t-h))}{\lambda(t+h) - \lambda(t-h)}.
\end{aligned} \tag{5.26}$$

We note that $p_j^\ell(\cdot)\colon J \to C_j^\natural(0)$ is continuous. From the locally uniform convergence in Lemma A.1 (i), we can see that

$$\begin{aligned}
&\frac{\phi_{t+h,0}(p_j^\ell(t+h)) - \phi_{t-h,0}(p_j^\ell(t+h))}{\lambda(t+h)-\lambda(t-h)} - (\tilde{\partial}_t^\lambda \phi_{t,0})(p_j^\ell(t)) \\
&= \left( \frac{\phi_{t+h,0}(p_j^\ell(t+h)) - \phi_{t-h,0}(p_j^\ell(t+h))}{\lambda(t+h)-\lambda(t-h)} - (\tilde{\partial}_t^\lambda \phi_{t,0})(p_j^\ell(t+h)) \right) \\
&\quad + \Big( (\tilde{\partial}_t^\lambda \phi_{t,0})(p_j^\ell(t+h)) - (\tilde{\partial}_t^\lambda \phi_{t,0})(p_j^\ell(t)) \Big) \\
&\to 0 \quad \text{as } h \to +0.
\end{aligned}$$

Hence, the first limit in the rightmost side of (5.26) is equal to $(\tilde{\partial}_t^\lambda \phi_{t,0})(p_j^\ell(t))$. Also, we see that the second limit is equal to $(\phi_{t,0} \circ \psi^{-1})'(\tilde{z}_j^\ell(t)) \cdot \tilde{\partial}_t^\lambda \tilde{z}_j^\ell(t)$. However, since $\phi_{t,0} \circ \psi^{-1}$ has a zero of the second order at $\tilde{z}_j^\ell(t)$ by Lemma 4.10, $(\phi_{t,0} \circ \psi^{-1})'(\tilde{z}_j^\ell(t)) = 0$. Thus, by (5.26) and (5.12) we have

$$\begin{aligned}
\tilde{\partial}_t^\lambda z_j^\ell(t) = (\tilde{\partial}_t^\lambda \phi_{t,0})(p_j^\ell(t)) &= \pi \int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,0}(p_j^\ell(t)), \xi)\, \nu_t(d\xi) \\
&= \pi \int_{\mathbb{R}} \Psi_{D_t}(z_j^\ell(t), \xi)\, \nu_t(d\xi). \qquad \square
\end{aligned}$$

**Remark 5.16** (Komatu–Loewner equation on annuli)**.** As is already mentioned in Remark 2.7, the Komatu–Loewner equation for the slits (5.24) and (5.25) describes the motion of moduli. On the other hand, Contreras, Diaz-Madrigal and Gumenyuk [22] constructed Loewner theory on annuli. In their theory, the moduli, i.e., the ratios $r(t)$ of the outer and inner radii of the underlying annuli $\mathbb{A}_{r(t)} = \{\, z\, ;\, r(t) < |z| < 1 \,\}$ form a monotone function of $t$, which is used as a new time-parameter. Since $r(t)$ itself play the role of time, Loewner theory on annuli does not need an evolution equation for moduli. See also Komatu [39], Zhan [61], and Fukushima and Kaneko [31].

## 5.5 Komatu–Loewner equation around inner boundaries

As a preparatory observation for Chapter 6, we rewrite the Komatu–Loewner equation (5.12) by compositing the function $\phi_{t,s}$ with such a local chart around the slits as was introduced in Section 4.3.1. For simplicity of notation, we treat only the case $\lambda(t) = 2t$. In this case, we have already seen that (5.12) reduces to (5.15) with $\dot{\lambda}(t) = 2$.

We first consider the equation "above or below the slits". Let $s \in [0, T)$ and $p \in R_j^+(s)$. Here, $R_j^+(s)$ is defined for the parallel slit half-plane $D_s$ in the same way as $R_j^+$ is defined in Section 4.3.1. We take $\delta > 0$ so that $\phi^{\natural}_{t,s}(p) \in R_j^+(t)$ for $t \in [s, s+\delta]$. We put $\tilde{z}(t; s, p) := \mathrm{pr}(\phi^{\natural}_{t,s}(p))$. By (5.15) with $\dot{\lambda}(t) = 2$, the local coordinate $\tilde{z}(t; s, p)$ satisfies

$$\frac{d\tilde{z}(t;s,p)}{dt} = 2\pi \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t)}((\mathrm{pr}|_{R_j^+(t)})^{-1}(\tilde{z}(t;s,p)), \xi)\, \nu_t(d\xi) \tag{5.27}$$

for Lebesgue a.e. $t \in [s, s+\delta]$. If $p \in R_j^-(s)$, then the same equation holds true with $+$ replaced by $-$.

Next, we consider the equation "near the endpoints of the slits." Let $s \in [0, T)$ and $p \in U_{z_j^\ell(s)}(l_{D_s}) \setminus \{z_j^\ell(s)\}$. Here, $U_{z_j^\ell(s)}$ and $l_{D_s}$ are defined in the same way as in Section 4.3.1. We take $r, \delta > 0$ so that $\phi^{\natural}_{t,s}(p) \in U_{z_j^\ell(t)}(r) \setminus \{z_j^\ell(t)\}$ for $t \in [s, s+\delta]$. We put $\tilde{z}(t; s, p) := \mathrm{sq}^\ell_{j,t}(\phi^{\natural}_{t,s}(p))$. Here, we define the local chart $\mathrm{sq}^\ell_{j,t}$ around $z_j^\ell(t)$ as in Section 4.3.1, replacing $z_j^\ell$ there by $z_j^\ell(t)$. The local coordinate $\tilde{z}(t; s, p)$ satisfies

$$\begin{aligned}\frac{d}{dt}\tilde{z}(t;s,p) &= \frac{1}{2\tilde{z}(t;s,p)}\left(\frac{\partial \phi_{t,s}(p)}{\partial t} - \frac{dz_j^\ell(t)}{dt}\right) \\ &= \frac{\pi}{\tilde{z}(t;s,p)} \int_{\mathbb{R}} \left(\Psi_{D_t}((\mathrm{sq}^\ell_{j,t})^{-1}(\tilde{z}(t;s,p)), \xi) - \Psi_{D_t}(z_j^\ell(t), \xi)\right) \nu_t(d\xi).\end{aligned} \tag{5.28}$$

Since the holomorphic function

$$\Psi^{\ell,j}_{D_t}[\nu_t](z) := \int_{\mathbb{R}} \left(\Psi_{D_t}((\mathrm{sq}^\ell_{j,t})^{-1}(z), \xi) - \Psi_{D_t}(z_j^\ell(t), \xi)\right) \nu_t(d\xi) \tag{5.29}$$

has a zero at 0 for each $t$, there exists a holomorphic function $H_t(z) = H^{\ell,j}(t, z)$ such that

$$\Psi^{\ell,j}_{D_t}[\nu_t](z) = zH^{\ell,j}(t,z),$$

and it is given by

$$H^{\ell,j}(t,z) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{\Psi^{\ell,j}_{D_t}[\nu_t](\zeta)}{\zeta(\zeta - z)}\, d\zeta, \quad z \in B(0,r). \tag{5.30}$$

Substituting $H^{\ell,j}(t,z)$ into the rightmost side of (5.28), we have

$$\frac{d}{dt}\tilde{z}(t;s,p) = \pi H^{\ell,j}(t, \tilde{z}(t;s,p)). \tag{5.31}$$

This equality holds for Lebesgue a.e. $t \in [s, s+\delta]$. The same equality holds true with the superscript $\ell$ replaced by $r$.

# Chapter 6

# Solutions to Komatu–Loewner equation

In this chapter, we investigate solutions to the Komatu–Loewner equation and families of univalent functions that such solutions form. We treat only the case $\lambda(t) = 2t$ for simplifying the argument.

## 6.1 Local existence of slit motion

We recall a general fact on ordinary differential equations (ODEs for short). Let $v(t, x)$ be an $\mathbb{R}^d$-valued function on an open set $G \subset \mathbb{R} \times \mathbb{R}^d$. We say that $v(t, x)$ enjoys the *Carathéodory condition* if it is Lebesgue measurable in $t$ for each $x$ and continuous in $x$ for each $t$, and for any compact set $K \subset G$, there exists a function $V_K \in L^1(I_K)$ with $I_K := \{\, t \,;\, (t, x) \in K \text{ for some } x \,\}$ such that

$$\|v(t, x)\| \le V_K(t), \quad (t, x) \in K.$$

**Theorem 6.1** (e.g. Theorem 5.1 in Chapter 1 of Hale [35]). *Suppose that $v(t, x)$ enjoys the Carathéodory condition on an open set $G \subset \mathbb{R} \times \mathbb{R}^d$. Then given an initial condition in $G$, the ODE*

$$\frac{dx}{dt}(t) = v(t, x(t)) \tag{6.1}$$

*has a local solution. More precisely, for any $(t_0, x_0) \in G$, there exists a interval $J$ containing $t_0$ and an absolutely continuous function $x(t)$ on $J$ with $x(t_0) = x_0$ such that* (6.1) *holds for Lebesgue a.e. $t \in J$.*

Let us return to the Komatu–Loewner equation for slits (5.24)–(5.25). Given a driving process $(\nu_t)_{t\ge0}$, i.e., a Lebesgue-measurable mapping $t \mapsto \nu_t$

from $[0,\infty)$ to $\mathcal{M}_{\leq 1}(\mathbb{R})$, we solve the equation to obtain the slit motion $\boldsymbol{s}(t)$. To this end, we introduce a suitable notation. For $\boldsymbol{s} \in \mathbf{Slit}$, $\nu \in \mathcal{M}_{\leq 1}(\mathbb{R})$, and $k = 1, 2, \dots, 3N$, we put

$$b_k(\nu, \boldsymbol{s}) := \begin{cases} 2\pi \int_{\mathbb{R}} \Im\Psi_{\boldsymbol{s}}(z_k^\ell, \xi)\, \nu(d\xi), & 1 \leq k \leq N \\ 2\pi \int_{\mathbb{R}} \Re\Psi_{\boldsymbol{s}}(z_{k-N}^\ell, \xi)\, \nu(d\xi), & N+1 \leq k \leq 2N \\ 2\pi \int_{\mathbb{R}} \Re\Psi_{\boldsymbol{s}}(z_{k-2N}^r, \xi)\, \nu(d\xi), & 2N+1 \leq k \leq 3N. \end{cases}$$

We note that $b_1(\nu, \boldsymbol{s}), \dots, b_N(\nu, \boldsymbol{s})$ are positive because $\Im\Psi_{\boldsymbol{s}}(\cdot, \xi)$ is the (harmonic extension of) BMD Poisson kernel $K^*_{D(\boldsymbol{s})}(\cdot, \xi)$. Let $\boldsymbol{b} := (b_k)_{k=1}^{3N}$. Then given a (Lebesgue) measurable process $[0,\infty) \ni t \to \nu_t \in \mathcal{M}_{\leq 1}(\mathbb{R})$, the equation (5.24)–(5.25) can be written as

$$\frac{d\boldsymbol{s}}{dt}(t) = \boldsymbol{b}(\nu_t, \boldsymbol{s}(t)). \tag{6.2}$$

Using Lemma 3.2, we can easily check that $\boldsymbol{b}(\nu_t, \boldsymbol{s})$ satisfies the Carathéodory condition when viewed as a function of variables $(t, \boldsymbol{s})$. By Theorem 6.1, we have the following:

**Proposition 6.2.** *Let $t_0 \in [0,\infty)$. For an initial condition $\boldsymbol{s}(t_0) = \boldsymbol{s}_0 \in$* **Slit***, the ODE* (6.2) *has a local solution $\boldsymbol{s}(t)$ within* **Slit***. For any local solution $\boldsymbol{s}(t)$ to* (6.2)*, the y-coordinates $y_j(t)$, $j = 1, \dots, N$, are non-decreasing in $t$.*

**Proposition 6.3.** *Let $t_0 \in [0,\infty)$. Suppose that, for some neighborhood $J$ of $t_0$, the set $\bigcup_{t\in J} \operatorname{supp} \nu_t$ is bounded. Then a solution $\boldsymbol{s}(t)$ to* (6.2) *with a condition $\boldsymbol{s}(t_0) = \boldsymbol{s}_0 \in$* **Slit** *is unique on $J$.*

*Proof.* The Lipschitz condition of $\boldsymbol{b}(\nu_t, \cdot)$ follows in the same way as in Chen and Fukushima [16, Lemma 4.1] if $\bigcup_{t\in J} \operatorname{supp} \nu_t$ is bounded. Hence the uniqueness follows from Theorem 5.3 in Chapter I of Hale [35]. $\square$

## 6.2 Solutions around moving boundaries

We fix a constant $T > 0$, driving process $[0,T) \ni t \mapsto \nu_t \in \mathcal{M}_{\leq 1}(\mathbb{R})$, and solution $\boldsymbol{s}(t)$ to (6.2) on $[0,T)$. For sufficiently small $T$, the existence of such a solution is guaranteed by Proposition 6.2. For the parallel slit half-planes $D_t := D(\boldsymbol{s}(t))$, we use the notation about the enlargement and analytic continuation on $D_t$ in the same way as in Sections 4.3 and 5.5.

We analyze the Komatu–Loewner equation (5.27) "above the slits." Let $t_0 \in [0,T)$ and $p_0 \in C_j^+(t_0)$. By the continuity of $\boldsymbol{s}(t)$, we can take a compact neighborhood $J$ of $t_0$ and an open neighborhood $U_{p_0}$ of $p_0$ so that $V_{p_0} := \mathrm{pr}|_{U_{p_0}}(U_{p_0})$ satisfies

$$V_{p_0} \subset R_j(t), \quad \emptyset \neq V_{p_0} \cap C_j(t) \subset C_j^\circ(t), \quad \text{and} \quad V_{p_0} \cap \bigcup_{k \neq j} C_k(t) = \emptyset$$

for any $t \in J$.

**Lemma 6.4.** (i) *The ODE*

$$\frac{d\tilde{z}(t)}{dt} = 2\pi \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t)}((\mathrm{pr}|_{R_j^+(t)})^{-1}(\tilde{z}(t), \xi)\, \nu_t(d\xi) \tag{6.3}$$

*has a unique local solution $\tilde{z}(t) = \tilde{z}(t; t_0, p_0)$ with initial value $\tilde{z}(t_0) = \mathrm{pr}(p_0)$.*

(ii) *$\tilde{z}(t; t_0, p_0) \in C_j^\circ(t)$ holds for $t$ near $t_0$.*

*Proof.* (i) We prove the Lipschitz condition of the vector field

$$H^+(t, z) := 2\pi \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t)}((\mathrm{pr}|_{R_j^+(t)})^{-1}(z), \xi)\, \nu_t(d\xi),$$
$$t \in J, \ z \in V_{p_0}.$$

More precisely, the condition is that, for every compact set $K \subset V_{p_0}$,

$$\sup_{t \in J} \sup_{z, w \in K} \frac{|H^+(t, z) - H^+(t, w)|}{|z - w|} < \infty.$$

To this end, it suffices to show that the family of holomorphic functions $z \mapsto H(t, z)$, $t \in J$, is locally bounded on $V_{p_0}$. Then the local boundedness of $\frac{\partial}{\partial z} H^+(t, z)$ follows from the Cauchy integral formula. However, we can check the local boundedness easily in the same way as in the proof of (5.3).
(ii) Since a local solution is unique, it suffices to construct a local solution $\hat{z}(t) = \hat{x}(t) + i\hat{y}(t)$ to (6.3) that satisfies $\hat{z}(t_0) = \mathrm{pr}(p_0)$ and $\hat{z}(t) \in C_j^\circ(t)$.

We note that, for each $t$,

$$\Im \Psi_{\boldsymbol{s}(t)}(p, \xi) = \Im \Psi_{\boldsymbol{s}(t)}(z_j^\ell(t), \xi), \quad p \in C_j^\natural(t), \ \xi \in \partial\mathbb{H}.$$

Clearly, the ODE

$$\frac{d\hat{y}(t)}{dt} = 2\pi \int_{\mathbb{R}} \Im \Psi_{\boldsymbol{s}(t)}(z_j^\ell(t), \xi)\, \nu_t(d\xi)$$

with initial condition $\hat{y}(t_0) = \Im \operatorname{pr}(p_0) = y_j(t_0)$ has a unique solution $\hat{y}(t) = y_j(t)$. Now, we consider the ODE

$$\frac{d\hat{x}(t)}{dt} = \Re H^+(t, \hat{x}(t) + iy_j(t)).$$

This ODE also has a unique local solution $\hat{x}(t)$ with initial condition $\hat{x}(t_0) = \Re \operatorname{pr}(p_0)$ by the Lipschitz condition of $H^+$.

$\hat{z}(t) := \hat{x}(t) + iy_j(t)$ is a local solution to (6.3) with initial condition $\hat{z}(t_0) = \operatorname{pr}(p_0)$, and $\hat{z}(t) \in C_j^\circ(t)$ on some neighborhood of $t_0$. $\square$

Similarly, we consider the Komatu–Loewner equation "below the slits." Let $p_0 \in C_j^-(t_0)$. We take $U_{p_0}$ and $V_{p_0}$ in the same way as above. Since the proof of the following lemma is quite similar to that of Lemma 6.4, we omit it:

**Lemma 6.5.** (i) *The ODE*

$$\frac{d\tilde{z}(t)}{dt} = 2\pi \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t)}((\operatorname{pr}|_{R_j^-(t)})^{-1}(\tilde{z}(t), \xi)\, \nu_t(d\xi) \tag{6.4}$$

*has a unique local solution $\tilde{z}(t) = \tilde{z}(t; t_0, p_0)$ with initial value $\tilde{z}(t_0) = \operatorname{pr}(p_0)$.*

(ii) *$\tilde{z}(t; t_0, p_0) \in C_j^\circ(t)$ holds for $t$ near $t_0$.*

We move to the analysis of the Komatu–Loewner equation (5.31) "near the endpoints of the slits." We use symbols like $U_{z_j^\ell(t)}(\cdot)$, $l_{D_t}$, $\operatorname{sq}_{j,t}^\ell$, $\Psi_{D_t}^{\ell,j}[\nu_t]$, and $H^{\ell,j}$ as they are used in the latter half of Section 5.5. Let $t_0 \in [0, T)$. By the continuity of $\boldsymbol{s}(t)$ and the definition of $U_{z_j^\ell(t)}(\cdot)$, we can take a compact neighborhood $J$ of $t_0$ and constant $r > 0$ so that $2r < \inf_{t \in J} l_{D_t}$.

**Lemma 6.6.** (i) *Let $p_0 \in U_{z_j^\ell(t_0)}(r)$ be fixed. The ODE*

$$\frac{d\tilde{z}(t)}{dt} = \pi H^{\ell,j}(t, \tilde{z}(t)) \tag{6.5}$$

*has a unique local solution $\tilde{z}(t) = \tilde{z}(t; t_0, p_0)$ with initial value $\tilde{z}(t_0; t_0, p_0) = \operatorname{sq}_{j,t_0}^\ell(p_0) \in B(0, r)$.*

(ii) *Let $p_0 = z_j^\ell(t_0)$ in* (i). *Then $\Im \tilde{z}(t; t_0, z_j^\ell(t_0)) = 0$ for $t$ near $t_0$.*

(iii) *The same results hold with the superscript $\ell$ replaced by $r$.*

*Proof.* (i) As in the proof of Lemma 6.4 (i), it suffices to show the local boundedness of $H^{\ell,j}(t,z)$ on $B(0,r)$. This is easily shown by (5.29) and (5.30) with a similar reasoning to the proof of (5.3).
(ii) By the definition of $\Psi^{\ell,j}_{D_t}[\nu_t]$ and $\mathrm{sq}^{\ell}_{j,t}$, we have

$$\Im\Psi^{\ell,j}_{D_t}[\nu_t]((\mathrm{sq}^{\ell}_{j,t})^{-1}(x)) = 0, \quad x \in B(0,r) \cap \partial\mathbb{H}$$

and thus $\Im H^{\ell,j}(x,t) = 0$ for $x \in B(0,r) \cap \partial\mathbb{H}$. Hence, the solution to (6.5) belongs to $\mathbb{R}$ if the initial value $\mathrm{sq}^{\ell}_{j,t_0}(p_0)$ is real. In particular, since $\mathrm{sq}^{\ell}_{j,t_0}(z^{\ell}_j(t_0)) = 0$, we have $\Im\tilde{z}(t;t_0,z^{\ell}_j(t_0)) = 0$.
(iii) The proof is the same, and therefore we omit it. □

## 6.3 Solutions inside moving domains

### 6.3.1 General case

We consider a solution $z(t) = z(t;s,z_0)$ to the ODE

$$\frac{dz(t)}{dt} = 2\pi \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t)}(z(t),\xi)\,\nu_t(d\xi) \tag{6.6}$$

with initial value $z(s) = z_0 \in D_s$ for each $s \in [0,T)$. We can check that (6.6) satisfies the local Lipschitz condition by showing the local boundedness of the vector field on the right-hand side of (6.6) as in the proof of Lemmas 6.4, 6.5 and 6.6. Hence a local solution $z(t;s,z_0)$ in the $(1+2)$-dimensional domain $\bigcup_{t\in[0,T)}\{t\}\times D_t$ exists uniquely on its maximal time interval $[s,\tau_{s,z_0}) \subset [0,T)$ of existence. We note that $\Im z(t;s,z_0)$ is non-decreasing in $t$ because

$$\Im\Psi_{\boldsymbol{s}(t)}(z(t;s,z_0),\xi) = K^*_{D(\boldsymbol{s}(t))}(z(t;s,z_0),\xi) \geq 0.$$

**Proposition 6.7.** *For any $s \in [0,T)$ and $z_0 \in D_s$, it holds that $\tau_{s,z_0} = T$.*

*Proof.* We assume $\tau_{s,z_0} < T$ and derive a contradiction. Since $\Psi_{\boldsymbol{s}(t)}(z(t;s,z_0),\xi)$ is bounded by Lemma 3.2, the limit

$$\begin{aligned}\tilde{z} &:= \lim_{t\nearrow\tau_{s,z_0}} z(t;s,z_0)\\ &= z_0 + 2\pi \lim_{t\nearrow\tau_{s,z_0}} \int_s^{\tau_{s,z_0}}\int_{\mathbb{R}} \mathbf{1}_{[s,t]}(u)\Psi_{\boldsymbol{s}(u)}(z(u;s,z_0),\xi)\,\nu_u(d\xi)\,du\end{aligned}$$

exists by the dominated convergence theorem. $\tilde{z}$ does not belong to $D_{\tau_{s,z_0}}$ by definition, and also it does not lie on $\partial\mathbb{H}$ because $\Im z(t;s,z_0)$ is non-decreasing in $t$. Thus, $\tilde{z} \in C_j(\tau_{s,z_0})$ for some $j$.

Suppose that $\tilde{z} = z_j^\ell(\tau_{s,z_0})$. By the continuity of $\boldsymbol{s}(t)$, we can take constants $r > 0$ and $\delta > 0$ such that

$$r < \min\{\, l_{D_t}\ ;\ \tau_{s,z_0} - \delta \le t \le \tau_{s,z_0} \,\} \quad \text{and}$$
$$z(t; s, z_0) \in B(z_j^\ell(t), r) \setminus C_j^\circ(t), \quad t \in [\tau_{s,z_0} - \delta, \tau_{s,z_0}).$$

By the same computation as in the latter half of Section 5.5, the function $\tilde{z}(t) := \mathrm{sq}_{j,t}^\ell(z(t; s, z_0))$ is a solution to the ODE (6.5) with $\tilde{z}(\tau_{s,z_0}) = \mathrm{sq}_{j,\tau_{s,z_0}}^\ell(z_j^\ell(\tau_{s,z_0})) = 0$. Hence $\Im \tilde{z}(t) = 0$ for $t$ near $\tau_{s,z_0}$ by Lemma 6.6. This implies $z(t; s, t_0) \in C_j(t)$ near $\tau_{s,z_0}$, a contradiction. Thus, $\tilde{z} \neq z_j^\ell(\tau_{s,z_0})$. Since $\tilde{z} \neq z_j^r(\tau_{s,z_0})$ for the same reason, we have $\tilde{z} \in C_j^\circ(\tau_{s,z_0})$. By the definition of $\tau_{s,z_0}$, there exists $\delta > 0$ such that $\Im z(t; s, z_0) - y_j(t)$ takes a constant sign on $[\tau_{s,z_0} - \delta, \tau_{s,z_0})$. Suppose that $\Im z(t; s, z_0) > y_j(t)$ on this interval. Then, $z(t; s, z_0)$ is a local solution to (6.3) near $\tau_{s,z_0}$ that satisfies $\lim_{t \nearrow \tau_{s,z_0}} z(t; s, z_0) = \tilde{z} \in C_j^+(\tau_{s,z_0})$. However, this contradicts Lemma 6.4 because $z(t; s, z_0) \notin C_j(t)$ for $t < \tau_{s,z_0}$. Similarly, the case $\Im z(t; s, z_0) < y_j(t)$ does not occur by Lemma 6.5. In conclusion, $\tilde{z} \notin C_j(t)$ and $\tau_{s,z_0} = T$. $\square$

For a fixed $t_0 \in (0, T)$, we also consider the solution $w(t) = w(t; t_0, z_0)$ to the backward equation

$$\frac{dw(t)}{dt} = -2\pi \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t_0-t)}(w(t), \xi)\, \nu_{t_0-t}(d\xi) \tag{6.7}$$

with the initial condition $w(t_0) = z_0 \in D_{t_0}$. By the same proof to that of Proposition 6.7, we can show that the explosion time $\tilde{\tau}_{t_0,z_0}$ of the solution $w(t; t_0, z_0)$ enjoys $\lim_{t \nearrow \tilde{\tau}_{t_0,z_0}} \Im w(t; t_0, z_0) = 0$ if $\tilde{\tau}_{t_0,z_0} < t_0$.

**Lemma 6.8.** *There exists a constant $\delta_0 > 0$ such that*

$$\tilde{\tau}_{t_0,z_0} \ge (2\delta_0) \wedge t_0$$

*for any $t_0 \in (0, T)$ and $z_0 \in D_{t_0} \cap \mathbb{H}_{\eta_{D_0}/2}$. Here, $\eta_{D_0} := \min\{\, \Im z\ ;\ z \in \mathbb{H} \setminus D_0 \,\}$.*

*Proof.* By Lemma 3.2, $\sup_{\xi \in \mathbb{R}} \Im \Psi_{\boldsymbol{s}(t_0-t)}(w, \xi)$ is bounded by a constant for any $t \in [0, t_0]$ and $w \in D_{t_0-t} \cap \mathbb{H}_{\eta_{D_0}/4}$. Hence the lemma is obvious. $\square$

**Theorem 6.9.** *Let $z(t) = z(t; s, z)$ be the solution to (6.6) with initial condition $z(s) = z \in D_s$. The two-parameter family of the mappings*

$$\phi_{t,s} \colon D_s \to D_t,\ z \mapsto z(t; s, z)$$

*parametrized by $(s, t) \in [0, T)^2_{\le}$ enjoys all the properties of an evolution family over $(D_t)_{t \in [0,T)}$ in Definition 5.1 except the existence of angular residue at infinity.*

*Proof.* It follows from the general theory on ODEs[1] that $\phi_{t,s}$ is holomorphic. Since a solution to (6.6) is unique, $z \neq w$ implies $\phi_{t,s}(z) = z(t;s,z) \neq z(t;s,w) = \phi_{t,s}(w)$, which means the univalence of $\phi_{t,s}$. The uniqueness of solution also yields the identity $\phi_{u,s} = \phi_{u,t} \circ \phi_{t,s}$ for $(s,t,u) \in [0,T)^3_{\le}$.

Let $\delta_0$ be the constant in Lemma 6.8. If $(s,t) \in [0,T)^2_{\le}$ satisfies $t-s \le \delta_0$, then $\phi_{t,s}(D_s)$ contains $D_s \cap \mathbb{H}_{\eta_0/2}$. Since conformal mappings preserve the degree of connectivity of domains, this inclusion implies that $D_t \setminus \phi_{t,s}(D_s)$ is a possibly unbounded $\mathbb{H}$-hull. In fact, the restriction $t-s \le \delta_0$ is unnecessary because, for any $(s,t) \in [0,T)^2_{\le}$, there exists a finite sequence $s = t_0 \le t_1 \le \cdots \le t_n = t$ such that $t_k - t_{k-1} \le \delta_0$ for $k = 1, \dots, n$. From the decomposition

$$\phi_{t,s} = \phi_{t_n,t_{n-1}} \circ \cdots \circ \phi_{t_2,t_1} \circ \phi_{t_1,t_0},$$

we can conclude that $D_t \setminus \phi_{t,s}(D_s)$ is a possibly unbounded $\mathbb{H}$-hull in the general case.

Let $(s,t) \in [0,T)^2_{\le}$ be fixed. By (6.6),

$$\phi_{t,s}(z) = z + 2\pi \int_s^t \int_{\mathbb{R}} \Psi_{\boldsymbol{s}(u)}(z(u;s,z),\xi)\, \nu_u(d\xi)\, du. \tag{6.8}$$

For any $z \in D_s \cap \mathbb{H}_{\eta_{D_0}}$ and $\xi \in \mathbb{R}$, we have $|\Psi_{\boldsymbol{s}(u)}(z(u;s,z),\xi)| \le 4(\pi\eta_{D_0})^{-1}$ by Lemma 3.2. Thus, the integral on the right-hand side of (6.8) is bounded in $z \in D_s \cap \mathbb{H}_{\eta_{D_0}}$. Letting $z \to \infty$ in (6.8), we see that $\phi_{t,s}(z) \to \infty$ as $z \to \infty$. Then we can use (6.8) again and the dominated convergence theorem to obtain

$$\begin{aligned} &\lim_{\substack{z\to\infty \\ \Im z > \eta_0}} (\phi_{t,s}(z) - z) \\ &= 2\pi \int_s^t \int_{\mathbb{R}} \lim_{\substack{z\to\infty \\ \Im z > \eta_{D_0}}} \Psi_{\boldsymbol{s}(u)}(\phi_{u,s}(z),\xi)\, \nu_u(d\xi)\, du = 0. \end{aligned}$$

Hence $\phi_{t,s}$ enjoys the hydrodynamic normalization at infinity. $\square$

### 6.3.2 Bounded-support case

Let us observe that, if the support of $\nu_t$ is bounded in Theorem 6.9, then $\phi_{t,s}$ is actually an evolution family. Suppose that there exists $a > 0$ such that

$$\operatorname{supp} \nu_t \subset [-a, a], \quad t \in [0,T). \tag{6.9}$$

[1]See, e.g., Exercise 3.3 in Chapter I of Hale [35] or Theorem 8.4 in Chapter 1 of Coddington and Levinson [19]. Although the vector fields on the right-hand side of ODEs are assumed to be jointly continuous in these references, proof can be modified easily.

In this case, the vector field $\int_{\mathbb{R}} \Psi_{\boldsymbol{s}(t)}(z,\xi)\,\nu_t(d\xi)$ is defined on the $(1+2)$-dimensional domain

$$\bigcup_{t\in[0,T)} (\{t\}\times(D_t\cup(\partial\mathbb{H}\setminus[-a,a])\cup\Pi_0 D_t))$$

and enjoys the local Lipschitz condition there. Hence (6.6) has a unique local solution passing through $(t_0,z_0)$ for any initial data $(t_0,z_0)$ in this domain. Since $\Im\Psi_{\boldsymbol{s}(t)}(x,\xi)=0$ for $x\in\partial\mathbb{H}\setminus\{\xi\}$, the following lemma is obvious:

**Lemma 6.10.** *If $t_0\in[0,T)$ and $x_0\in\partial\mathbb{H}\setminus[-a,a]$, then $\Im z(t;t_0,x_0)=0$ for all $t$ (in its maximal interval of existence).*

For $t_0\in[0,T)$ and $x_0\in\partial\mathbb{H}\setminus[-a,a]$, we put

$$\sigma_{t_0,x_0} := T\wedge\sup\{\,t\in(t_0,T)\ ;\ z(t;t_0,x_0)\in\partial\mathbb{H}\setminus[-a,a]\,\}.$$

**Lemma 6.11.** *There exists a constant $\delta_1$ such that*

$$\sigma_{t_0,x_0}\ge(2\delta_1)\wedge(T-t_0)$$

*for any $t_0\in[0,T)$ and $x_0\in\mathbb{R}\setminus[-2a,2a]$.*

*Proof.* By Lemma 3.2, we have

$$|\Psi_{\boldsymbol{s}(t)}(x,\xi)|\le\frac{4}{\pi(\eta_{D_0}\wedge a)},\quad x\in\partial\mathbb{H}\setminus[-2a,2a],\ \xi\in[-a,a].$$

Hence the conclusion easily follows from (6.6). □

**Theorem 6.12.** *The family $(\phi_{t,s})_{(s,t)\in[0,T)^2_{\le}}$ defined in Theorem 6.9 is an evolution family over $(D_t)_{t\in[0,T)}$ with $\mu(\phi_{t,0};\mathbb{R})=2t$ under the assumption* (6.9). *$(\phi_{t,s})$ is an unique evolution family that satisfies* (5.12) *with $\lambda(t)=2t$.*

*Proof.* We fix $(s,t)\in[0,T)^2_{\le}$ such that $t-s\le\delta_1$. Here, $\delta_1$ is the constant in Lemma 6.11. For each $u\in[s,t]$, the function $\phi_{u,s}(z):=z(u;s,z)$ is defined on $D_s\cup(\partial\mathbb{H}\setminus[-2a,2a])\cup\Pi_0 D_s$ and univalent there. We can show that $D_u\setminus\phi_{u,s}(D_s)$ is bounded uniformly in $u\in[s,t]$ as follows: Let $r^{\mathrm{out}}_{D_u}:=\sup\{\,|z|\ ;\ z\in\mathbb{H}\setminus D_u\,\}$. Then $\max_{u\in[s,t]} r^{\mathrm{out}}_{D_u}<\infty$ by the continuity of $\boldsymbol{s}(u)$. We choose a constant $r$ so that

$$r>\eta_0+2a+\frac{8\delta_1}{\eta_{D_0}\wedge a}+\max_{u\in[s,t]} r^{\mathrm{out}}_{D_u}.$$

By Lemma 3.2,

$$|\Psi_{\boldsymbol{s}(u)}(w,\xi)| \le \frac{4}{\pi(\eta_{D_0}\wedge a)} \tag{6.10}$$
$$\text{for } w \in B(0,r)^c,\ \xi \in [-a,a],\ u \in [s,t].$$

Hence, by considering the backward equation as in Section 6.3.1, we can show that $\phi_{u,s}(D_s)$ contains $B(0,2r)^c$ for every $u \in [s,t]$. In other words, $D_u \setminus \phi_{u,s}(D_s) \subset B(0,2r)$. It follows from a similar reasoning that

$$\phi_{u,s}(B(0,2r)^c) \subset B(0,r)^c, \quad u \in [s,t]. \tag{6.11}$$

Thus, (6.8), (6.10) and (6.11) imply

$$\lim_{z\to\infty} \phi_{u,s}(z) = \infty, \quad u \in [s,t],$$

and

$$\begin{aligned}&\lim_{z\to\infty}(\phi_{t,s}(z) - z)\\ &= 2\pi \int_s^t \int_{\mathbb{R}} \lim_{z\to\infty} \Psi_{\boldsymbol{s}(u)}(\phi_{u,s}(z),\xi)\,\nu_u(d\xi)\,du = 0\end{aligned} \tag{6.12}$$

owing to the dominated convergence theorem.

The residue at infinity is obtained as follows: Recall from Proposition 3.6 that the function

$$w \mapsto -\frac{1}{w}\Psi_{\boldsymbol{s}(u)}\left(-\frac{1}{w},\xi\right)$$

is holomorphic around the origin. By the maximum principle and (6.10), we have

$$\begin{aligned}\sup_{z\in B(0,r)^c} |z\Psi_{\boldsymbol{s}(u)}(z,\xi)| &= \sup_{w\in B(0,r^{-1})} \left|-\frac{1}{w}\Psi_{\boldsymbol{s}(u)}\left(-\frac{1}{w},\xi\right)\right|\\ &\le \frac{4r}{\pi(\eta_{D_0}\wedge a)}, \quad \xi\in[-a,a],\ u\in[s,t].\end{aligned} \tag{6.13}$$

Now, we consider the identity

$$\begin{aligned}z\Psi_{\boldsymbol{s}(u)}(\phi_{u,s}(z),\xi) = \phi_{u,s}(z)&\Psi_{\boldsymbol{s}(u)}(\phi_{u,s}(z),\xi)\\ &- (\phi_{u,s}(z) - z)\Psi_{\boldsymbol{s}(u)}(\phi_{u,s}(z),\xi).\end{aligned} \tag{6.14}$$

The right-hand side of (6.14) is bounded in $z \in B(0,2r)^c$ and $u \in [s,t]$ by (6.8), (6.10) and (6.11). Moreover, it converges to $-1/\pi$ as $z \to \infty$ by

Proposition 3.6 and (6.12). Thus, the dominated convergence theorem yields

$$\begin{aligned}&\lim_{z\to\infty} z(\phi_{t,s}(z)-z)\\&=2\pi\int_s^t\int_{\mathbb{R}}\lim_{z\to\infty} z\Psi_{\boldsymbol{s}(u)}(\phi_{u,s}(z),\xi)\,\nu_u(d\xi)\,du=-2(t-s).\end{aligned}$$

Finally, given a general pair $(s,t)\in[0,T)^2_{\leq}$, we can decompose $\phi_{t,s}$ along a finite sequence $s=t_0\leq t_1\leq\cdots\leq t_n=t$ with $t_k-t_{k-1}\leq\delta_1$ as

$$\phi_{t,s}=\phi_{t_n,t_{n-1}}\circ\cdots\circ\phi_{t_2,t_1}\circ\phi_{t_1,t_0}.$$

It is easy to derive the conclusion from this decomposition. The uniqueness of an evolution family is just a consequence of the uniqueness of solutions to (6.2) and (6.6). □

**Theorem 6.13.** *Let $\boldsymbol{s}_0\in\mathbf{Slit}$ and $[0,\infty)\ni t\mapsto\nu_t\in\mathcal{M}_{\leq 1}(\mathbb{R})$ be a Lebesgue measurable process. Suppose that, for any $T>0$, there exists a constant $a>0$ such that* (6.9) *holds. Then there exists an unique evolution family $(\phi_{t,s})_{(s,t)\in[0,\infty)^2_{\leq}}$ over a family $(D_t)_{t\in[0,\infty)}$ of parallel slit half-planes with $D_0=D(\boldsymbol{s}_0)$ such that* (5.12), (5.24) *and* (5.25) *hold with $\lambda(t)=2t$.*

*Proof.* We assume that a solution $\boldsymbol{s}(t)$ to (6.2) with initial value $\boldsymbol{s}(0)=\boldsymbol{s}_0$ explodes at $T<\infty$. This is a unique solution by Proposition 6.3. By Theorem 6.12, there exists a unique evolution family $(\phi_{t,s})_{(s,t)\in[0,T)^2_{\leq}}$ driven by $\nu_t$ and $\boldsymbol{s}(t)$. By Proposition 5.9, this family is extended to that on $[0,T]^2_{\leq}$. In particular, it follows that $\lim_{t\nearrow T}\boldsymbol{s}(t)\in\mathbf{Slit}$, which contradicts the definition of $T$. Hence (6.2) has a unique global solution. By applying Theorem 6.12 to any bounded interval $[0,T]$ again, we can get the desired evolution family. □

# Chapter 7

# Application

We apply the present and previous results of the author to illustrate the way in which driving processes reflect the geometry and continuity of Loewner chains.

## 7.1 Conditions equivalent to local growth property

Let $(F_t)_{t\in[0,T]}$ be a family of *bounded* $\mathbb{H}$-hulls growing in a parallel slit half-plane $D = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j$. Note that it is assumed to be strictly increasing; $s < t$ implies $F_s \subsetneq F_t$. The BMD half-plane capacity $\mathrm{hcap}^D(F_t)$ is hence strictly increasing in $t$. Let $g_t \colon D \setminus F_t \to D_t$ be the mapping-out function of $F_t$, that is, a unique conformal mapping onto a parallel slit half-plane $D_t$ with $\lim_{z\to\infty}(g_t(z) - z) = 0$. The family $(g_t^{-1})_{t\in[0,T]}$ is a *reversed* Loewner chain, i.e., $(g_{T-t}^{-1})_{t\in[0,T]}$ is a Loewner chain. Thus, the general form of Komatu–Loewner equation (5.21) applies. We consider necessary and sufficient conditions on $(F_t)_{t\in[0,T]}$ in order that $g_t(z)$ obeys a differential equation of a particular form (2.11).

**Definition 7.1.** $(F_t)_{t\in[0,T]}$ is said to have the *local growth property* if, for $\varepsilon > 0$, there exists a constant $\delta \in (0, T)$ with the following property: For each $t \in [0, T - \delta]$, some cross-cut $C$ of $D \setminus F_t$ with $\mathrm{diam}(C) < \varepsilon$ separates the increment $F_{t+\delta} \setminus F_t$ from the point at infinity in $D \setminus F_t$. Here, by a *cross-cut* of $D \setminus F_t$, we mean the trace of a simple curve $c \colon [0, 1] \to \overline{D \setminus F_t}$ with $c(0), c(1) \in \partial(D \setminus F_t)$ and $c(0, 1) \subset D \setminus F_t$.

In Definition 7.1, $(F_t)_{t\in[0,T]}$ has the "uniform continuity" in terms of the diameter of cross-cuts. This will be clearer if we rephrase the local growth property as follows:

For any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $0 \le t - s \le \delta$, then some cross-cut $C$ of $D \setminus F_s$ with $\mathrm{diam}(C) < \varepsilon$ separates $F_t \setminus F_s$ from $\infty$ in $D \setminus F_s$.

Here, even if $s > T - \delta$, the difference $F_t \setminus F_s (\subset F_T \setminus F_{T-\delta})$ is separated from $\infty$ in $D \setminus F_{T-\delta}$ by a cross-cut $C$ of $D \setminus F_{T-\delta}$. By definition, $C$ does not intersect $F_s \setminus F_{T-\delta} (\subset F_T \setminus F_{T-\delta})$ except at its endpoints. Thus, it is also a cross-cut of $D \setminus F_s = (D \setminus F_{T-\delta}) \setminus (F_s \setminus F_{T-\delta})$.

Pommerenke [51] proved that the local growth property holds if and only if the driving process reduces to a continuous function for the radial Loewner equation (1.1). In the SLE context, Lawler, Schramm and Werner [46] proved this equivalence for the chordal Loewner equation (2.7), and Zhan [61] mentioned the annulus case. Böhm [9] proved this fact for the radial Komatu–Loewner equation on circularly slit disks. As he pointed out, we can always assume that the endpoints of the cross-cut $C$ lie on the outer boundary $\partial(\mathbb{H} \setminus F_t)$ of $D \setminus F_t$ in Definition 7.1.

**Proposition 7.2.** *Suppose that $(F_t)_{t \in [0,T]}$ is of local growth.*

(i) *The function $t \mapsto \mathrm{hcap}^D(F_t)$ is continuous.*

(ii) *There exists a continuous function $\xi \colon [0,T] \to \mathbb{R}$ such that*

$$\bigcap_{\delta > 0} \overline{g_t(F_{t+\delta} \setminus F_t)} = \{\xi(t)\}, \quad t \in [0,T). \tag{7.1}$$

We prove Proposition 7.2 in Section 7.2 below. The property (7.1) is important in that $\xi(t)$ is to be the driving function in what follows. As it is called the right continuity with limit $\xi(t)$ in Lawler's monograph [44], it greatly reflects the "right continuity" of $(F_t)_{t \in [0,T]}$. In order to describe the "left continuity", we recall a classical concept concerning the convergence of domains from Section 5, Chapter V of Goluzin [33].

**Definition 7.3.** (i) Let $G_n$, $n \in \mathbb{N}$, be domains in $\mathbb{C}$ and $a \in \mathbb{C}$. The kernel $\ker_a(G_n)_{n \in \mathbb{N}}$ with respect to $a$ is defined as the component containing $a$ of the set $\{\, z \in \mathbb{C} \ ; \ B(z; r) \subset \bigcap_{n \ge N} G_n$ for some $r > 0$ and some $N \in \mathbb{N} \,\}$.

(ii) Let $I$ be an interval, $t_0 \in I$, and $a \in \mathbb{C}$. Let $G_t$, $t \in I$, be domains in $\mathbb{C}$. We say that $G_t$ converges to $G_{t_0}$ as $t \to t_0$ in the sense of kernel (or in Carathéodory's sense) with respect to $a$ if $\ker_a(G_{s_n})_{n \in \mathbb{N}} = G_{t_0}$ for every sequence $(s_n)_{n \in \mathbb{N}}$ of $I$ with $s_n \to t_0$.

(iii) $(F_t)_{t\in[0,T]}$ is said to be *left continuous* at $t_0 \in (0,T]$ (in the sense of kernel convergence or in Carathéodory's sense) if the domain $D \setminus F_t$ converges to $D \setminus F_{t_0}$ as $t$ increases to $t_0$ in the sense of kernel with respect to some (any) $a \in D \setminus F_T$.

Since $(F_t)_{t\in[0,T]}$ is increasing, it automatically holds that $D \setminus F_{t_0} \subset \ker_a(D \setminus F_{s_n})_{n\in\mathbb{N}}$ for any sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \uparrow t_0$. Our left continuity requires that this inclusion should be equality. We can define the right continuity of $(F_t)_{t\in[0,T]}$ in the same manner. In particular, our right continuity follows from the property (7.1).

**Lemma 7.4** (Murayama [49, Lemma 4.4])**.** *If $(F_t)_{t\in[0,T]}$ is continuous (in the sense of kernel convergence), then $t \mapsto \operatorname{hcap}^D(F_t)$ is continuous.*

The author proved in the previous paper [49] that the property (7.1) and left continuity hold if and only if the chordal Komatu–Loewner equation (2.11) holds. We relate these conditions to the local growth property as follows:

**Theorem 7.5.** *Let $\ell(t) := \operatorname{hcap}^D(F_t)$. The following three conditions are mutually equivalent:*

(i) *$(F_t)_{t\in[0,T]}$ has the local growth property;*

(ii) *The property* (7.1) *holds for some continuous function $\xi(t)$ on $[0,T]$, and $(F_t)_{t\in[0,T]}$ is left continuous on $(0,T]$ in the sense of Definition 7.3* (iii)*;*

(iii) *$\ell(t)$ is continuous, and there exists a continuous function $\xi(t)$ on $[0,T]$ such that*

$$\frac{\partial g_t(z)}{\partial \ell(t)} := \lim_{h\to 0} \frac{g_{t+h}(z) - g_t(z)}{\ell(t+h) - \ell(t)} = -\pi \Psi_{D_t}(g_t(z), \xi(t)) \tag{7.2}$$

*for every $z \in D \setminus F_t$ and $t \in [0,T]$.*

*Proof.* By Proposition 7.2 (i) and Lemma 7.4, the function $\ell(t)$ is increasing and continuous if one of the three conditions holds. In particular, if we take any increasing and continuous function $\theta(t)$ on $[0,T]$ and perform time-change as $\tilde{F}_t := F_{\theta^{-1}(t)}$, $\tilde{g}_t := g_{\theta^{-1}(t)}$, $\tilde{D}_t := D_{\theta^{-1}(t)}$, and $\tilde{\xi}(t) := \xi(\theta^{-1}(t))$, then the conditions (ii) and (iii) on $(F_t)_{t\in[0,T]}$ are equivalent to those on $(\tilde{F}_t)_{t\in[0,\theta(T)]}$, respectively. (See also (5.16).) We can easily prove, using the uniform continuity of $\theta$ on $[0,T]$, that this is also the case for (i). Therefore, we may reparametrize $(F_t)_{t\in[0,T]}$ whenever it is necessary to make our setting consistent to those of previous studies.

In addition to the independence from reparametrization, we note that the conditions (i) and (ii) are independent of whether parallel slits exist or not. Namely, if $(F_t)_{t\in[0,T]}$ enjoys (i) or (ii) as a family of $\mathbb{H}$-hulls *in* $D$, then so does it, respectively, as a family of $\mathbb{H}$-hulls *in* $\mathbb{H}$, and vice versa. This is clear from definition (see [49, Proposition 4.7] for example).

Lawler, Schramm and Werner [46, Theorem 2.6] showed that (i) is equivalent to (iii) as long as $(F_t)_{t\in[0,T]}$ is regarded as a family of hulls in $\mathbb{H}$. We also know from Murayama [49, Theorem 4.6] that (ii) and (iii) are equivalent both in $\mathbb{H}$ and in $D$, but this result applies to a right-open interval $[0, T)$ only. If we can extend it to $t = T$, then proof is complete.

The implication (iii) $\Rightarrow$ (ii) at $t = T$ is trivial, because Lemma 5.4 shows the continuity of corresponding slit motion $\boldsymbol{s}(t)$. We observe (ii) $\Rightarrow$ (iii) at $t = T$. First, the (left) continuity of $\ell(t)$ at $t = T$ follows from the left continuity of $(F_t)_{t\in[0,T]}$ [49, Lemma 4.4 (i)]. It remains to show

$$g_T(z) = z - \pi \int_0^T \Psi_{D_t}(g_t(z), \xi(t))\, dt \tag{7.3}$$

for every $z \in D \setminus F_T$. By the left continuity of $(F_t)_{t\in[0,T]}$ again and the kernel theorem [49, Theorem 3.8], $\boldsymbol{s}(t) \to \boldsymbol{s}(T)$ and $g_t(z) \to g_T(z)$, $z \in D \setminus F_T$, hold as $t \uparrow T$. Then (7.3) follows from Lebesgue's dominated convergence theorem (cf. Step 1 in the proof of Proposition 5.9). $\square$

## 7.2 Direct derivation from local growth property

By Theorem 7.5, the local growth property implies that the driving process reduces to a real-valued continuous function. However, the proof of this theorem in Section 7.1 has told us little about the mechanism through which the local growth property determines the support $\{\xi(t)\}$ of the measure-valued driving process $\nu_t = \delta_{\xi(t)}$. In order to observe this mechanism, it is helpful to prove Proposition 7.2 along the same line[1] as that of Lawler, Schramm and Werner [46, Theorem 2.6] and to show the implication (i) $\Rightarrow$ (iii) in Theorem 7.5. Following the original proof [46, Theorem 2.6] with suitable modification, we shall below give entire proofs of Proposition 7.2 and the implication (i) $\Rightarrow$ (iii) in Theorem 7.5.

We begin with recalling the definition of extremal length. Let $\Gamma$ be a path

[1]See also Pommerenke's original argument [51].

family in a planar domain, that is, a set consisting of rectifiable paths[2]. The *extremal length* of $\Gamma$ is defined by

$$\mathrm{EL}(\Gamma) := \sup_{\rho} \frac{\inf_{\gamma\in\Gamma} \int_\gamma \rho(z)\,|dz|}{\int_U \rho(z)^2\,dx\,dy}, \quad z = x + iy.$$

Here, we fix any domain $U$ containing all paths of $\Gamma$, and the supremum is taken over every non-negative Borel measurable function $\rho$ on $U$ with $0 < \int_U \rho^2\,dx\,dy < \infty$. We can deduce that $\mathrm{EL}(\Gamma)$ is independent of the choice of $U$ and moreover conformally invariant. We refer the reader to Chapter 4 of Ahlfors [1] or Chapter IV of Garnett and Marshall [32] for the property of extremal length.

*Proof of Proposition 7.2.* Let $(F_t)_{t\in[0,T]}$ be a family of $\mathbb{H}$-hulls with local growth in a parallel slit half-plane $D = \mathbb{H} \setminus \bigcup_{j=1}^N C_j$, and put $L := \sup\{\,|z|\,;\, z \in F_T \cup \bigcup_{j=1}^N C_j\,\}$. We fix $\varepsilon \in (0,1)$ such that $2\sqrt{\varepsilon} < d^{\mathrm{Eucl}}(F_T, \bigcup_{j=1}^N C_j)$. Then we can take $\delta > 0$ with the following property: For every $(s,t) \in [0,T]^2_<$ with $t - s \le \delta$, a cross-cut $C$ with $\mathrm{diam}(C) < \varepsilon$ separates $F_t \setminus F_s$ from $\infty$ in $D \setminus F_s$. Using this cross-cut $C$, we give an upper bound of the extremal length of the set $\Gamma$ of rectifiable paths which separate $F_t \setminus F_s$ from $B(0, L+2)^c$ in $D \setminus F_s$ as follows: For a fixed $z_0 \in C$, let $\Gamma'$ be the set of rectifiable paths separating the inner and outer boundaries of the annulus $\mathbb{A}(z_0; \varepsilon, \sqrt{\varepsilon}) := \{\, z \in \mathbb{C}\,;\, \varepsilon < |z - z_0| < \sqrt{\varepsilon}\,\}$. Any $\gamma' \in \Gamma'$ contains some path $\gamma \in \Gamma$ in the sense that $\gamma \subset \gamma'$. It follows from [1, Theorem 4.1] or [32, Eq. (3.2)] that

$$\mathrm{EL}(\Gamma) \le \mathrm{EL}(\Gamma') = \frac{4\pi}{\log(1/\varepsilon)}.$$

Here, see Section 1, Chapter IV of [32] for computing the value of $\mathrm{EL}(\Gamma')$. Moreover, by the conformal invariance of extremal length,

$$\mathrm{EL}(g_s(\Gamma)) = \mathrm{EL}(\Gamma) \le \frac{4\pi}{\log(1/\varepsilon)}. \tag{7.4}$$

We notice that $g_s(B(0, L+2) \cap (D \setminus F_s))$ is bounded. Indeed, a consequence [49, Lemma 3.9] from the hydrodynamic normalization implies that

$$\mathrm{diam}(g_s(B(0, L+2) \cap (D \setminus F_s))) \le 4(L+2).$$

Thus, by (7.4) and the definition of extremal length,

$$\inf_{\gamma\in g_s(\Gamma)} \left(\int_\gamma |dz|\right)^2 \le \frac{4\pi}{\log(1/\varepsilon)} \int_{g_s(B(0,L+2)\cap(D\setminus F_s))} dx\,dy \le \frac{32\pi^2(L+2)^2}{\log(1/\varepsilon)}.$$

[2] In general, the definition of path family allows an element $\gamma \in \Gamma$ to be a countable union of curves, but in what follows, we treat only connected paths.

Since any $\gamma \in g_s(\Gamma)$ is connected and separates $g_s(F_t \setminus F_s)$ from $\infty$ in $D_s$, we have

$$\mathrm{diam}(g_s(F_t \setminus F_s)) \leq \frac{4\sqrt{2}\pi(L+2)}{\sqrt{\log(1/\varepsilon)}}. \tag{7.5}$$

We write the right-hand side of (7.5) as $r(\varepsilon)$.

We now put $g_{t,s} = g_s \circ g_t^{-1}$. From the same reasoning as above and the boundary correspondence, it follows that

$$\mathrm{supp}[\mu(g_{t,s};\cdot)] \subset [\xi(t) - r(\varepsilon), \xi(t) + r(\varepsilon)]. \tag{7.6}$$

Here, $\mu(g_{t,s};\cdot)$ is the measure defined in Theorem 4.3 and the paragraph after Remark 4.4. Since

$$\mathrm{hcap}^{D_s}(g_s(F_t \setminus F_s)) = \pi^{-1} \int_{\mathrm{supp}[\mu(g_{t,s};\cdot)]} \Im g_{t,s}(\xi)\, d\xi,$$

we have

$$\ell(t) - \ell(s) = \mathrm{hcap}^{D_s}(g_s(F_t \setminus F_s)) \leq 2r(\varepsilon)^2$$

by (7.5) and (7.6). This inequality proves the uniform continuity of $\ell(t)$ and hence (i).

By (7.5), there exists a point $\xi(t) \in \partial\mathbb{H}$ such that $\bigcap_{\delta>0} \overline{g_t(F_{t+\delta} \setminus F_t)} = \{\xi(t)\}$ for every $t \in [0,T)$. The proof of (ii) is thus complete if we prove the uniform continuity of $\xi(t)$ on $[0,T)$. Recall from (4.1) that

$$g_{t,s}(z) = z + \pi \int_{\mathrm{supp}[\mu(g_{t,s};\cdot)]} \Psi_{D_t}(z,\xi) \cdot \pi^{-1} \Im g_{t,s}(\xi)\, d\xi. \tag{7.7}$$

Let $0 < r' < \eta_{D_T} (= \min\{\, \Im z \ ;\ z \in \mathbb{H} \setminus D_T \,\})$. The representation (7.7), combined with (3.12) and (7.6), implies that, for any $z \in D_t \setminus \bar{B}(\xi(t), r(\varepsilon)+r')$,

$$|g_{t,s}(z) - z| \leq \frac{4}{r'}(\ell(t) - \ell(s)).$$

Then applying an argument using cross-cuts, which is similar to the above one, to the conformal mapping $g_{t,s}$, we have

$$|\xi(s) - \xi(t)| \leq r(\varepsilon) + r' + \frac{4}{r'}(\ell(t) - \ell(s)).$$

Since the right-hand side depends only on $\delta$, not on $s, t$, it holds that

$$\limsup_{\delta \downarrow 0} \sup_{0<t-s\leq\delta} |\xi(s) - \xi(t)| \leq r'.$$

Letting $r' \downarrow 0$, we obtain the uniform continuity of $\xi(t)$. $\square$

By Proposition 7.2 and its proof, we can show the implication (i) $\Rightarrow$ (iii) in Theorem 7.5 as follows:

*Proof of* (i) $\Rightarrow$ (iii) *in Theorem 7.5.* We put $\phi_{u,s} := g_{T-s,T-u} = g_{T-u} \circ g_{T-s}^{-1}$ for $(s,u) \in [0,T]^2_{\leq}$ and $\lambda(t) := \mathrm{hcap}^D(F_T) - \mathrm{hcap}^D(F_{T-t})$ for $t \in [0,T]$. Then (7.6) implies that $\mu(\phi_{u,s};\cdot)/\mu(\phi_{u,s};\mathbb{R})$ converges weakly to $\delta_{\xi(T-t)}(\cdot)$ as $s,u \to t$. By Corollary 5.8, we have

$$\begin{aligned}\partial_t^\lambda \phi_{t,s}(z) &= \pi \int_{\mathbb{R}} \Psi_{D_{T-t}}(\phi_{t,s}(z), \xi')\, \delta_{\xi(T-t)}(d\xi') \\ &= \Psi_{D_{T-t}}(\phi_{t,s}(z), \xi(T-t)).\end{aligned}$$

Hence, substituting $g_{T-s}(z)$ into the $z$ in this equation and taking time-reversal, we get the conclusion. $\square$

## 7.3 Case of multiple paths

Let $D$ be a parallel slit half-plane and $\gamma_k\colon [0,T] \to \overline{D}$, $k = 1, \dots, n$, be $n$ disjoint simple curves with $\gamma_k(0) \in \partial\mathbb{H}$ and $\gamma_k(0,T] \subset D$. We put $F_t := \bigcup_{k=1}^n \gamma_k(0,t]$ and consider the mapping-out function $g_t\colon D \backslash F_t \to D_t$. For each $k$ and $t$, There exists a unique point $\xi_k(t) \in \partial\mathbb{H}$ such that $\lim_{z \to \xi_k(t)} g_t(z) = \gamma_k(t)$ by the boundary correspondence.

**Proposition 7.6.** (i) *$\ell(t) := \mathrm{hcap}^D(F_t)$ and $\xi_k(t)$, $k = 1, \dots, n$, are continuous in $t$.*

(ii) *There exist an $m_\ell$-null set $N \subset [0,T]$ and $c_1(t), \dots, c_n(t) \geq 0$ with $\sum_{k=1}^n c_k(t) = 1$ such that*

$$\tilde{\partial}_t^\ell g_t(z) = -\pi \sum_{k=1}^n c_k(t) \Psi_{D_t}(g_t(z), \xi_k(t)) \tag{7.8}$$

*holds for every $t \in [0,T] \setminus N$ and $z \in D \setminus F_t$.*

*Proof.* Since this proposition can be proved along the same line as in Section 7.2, we omit the detail. We just note two things. Firstly, even if the support $\mathrm{supp}[\mu(\phi_{u,s};\cdot)]$ for $\phi_{u,s} = g_{T-u} \circ g_{T-s}$ shrinks to the $n$-point set $\{\xi_1(T-t), \dots, \xi_n(T-t)\}$ as $s,u \to t$, the normalized measure $\mu(\phi_{u,s};\cdot)/\mu(\phi_{u,s};\mathbb{R})$ does not necessarily converge weakly. The mass on a neighborhood of each $\xi_k(T-t)$ may oscillate. For this reason, we use Theorem 5.7 instead of Corollary 5.8 in the present case. Secondly, (i) follows from Lemmas 2.38 and 2.43 of Böhm [9] as well. $\square$

**Remark 7.7** (Branch points)**.** In contrast to (7.2) in Theorem 7.5, one has not formulated so far any explicit condition on $(F_t)_{t\in[0,T]}$ that is equivalent to (7.8). For example, we can replace disjoint paths in Proposition 7.6 by disjoint hulls of local growth. See Starnes [59]. However, this replacement does not give a necessary condition for (7.8). We also have to consider the more complicated situation in which one path or hull touches other one. In fact, Böhm and Schleißinger [12] studied the $t$-differentiability of the mapping-out function $g_t(z)$ for the union of two paths $\gamma_1(0,t]$ and $\gamma_2(0,t]$ such that $\gamma_1(0) = \gamma_2(0)$. They gave a condition of $\gamma_1$ and $\gamma_2$ sufficient for $g_t(z)$ to be (right-)differentiable at $t = 0$ [12, Theorem 1.5], while constructing an example of a pair $(\gamma_1, \gamma_2)$ for which $t \mapsto g_t(z)$ is not differentiable at $t = 0$.

# Chapter 8

# Concluding remarks

We add some remarks related to previous and future works.

## 8.1 Remaining problems

We recall that a solution to the Komatu–Loewner equation for slits (6.2) is not shown to be unique. However, from the viewpoint described in Section 1.1, *the uniqueness of slit motion is plausible.* A possible way for proving the uniqueness is to establish a result on the local Lipschitz continuity of the function $\mathbf{Slit} \ni \boldsymbol{s} \mapsto \Psi_{\boldsymbol{s}}(z, \xi)$ stronger than Chen, Fukushima and Rohde [17, Theorem 9.1]. If *the Lipschitz constant turns out to be independent of* $\xi \in \mathbb{R}$, then we can drop the assumption that $\bigcup_{t \in J} \operatorname{supp} \nu_t$ is bounded in Proposition 6.3. In order to show this independence, we may need to elaborate the interior variation method developed in Section 12 of Chen, Fukushima and Rohde [17].

Chapter 6 contains another problem. From the same viewpoint as in the preceding paragraph, we believe that *the univalent function* $\phi_{t,s}$ *in Theorem 6.9 should have a finite angular residue at infinity.* An obstacle to a proof of this property is the dependence on $D$ of the estimate (3.16). If (3.16) is strengthened so that *it is locally uniform with respect to the variation of* $D$, then we can obtain the existence of finite angular residue. Nevertheless, it seems that some new ideas are required for the improvement of (3.16) at the present moment.

Although digressing from our subject, Section 4.2 contains one more remaining problem. In Theorem 4.3, we have assumed the univalence of $f$ in advance to obtain the representation formula (4.1). Conversely, *what condition of the measure* $\mu$ *ensures that the holomorphic function* $f$ *given by* (4.1) *is univalent*? The absolute continuity of $\mu$ with respect to the Lebesgue measure is necessary by Theorem 4.3, but we do not know whether it is also sufficient. This question will be of independent interest in geometric function theory.

In addition to the above-mentioned problems appearing in this thesis, it is a natural problem to construct analogous theories on circularly slit disk (radial case) and on circularly sit annuli (bilateral case). See Bauer and Friedrich [5, 6, 7], Fukushima and Kaneko [31], Böhm and Lauf [10], and Böhm [9] for previous studies on Komatu–Loewner equations in these cases.

## 8.2 Time-reversal, explosion of slit motion, and SLE on multiply connected domains

In Chapters 2 and 7, we have considered reversed evolution families and reversed Loewner chains. As is illustrated in Chapter 7, we can derive a differential equation for reversed families without any additional effort. This corresponds to the arrows from Loewner chains and from evolution families to driving processes in Figure 1.1. However, matters are different for the opposite arrows. We can see it from the reversed Komatu–Loewner equation for slits[1] $\dot{\boldsymbol{s}}(t) = -\boldsymbol{b}(\nu_t, \boldsymbol{s}(t))$. In this case, the $y$-coordinates $y_j(t)$ $(1 \leq j \leq n)$ are decreasing in $t$ whereas they are increasing in Section 6.1. Thus, even if $\operatorname{supp} \nu_t \subset [-a, a]$ for some $a > 0$, the motion $\boldsymbol{s}(t)$ may explode in the sense that $\lim_{t\uparrow\zeta} \min_{1\leq j\leq n} y_j(t) = 0$ for some $\zeta \in (0, \infty)$. In other words, the slits of $D_t$ may be absorbed by the outer boundary $\partial\mathbb{H}$ at $t = \zeta$. For a reversed Loewner chain $(f_t)_{t\in[0,T]}$ with $f_0(D_0) = D$, such explosion is closely related to the phenomenon that the hulls $F_t = D \setminus f_t(D_t)$ "swallow" some part of the slits of $D$.

The author studied the case $\zeta < \infty$ for (7.2) in the previous work [50]. In that paper, we can obtain a certain result on the motion $\boldsymbol{s}(t)$ around $t = \zeta$. On the other hand, we can say little about the behavior of $F_t$ around $t = \zeta$. In particular, it remains to be discussed how the "limit" of $F_t$ as $t \uparrow \zeta$ is constructed. It is reasonable to believe the following: We can define the limit hull $F_\zeta$ in such a way that $\lim_{t\uparrow\zeta} y_j(t) = 0$ if and only if $C_j \cap F_\zeta \neq \emptyset$. Here, note that, even if $(g_t)_{t\in[0,\zeta)}$ obeys (7.2), the local growth property cannot be expected at $t = \zeta$ anymore. We cannot exclude the possibility that the driving function $\xi(t)$ diverges as $t \uparrow \zeta$. The author hopes that the present work will help to treat such a subtle situation.

The study on reversed Loewner chains that is discussed in the preceding two paragraphs plays a role in defining and analyzing extensions of $\mathrm{SLE}_\kappa$ to

[1] Compare to (6.2).

multiply connected domains. SKLE in Section 2.3.1 is one such extension, which has the most direct connection with the Komatu–Loewner equation. As we have mentioned in Remark 2.7, a similar equation also appeared in Zhan's study [60] of harmonic random Loewner chains, another extension of SLE. On the other hand, Lawler [45] and Jahangoshahi and Lawler [37] studied other ways of extending SLE, respectively, without using the Komatu–Loewner equation. From this context, the following question arises naturally: *How are these different extensions of SLE related to each other?* Answering this question will make the theory of Komatu–Loewner equation applicable to problems that have been studied by other methods. Such application in the study of SLE on multiply connected domains is yet to be investigated.

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# Appendix A

# One-parameter families of holomorphic functions

The contents of this appendix is analogous to the classical arguments on a.e. differentiability in the proof of Pommerenke [52, Theorem 6.2] and Goryainov and Ba [34, Theorem 3]. Since we need more general results in this thesis, we provide a self-contained proof of each statement for the sake of completeness.

## A.1 Absolute continuity and almost everywhere differentiability

Let $I$ be an interval equipped with a non-atomic Radon measure $\mu$ and $f_t$ be a holomorphic function on a Riemann surface $X$ for each $t \in I$. We consider the following properties:

$(\mathrm{AC})_\mu$ For any compact subset $K$ of $X$, there exists a measure $\nu_K$ on $I$ which is absolutely continuous with respect to $\mu$ and satisfies

$$\sup_{p \in K} |f_t(p) - f_s(p)| \leq \nu_K((s,t]) \quad \text{for } (s,t) \in I^2_{\leq}.$$

$(\mathrm{Lip})_\mu$ For any compact subset $K$ of $X$, there exists a constant $L_K$ such that

$$\sup_{p \in K} |f_t(p) - f_s(p)| \leq L_K \mu((s,t]) \quad \text{for } (s,t) \in I^2_{\leq}.$$

Obviously $(\mathrm{Lip})_\mu$ implies $(\mathrm{AC})_\mu$, and if $(\mathrm{AC})_\mu$ holds, then $t \mapsto f_t$ is continuous in $\mathrm{Hol}(X; \mathbb{C})$, the space of holomorphic functions on $X$ equipped with the topology of locally uniform convergence. $(\mathrm{AC})_\mu$ also implies that, for each

$p \in X$, the set function $\kappa_p((s,t]) := f_t(p) - f_s(p)$ on the set of left half-open intervals extends to a complex measure on every compact subinterval of $I$ which is absolutely continuous with respect to $\mu$. By the generalized Lebesgue's differentiation theorem [8, Theorem 5.8.8], the limit

$$\tilde{\partial}_t^{\mu} f_t(p) := \lim_{\delta \downarrow 0} \frac{f_{t+\delta}(p) - f_{t-\delta}(p)}{\mu((t-\delta, t+\delta))}$$

exists for a.a. $t \in I$ and is a version of the Radon–Nikodym derivative $d\kappa_p/d\mu$. If $\mu$ is associated with a continuous non-decreasing function $F$ on $I$ by the relation $\mu((s,t]) = F(t) - F(s)$ (i.e., $\mu = m_F$), then we designate the properties $(\mathrm{AC})_{\mu}$ and $(\mathrm{Lip})_{\mu}$ by $(\mathrm{AC})_F$ and by $(\mathrm{Lip})_F$, respectively, and the derivative $\tilde{\partial}_t^{\mu} f_t(p)$ by $\tilde{\partial}_t^{F} f_t(p)$ as well.

In general, the $\mu$-null set on which $\tilde{\partial}_t^{\mu} f_t(p)$ does not exist depends on $p$. However, $(\mathrm{AC})_{\mu}$ enables us to choose this exceptional set $N$ independently of $p$, as shown in the following proposition:

**Proposition A.1.** *Suppose that a family $(f_t)_{t \in I}$ of holomorphic functions on a Riemann surface $X$ satisfies* $(\mathrm{AC})_{\mu}$.

(i) *There exists a $\mu$-null set $N \subset I$ such that, for each $t \in I \setminus N$, the convergence*

$$\frac{f_{t+\delta}(p) - f_{t+\delta}(p)}{\mu((t-\delta, t+\delta))} \to \tilde{\partial}_t^{\mu} f_t(p) \quad \text{as } \delta \to +0$$

*occurs locally uniformly in $p \in X$, and hence $\tilde{\partial}_t^{\mu} f_t$ is a holomorphic function on $X$.*

(ii) *If $(f_t)_{t \in I}$ further satisfies* $(\mathrm{Lip})_{\mu}$, *then we can choose a null-set $N$ in* (i) *as follows: For any countable set $A \subset X$ having an accumulation point in $X$,*

$$N = \bigcup_{p \in A} \{\, t \in I \,;\, \tilde{\partial}_t^{\mu} f_t(p) \text{ does not exists} \}$$

*Proof.* (i) We take an exhaustion sequence $(X_n)_{n \in \mathbb{N}}$ of $X$; that is, all $X_n$'s are relatively compact subdomains of $X$ with $\bigcup_{n=1}^{\infty} X_n = X$. It suffices to show that, for each $n \in \mathbb{N}$, there exists a $\mu$-null set $N_n \subset I$ such that $\tilde{\partial}_t^{\mu} f_t(p)$ exists and is holomorphic on $X_n$ for each $t \in I \setminus N_n$. Indeed, we can conclude from this auxiliary assertion that $\tilde{\partial}_t^{\mu} f_t(p)$ exists and is holomorphic on $X$ for each $t \in I \setminus N$ with $N := \bigcup_{n \in \mathbb{N}} N_n$. Therefore, we fix $n \in \mathbb{N}$ and prove the proposition on $X_n$.

$\overline{X_n}$ is a compact subset of $X$, and hence there exists a measure $\nu_n \ll \mu$ on $I$ such that $|f_t(p) - f_s(p)| \leq \nu_n((s,t])$ for any $p \in X_n$ and $(s,t) \in I^2_{\leq}$. Let $A \subset X_n$ be a countable set having an accumulation point in $X_n$. Since $\tilde{\partial}^\mu_t f_t(p)$ exists at $\mu$-a.a. $t$ for each fixed $p \in A$, there exists a null set $N_n \subset I$ such that

$$\tilde{\partial}^\mu_t f_t(p) \ (p \in A) \quad \text{and} \quad D_\mu \nu_n(t) := \lim_{\delta \downarrow 0} \frac{\nu_n((t-\delta, t+\delta))}{\mu((t-\delta, t+\delta))}$$

all exist at every $t \in I \setminus N_n$. We fix such $t$. By $(\mathrm{AC})_\mu$ we have

$$\frac{|f_{t-\delta}(p) - f_{t+\delta}(p)|}{\mu((t-\delta, t+\delta))} \leq \frac{\nu_n((t-\delta, t+\delta))}{\mu((t-\delta, t+\delta))}. \tag{A.1}$$

The left-hand side in this inequality is bounded in $p \in X_n$ and $\delta > 0$ because the right-hand side converges to $D_\mu \nu_n(t)$ as $\delta \downarrow 0$. Moreover, $(f_{t-\delta}(p) - f_{t+\delta}(p))/\mu((t-\delta, t+\delta))$ converges to $\tilde{\partial}^\mu_t f_t(p)$ as $\delta \downarrow 0$ for each $p \in A$. Thus, this divided difference converges as $\delta \downarrow 0$ locally uniformly on $X_n$ by Vitali's theorem (see, e.g., Chapter 7, Section 2 of [56]), which implies that $\tilde{\partial}^\mu_t f_t(p)$ exists and is holomorphic on $X_n$.
(ii) Let $A$ and $N$ be as in the statement of (ii). Then the left-hand side of (A.1) is bounded by $L_K$ on every compact subset $K$. Hence it is locally uniformly bounded on $X$. Vitali's theorem thus implies that $\tilde{\partial}^\mu_t f_t(p)$ exists for every $t \in I \setminus N$ and $p \in X$. Note that we do not need to take an exhaustion sequence $(X_n)_n$ in this case. □

**Remark A.2.** In the case where $\mu$ coincides with the Lebesgue measure **Leb** on $I$, Bracci, Contreras and Diaz-Madrigal [14] and Contreras, Diaz-Madrigal and Gumenyuk [20] considered a condition broader than $(\mathrm{AC})_{\mathbf{Leb}}$ and $(\mathrm{Lip})_{\mathbf{Leb}}$. Roughly speaking, they say that a family $(f_t)_{t \in I}$ is of order $d \in [1, \infty]$ if, for every compact subset $K$, there exists a function $k_K \in L^d(I)$ such that

$$\sup_{p \in K} |f_t(p) - f_s(p)| \leq \int_s^t k_K(u)\, du, \quad (s,t) \in I^2_{\leq}.$$

According to this definition, $(f_t)_{t \in I}$ satisfies $(\mathrm{AC})_{\mathbf{Leb}}$ if it is of order $d$ for some $d \in [1, \infty]$, and in particular, $(\mathrm{Lip})_{\mathbf{Leb}}$ holds if and only if $d = \infty$. From this viewpoint, Lemma 5.4 shows that, given an evolution family $(\phi_{t,s})$, we can always assume $d = \infty$ if we replace **Leb** by the measure $m_\lambda$ associated with $(\phi_{t,s})$. This fact makes our argument easier, for example, in Lemma 5.6 and Proposition A.1 (ii).

## A.2 Descent to spatial derivatives and inverse functions

In this and next sections, we discuss only the case in which $X$ is a planar domain $D \subset \mathbb{C}$.

**Proposition A.3.** *Let $(f_t)_{t\in I}$ be a family of holomorphic functions on a planar domain $D$.*

(i) *If $(f_t)_{t\in I}$ is continuous in $\mathrm{Hol}(D;\mathbb{C})$, then so is the family $(f_t^{(n)})_{t\in I}$ of the $n$-th order $z$-derivatives for any $n \in \mathbb{N}$.*

(ii) *If $(f_t)_{t\in I}$ satisfies $(\mathrm{Lip})_\mu$, then so is $(f_t^{(n)})_{t\in I}$ for any $n$.*

*Proof.* (i) is just a standard fact in complex analysis. We prove (ii) here.

Assume that $(f_t)_{t\in I}$ satisfies $(\mathrm{Lip})_\mu$. Without loss of generality, we may and do assume that $D = \mathbb{D}$. Let $r$ and $\delta$ be two arbitrary positive numbers such that $r+\delta < 1$. We take the constant $L_K$ in $(\mathrm{Lip})_\mu$ with $K := \partial B(0, r+\delta)$. Using Cauchy's integral formula, we have

$$\begin{aligned}\sup_{|z|\le r}|f_t^{(n)}(z) - f_s^{(n)}(z)| &\le \frac{1}{2\pi}\sup_{|z|\le r}\int_{|\zeta|=r+\delta}\frac{|f_t(\zeta)-f_s(\zeta)|}{|\zeta - z|^{n+1}}\,|d\zeta| \\ &\le \frac{r+\delta}{\delta^{n+1}}L_K\mu((s,t])\end{aligned}$$

for any $(s,t) \in I^2_\le$, which yields Property $(\mathrm{Lip})_\mu$ of $(f_t^{(n)})_{t\in I}$. □

If $f_t$'s are univalent and satisfy $(\mathrm{Lip})_\mu$, then their inverse functions satisfy the same property locally in time and space, which is a conclusion from the following Lagrange inversion formula:

**Lemma A.4.** *Let $f$ be a univalent function on a planar domain $D$, $w$ be a point of $f(D)$ and $C$ be a simple closed curve in $D$ surrounding $f^{-1}(w)$ such that $\operatorname{ins} C \subset D$. Then the equality*

$$f^{-1}(w) = \frac{1}{2\pi i}\int_C \frac{\zeta f'(\zeta)}{f(\zeta) - w}\,d\zeta$$

*holds.*

*Proof.* The function $zf(z)/(f(z)-w)$ of $z$ has a pole of the first order at $z = f^{-1}(w)$, and its residue is

$$\lim_{z\to f^{-1}(w)}(z - f^{-1}(w))\frac{zf'(z)}{f(z)-w} = f^{-1}(w).$$

Hence the conclusion follows from the residue theorem. □

**Proposition A.5.** *Suppose that a family $(f_t)_{t\in I}$ of univalent functions is continuous in* $\mathrm{Hol}(D;\mathbb{C})$. *Let $t_0 \in I$ and $U$ be a bounded domain with $\overline{U} \subset f_{t_0}(D)$. Then there exists a neighborhood $J$ of $t_0$ in $I$ such that $\overline{U} \subset \bigcap_{t\in J} f_t(D)$. For any such pair $(J, U)$, the family $(f_t^{-1})_{t\in J}$ of the inverse functions is continuous in* $\mathrm{Hol}(U;\mathbb{C})$. *If $(f_t)_{t\in I}$ further satisfies* $(\mathrm{Lip})_\mu$ *on $D$, then so is $(f_t^{-1})_{t\in J}$ on $U$.*

*Proof.* Owing to the compactness of $\overline{U}$, it suffices to prove that for any fixed $w_0 \in f_{t_0}(D)$, the proposition holds with $U$ replaced by a sufficiently small disk $B(w_0, r_0)$. We assume $f_{t_0}^{-1}(w_0) = 0 \in D$ for the simplicity of notation.

We choose such a small $r_0$ that $f_{t_0}^{-1}(\overline{B(w_0, r_0)}) \subset B(0, r) \subset \overline{B(0, r)} \subset D$ holds for some $r > 0$. Set $\epsilon := d^{\mathrm{Eucl}}(f_{t_0}(\partial B(0, r)), \partial B(w_0, r_0)) > 0$. Since $(f_t)_{t\in I}$ is continuous in the topology of locally uniform convergence, there exists a closed neighborhood $J = [\alpha, \beta]$ of $t_0$ such that $|f_t(z) - f_{t_0}(z)| < \epsilon/2$ holds for $z \in \overline{B(0, r)}$ and $t \in J$. This inequality implies that $\overline{B(w_0, r_0)} \subset \bigcap_{t\in J} f_t(B(0, r)) \subset \bigcap_{t\in J} f_t(D)$.

Next, we show that $(f_t^{-1})_{t\in J}$ satisfies $(\mathrm{Lip})_\mu$ on $U$, assuming that $(f_t)_{t\in I}$ satisfies $(\mathrm{Lip})_\mu$ on $D$. Since the continuity of $(f_t^{-1})_{t\in J}$ in $\mathrm{Hol}(U;\mathbb{C})$ is proved in a similar way, we omit it. By Proposition A.3 (ii), we can take two constants $L_0$ and $L_1$ such that $\sup_{|z|\le r} |f_t^{(n)}(z) - f_s^{(n)}(z)| \le L_n \mu((s, t])$, $n = 0, 1$, holds for any $(s, t) \in I^2_{\le}$. In particular, we have

$$\begin{aligned} M_n &:= \max\{\, |f_t^{(n)}(z)| \,;\, |z| = r,\ t \in J \,\} \\ &\le \max_{|z|=r} |f_{t_0}^{(n)}(z)| + L_n \mu((\alpha, \beta]) < \infty, \quad n = 0, 1. \end{aligned}$$

Now, using Lemma A.4 we have

$$\begin{aligned} &f_t^{-1}(w) - f_s^{-1}(w) \\ &= \frac{1}{2\pi i} \int_{|\zeta|=r} \left( \frac{\zeta f_t'(\zeta)}{f_t(\zeta) - w} - \frac{\zeta f_s'(\zeta)}{f_s(\zeta) - w} \right) d\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\zeta \{ f_t'(\zeta)(f_s(\zeta) - w) - f_s'(\zeta)(f_s(\zeta) - w)\}}{(f_t(\zeta) - w)(f_s(\zeta) - w)} \, d\zeta \end{aligned}$$

for $w \in \overline{B(w_0, r_0)}$ and $(s, t) \in J^2_{\le}$. Hence

$$\begin{aligned} &|f_t^{-1}(w) - f_s^{-1}(w)| \\ &\le \frac{2r}{\pi \epsilon^2} \int_{|\zeta|=r} (|f_t(\zeta) - w||f_t'(\zeta) - f_s'(\zeta)| + |f_t(\zeta) - f_s(\zeta)||f_s'(\zeta)|)\ |d\zeta| \\ &\le \frac{4r^2((M_0 + |w_0| + r_0) \vee M_1)}{\epsilon^2} (L_0 + L_1) \mu((s, t]), \end{aligned}$$

which yields Property $(\mathrm{Lip})_\mu$ of $(f_t^{-1})_{t\in J}$ on $B(w_0, r_0)$. ∎

## A.3 Implicit function theorem

**Proposition A.6.** *Let $(f_t)_{t\in I}$ be a family of holomorphic functions that satisfies* $(\mathrm{Lip})_\mu$ *on a planar domain $D$. Suppose that a point $(t_0, z_0) \in I \times D$ enjoys the conditions*

$$f_{t_0}(z_0) = 0 \quad \text{and} \quad f'_{t_0}(z_0) \neq 0.$$

*Then there exist some neighborhood $J$ of $t_0$ in $I$, neighborhood $U$ of $z_0$ in $D$ and function $\tilde{z}\colon J \to U$ such that*

- *$z = \hat{z}(t)$ is a unique zero of the holomorphic function $f_t$ in $U$, which is of the first order, for any $t \in J$;*
- *$\hat{z}(t)$ is Lipschitz continuous with respect to $\mu$ in the sense that*

  $$|\hat{z}(t) - \hat{z}(s)| \leq \tilde{L}\mu((s,t]), \quad (s,t) \in J^2_{\leq},$$

  *holds for some constant $\tilde{L}$. In particular, the complex measure $\tilde{\kappa}$ induced from $\tilde{z}(t)$ on every compact subinterval of $I$ is absolutely continuous with respect to $\mu$.*

*Proof.* As $f'_{t_0}(z_0) \neq 0$, there exists $r_0 > 0$ such that $f_{t_0}$ is univalent on $B(z_0, r_0)$. We take $r_1 \in (0, r_0)$ and set $m_{r_1} := \min_{|z-z_0|=r_1}|f_{t_0}(z)| > 0$. Since $f_{t_0}(z_0) = 0$, there exists $r \in (0, r_1)$ such that $\sup_{z\in B(z_0,r)}|f_{t_0}(z)| < m_{r_1}/4$. Moreover, there exists $\delta_{r_1} > 0$ such that

$$\sup_{z\in B(z_0,r_1)} |f_t(z) - f_{t_0}(z)| < \frac{m_{r_1}}{4}$$

holds if $|t - t_0| < \delta_{r_1}$. We see that, if $|t - t_0| < \delta_{r_1}$, then $f_t(B(z_0, r)) \subset B(0, m_{r_1}/2)$. Assuming $|w| < m_{r_1}/2$, $|z - z_0| \leq r$ and $|t - t_0| < \delta_{r_1}$, we have

$$|(f_t(z) - w) - (f_{t_0}(z) - w)| < \frac{m_{r_1}}{2} \leq |f_{t_0}(z)| - \frac{m_{r_1}}{2} \leq |f_{t_0}(z) - w|.$$

Since $f_{t_0}$ is univalent on $B(z_0, r_0)$, it takes each value $w \in B(0, m_{r_1}/2)$ at most once, counting multiplicities, on $B(z_0, r_1)$, and so does $f_t$ if $|t - t_0| < \delta_{r_1}$ by Rouché's theorem. In this way, we see that $f_t$ is univalent on $B(z_0, r)$ if $|t - t_0| < \delta_{r_1}$. It is now clear from Proposition A.5 that a desired triplet $(J, U, \hat{z}(t))$ is given by $J = (t_0 - \delta_{r_1}, t_0 + \delta_{r_1}) \cap I$, $U = B(z_0, r)$ and $\tilde{z}(t) = f_t^{-1}(0)$. □

**Remark A.7.** Let us refer to one of the following two conditions, which one can prove to be equivalent to each other, as (CD):

- For each $t \in I$, $(f_{t+h} - f_t)/h$ converges in $\mathrm{Hol}(X;\mathbb{C})$, and the family of the limits $\dot{f}_t$, $t \in I$, is also continuous in $\mathrm{Hol}(X;\mathbb{C})$.

- For each $p \in X$, the function $t \mapsto f_t(p)$ is $C^1$, and the family of the $t$-derivatives $\dot{f}_t$, $t \in I$, is locally bounded on $X$.

Then Propositions A.3 and A.5 are valid with $(\mathrm{Lip})_\mu$ replaced by (CD). Proposition A.6 also holds with the following replacement: $(\mathrm{Lip})_\mu$ in the assumption is replaced by (CD), and the Lipschitz continuity of $\hat{z}(t)$ is replaced by the continuous differentiability of $\hat{z}(t)$ in $t$. Although we do not use these facts in this paper, one can see that such considerations make the argument in Section 2 of Chen and Fukushima [16] slightly simpler.

# Appendix B

# Hyperbolic and quasi-hyperbolic distances

For the detail of the contents of this appendix, see Sections 5 and 7 in Chapter 1 of Ahlfors [1] for example.

The hyperbolic distance on $\mathbb{D}$ is defined by

$$d_{\mathbb{D}}^{\mathrm{Hyp}}(z, w) := \inf_C \int_C \frac{2}{1 - |\zeta|^2} \, |d\zeta|.$$

Here, the infimum is taken over all piecewise smooth curves $C$ connecting $z$ and $w$. We can easily observe from the Schwarz–Pick lemma that, if $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$d_{\mathbb{D}}^{\mathrm{Hyp}}(f(z), f(w)) \leq d_{\mathbb{D}}^{\mathrm{Hyp}}(z, w). \tag{B.1}$$

Since the universal covering space of any proper subdomain $D$ of $\mathbb{C}$ is $\mathbb{D}$, the covering map $p\colon \mathbb{D} \to D$ induces the hyperbolic distance $d_D^{\mathrm{Hyp}}$ on $D$ from $d_{\mathbb{D}}^{\mathrm{Hyp}}$. The following contraction principle is a consequence of (B.1):

**Proposition B.1.** *Let $f\colon D \to \tilde{D}$ be a holomorphic function. Then*

$$d_{\tilde{D}}^{\mathrm{Hyp}}(f(z), f(w)) \leq d_D^{\mathrm{Hyp}}(z, w), \quad z, w \in D.$$

*In particular, if $D \subset \tilde{D}$, then $d_{\tilde{D}}^{\mathrm{Hyp}}(z, w) \leq d_D^{\mathrm{Hyp}}(z, w)$ for any $z, w \in D$.*

Let $\delta_D(z) := d^{\mathrm{Eucl}}(z, \partial D)$ for a proper subdomain $D$ of $\mathbb{C}$. We define the quasi-hyperbolic distance on $D$ by

$$d_D^{\mathrm{QH}}(z, w) := \inf_C \int_C \frac{2}{\delta_D(\zeta)} \, |d\zeta|.$$

The infimum is taken over all piecewise smooth curves $C$ connecting $z$ and $w$.

**Proposition B.2.** $d_D^{\mathrm{Hyp}} \leq d_D^{\mathrm{QH}}$ *holds for every proper subdomain* $D$ *of* $\mathbb{C}$.

We give an easy estimate on $d_D^{\mathrm{QH}}$, which is used in the proof of Proposition 5.5.

**Lemma B.3.** *Let* $C$ *be a convex subset of* $D$ *such that* $d^{\mathrm{Eucl}}(C, \partial D) > 0$. *Then*

$$d_D^{\mathrm{QH}}(z, w) \leq \frac{2|z - w|}{d^{\mathrm{Eucl}}(C, \partial D)}, \quad z, w \in C. \tag{B.2}$$

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