Integrable deformations in 2D dilaton gravity models

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Abstract

In this thesis, we discuss the relationship between a gravitational perturbation and a $T\bar{T}$-deformation in 2D dilaton gravity systems. The $T\bar{T}$-deformation is an irrelevant deformation of two-dimensional field theories generated by the determinant operator of the energy-momentum tensor. As a notable feature, the $T\bar{T}$-deformation is solvable and many properties such as the spectrum can be investigated. In a pioneering work by Dubovsky et al. , it has been shown that a gravitational perturbation around the flat space-time can be interpreted as a $T\bar{T}$-deformation of an original matter action. In this paper, we extend this discussion to the case of AdS_2 space-time.

To obtain solvable 2D dilaton gravity systems, we employ the technique of the Yang-Baxter deformations which is a systematical method of integrable deformations of non-linear sigma models. We analytically derive general solutions of the deformations of the Jackiw-Teitelboim (JT) model and obtain the deformed black hole solution. We calculate the entropy of the deformed black hole and show that it is reproduced from the physical quantities on the singularity surface generated by the deformation. This result is a non-trivial evidence providing that the holographic principle is also valid on the deformed space-time. In addition, we find that the deformed gravity model is classically equivalent to the Liouville dilaton gravity with a negative cosmological constant term.

Then, we study the relationship between a gravitational perturbation and a $T\bar{T}$-deformation. First, we consider general dilaton gravity models coupled to arbitrary matters and derive a quadratic action. Then, we find certain conditions under which a gravitational perturbation can be interpreted as a $T\bar{T}$-deformation of a matter action. We further show that, in the case of the Liouville gravity theory, a gravitational perturbation can be interpreted as a $T\bar{T}$-deformation on AdS_2. We also construct a gravitational solution coupled to general matter fields.

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Chapter 1

Introduction

One of the most significant issues to seek quantum gravity is to understand the behavior of gravity in the ultraviolet region. Since gravity is an irrelevant interaction, it is important to understand the irrelevant perturbation of field theories. However, the irrelevant deformation induces an infinite number of interaction terms in the ultraviolet region, which cannot be renormalized, and its general properties cannot be understood. Recently, in the case of two dimensions, an irrelevant but integrable deformation called a $T\bar{T}$-deformation was proposed [1, 2], and the relationship with two-dimensional gravity has been actively discussed.

A infinitesimal $T\bar{T}$-deformation is triggered by the $T\bar{T}$-operator $\mathcal{O}_{T\bar{T}}$, such that

$$\mathcal{O}_{T\bar{T}} = -\frac{1}{4}\mathrm{det}T_{\mu\nu} = T_{zz}T_{\bar{z}\bar{z}} - T_{z\bar{z}}^2 \,. \tag{1.0.1}$$

Here $T_{\mu\nu}$ is the energy momentum tensor and $(z, \bar{z})$ are (anti-)holomorphic coordinates. This composite operator was first introduced by Sasha Zamolodchikov [3] for two-dimensional field theories. Since it was originally discussed as a composite operator of CFT_2, this operator is customarily called the $T\bar{T}$-operator even though the trace part Θ is zero.

From now on, we describe the undeformed Lagrangian as $\mathcal{L}^{(0)}$. The Lagrangian labeled with the deformation parameter τ is scripted as $\mathcal{L}^{(\tau)}$. The infinitesimally deformed Lagrangian $\mathcal{L}^{(\tau+\delta\tau)}$ is given by

$$\mathcal{L}^{(\tau+\delta\tau)} = \mathcal{L}^{(\tau)} - \delta\tau\mathcal{O}_{T\bar{T}}^{(\tau)} \,. \tag{1.0.2}$$

It should be noted that the energy-momentum tensor on the right-hand side is defined by the deformed Lagrangian $\mathcal{L}^{(\tau)}$. The mass dimension of the parameter τ is $(\text{length})^2$, and then this deformation is actually an irrelevant deformation. This gives a finite flow in theory space.

$$\partial_\tau \mathcal{L}^{(\tau)} = -\mathcal{O}^{(\tau)}_{T\bar{T}} \,. \tag{1.0.3}$$

The $T\bar{T}$-deformation is independently proposed by [1, 2]. It has been discussed the deformation as random geometry [4] and a deformed partition function [5]. The deformations has been presented in closed form [12]. [1] For a nice review on the $T\bar{T}$-deformation, see [13].

The most notable feature of the $T\bar{T}$-deformation is the "integrability" in the following sense. For example, the Lax pair of the deformed theory was also constructed explicitly [2, 14, 15]. That means that the infinite number of the classical conserved charges are guaranteed and the deformed theories are classically integrable. In fact, the energy spectrum of $T\bar{T}$-deformed theories is known to follow the inviscid Burgers' equation and can be calculated. Another argument is about the deformation of the S-matrix. In 2D integrable QFT, it is known that N-body S-matrix is factorized into a product of 2-body S-matrixes. This factorization property is the onset of the quantum integrability. The S-matrix is determined by Lorentz symmetry, crossing symmetry, and unitarity, up to the phase factor called Castillejo-Dalitz-Dyson (CDD) factor [16]. The $T\bar{T}$-deformation is a irrelevant deformation and then modifies only the CDD factor. Thus, the factorization property of S-matrix still holds after the deformation. In this sense as well, the $T\bar{T}$-deformation may be called integrable deformation. Thus, the $T\bar{T}$-deformation is solvable by the "integrability" behind it even though it is an irrelevant perturbation.

The simplest and the most interesting example of the $T\bar{T}$-deformation is the case of massless bosons. Specifically, solving the flow equation, the deformed Lagrangian becomes the Nambu-Goto (NG) action fixed to a static gauge [2]. The NG action is, of course, known as the action of the string theory. In fact, the energy spectrum and the entropy of the $T\bar{T}$-deformed theory show the Hagedorn behavior which is characteristic feature of a non-local theory like string theory. From these facts, it is expected that a $T\bar{T}$-flow will realize a string-like theory or a gravity theory in the ultraviolet region.

[1] There are many kinds of generalization of $T\bar{T}$-deformations, for example, the deformations on curved space-time [6, 7] and the deformation of super symmetric models [8–11]

One achievement on the correspondence between a $T\bar{T}$-deformation and a gravitational perturbation was provided by Dubovsky, Gorbenko, and Mirbabayi [17]. They considered a gravitational perturbation around a vacuum solution to the flat-space Jackiew-Teitelboim gravity. The Jackiw-Teitelboim (JT) gravity [18, 19] is originally one of 2D dilaton gravity models, which supports the AdS_2 space-time as a vacuum solution. Dubovsky et al. considered the flat space limit of the JT model coupled to arbitrary matter fields and showed that a gravitational perturbation can be reinterpreted as a $T\bar{T}$-deformation of the original matter action. They also showed that the matter theory gets a gravitationally dressing factor in front of the S-matrix due to the perturbation. This result indicates that the classical gravitational perturbations can be seen as a non-perturbative quantum effect to the matter field theory. However, this result has been shown only in the flat-space JT gravity.

In this thesis, we discuss the generalization of the work by Dubovsky et al. to the cases on curved space-times, especially the AdS space-time based on our works [20–23]. We find the condition for dilaton potentials so that a gravitational perturbation can be interpreted as a $T\bar{T}$-deformation in general dilaton gravity systems. An important example is the case of the Liouville gravity with a negative cosmological constant. This model also supports AdS space-time as a vacuum solution, and at the same time, the gravitational perturbation can be reinterpreted as a finite $T\bar{T}$-deformation of a matter action.

This thesis is organized as follow. In chapter 2, we review the basic properties of the $T\bar{T}$-deformation. First, we give the definition of $T\bar{T}$-operator and $T\bar{T}$-deformation and show some concrete examples. Then, we demonstrate that the deformation of a scalar field theory becomes the Nambu-Goto action in a static gauge. After confirming that this operator is quantum-mechanically well-define, we give a proof of Zamolodchikov's factorization theorem. As an important consequence of this theorem, we make sure that the energy spectrum of the deformed theories follows the inviscid Burgers' equation. In particular, we focus on the deformation of CFT_2 and discuss its properties. Lastly, from the perspective of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, we list and comment on some suggestions for the holographic dual of the deformed CFT_2.

In chapter 3, we considers the two-dimensional dilaton gravity theory. First, we will introduce a general system consisting only of metric and dilaton fields. In this paper, we pay particular attention to the JT gravity and summarize the vacuum solution and background space-times. It is point out that the thermodynamic quantities of the AdS black hole is holographically reproduced from the physical quantity on the boundary of the AdS space-time originally discussed in [24]. Next, we consider the deformation of the JT model by employing the technique of the Yang-Baxter (YB) deformations [25–27]. The YB deformations is a systematical integrable deformation that recently attracts great attention in the study of the string sigma models. (About recent developments, see reviews [28, 29].) We apply this technique to obtain another interesting gravity model to replace the JT model, and we derive the deformed JT model which has a hyperbolic-type dilaton potential. As an interesting solution, we find the deformed black hole solution and calculate its thermodynamic quantities. We also make sure that these thermodynamic quantities are reproduced from the physical quantities on the singularity surface generated by the deformation. We point out that the deformed dilaton gravity model is equivalent to the Liouville gravity model.

In chapter 4, we discusses the relationship between a $T\bar{T}$-deformation and a gravitational perturbation. First, we review the model by Dubovsky et al. with the flat space-time background. This result is extended to general dilaton gravity models. Especially in the case of AdS and dS space-time, it is confirmed that a gravitational perturbation can be interpreted as a $T\bar{T}$-deformation for a infinitesimal deformation. Finally, in the case of the Liouville gravity theory discussed in chapter 2, we see that gravitational perturbation can be recast as a $T\bar{T}$-deformation for a finite deformation parameter. In chapter 5, we summarizes this thesis and discuss future problems.

Chapter 2

A review of the $T\bar{T}$-deformation

In this chapter, we review basic properties of $T\bar{T}$-deformations of 2D QFTs. This deformation is originally introduced by [1, 2]. In section 1, we introduce $T\bar{T}$-deformations and give some simple examples. In particular, we demonstrate that the deformation of the action of massless bosons provides the Nambu-Goto action. In section 2, we discuss the spectrum of deformed theories. We focus on the deformed CFT_2 and point out some properties. In section 3, we give some comments for proposals for a holographic dual of the deformed CFT_2

2.1 A definition of the $T\bar{T}$-deformation

First of all, let us give a explicit definition of a $T\bar{T}$-deformation. We consider a trajectory in the field theory space parametrized by τ. We denote the Lagrangian at each point of the trajectory by $\mathcal{L}^{(\tau)}$. The undeformed Lagrangian is expressed as $\mathcal{L}^{(0)}$. The flow for the theories on he trajectory is triggered by the determinant operator, so-called $T\bar{T}$-operator

$$\mathcal{L}^{(\tau+\delta\tau)} = \mathcal{L}^{(\tau)} - \frac{\delta\tau}{4}\,\mathrm{det}T^{(\tau)}_{\mu\nu} = \mathcal{L}^{(\tau)} - \delta\tau\mathcal{O}^{(\tau)}_{T\bar{T}}\,. \tag{2.1.1}$$

In the following, let us give the explicit definition.

In this chapter, we work in the holomorphic and anti-holomorphic coordinates $z = (z, \bar{z})$ defined by two dimensions Cartesian coordinates (x, y),

$$z = x + iy, \qquad \bar{z} = x - iy\,. \tag{2.1.2}$$

The components of the stress energy tensor are related by

$$\begin{aligned} T_{zz} &= \frac{1}{4}(T_{xx} - T_{yy} - 2iT_{xy})\,, \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{xx} - T_{yy} + 2iT_{xy})\,, \\ T_{z\bar{z}} &= \frac{1}{4}(T_{xx} + T_{yy})\,. \end{aligned} \tag{2.1.3}$$

We define T, $\bar{T}$ and Θ as

$$T = T_{zz}\,, \qquad \bar{T} = T_{\bar{z}\bar{z}}\,, \qquad \Theta = T_{z\bar{z}}\,. \tag{2.1.4}$$

Then, The $T\bar{T}$-operator is given by

$$\mathcal{O}^{(\tau)}_{T\bar{T}} = -\frac{1}{4}\mathrm{det}T_{\mu\nu} = (T_{zz}T_{\bar{z}\bar{z}} - T^2_{z\bar{z}}) = T\bar{T} - \Theta^2\,. \tag{2.1.5}$$

The mass dimension of the operator $\mathcal{O}^{(\tau)}_{T\bar{T}}$ is $(\text{length})^{-4}$. Therefore the dimension of t is $(\text{length})^2$ and the deformation is an irrelevant deformation. To consider the quantum flow generated by $T\bar{T}$-operator, we should confirm whether this operator is locally well-defined or not because generally, such a composite operator locally diverges. Later we will see that the local divergence is canceled and the operator is well-defined, but before that we give some simple examples.

Some examples

Free massless boson

Let us consider a free massless boson case. It is the simple but a suggestive example. We denote the scalar field as Φ. The Lagrangian is given by

$$\mathcal{L}^{(0)}_{FB} = \partial\Phi\bar{\partial}\Phi\,. \tag{2.1.6}$$

Here we use the short-hand notation for the holomorphic and anti-holomorphic derivatives

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_x - i\partial_y)\,, \qquad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)\,. \tag{2.1.7}$$

We will show that the deformed Lagrangian is interpreted as the Nambu-Goto action on 3-dimensional target space.

$$\mathcal{L}^{(\tau)}_{FB} = \frac{1}{2\tau}\left(\sqrt{4\tau\partial\Phi\bar{\partial}\Phi + 1} - 1\right) = -\frac{1}{2\tau} + \mathcal{L}_{\text{Nambu}-\text{Goto}}\,. \tag{2.1.8}$$

$\mathcal{L}_{\text{Nambu-Goto}}$ is the Nambu-Goto action in the static gauge

$$\mathcal{L}_{\text{Nambu-Goto}} = \frac{1}{2\tau}\sqrt{\det(\partial_\alpha X \cdot \partial_\beta X)}\,. \tag{2.1.9}$$

Here the target space coordinates X is fixed as

$$X_1 = x\,, \qquad X_2 = y\,, \qquad X_3 = \frac{\sqrt{\tau}\Phi}{2}\,. \tag{2.1.10}$$

Let us start to derive the deformed Lagrangian. The canonical stress-energy tensor is given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi - \eta^{\mu\nu}\mathcal{L}\,. \tag{2.1.11}$$

Each component of the stress tensor is

$$\begin{aligned} T &= \frac{\partial \mathcal{L}}{\partial(\bar{\partial}\phi)}\partial\phi\,, \\ \bar{T} &= \frac{\partial \mathcal{L}}{\partial(\partial\phi)}\bar{\partial}\phi\,, \\ \Theta &= -\frac{1}{2}\left(\frac{\partial \mathcal{L}}{\partial(\partial\phi)}\partial\phi + \frac{\partial \mathcal{L}}{\partial(\bar{\partial}\phi)}\bar{\partial}\phi - 2\mathcal{L}\right)\,. \end{aligned} \tag{2.1.12}$$

The flow equation to define the finite $T\bar{T}$-deformation is given by

$$\partial_\tau \mathcal{L}^{(\tau)} = -\mathcal{O}^{(\tau)}_{T\bar{T}}\,. \tag{2.1.13}$$

Please note that the energy-momentum tensor in the flow equation is defined by the deformed Lagrangian.

To solve the flow equation, let us consider the τ-expansion and find the coefficients order by oder

$$\mathcal{L}^{(\tau)} = \sum_{j=0}^{\infty} \tau^j\, L_j\,. \tag{2.1.14}$$

The $(j+1)$-th order of τ takes the form

$$L_{j+1} = -\frac{1}{j+1}\sum_{k=0}^{j}(T^{(k)}\bar{T}^{(j-k)} - \Theta^{(k)}\Theta^{(j-k)})\,. \tag{2.1.15}$$

Each component of the energy momentum for k-th order is defined by k-th order Lagrangian

$$\begin{aligned}
T^{(k)} &= \frac{\partial L_k}{\partial(\bar{\partial}\Phi)}\partial\Phi\,, \\
\bar{T}^{(k)} &= \frac{\partial L_k}{\partial(\partial\Phi)}\bar{\partial}\Phi\,, \\
\Theta^{(k)} &= -\frac{1}{2}\left(\frac{\partial L_k}{\partial(\partial\Phi)}\partial\Phi + \frac{\partial L_k}{\partial(\bar{\partial}\Phi)}\bar{\partial}\Phi - 2L_k\right)\,.
\end{aligned} \tag{2.1.16}$$

As an initial condition, we set the undeformed variables

$$T^{(0)} = (\partial\Phi)^2\,, \qquad \bar{T}^{(0)} = (\bar{\partial}\Phi)^2\,, \qquad \Theta^{(0)} = 0\,. \tag{2.1.17}$$

From the initial variables, we find the first coefficient L_1

$$L_1 = -(T^{(0)}\bar{T}^{(0)} - {\Theta^{(0)}}^2) = -(\partial\Phi)^2(\bar{\partial}\Phi)^2\,, \tag{2.1.18}$$

and the energy-momentum tensor is obtain by

$$T^{(1)} = -2(\partial\Phi)^3(\bar{\partial}\Phi)\,, \qquad \bar{T}^{(1)} = -2(\partial\phi)(\bar{\partial}\Phi)^3\,, \qquad \Theta^{(1)} = (\partial\Phi)^2(\bar{\partial}\Phi)^2\,. \tag{2.1.19}$$

Again, we can obtain the second order Lagrangian.

$$\begin{aligned}
L_2 &= -\frac{1}{2}(T^{(0)}\bar{T}^{(1)} - \Theta^{(0)}\Theta^{(1)} + T^{(1)}\bar{T}^{(0)} - \Theta^{(0)}\Theta^{(1)}) \\
&= -\frac{1}{2}\left(-2(\partial\Phi)^3(\bar{\partial}\Phi)^3 - 2(\partial\Phi)^3(\bar{\partial}\Phi)^3\right) \\
&= 2(\partial\Phi)^3(\bar{\partial}\Phi)^3\,.
\end{aligned} \tag{2.1.20}$$

Then one can derive the Lagrangian and the energy-momentum tensor order by order. The general form of the i-th order Lagrangian is given by

$$\begin{aligned}
L_0 &= \mathcal{L}_{FB} \\
L_j &= (-1)^j 4^j \frac{(1/2)_j}{(2)_j}(L_0)^{j+1}\,.
\end{aligned} \tag{2.1.21}$$

Here $(a)_n$ is the Pochhammer symbol

$$(a)_n = \prod_{k=0}^{n-1}(a-k)\,. \tag{2.1.22}$$

One can confirm each coefficient of τ-expansion of (2.1.8) is equal to (2.1.21).

In general case, it is difficult to read off the full deformed Lagrangian from the coefficient of the τ-expansion. As an another way to solve the flow equation, one can assume the ansatz and obtain the deformed Lagrangian. Let us start from the following ansatz

$$\mathcal{L}^{(\tau)} = \frac{1}{\tau} F(\tau\, \partial\Phi\bar{\partial}\Phi)\,. \tag{2.1.23}$$

Using the ansatz, each component of the energy-momentum is given by

$$\begin{aligned} T &= \frac{\partial \mathcal{L}}{\partial(\bar{\partial}\Phi)} \partial\Phi = F'(x)(\partial\Phi)^2\,, \\ \bar{T} &= \frac{\partial \mathcal{L}}{\partial(\partial\Phi)} \bar{\partial}\Phi = F'(x)(\bar{\partial}\Phi)^2\,, \\ \Theta &= -\frac{1}{2}\left[2F'(x)(\partial\Phi\bar{\partial}\Phi) - 2\frac{F(x)}{\tau}\right] = -\frac{x\,F'(x)}{\tau} + \frac{F(x)}{\tau}\,. \end{aligned} \tag{2.1.24}$$

where $x = \tau\, \partial\Phi\bar{\partial}\Phi$. The derivative of the deformed Lagrangian is written as

$$\partial_\tau \mathcal{L}^{(\tau)} = -\frac{F(x)}{\tau^2} + \frac{x\,F'(x)}{\tau^2}\,. \tag{2.1.25}$$

We obtain the following the differential equation

$$F^2 - 2x\,F'F - x\,F' + F = 0\,. \tag{2.1.26}$$

One can easily solve the equation with the initial condition $\mathcal{L}^{(0)}$ and obtain

$$F(x) = \frac{1}{2}(\sqrt{1+4x} - 1)\,. \tag{2.1.27}$$

It is nothing but the Nambu-Goto action in the static gauge.

It is remarkable that the deformation parameter τ plays a role as a string tension α'. In this sense, it seems natural to consider a positive sign case $\tau > 0$. Actually, a negative sign case seems problematic because the spectrum of the deformed theories becomes complex in the high energy region as we will see in the next section.

N massless bosons

In the above discussion, we derived the $T\bar{T}$-deformed Lagrangian of the single massless scalar field in two ways. One can easily extend this discussion for the N scalar fields case. Let us consider the $O(N)$ vector model as the undeformed Lagrangian

$$\mathcal{L}^{(0)}(\vec{\phi}) = \partial\vec{\Phi}\cdot\bar{\partial}\vec{\Phi}\,, \qquad \vec{\Phi} = (\Phi_1(z,\bar{z}), \cdots, \Phi_N(z,\bar{z}))\,. \tag{2.1.28}$$

Setting the similar ansatz, one can solve the flow equation and find the deformed Lagrangian

$$\begin{aligned}\mathcal{L}^{(t)}(\vec{\phi}) &= \frac{1}{2\tau}\left(-1+\sqrt{1+4\tau\mathcal{L}^{(0)}(\vec{\phi})-4\tau^2\mathcal{B}}\right)\\ &= \frac{1}{2\tau}\left(-\sqrt{\det\eta_{\mu\nu}}+\sqrt{\det(\eta_{\mu\nu}+\tau\, h_{\mu\nu})}\right)\end{aligned} \tag{2.1.29}$$

Here $\eta_{\mu\nu} = diag(1,1)$ is a metric on the flat space and $h_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi$. $\mathcal{B}$ is defined as

$$\mathcal{B} = |\partial\vec{\Phi}\times\bar{\partial}\vec{\Phi}|^2 = -\frac{1}{4}\det h_{\mu\nu}\,. \tag{2.1.30}$$

The deformed Lagrangian is the Nambu-Goto action in the $(N+2)$ dimensional target space with a static gauge.

Interacting scalar

Finally, we comment on the interacting scalar case. Let us consider the general potential $V(\Phi)$

$$\mathcal{L}_X^{(0)} = \frac{1}{2}X + V(\Phi)\,. \tag{2.1.31}$$

Here X is the kinetic term of the scalar field $X = \partial\Phi\bar{\partial}\Phi$. The deformed Lagrangian is obtained as

$$\mathcal{L}_X^{(\tau)} = -\frac{1}{2\tau}\frac{1-2\tau V}{1-\tau V} + \frac{1}{2\tau}\sqrt{\frac{(1-2\tau V)^2}{(1-\tau V)^2} + 2t\frac{X+2V}{1-\tau V}}\,. \tag{2.1.32}$$

For example, massive scalar and the sine-Gordon model are derived and discussed [12].

2.2 Spectrum of the deformed theories

In this section, we derive the deformed spectrum of the $T\bar{T}$-deformed theories. The key property to derive the spectrum is Zamolodchikov's factorization formula

$$\langle n|\,\mathcal{O}_{T\bar{T}}\,|n\rangle = \langle n|T|n\rangle\,\langle n|\bar{T}|n\rangle - \langle n|\Theta|n\rangle\,\langle n|\Theta|n\rangle\,. \tag{2.2.1}$$

Here $|n\rangle$ is denoted as an eigenstate of the Hamiltonian.

First we show that the composite operator $\mathcal{O}_{T\bar{T}}$ is locally well-defined. After that, we sketch the proof of the Zamolodchikov's factorization formula. We derive the inviscid Burgers' equation and provide the deformed spectrum. In this section, we consider QFTs on a cylinder with radius R.

2.2.1 Finiteness of the $\mathcal{O}_{T\bar{T}}$

Let us define the $T\bar{T}$-operator at the quantum level. In the $(z, \bar{z})$ coordinates, the conservation law of the energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$ is written as

$$\bar{\partial} T(z) = \partial \Theta(z) \,, \qquad \partial \bar{T}(z) = \bar{\partial} \Theta(z) \,. \tag{2.2.2}$$

To define the operator, we consider the following limit

$$\lim_{z \to w} \left(T(z) \bar{T}(w) - \Theta(z) \Theta(w) \right) \,. \tag{2.2.3}$$

In the case of CFT, this limit is well-defined because T and $\bar{T}$ are holomorphic and anti-holomorphic function respectively and the trace part Θ vanishes. However, in general, it is non-trivial that the limit is well-defined.

Let us take the derivative $\partial_{\bar{z}}$

$$\begin{aligned}
&\partial_{\bar{z}} \left(T(z) \bar{T}(w) - \Theta(z) \Theta(w) \right) \\
&= \partial_{\bar{z}} T(z) \bar{T}(w) - \partial_{\bar{z}} \Theta(z) \Theta(w) \\
&= \partial_z \Theta(z) \bar{T}(w) - \partial_{\bar{z}} \Theta(z) \Theta(w) \\
&= \partial_z \Theta(z) \bar{T}(w) - \partial_{\bar{z}} \Theta(z) \Theta(w) + \left(\Theta(z) \partial_w \bar{T}(w) - \Theta(z) \partial_{\bar{w}} \Theta(w) \right) .
\end{aligned} \tag{2.2.4}$$

Here in the third line we used the conservation law, and in the last line we added the bracket term which is equal to zero by the conservation law. Thus, (2.2.4) can be written as

$$\partial_{\bar{z}} \left(T(z) \bar{T}(w) - \Theta(z) \Theta(w) \right) = (\partial_z + \partial_w) \Theta(z) \bar{T}(w) - (\partial_{\bar{z}} + \partial_{\bar{w}}) \Theta(z) \Theta(w) \,. \tag{2.2.5}$$

Similarly, one can find out that

$$\partial_z \left(T(z) \bar{T}(w) - \Theta(z) \Theta(w) \right) = (\partial_z + \partial_w) T(z) \bar{T}(w) - (\partial_{\bar{z}} + \partial_{\bar{w}}) T(z) \Theta(w) \,. \tag{2.2.6}$$

Let us consider the OPE. For example, the OPE of $\Theta(z) \bar{T}(\omega)$ is written as

$$\Theta(z) \bar{T}(w) = \sum_i c^i (z - w) \mathcal{O}_i(w) \,. \tag{2.2.7}$$

where the sum is over all operators. It is clear that $(\partial_z + \partial_w)$ and $(\partial_{\bar{z}} + \partial_{\bar{w}})$ annihilate the coefficient $c^i(z-w)$, so it is only acts on the operators. (2.2.4) and (2.2.6) are written as

$$\begin{aligned}
\partial_z \left(T(z)\bar{T}(w) - \Theta(z)\Theta(w)\right) &= (\partial_z + \partial_w)T(z)\bar{T}(w) - (\partial_{\bar{z}} + \partial_{\bar{w}})T(z)\Theta(w) \\
&= \sum_i A^i(z-w)\partial_w \mathcal{O}_i(w) + \sum_i B^i(z-w)\partial\bar{w}\mathcal{O}_i(w) \\
\partial_{\bar{z}} \left(T(z)\bar{T}(w) - \Theta(z)\Theta(w)\right) &= (\partial_z + \partial_w)\Theta(z)\bar{T}(w) - (\partial_{\bar{z}} + \partial_{\bar{w}})\Theta(z)\Theta(w) \\
&= \sum_i C^i(z-w)\partial_w \mathcal{O}_i(w) + \sum_i D^i(z-w)\partial\bar{w}\mathcal{O}_i(w) \qquad (2.2.8)
\end{aligned}$$

The exact form of $A_i, \cdots, D^i$ are not important. The important point is that the operators takes the form of derivatives of some operators. This gives a constraint for the form of OPE. Let us write the OPE as

$$T(z)\bar{T}(w) - \Theta(z)\Theta(w) = \sum_i \mathcal{C}_n^i(z-w)\mathcal{O}_n(w) \,. \qquad (2.2.9)$$

If an operator is not the derivative of another operator, the corresponding coefficient C_i must be a constant. Suppose we have the term $C_n(z-w)\mathcal{O}_n(w)$ in the OPE where $\mathcal{O}_n$ is not the derivative of another operator

$$T(z)\bar{T}(w) - \Theta(z)\Theta(w) = C^n(z-w)\mathcal{O}_n(w) + \cdots \,. \qquad (2.2.10)$$

Taking derivative on the both sides, one can obtain

$$\partial_z(T(z)\bar{T}(w) - \Theta(z)\Theta(w)) = \partial_z \mathcal{C}^n(z-w)\mathcal{O}_n(w) + \mathcal{C}^n(z-w)\partial_z\mathcal{O}_n(w) + \cdots \,. \qquad (2.2.11)$$

Note that the first term is not compatible with (2.2.8) unless $\partial_z \mathcal{C}^n = 0$. Absorbing the constant coefficient int the definition of the local operator, we conclude that

$$\lim_{z\to w} \left(T(z)\bar{T}(w) - \Theta(z)\Theta(w)\right) = \mathcal{O}_{T\bar{T}}(w) + \text{derivatives} \,. \qquad (2.2.12)$$

Thus we defined the operator up to total derivatives.

2.2.2 The factorization property

Next, we give a proof of Zamoldchikov's factorization formula following his original work [3]. He set up 4 assumptions to prove the theorem.

1. *Local translational and rotational symmetry.* The local energy-momentum tensor $T_{\mu\nu}$ exists and it is symmetric $T_{\mu\nu} = T_{\nu\mu}$. It also satisfied the conservation law $\partial_\mu T^{\mu\nu} = 0$.

2. *Global translational symmetry.* For any local field $\mathcal{O}(z)$ the expectation value $\langle \mathcal{O}(z) \rangle$ is a constant and independent of z. From assumption 1, the two-point correlation function depends only on the separations

$$\langle \mathcal{O}_i(z)\mathcal{O}_j(z')\rangle = G_{ij}(z - z')\,.$$

3. *Infinite separations.* At least one Euclidean vector $e = (e, \bar{e})$ exists, such that for any $\mathcal{O}_i$ and $\mathcal{O}_j$

$$\lim_{t\to\infty} \langle \mathcal{O}_i(z + et)\mathcal{O}_j(z')\rangle = \langle \mathcal{O}_i(z)\rangle \langle \mathcal{O}_j(z')\rangle \ .$$

The assumptions 2 and 3 imply 2D space is either an infinite plane or an infinitely long cylinder.

4. *CFT limit at short distances.* The short-distance behavior is governed by CFT.

As the first step, let us consider the following quantity

$$C(z, w) = \langle T(z)\bar{T}(w)\rangle - \langle \Theta(z)\Theta(w)\rangle \ . \tag{2.2.13}$$

Taking the derivative $\partial_{\bar{z}}$,

$$\begin{aligned} \partial_{\bar{z}} C(z, w) \ &= \langle \partial_{\bar{z}} T(z)\bar{T}(w)\rangle - \langle \partial_{\bar{z}}\Theta(z)\Theta(w)\rangle \\ &= \langle \partial_z \Theta(z)\bar{T}(w)\rangle + \langle \Theta(z)\partial_{\bar{w}}\Theta(w)\rangle \\ &= -\langle \Theta(z)\partial_w \bar{T}(w)\rangle + \langle \Theta(z)\partial_{\bar{w}}\Theta(w)\rangle = 0\,. \end{aligned} \tag{2.2.14}$$

Here we used the conservation law and the translational invariance to move the derivative from one operator to the other. Similarly, one can show $\partial_z C(z, \bar{z}) = 0$. Then $C(z, \bar{z})$ is a constant.

Now we can write it in two different ways. Taking the two points infinitely separated from each other, by cluster decomposition theorem, the two-point function can be factorized and obtained as

$$C = \lim_{|z-w|\to\infty} C(z, w) = \langle T\rangle \langle \bar{T}\rangle - \langle \Theta\rangle \langle \Theta\rangle \ . \tag{2.2.15}$$

On the other hand, we can take the coinciding limit and find that

$$C = \lim_{|z-w|\to 0} C(z,w) = \langle \mathcal{O}_{T\bar{T}} \rangle \ . \tag{2.2.16}$$

From the two expression, we prove the factorization formula for the vacuum state $|n\rangle = |0\rangle$. To complete the proof, we should consider the generic exited state.

Let us define the similar function

$$C_n(z,w) = \langle n|T(z)\bar{T}(w)|n\rangle - \langle n|\Theta(z)\Theta(w)|n\rangle \ . \tag{2.2.17}$$

one can show that $C_n(z,w)$ is also a constant as before. However the asymptotic factorization property no longer holds. We need to consider the contributions from the intermediate states. This can be seen from the spectral expression

$$\langle n|T(z)\bar{T}(z')|n\rangle = \sum_m \langle n|T(z)|m\rangle \langle m|\bar{T}(z')|n\rangle \, \mathrm{e}^{(E_n-E_m)|y-y'|+i(P_n-P_m)|x-x'|} \ , \tag{2.2.18}$$

and similar spectral expression for $\langle n|\Theta(z)\Theta(z')|n\rangle$. Now $C_n(z,w)$ is a constant and independent of any coordinates. Thus all non-diagonal matrix elements should cancel and only diagonal terms are left.

$$\langle n| \mathcal{O}_{T\bar{T}} |n\rangle = \langle n|T|n\rangle \langle n|\bar{T}|n\rangle - \langle n|\Theta|n\rangle \langle n|\Theta|n\rangle \ . \tag{2.2.19}$$

This is the factorization property for the general exited states.

2.2.3 The inviscid Burgers' equation and the deformed spectrum

Using the factorization formula, let us consider the energy spectrum of the deformed theory. For QFTs on a cylinder with radius R, expectation values of each component of the energy-momentum tensor are interpreted as

$$\begin{aligned} E_n(R,\tau) &= -R \langle n|T_{yy}|n\rangle \\ \partial_R E_n(R,\tau) &= -\langle n|T_{xx}|n\rangle \\ P_n &= -iR \langle n|T_{xy}|n\rangle \end{aligned} \tag{2.2.20}$$

Here $E_n(R,\tau)$ is the spectrum of the deformed theory and P_n is a momentum along the compact direction. Using the factorization property, one can rewrite the right-hand side of the flow equation in terms of these variables.

On the other hand, from the definition of the $T\bar{T}$-deformation, one can obtain

$$\partial_\tau E_n(R,\tau) = -R \langle n|\det(T_{\mu\nu})|n\rangle \,. \tag{2.2.21}$$

Therefore one find that

$$\partial_\tau E_n(R,\tau) = E_n(R,\tau)\partial_R E_n(R,\tau) + \frac{1}{R}P_n(R)^2 \,. \tag{2.2.22}$$

This is called the inviscid Burgers' equation. If we give an initial condition $E_n(R,0) = E_n(R)$ which is usually the spectrum of an undeformed theory, we can solve this equation and give a deformed spectrum.

The spectrum and the entropy of deformed CFT$_2$

For example, let us consider the case that an undeformed theory is CFTs and we have

$$E_n(R) = \frac{1}{R}\left(n + \bar{n} - \frac{c}{12}\right) , \qquad P_n(R) = \frac{1}{R}(n - \bar{n}) \,. \tag{2.2.23}$$

Here n and $\bar{n}$ are the eigenvalues of the Virasoro generator L_0 and $\bar{L}_0$. The solution of the inviscid Burgers' equation is given by

$$E_n(R,\tau) = \frac{R}{2\tau}\left(\sqrt{1 + \frac{4\tau\, E_n}{R} + \frac{4\tau^2\, P_n^2}{R^2}} - 1\right) . \tag{2.2.24}$$

It is remarkable that the first term of the deformed spectrum is a square root and is not sure whether that is positive definite or not.

Let us discuss the some features of the deformed theories in detail. For simplicity, in the following, we consider $n = \bar{n}$ ($P_n = 0$) case. It is useful to introduce the following dimensionless variables

$$b \equiv \frac{4\pi\tau}{R^2}, \qquad \mathcal{E}_n \equiv \frac{E_n R}{2\pi}, \qquad M \equiv 2n - \frac{c}{12} \,. \tag{2.2.25}$$

The deformed spectrum takes the form

$$\mathcal{E}_n(b,M) = -\frac{1}{b} + \sqrt{\frac{1}{b^2} + \frac{2M}{b}} \,. \tag{2.2.26}$$

In the following, we discuss each case of a signature of the parameter τ.

Positive b case

First, let us consider a positive sign case. In the region $|M| \ll 1/b$ which is a weak coupling and a low energy region, energies are little changed by the perturbation and are given to a good approximation $\mathcal{E} \sim M$. On the other hand, in the high energy region $|Mb| \gg 1$, the behavior of the spectrum depends on the signature of the parameter τ. In the positive sign case $\tau > 0$, the spectrum is positive definite and one find

$$\mathcal{E}_n(b, M) \simeq \sqrt{\frac{2M}{b}}, \qquad E \simeq \sqrt{\frac{2\pi M}{\tau}}\,. \tag{2.2.27}$$

In this limit, the spectrum becomes R independent.

It is also interesting to consider the case of the ground state $n = 0$, $M = -c/12$. In this case, one finds that the energy becomes complex for $bc > 6$. One can think of bc as 't Hooft coupling of the deformed theories. For a large c, it corresponds to the region of weak coupling $b \gg 1$

The entropy $S(\mathcal{E})$ is given by the usual Cardy formula. The entropy of CFT_2 is approximated as

$$S(M) \simeq 2\pi\sqrt{\frac{c\,M}{3}}\,. \tag{2.2.28}$$

This formula is valid for $M \gg 1$. By using (2.2.26), the Cardy entropy of the deformed theories can be written as

$$S(\mathcal{E}) \simeq 2\pi\sqrt{\frac{c}{6}(2\mathcal{E} + b\mathcal{E}^2)}\,. \tag{2.2.29}$$

This is expected to be valid for $\mathcal{E} \gg 1$. For $1 \ll \mathcal{E} \ll 1/b$, the undeformed entropy is reproduced. On the other hand, for $\mathcal{E} \gg 1/b$, one fines

$$S \simeq 2\pi\sqrt{\frac{bc}{6}}\mathcal{E}\,. \tag{2.2.30}$$

This is nothing but a Hagedorn entropy $S = \beta_H E$ with Hagedorn temperature

$$\beta_H = \sqrt{\frac{2\pi c\tau}{3}}\,. \tag{2.2.31}$$

When β_H is equal to the radius R, it means $bc = 6$ and the ground state energy becomes complex.

Negative b case

If the parameter τ is negative ($\tau < 0,\ b < 0$), the spectrum become complex in the high energy region

$$M \geqq -\frac{1}{2b}\,. \tag{2.2.32}$$

This bound $M = 1/2|b|$ is interpreted as a UV cut-off in the deformed theories.

Again, the entropy is obtained by the Cardy formula

$$S(\mathcal{E}) \simeq 2\pi\sqrt{\frac{c}{6}(2\mathcal{E} - |b|\mathcal{E}^2)}\,. \tag{2.2.33}$$

At the high energy bound $M = 1/2|b|$, the entropy get a maximal value

$$S = 2\pi\sqrt{\frac{c}{6|b|}}\,. \tag{2.2.34}$$

In the previous section, we pointed out that the deformation parameter may relate to a string scale α'. In this sense, a negative-sign deformation seems pathological. In the last section of this chapter, we will consider some proposals of this problem.

2.3 Holographic dual for $T\bar{T}$-deformed CFT$_2$

One of the most interesting problem is to confirm the relation between 3 dimensional gravity theories and the $T\bar{T}$-deformed CFT$_2$. That is a non-trivial generalization of AdS$_3$/CFT$_2$ correspondence and it is important to understand how the gauge/gravity duality realizes for an irrelevant perturbation from conformal field theories. In the previous section, we derive the spectrum and the entropy of the deformed CFT$_2$. In this section, we show some proposals for a gravity dual of the $T\bar{T}$-deformed CFT and compare physical variables of both sides.

A cut-off AdS$_3$

One of the candidates of the holographic dual of the $T\bar{T}$-deformed CFT$_2$ is a cut-off AdS$_3$ geometry [30]. One imposes the Dirichlet boundary conditions on finite radius cut-off $r = r_c$, where r is a radius coordinate of AdS$_3$ and r_c is a constant.

Let us compare the energy and the entropy of a BTZ black hole with ones of the $T\bar{T}$-deformed CFT$_2$. The gravitational action for 3D space-time with a negative cosmological term is

$$S_3 = \frac{1}{16\pi G_3}\int \mathrm{d}^3x\sqrt{-g_3}(R_3+2) - \frac{1}{8\pi G_3}\int_B \mathrm{d}^2x\sqrt{-g}(K+1)\,. \tag{2.3.1}$$

Here G_3 is a Newton constant in 3 dimension and g_3 is the metric on 3D space-times. with a time-like boundary B. K denotes the extrinsic curvature of the boundary. From now on, we consider a rotating BTZ black hole solution with a static boundaty $B=\{r=r_c\}$.

The metric of the BTZ black hole with mass M and angular momentum J is written as

$$\begin{aligned} \mathrm{d}^2s &= -f^2(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta-\omega(r)\mathrm{d}t)^2, \\ f^2(r) &= r^2 - 8G_3M + \frac{16{G_3}^2J^2}{r^2}\,, \qquad \omega(r) = \frac{4GJ}{r^2}\,. \end{aligned} \tag{2.3.2}$$

The inner horizon r_- and the outer horizon r_+ are related to M and J through

$$M = \frac{r_+^2+r_-^2}{8G_3}\,, \qquad J = \frac{r_+r_-}{4G_3}\,. \tag{2.3.3}$$

The left- and right- inverse temperature $\beta_\pm$ can be written as

$$\beta_\pm = \frac{2\pi}{r_+ \mp r_-}\,. \tag{2.3.4}$$

We impose the Dirichlet boundary conditions, by fixing the form of the boundary metric

$$\mathrm{d}^2s|_B = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = -N^2\mathrm{d}t^2 + \mathrm{e}^{2\varphi}(\mathrm{d}\theta-\omega(r)\mathrm{d}t)^2\,. \tag{2.3.5}$$

By solving the equations of motions, one can obtain the on-shell action $S[g]$.

The gravitational energy of the black hole is defined in terms of the variation of the action $S[g]$

$$\delta S = \int_B \mathrm{d}^2x\,\pi^{\alpha\beta}\delta g_{\alpha\beta} = \int \mathrm{d}^2x\sqrt{-g}\,(-\epsilon\delta N - j\delta\omega + p\,\delta\varphi)\,. \tag{2.3.6}$$

Here ϵ, j, and p are interpreted as gravitational energy density, momentum density, and pressure respectively, as measured on the boundary. The total energy can be obtained by

$$E = \oint \mathrm{d}\theta\,\mathrm{e}^{\varphi}\epsilon\,. \tag{2.3.7}$$

The quasi-local energy of a rotating BTZ black hole is written as

$$E = \frac{r_c}{4G_3}\left[1 - \sqrt{1 - \frac{8G_3 M}{r_c^2} + \frac{16G_3^2 J^2}{r_c^4}}\right] . \tag{2.3.8}$$

This energy perfectly matches with the energy spectrum of the $T\bar{T}$-deformed CFT_2 (2.2.26) and one can identify

$$b = -\frac{4G_3}{r_c^2} = -\frac{6}{c\, r_c^2} . \tag{2.3.9}$$

Then, the deformation parameter b is related to a cut-off radius r_c.

The Bekenstein-Hawking entropy can be evaluated by using the outer horizon r_+

$$S = \frac{\pi r_+}{2G_3} . \tag{2.3.10}$$

Now, the energy and the entropy are saturated in the limit that the BTZ black hole completely fills out the space-time inside the wall $r = r_c$. Then, the maximal value of the energy $E_{\max}$ and the entropy $S_{\max}$ are given by

$$E_{\max} = \frac{r_c}{4G_c} , \qquad S_{\max} = \frac{\pi r_c}{2G_3} . \tag{2.3.11}$$

This is the proposal to the interpretation of the bound of the deformed spectrum in the view point of the cut-off geometry.

In the cut-off AdS conjecture, the background is fixed to AdS_3 and it is easy to evaluate physical quantities. Then, there are many works to verify this conjecture. In the original work [30], authors have already discussed signal propagation speed and holographic renormalization group. The deformed correlation functions [31] and renormalization group flow [32] have been discussed. Another aspect discussed well is a holographic entanglement entropy [33–37].

However, there are some problems in this conjecture. First, it is difficult to reproduce the deformed correlation function from the cut-off AdS_3 by applying GKP-Witten relation. Moreover, in the standard interpretation of the AdS/CFT correspondence, one considers that the radius direction of AdS space corresponds to the energy scale of a holographic dual of field theories. The boundary and the center of AdS correspond to UV and IR region, respectively. In this sense, the cut-off AdS geometry seems to correspond to the flow from UV region to IR region. On the other hand, the $T\bar{T}$-deformation is an irrelevant deformation and this deformation determine the flow from IR to UV. Thus, the direction of the RG flow looks the other way around.

It is also remarkable that, the physical quantities perfectly match only in the case of a negative sign $T\bar{T}$-deformation with $b < 0$. In this scenario, one cannot interpret the origin of the Hagedorn behavior of the spectrum in the case of a positive sign $T\bar{T}$-deformation with $b > 0$.

One of the possibility to resolve the problems of the cut-off conjecture is a mixed boundary conjecture or a "mirage" conjecture [38]. In this conjecture, it is proposed that one need to impose not the Dirichlet boundary conditions on the AdS boundary but the non-linear mixed boundary condition. An advantage of this idea is that one can deal with a $T\bar{T}$-deformation of both cases of the signature of the deformation parameter. As an example of recent success, the $T\bar{T}$-deformed WZW model has been reproduced as a boundary theory from 3D Chern-Simons gravity imposing a mixed boundary condition [39].

Another possibility is a random boundary geometry scenario proposed by [40]. It is known that a $T\bar{T}$-deformation can be understood in terms of random geometry [4], and is expected that a holographic dual of $T\bar{T}$-deformation is described by random boundary geometry. Evaluating variables at a saddle point, the spectrum and the thermodynamical quantities can be reproduced [40]. Surprisingly, correlation functions are also reproduced in [40, 41] unlike other conjectures.

A little string theory

Naively, it is expected that a holographic dual of a positive-sign $T\bar{T}$-deformed CFT_2 is as a kind of string in the UV region. Such a background was proposed by [42]. Let us summarize the core point of these discussions.

Let us assume that the full background takes the form

$$\mathcal{M}_3 \times \mathcal{N} \,. \tag{2.3.12}$$

Here $\mathcal{N}$ is a compact CFT and is a 3D background $\mathcal{M}_3$. A well studies example is superstring theory with $\mathcal{M}_3 = \mathrm{AdS}_3$ and $\mathcal{N} = S^3 \times T^4$, which describe the near-horizon geometry pf k NS5-branes wrapped on $S^1 \times T^4$ and p fundamental strings wrapped on S^1.

In [42], authors considered the following three-dimensional background $\mathcal{M}_3$:

$$\begin{aligned} \mathrm{d}^2 s \ &= f_1^{-1} l_s^2 \mathrm{d}\gamma \mathrm{d}\bar{\gamma} + k l_s^2 \mathrm{d}\phi^2 \,, \\ \mathrm{e}^{2\Phi} \ &= \frac{v}{p} \mathrm{e}^{-2\phi} f_1^{-1} \,, \\ \mathrm{d}B \ &= 2i \mathrm{e}^{-2\phi} f_1^{-1} \epsilon_3 \,. \end{aligned} \tag{2.3.13}$$

Here $l_s \gamma = x^1 + x^0$, $l_s \bar{\gamma} = x^1 - x^0$ and f_1 is defined as

$$f_1 = 1 + \frac{1}{k} \mathrm{e}^{-2\phi} \,, \tag{2.3.14}$$

and v is a constant associated with the compact CFT $\mathcal{N}$. The x^1 is periodically identified, $x^1 \sim x^1 + R$ This background interpolates between the little string theory and near the boundary $\phi \to \infty$ and AdS_3 background in the infrared $\phi \to -\infty$. It describes as RG flow from a non-local theory with a Hagedorn spectrum in the UV, to a standard CFT_2 in the IR. Based on this conjecture, the deformation parameter corresponds to a string length l_s

$$b = \frac{4\pi l_s^2}{R^2} \,. \tag{2.3.15}$$

For example, the thermodynamics has discussed and seems nicely to match one of $T\bar{T}$-deformed systems. (About further discussion, see [43–48].)

It seems that the parameter should be positive $b > 0$ and it looks that one cannot deal with a negative-sign $T\bar{T}$-deformation. However, a negative string background is introduced by replacing to $l_s^2 \to -l_s^2$ and discussed [49]. Even though the background has a singularity surface and seems pathological, one can analyze the dynamics of a probe string and the string can go through the singularity surface. It may be interesting to investigate such a background as a candidate of a holographic dual of deformed theories.

However, as it has been already mentioned in the original work [42], we should emphasize that a boundary dual is different form a $T\bar{T}$-deformation discussed in this chapter. Let us explain the reason. A boundary CFT of the $\mathrm{AdS}_5 \times S^3 \times T^4$ at large p is expected to have the form of a symmetric product

$$\mathcal{M}^p / S_p \,. \tag{2.3.16}$$

Here $\mathcal{M}$ is a CFT with central charge $c_M = 6k$. Since for large p string theory is weakly coupled, one can use world sheet technique to study it. One can construct vertex operator s that describe and think of these operator as "single-trace" operators in the symmetric orbifold CFT. The correspond to operators of the form

$$\sum_i^p \mathcal{O}_i(x)\,. \tag{2.3.17}$$

Here $\mathcal{O}_i$ correspond to a particular operator living in the i'th factor of $\mathcal{M}^p/S_p$, with the same over i imposing S_p invariance. A holographic dual of this background should be a "single-trace" $T\bar{T}$-deformation of CFT. On the other hand, we have considered a "double-trace" deformation as originally introduced by [3]. Then, strictly they are different deformations.

However, it is still interesting to investigate their correspondence because a single-trace deformation is related to another kind of integrable deformations, called the Yang-Baxter deformations. For example, in [50,51], it has been discussed the relation between the single-trace $T\bar{T}$-deformation and the YB-deformation. For related works, see [52, 53]. The YB-deformation is one of an integrable deformation for non- linear sigma model and it is known that they correspond to a certain duality transformation. The background (2.3.13) has been provided by TsT-transformation [54,55] and the deformed spectrum has been computed [56]. In chatper 3 and 4, we will investigate the relationship between the YB-deformation and the double-trace $T\bar{T}$-deformation.

Chapter 3

2D dilaton gravity models and the Yang-Baxter deformations

In two-dimension cases, the Ricci scalar is written as a total derivative and the Einstein-Hilbert action is completely determined by a topological number of a space-time. Thus, one has considered the dilaton field which has a non-minimal coupling to the Ricci scalar. Historically, such 2D dilaton gravity models have played an important roles in the study of the string theory and quantum gravity. (See reviews [57, 58].)

Recently, 2D dilaton gravity models are refocused in the context of $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence which is a model in the lowest dimensions to realize the holographic principle. In the pioneering work by [24], the Jackiw-Teitelboim (JT) model is regarded as a candidate of a holographic dual of a 1D quantum mechanics. About further discussion, see [59, 60].

On the other hand, as we have mentioned in the previous chapter, the (single trace) $T\bar{T}$-deformation of CFT_2 is related to a TsT-transformation of 3-dimensional space-times. TsT-transformations can be understood from the perspective of the Yang-Baxter (YB) deformations. The YB-deformations are systematical techniques of integrable deformations for 2D non-linear sigma models [25–27]. (About the application of the YB-deformation for the superstring sigma models and (generalized) SUGRA, see reviews [28, 29].) Moreover, it has been proposed that various types of integrable systems included the YB-deformed sigma models can be reproduced form 4D Chern-Simons theories [61–65]. Thus, we expect that the YB-deformations are useful to derive a 2D dilaton gravity model closely related to a $T\bar{T}$-deformation.

3.1 2D dilaton gravity system and the JT model

In this section, let us prepare the 2D dilaton gravity system and some terminologies used in the following discussion. The dilaton gravity system in 2 dimensions is composed of the metric g_{ab} $(a, b = 0, 1)$ and the dilaton ϕ. The coordinates are parametrized as $x^a = (x^0, x^1) = (t, z)$. In this chapter, we will work with the Lorentzian signature.

The classical action S is given by

$$\begin{aligned} S &= S_{g,\phi} + S_{\text{matter}} \,, \\ S_{g,\phi} &= \frac{1}{16\pi G}\int d^2x\,\sqrt{-g}\,(\phi R - U(\phi)) \,, \\ S_{\text{matter}} &= \frac{1}{32\pi G}\int d^2x\,\sqrt{-g}\,\Omega(\phi)\,(\nabla f)^2 \,. \end{aligned} \tag{3.1.1}$$

Here G is the Newton constant in 2 dimensions and $U(\phi)$ is the dilaton potential. f is a scalar field and Ω is a function of the dilaton field.

In this chapter, we will work with the metric in the conformal gauge,

$$\mathrm{d}s^2 = -\mathrm{e}^{2\omega(x^+,x^-)}\mathrm{d}x^+\mathrm{d}x^- \,, \tag{3.1.2}$$

where the light-cone coordinates are defined as

$$x^\pm \equiv t \pm z \,. \tag{3.1.3}$$

Then the equations of motion are given by

$$\begin{aligned} \partial_+(\Omega\partial_- f) + \partial_-(\Omega\partial_+ f) &= 0 \,, \\ 4\partial_+\partial_-\phi - \mathrm{e}^{2\omega}U(\phi) &= 0 \,, \\ 2\partial_+(\mathrm{e}^{-2\omega}\partial_-\mathrm{e}^{2\omega}) - \frac{1}{2}e^{2\omega}\partial_\phi U(\phi) &= (\partial_\phi\Omega)\partial_+ f\partial_- f \,, \\ -\mathrm{e}^{2\omega}\partial_+(\mathrm{e}^{-2\omega}\partial_+\phi) &= \frac{\Omega}{2}\partial_+ f\partial_+ f \,, \\ -\mathrm{e}^{2\omega}\partial_-(\mathrm{e}^{-2\omega}\partial_-\phi) &= \frac{\Omega}{2}\partial_- f\partial_- f \,. \end{aligned} \tag{3.1.4}$$

The energy-momentum tenser for the matter field f is normalized as

$$\begin{aligned}(T_{\rm matter})_{ab} &\equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{ab}} \\ &= -\frac{\Omega(\phi)}{16\pi G}\left(\partial_a f\,\partial_b f - \frac{1}{2}g_{ab}\,\partial^c f\partial_c f\right)\,. \end{aligned} \tag{3.1.5}$$

This expression (3.1.5) is valid for the general form of $\Omega(\phi)$.

The JT model

The JT model corresponds to a special case of 2 D dilaton gravity specified by the following $U(\phi)$ and $\Omega(\phi)$:

$$U(\phi) = 2 - 2\phi, \qquad \Omega(\phi) = 1\,. \tag{3.1.6}$$

This model exhibits nice properties. Among them, we are concerned with the vacuum solution of this model. For our later convenience, we shall give a brief review of the work [24] by focusing upon the vacuum solution in the following.

The general vacuum solution is given by

$$\mathrm{d}s^2 = \frac{1}{z^2}(-\mathrm{d}t^2 + \mathrm{d}z^2)\,, \tag{3.1.7}$$

$$\phi = 1 + \frac{a + b\,t + c\,(-t^2 + z^2)}{z}\,, \tag{3.1.8}$$

and depends on three real constants a, b and c. This three-parameter family contains interesting solutions as specific examples. For example, the case with $a = 1/2$, $b = 0$ and $c = 0$ corresponds to a renormalization group flow solution from a conformal Lifshitz spacetime to AdS_2 [24], with an appropriate lift-up to higher dimensions.

Another intriguing example is a black hole solution specified with $a = 1/2$, $b = 0$ and $c = \mu/2$, where μ is a real positive constant. Then by performing a coordinate transformation,

$$x^{\pm} = \frac{1}{\sqrt{\mu}}\tanh\left(\sqrt{\mu}\,(T \pm Z)\right)\,, \tag{3.1.9}$$

the solution is rewritten into the following form:

$$\begin{aligned}\mathrm{d}s^2 &= \frac{4\,\mu}{\sinh(2\sqrt{\mu}\,Z)}(-\mathrm{d}T^2 + \mathrm{d}Z^2)\,, \\ \phi &= 1 + \sqrt{\mu}\coth(2\sqrt{\mu}\,Z)\,. \end{aligned} \tag{3.1.10}$$

The new coordinates T and Z cover a smaller region which is in the inside of the entire Poincaré AdS_2 .

The background (3.1.10) indeed describes a black hole geometry, but it may not be so manifest. To figure out the black hole geometry, it is nice to move to the Schwarzschild coordinates by performing a further coordinate transformation,

$$Z = \frac{1}{2\sqrt{\mu}}\mathrm{arccoth}\left(\frac{\rho}{\sqrt{\mu}}\right) . \tag{3.1.11}$$

Then the background (3.1.10) can be rewritten as[1]

$$\mathrm{d}s^2 = -4(\rho^2-\mu)\mathrm{d}t^2 + \frac{\mathrm{d}\rho^2}{\rho^2-\mu}\,, \qquad \phi = 1+\rho\,. \tag{3.1.12}$$

In this metric, the black hole horizon is located at $\rho = \sqrt{\mu}$, and the Hawking temperature T_{H} can be evaluated in the standard manner as

$$T_{\mathrm{H}} = \frac{1}{4\,\pi}\partial_\rho\sqrt{\frac{-g_{tt}}{g_{\rho\rho}}}\Bigg|_{\rho=\sqrt{\mu}} = \frac{\sqrt{\mu}}{\pi}\,. \tag{3.1.13}$$

Thus one can see that the background (3.1.10) describes a black hole whose horizon is located at $Z=\infty$.

The Bekenstein-Hawking entropy can also be computed as

$$S_{\mathrm{BH}} = \frac{A}{4G_{\mathrm{eff}}}\bigg|_{Z\to\infty} = \frac{\phi}{4G}\bigg|_{\sqrt{\mu}=\pi\,T_H} = \frac{1+\pi\,T_{\mathrm{H}}}{4G}\,. \tag{3.1.14}$$

Here the area A is taken as $A=1$ because the horizon is just a point, and the effective Newton constant G_{eff} can be read off from the classical action as

$$\frac{1}{G_{\mathrm{eff}}} = \frac{\phi}{G}\,. \tag{3.1.15}$$

On the other hand, the holographic entropy can be computed by using the renormalized boundary stress tensor. For the detailed computation like the regularization and the counter term, see [24]. As a result, the renormalized boundary stress tensor is evaluated as

$$\langle \hat{T}_{tt}\rangle = \frac{\mu}{8\pi G} \equiv E\,. \tag{3.1.16}$$

[1]The factor 4 is included so that the Bekenstein-Hawking entropy should match with the holographic computation. This normalization guarantees the matching of the bulk and boundary times (or temperatures).

Then by using the thermodynamic relation

$$dS = \frac{dE}{T_{\rm H}}\,, \tag{3.1.17}$$

the entropy is obtained as

$$S = \frac{\pi T_{\rm H}}{4G} + S_{T_{\rm H}=0}\,, \tag{3.1.18}$$

where $S_{T_{\rm H}=0}$ is an integration constant. Thus the holographic entropy agrees with the Bekenstein-Hawking entropy, up to the temperature-independent constant.

3.2 Yang-Baxter deformations of AdS$_2$

In this section, we consider the most general Yang-Baxter deformation of the AdS$_2$ metric. First of all, we briefly describe a coset construction of the Poincaré AdS$_2$ Then we study the most general Yang-Baxter deformation of Poincaré AdS$_2$. As a result, we obtain a three-parameter family of deformed AdS$_2$ spaces.

3.2.1 Coset construction of AdS$_2$

Let us recall a coset construction of the Poincaré AdS$_2$ metric (For the detail of the coset construction, for example, see [66]).

The starting point is that the AdS$_2$ geometry is represented by a coset

$$\mathrm{AdS}_2 \quad = \quad SL(2)/U(1)\,. \tag{3.2.1}$$

By using the coordinates t and z, a coset representative g is parametrized as

$$g = \exp\left[tH\right]\exp\left[(\log z)D\right]\,, \tag{3.2.2}$$

where H and D are the time translation and dilatation generators, respectively. By involving the special conformal generator C, the $\mathfrak{sl}(2)$ algebra in the conformal basis is spanned as

$$[D,H] = H\,, \qquad [C,H] = 2D\,, \qquad [D,C] = -C\,. \tag{3.2.3}$$

These generators can be represented by the $\mathfrak{so}(1,2)$ ones $T_I \;\; (I=0,1,2)$ like

$$H \equiv T_0 + T_2\,, \qquad C \equiv T_0 - T_2\,, \qquad D \equiv T_1\,, \tag{3.2.4}$$

where T_I's satisfy the commutation relations:

$$[T_0, T_1] = -T_2\,, \qquad [T_1, T_2] = T_0\,, \qquad [T_2, T_0] = -T_1\,. \tag{3.2.5}$$

In the following, we will work with T_I's in the fundamental representation,

$$T_0 = \frac{i}{2}\sigma_1\,, \qquad T_1 = \frac{1}{2}\sigma_2\,, \qquad T_2 = \frac{1}{2}\sigma_3\,, \tag{3.2.6}$$

where $\sigma_i \; (i=1,2,3)$ are the standard Pauli matrices.

Note here that the coset (3.2.1) is symmetric as one can readily understand from (3.2.5) . When vector spaces $\mathfrak{h}$ and $\mathfrak{m}$ are spanned as

$$\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\langle\, T_2\,\rangle\,, \qquad \mathfrak{m} = \mathrm{span}_{\mathbb{R}}\langle T_0, T_1\rangle\,,$$

the $\mathbb{Z}_2$-grading structure is expressed as

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}\,, \qquad [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}\,, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,. \tag{3.2.7}$$

When representing the $\mathfrak{sl}(2)$ algebra by a direct product (as vector spaces) ,

$$\mathfrak{sl}(2) = \mathfrak{h} \oplus \mathfrak{m}\,,$$

the projection operator $P: \;\mathfrak{sl}(2) \;\to\; \mathfrak{m}$ can be defined as

$$P(X) \equiv \frac{\mathrm{Tr}(X\;T_0)}{\mathrm{Tr}(T_0\;T_0)}T_0 + \frac{\mathrm{Tr}(X\;T_1)}{\mathrm{Tr}(T_1\;T_1)}T_1\,, \qquad X \in \mathfrak{sl}(2)\,. \tag{3.2.8}$$

Now the Poincaré AdS_2 metric can be computed by performing coset construction. The left invariant one-form $J = g^{-1}\mathrm{d}g$ is expanded as

$$J = e^0\, T_0 + e^1\, T_1 + \frac{1}{2}\omega^{01}\, T_2\,.$$

Here e^0 and e^1 are the zweibeins, and ω^{01} is the spin connection. With the parametrization (3.2.2) , the zweibeins are given by

$$e^0 = \frac{\mathrm{d}t}{z}\,, \qquad e^1 = \frac{\mathrm{d}z}{z}\,.$$

By using the projection operator P in (3.2.8) and the explicit expressions of the zweibeins e^0 and e^1 , the resulting metric is obtained as

$$
\begin{aligned}
\mathrm{d}s^2 &= 2\mathrm{Tr}\left[JP(J)\right] = -e^0 e^0 + e^1 e^1 \\
&= \frac{-\mathrm{d}t^2 + \mathrm{d}z^2}{z^2}\,.
\end{aligned}
\tag{3.2.9}
$$

This is nothing but the AdS_2 metric in the Poincaré coordinates.

In the light-cone coordinates (3.1.3), then the metric is rewritten as

$$
\mathrm{d}s^2 = -\mathrm{e}^{2\omega(x^+,x^-)}\,\mathrm{d}x^+\mathrm{d}x^- = -\frac{4\mathrm{d}x^+\mathrm{d}x^-}{(x^+ - x^-)^2}\,. \tag{3.2.10}
$$

The exponential factor will play an important role in later discussion.

3.2.2 The general Yang-Baxter deformation

Let us next consider Yang-Baxter deformations of the AdS_2 metric (3.2.9) . Then the anti-symmetric two-form is also involved as well as the metric. Here we will concentrate on the metric part only.

The prescription of the deformation is very simple. It is just to insert a factor as follows:

$$
\mathrm{d}s^2 = 2\mathrm{Tr}\left[J\,\frac{1}{1-2\eta R_g \circ P}\,P(J)\right]\,. \tag{3.2.11}
$$

Here η is a constant parameter which measures the deformation. Then R_g is defined as a chain of operation like

$$
R_g(X) \equiv g^{-1} \circ R(gXg^{-1}) \circ g\,, \tag{3.2.12}
$$

where g is the group element in (3.2.2) . The key ingredient is a linear operator $R:\ \mathfrak{sl}(2) \to \mathfrak{sl}(2)$, and satisfy the (modified) classical Yang-Baxter equation [(m)CYBE]:

$$
[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = c \cdot [X, Y] \qquad (X\,, Y \in \mathfrak{sl}(2))\,. \tag{3.2.13}
$$

Here c is a real constant parameter. The case with $c \neq 0$ is the mCYBE and the case with $c = 0$ is the homogeneous CYBE.

We consider the most general deformations with the following R-operator

$$R(T_I) = \widetilde{\Omega}_{IJ} M^{JK}\, T_K\,, \tag{3.2.14}$$

where $\widetilde{\Omega}_{IJ}$ and M^{IJ} are defined as

$$\widetilde{\Omega}_{IJ} \equiv \mathrm{Tr}(T_I T_J) = \frac{1}{2}\eta_{IJ}\,, \qquad M^{IJ} \equiv \begin{pmatrix} 0 & m_1 & m_2 \\ -m_1 & 0 & m_3 \\ -m_2 & -m_3 & 0 \end{pmatrix}\,, \tag{3.2.15}$$

Putting the ansatz (3.2.14) into the (m)CYBE (3.2.13) leads to an algebraic relation,

$$-m_1^2 - m_2^2 + m_3^2 = 4c\,. \tag{3.2.16}$$

After evaluating the expression (3.2.11) with the general ansatz (3.2.14) , one can obtain the following metric:

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}z^2}{z^2 - \eta^2 \left(\alpha + \beta t + \gamma(-t^2 + z^2)\right)^2}\,. \tag{3.2.17}$$

Here α , β and γ are defined as linear combinations of m_p $(p = 1, 2, 3)$ as follows:

$$\alpha \equiv \frac{1}{2}(m_1 + m_3)\,, \qquad \beta \equiv -m_2\,, \qquad \gamma \equiv \frac{1}{2}(m_1 - m_3)\,. \tag{3.2.18}$$

When $\eta = 0$, the undeformed metric (3.2.9) is reproduced. Note here that the four constant parameters m_p $(p = 1, 2, 3)$ and c appear in our discussion. Then a constraint (3.2.16) , which comes from the (m)CYBE, is imposed. Hence, three of them are independent each other.

The Ricci scalar curvature of the metric (3.2.17) is

$$R = -2\left(1 - \widetilde{\omega}\eta^2\right) \frac{z^2 + \eta^2 \left(\alpha + \beta t + \gamma(-t^2 + z^2)\right)^2}{z^2 - \eta^2 \left(\alpha + \beta t + \gamma(-t^2 + z^2)\right)^2}\,, \tag{3.2.19}$$

where we have introduced a new quantity,

$$\begin{aligned} \widetilde{\omega} &\equiv \beta^2 + 4\alpha\gamma = m_1^2 + m_2^2 - m_3^2 \\ &= -4c\,. \end{aligned} \tag{3.2.20}$$

At the last equality, the (m)CYBE (3.2.13) has been utilized. The scalar curvature (3.2.19) changes (even its sign) depending on the values of parameters and coordinates, while it becomes a constant -2 in the undeformed limit $\eta \to 0$. The expression (3.2.19) indicates that the deformed geometry contains both AdS and dS in general.

3.3 The deformed JT model

Let us consider deforming the JT model so that the deformed AdS_2 metric (3.2.17) is supported as a solution. For simplicity, the matter fields are turned off hereafter. Along this line, as well as the dilaton itself, the dilaton potential also has to be deformed from a simple linear one (3.1.6) to a hyperbolic function, similarly to integrable deformations.

The deformed metric

Before discussing the dilaton and the dilaton potential, it is helpful to rewrite the deformed metric (3.2.17) as

$$\begin{aligned} \mathrm{d}s^2 &= \frac{-\mathrm{d}t^2+\mathrm{d}z^2}{z^2-\eta^2\left(\alpha+\beta t+\gamma(-t^2+z^2)\right)^2} \\ &= \frac{1}{1-\eta^2(X\cdot P)^2}\frac{-\mathrm{d}t^2+\mathrm{d}z^2}{z^2}\,. \end{aligned} \tag{3.3.1}$$

Here we have introduced new quantities: a coordinate vector X^I and a parameter vector P_I defined as

$$\begin{aligned} X^I &\equiv \frac{1}{z}\left(t\ ,\ \frac{1}{2}(1+t^2-z^2)\ ,\ \frac{1}{2}(1-t^2+z^2)\right)\,, \\ P_I &\equiv (\beta\ ,\ \alpha-\gamma\ ,\ \alpha+\gamma) \qquad (I,J=1,2,3)\,. \end{aligned} \tag{3.3.2}$$

The metric of the embedding space $\mathbb{M}^{2,1}$ is taken as $\eta_{IJ}=\mathrm{diag}(-1,+1,-1)$. The inner products are defined as

$$X\cdot P \equiv X^I P_I = \frac{\alpha+\beta\,t+\gamma\,(-t^2+z^2)}{z}\,, \tag{3.3.3}$$

$$X\cdot X \equiv \eta_{IJ}X^I X^J = -1\,, \qquad P\cdot P \equiv \eta^{IJ}P_I P_J = -\tilde{\omega}\,. \tag{3.3.4}$$

These three products $X\cdot X$, $P\cdot P$ and $X\cdot P$ are transformed as scalars under the $SL(2,\mathbb{R})$ transformation. This $SL(2,\mathbb{R})$ transformation is the usual one generated by three transformations, 1) time translation, 2) dilatation and 3) special conformal transformation. For example, $X\cdot P$ is transformed as $X\cdot P=\tilde{X}\cdot\tilde{P}$, where $\tilde{X}$ and $\tilde{P}$ are new coordinate and parameter vectors, respectively.

It is worth noting that the conformal factor of the metric $\mathrm{e}^{2\omega}$ can be expressed as a Schwarzian derivative of the dilaton

$$\sqrt{\frac{1}{2}\mathrm{Sch}\{\phi,\,\eta(X\cdot P)\}} = \frac{z^2}{1+\eta^2P^2}\,\mathrm{e}^{2\omega}\,. \tag{3.3.5}$$

Here $\mathrm{Sch}\{X,x\}$ denotes the Schwarzian derivative defined as

$$\mathrm{Sch}\{X,x\} \equiv \frac{X'''}{X'} - \frac{3}{2}\left(\frac{X''}{X'}\right)^2 . \tag{3.3.6}$$

Using the $SL(2,\mathbb{R})$ transformation, we can choose the vector $\tilde{P}$ freely as long as it satisfies the relation $\tilde{P}\cdot\tilde{P} = P\cdot P = -\tilde{\omega}$. Note that only the warped factor of the metric changes like

$$\mathrm{d}s^2 = \frac{1}{1-\eta^2(\tilde{X}\cdot\tilde{P})^2}\frac{-\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{z}^2}{\tilde{z}^2} \tag{3.3.7}$$

because the rigid AdS_2 part is invariant under the $SL(2,\mathbb{R})$ transformation.

The dilaton sector

Given the deformed metric (3.2.17) [or equivalently (3.3.1)] , by solving the equations of motion (3.1.4) without the matter fields, the dilaton ϕ is determined as

$$\begin{aligned} \phi &= \frac{c_1}{2\eta}\log\left|\frac{z+\eta\,(\alpha+\beta t+\gamma(-t^2+z^2))}{z-\eta\,(\alpha+\beta t+\gamma(-t^2+z^2))}\right| + c_2 \\ &\equiv \frac{c_1}{2\eta}\log\left|\frac{1+\eta(X\cdot P)}{1-\eta(X\cdot P)}\right| + c_2 \,, \end{aligned} \tag{3.3.8}$$

when the dilaton potential is deformed as[2]

$$U(\phi) = \begin{cases} -\,(1-\tilde{\omega}\,\eta^2)\dfrac{c_1}{\eta}\sinh\left[\dfrac{2\eta}{c_1}(\phi-c_2)\right] & (\text{for} \;\; 1 > |\eta(X\cdot P)|) \\ +\,(1-\tilde{\omega}\,\eta^2)\dfrac{c_1}{\eta}\sinh\left[\dfrac{2\eta}{c_1}(\phi-c_2)\right] & (\text{for} \;\; 1 < |\eta(X\cdot P)|) \end{cases} . \tag{3.3.9}$$

Here c_1 and c_2 are arbitrary constants

In the undeformed limit $\eta \to 0$, the dilaton (3.3.8) is reduced to

$$\phi = c_2 + c_1\,\frac{\alpha+\beta\,t+\gamma(-t^2+z^2)}{z} \,,$$

and thus the dilaton (3.1.8) in the JT model has been reproduced when $c_1 = 1$ and $c_2 = 1$. Remarkably, the three parameters α , β and γ correspond to a , b and c in (3.1.8) , respectively. Similarly, as $\eta \to 0$, the upper branch of the potential (3.3.9) reduces to

$$U(\phi) = 2(c_2 - \phi) \,,$$

[2]According to an interesting paper [68], this dilaton potential leads to a q-deformation of $\mathfrak{sl}(2)$. This result should be closely related to the Yang-Baxter deformation, e.g., [67].

while the lower branch vanishes. Thus the dilaton potential of the JT model is reproduced when $c_2 = 1$. In total, the case with $c_1 = c_2 = 1$ is associated with the JT model and hence we will work with $c_1 = c_2 = 1$ hereafter.

The vacuum solution in the deformed JT model

In summary, the deformed JT model is specified by the deformed dilaton potential,

$$U(\phi) = \begin{cases} -(1-\tilde{\omega}\,\eta^2)\dfrac{1}{\eta}\sinh\left[2\eta(\phi-1)\right] & (\text{for} \ \ 1 > |\eta(X\cdot P)|) \\ \\ +(1-\tilde{\omega}\,\eta^2)\dfrac{1}{\eta}\sinh\left[2\eta(\phi-1)\right] & (\text{for} \ \ 1 < |\eta(X\cdot P)|) \end{cases},$$

and the vacuum solution is given by

$$\mathrm{d}s^2 = \frac{1}{1-\eta^2(X\cdot P)^2}\frac{-\mathrm{d}t^2+\mathrm{d}z^2}{z^2}\,, \qquad \phi = \frac{1}{2\eta}\log\left|\frac{1+\eta(X\cdot P)}{1-\eta(X\cdot P)}\right| + 1\,, \tag{3.3.10}$$

where

$$X\cdot P = \frac{\alpha+\beta\,t+\gamma\,(-t^2+z^2)}{z}\,.$$

A deformed black hole solution

In this subsection, we study a deformed black hole solution contained as a special case of the general vacuum solution (obtained in the previous subsection). This solution can be regarded as a deformation of the black hole solution presented in [24].

In the following, instead of $\tilde{\omega}$, we use a new parameter μ defined as

$$\mu \equiv -\tilde{P}\cdot\tilde{P} = \tilde{\omega} = -4c\,,$$

By performing the same coordinate transformation as in the undeformed case like

$$x^{\pm} = \frac{1}{\sqrt{\mu}}\tanh(\sqrt{\mu}\,(T\pm Z))\,, \tag{3.3.11}$$

the deformed black hole solution is obtained as

$$\begin{aligned} \mathrm{d}s^2 &= \frac{4\mu}{-\eta^2\mu+(1-\eta^2\mu)\sinh^2(2\sqrt{\mu}\,Z)}\left(-\mathrm{d}T^2+\mathrm{d}Z^2\right)\,, \\ \phi &= 1+\frac{1}{2\eta}\log\left|\frac{1+\eta\sqrt{\mu}\,\coth(\sqrt{\mu}\,Z)}{1-\eta\sqrt{\mu}\,\coth(\sqrt{\mu}\,Z)}\right|\,. \end{aligned} \tag{3.3.12}$$

In this coordinate, the Ricci scalar (3.2.19) is rewritten as

$$R = -(1-\eta^2\mu)\frac{1-\eta^2\mu-(1+\eta^2\mu)\cosh(4\sqrt{\mu}Z)}{\eta^2\mu-(1-\eta^2\mu)\sinh^2(2\sqrt{\mu}Z)}\,. \tag{3.3.13}$$

In the following, we impose that

$$\eta^2 < \frac{1}{\mu} \tag{3.3.14}$$

so as to ensure the existence of the undeformed limit. Otherwise, it is not possible to take the undeformed limit $\eta \to 0$ because $\eta^2 > 1/\mu$. Note here that this background has a naked singularity at $Z = Z_0$, where

$$Z_0 \equiv \frac{1}{2\sqrt{\mu}}\,\mathrm{arctanh}(\eta\sqrt{\mu})\,. \tag{3.3.15}$$

This is a peculiar feature of the Yang-Baxter deformed geometry based on the modified CYBE like the η-deformation of AdS_5 [71]. From (3.2.19), in the region with $Z > Z_0$ the Ricci scalar takes negative values, while for $0 < Z < Z_0$, it has positive values. In the undeformed limit $\eta \to 0$, Z_0 is sent to zero and the singularity disappears because the undeformed spacetime is just AdS_2. In the following discussion, we focus upon the negative-curvature region $(Z > Z_0)$. Therefore, we are concerned with only the upper branch of the potential (3.3.9).

By performing the following coordinate transformation,

$$r = \frac{1}{\eta}\mathrm{arctanh}\big(\eta\sqrt{\mu}\,\coth(2\sqrt{\mu}\,Z)\big)\,, \tag{3.3.16}$$

the metric takes a Schwarzschild-like form

$$\mathrm{d}s^2 = -4F(r)\,\mathrm{d}T^2 + \frac{\mathrm{d}r^2}{F(r)}\,, \tag{3.3.17}$$

The scalar function $F(r)$ is defined as

$$F(r) \equiv \frac{-1-\eta^2\mu+(1-\eta^2\mu)\cosh(2\,\eta\,r)}{2\eta^2}\,. \tag{3.3.18}$$

In this coordinate system, the dilaton takes the simplest form,

$$\phi = 1 + r\,. \tag{3.3.19}$$

The locations of the boundary and black hole horizon are

$$\text{boundary}:\quad r=\infty\,,\qquad \text{BH horizon}:\quad r = r^* \equiv \frac{1}{\eta}\mathrm{arctanh}(\eta\sqrt{\mu})\,. \tag{3.3.20}$$

Bekenstein-Hawking entropy

Let us compute the Bekenstein-Hawking entropy of the deformed black hole by utilizing the coordinate system (3.3.17) .

The Hawking temperature T_{H} is given by the standard formula:

$$T_{\mathrm{H}} = \frac{1}{4\pi}\partial_r\sqrt{-\frac{g_{tt}}{g_{rr}}}\bigg|_{r=r^*} = \frac{\sqrt{\mu}}{\pi}\,. \tag{3.3.21}$$

This is the same result as the undeformed case.

Here it may be worth noting that the black hole temperature is related to the modification of the CYBE. The zero temperature case corresponds to the homogeneous CYBE and the temperature is measured by negative values of c. Solutions of the mCYBE with negative (positive) c are called the split (non-split) type. The well-known example of the non-split type is the q-deformation of AdS_5 [69], while the split type has gotten little attention. For the recent progress on the split type, see [70]. It may be interesting to seek some connection between black hole geometries and solutions of split type.

By assuming that the horizon area A is 1 and using the effecting Newton constant G_{eff} in (3.6.17) , the Bekenstein-Hawking entropy S_{BH} can be computed as

$$S_{\mathrm{BH}} = \frac{A}{4G_{\mathrm{eff}}}\bigg|_{r=r^*} = \frac{\mathrm{arctanh}(\pi\, T_{\mathrm{H}}\,\eta)}{4G\eta} + \frac{1}{4G}\,. \tag{3.3.22}$$

In the undeformed limit $\eta \to 0$, the entropy is reduced to

$$S_{\mathrm{BH}}^{(\eta=0)} = \frac{\pi T_{\mathrm{H}}}{4G} + \frac{1}{4G}\,,$$

and thus the result of JT model has been reproduced.

3.3.1 The boundary computation of entropy

In this subsection, we compute the entropy of the deformed black hole by evaluating the renormalized boundary stress tensor. Now that the boundary structure is drastically changed, the first thing is to determine the location of the holographic screen. In the following, we take the screen on the singularity by following the proposal of [72]. More precisely, by introducing a UV cut-off ϵ, the boundary is taken just before the singularity $(Z = Z_0 + \epsilon)$.

In the conformal gauge, the total action including the Gibbons-Hawking term can be rewritten as

$$\begin{aligned} S_{g,\phi} &= \frac{1}{16\pi G}\int \mathrm{d}^2x\,\sqrt{-g}\,[\phi R - U(\phi)] + \frac{1}{8\pi G}\int \mathrm{d}t\,\sqrt{-\gamma}\,\phi K \\ &= \frac{1}{8\pi G}\int \mathrm{d}^2x \left[-4\partial_{(+}\phi\partial_{-)}\omega - \frac{1}{2}U(\phi)\,\mathrm{e}^{2\omega}\right]. \end{aligned} \tag{3.3.23}$$

K is the extrinsic curvature and γ is the extrinsic metric. By using the explicit expression of the deformed black hole solution in (3.3.12), the on-shell bulk action can be evaluated on the boundary,

$$S_{g,\phi} = \int \mathrm{d}t\ \left.\frac{-\mu}{2\pi G(1+\eta^2\mu + (-1+\eta^2\mu)\cosh(4\mu^{\frac{1}{2}}Z))}\right|_{Z\to Z_0}. \tag{3.3.24}$$

Recall that the regulator ϵ is introduced such that $Z - Z_0 = \epsilon$, the on-shell action can be expanded as

$$S_{g,\phi} = \int \mathrm{d}t\,\left[\frac{1}{16\pi G\,\eta\,\epsilon} - \frac{1+\eta^2\mu}{16\pi G\,\eta^2}\,\epsilon^0 + \frac{(3+\eta^2\mu(-2+3\eta^2\mu))\,\epsilon}{48\pi G\,\eta^3} + O(\epsilon^2)\right]. \tag{3.3.25}$$

To cancel the divergence that occurs as the bulk action approaches the boundary, it is appropriate to add the following counter term:

$$S_{\rm ct} = -\frac{1}{8\pi G}\int \mathrm{d}t\,\sqrt{-\gamma_{tt}}\,\sqrt{F(\phi-1) - \frac{1}{\eta^2}\log(1-\eta^2\mu)}\,. \tag{3.3.26}$$

Here the scalar function F is already given in (3.3.18) and hence

$$F(\phi-1) = \frac{-1-\eta^2\mu + (1-\eta^2\mu)\cosh(2\eta(\phi-1))}{2\eta^2} \tag{3.3.27}$$

Note that the inside of the root of (3.3.26) is positive due to the condition (3.3.14). The dual-theory interpretation of it is not so clear because it cotains an infinite number of polynomials and also depends on the temperature explicitly. Another counter term may be allowed and it would be nice to seek for it by following the procedure in [73].

The extrinsic metric γ_{tt} on the boundary is obtained as

$$\gamma_{tt} = -\mathrm{e}^{2\omega}\Big|_{Z\to Z_0}.$$

In the undeformed limit $\eta \to 0$, this counter term is reduced to

$$S^{(\eta=0)}_{\rm ct} = \frac{1}{8\pi G}\int \mathrm{d}t\,\sqrt{-\gamma_{tt}}\,(1-\phi)\,, \tag{3.3.28}$$

because $\phi - 1 > 0$. This is nothing but the counter term utilized in the JT model [24].

It is straightforward to check that the sum $S = S_{g,\phi} + S_{\rm ct}$ becomes finite on the boundary by using the expanded form of the counter term (3.5.39):

$$S_{\rm ct} = \int {\rm d}t \left[\frac{-1}{16\pi G\eta\,\epsilon} + \frac{1+\eta^2\mu + 2\log(1-\eta^2\mu)}{16\pi G\,\eta^2}\,\epsilon^0 + O(\epsilon) \right] . \tag{3.3.29}$$

Around the boundary, the warped factor of the metric in (3.3.12) can be expanded as

$${\rm e}^{2\omega} = \frac{1}{\eta\,\epsilon} - \left[\frac{1}{\eta^2} + \mu \right] \epsilon^0 + O(\epsilon) \,. \tag{3.3.30}$$

Hence, by normalizing the boundary metric as

$$\hat{\gamma}_{tt} = \eta\,\epsilon\,\gamma_{tt}\,,$$

the boundary stress tensor can be defined as

$$\langle \hat{T}_{tt} \rangle \equiv \frac{-2}{\sqrt{-\hat{\gamma}_{tt}}}\frac{\delta S}{\delta \hat{\gamma}^{tt}} = \lim_{\epsilon\to 0} \sqrt{\eta\,\epsilon}\,\frac{-2}{\sqrt{-\gamma_{tt}}}\frac{\delta S}{\delta \gamma^{tt}} \,. \tag{3.3.31}$$

After all, $\langle \hat{T}_{tt} \rangle$ has been evaluated as

$$\langle \hat{T}_{tt} \rangle = -\frac{\log(1-\eta^2\mu)}{8\pi G\,\eta^2} \,. \tag{3.3.32}$$

To compute the associated entropy, $\langle \hat{T}_{tt} \rangle$ should be identified with energy E like

$$E = -\frac{\log(1-\pi^2 T_{\rm H}^2\eta^2)}{8\pi G\,\eta^2} \,, \tag{3.3.33}$$

where we have used the expression of the Hawking temperature (3.6.16). Then by solving the thermodynamic relation (3.1.17) again, the entropy is obtained as

$$S = \frac{{\rm arctanh}(\pi T_{\rm H}\eta)}{4G\eta} + S_{T_{\rm H}=0} \,. \tag{3.3.34}$$

Here $S_{T_{\rm H}=0}$ has appeared as an integration constant that measures the entropy at zero temperature. Thus the resulting entropy precisely agrees with the Bekenstein-Hawking entropy (3.6.18), up to the temperature-independent constant.

Finally, it should be remarked that this agreement is quite non-trivial because the deformation changes the UV region of the geometry drastically. Hence the location of the holographic screen and the choice of the counter term are far from trivial. Although the holographic screen was supposed to be the singularity, inversely speaking, this agreement of the entropies here supports that the proposal in [72] would work well. As for the geometrical meaning of the counter term (3.5.39), we have no definite idea. It is significant to figure out a systematic prescription to produce the counter term (3.5.39) .

3.4 General vacuum solutions of the deformed JT model

In section 3.3, we discussed the YB-deformed AdS_2 and the dilaton (3.3.10) is one of the solutions of the deformed JT model. In this section, we will provide general vacuum solution. We will introduce a couple of the linear combination of the conformal factor ω and the dilaton ϕ. In terms of there new variables, the eom are rewritten as two Liouville equations with constraint conditions. To solve there equations, we can provide general solutions of the deformed JT model.

3.4.1 A symmetry of the vacuum solution

Let us first rewrite the metric into the following form:

$$\mathrm{d}s^2 = \mathrm{e}^{2\omega}\tilde{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{3.4.1}$$

Then the classical action can be rewritten as

$$S_\phi = \frac{1}{16\pi G}\int \mathrm{d}^2x\sqrt{-\tilde{g}}\left[\phi\tilde{R} + 2\tilde{\nabla}\phi\tilde{\nabla}\omega + \frac{\mathrm{e}^{2\omega}}{\eta}\sinh\left(2\eta(\phi-1)\right)\right] . \tag{3.4.2}$$

In order to simplify this expression, it is helpful to introduce a couple of new valuables:

$$\omega_1 \equiv \omega + \eta(\phi-1)\,, \qquad \omega_2 \equiv \omega - \eta(\phi-1)\,. \tag{3.4.3}$$

Then the action (3.4.2) becomes the sum of two Liouville systems:

$$\begin{aligned} S_\phi &= \frac{1}{32\pi G\eta}\int \mathrm{d}^2x\sqrt{-\tilde{g}}\left[\left(\omega_1\tilde{R} + (\tilde{\nabla}\omega_1)^2 + \mathrm{e}^{2\omega_1}\right) - \left(\omega_2\tilde{R} + (\tilde{\nabla}\omega_2)^2 + \mathrm{e}^{2\omega_2}\right)\right] \\ &\equiv S_{\omega_1} + S_{\omega_2}\,. \end{aligned} \tag{3.4.4}$$

By taking variations of the action (3.4.4) with respect to ω_1 and ω_2 , it is easy to derive the following equations of motion:

$$\begin{aligned}
\tilde{R} - 2(\tilde{\nabla}\omega_1)^2 + 2\mathrm{e}^{2\omega_1} &= 0\,, \\
\tilde{R} - 2(\tilde{\nabla}\omega_2)^2 + 2\mathrm{e}^{2\omega_2} &= 0\,.
\end{aligned} \tag{3.4.5}$$

Taking a variation with $\tilde{g}_{\mu\nu}$ gives rise to the constraints

$$\tilde{T}^{(1)}_{\mu\nu} + \tilde{T}^{(2)}_{\mu\nu} = 0\,, \tag{3.4.6}$$

where $\tilde{T}^{(1)}_{\mu\nu}$ and $\tilde{T}^{(2)}_{\mu\nu}$ are the energy-momentum tensors defined as, respectively,

$$\tilde{T}^{(1)}_{\mu\nu} \equiv \frac{-2}{\sqrt{-\tilde{g}}}\frac{\delta S_{\omega_1}}{\delta \tilde{g}^{\mu\nu}}\,, \qquad \tilde{T}^{(2)}_{\mu\nu} \equiv \frac{-2}{\sqrt{-\tilde{g}}}\frac{\delta S_{\omega_2}}{\delta \tilde{g}^{\mu\nu}}\,, \tag{3.4.7}$$

and the explicit forms are given by

$$\begin{aligned}
\tilde{T}^{(1)}_{\mu\nu} &= \frac{1}{16\pi G\eta}\left[\tilde{\nabla}_\mu\tilde{\nabla}_\nu\omega_1 - (\tilde{\nabla}_\mu\omega_1)(\tilde{\nabla}_\nu\omega_1) - \frac{1}{2}\tilde{g}_{\mu\nu}(2\tilde{\nabla}^2\omega_1 - (\tilde{\nabla}\omega_1)^2 - \mathrm{e}^{2\omega_1})\right]\,, \\
\tilde{T}^{(2)}_{\mu\nu} &= \frac{-1}{16\pi G\eta}\left[\tilde{\nabla}_\mu\tilde{\nabla}_\nu\omega_2 - (\tilde{\nabla}_\mu\omega_2)(\tilde{\nabla}_\nu\omega_2) - \frac{1}{2}\tilde{g}_{\mu\nu}(2\tilde{\nabla}^2\omega_2 - (\tilde{\nabla}\omega_2)^2 - \mathrm{e}^{2\omega_2})\right]\,.
\end{aligned} \tag{3.4.8}$$

Thus, by employing the new variables ω_1 and ω_2 , the deformed system (4.1.1) has been simplified drastically.

Then the equations of motion obtained from (3.4.2) are given by

$$4\partial_+\partial_-\phi + \frac{1}{\eta}\mathrm{e}^{2\omega}\sinh\left(2\eta(\phi-1)\right) = 0\,, \tag{3.4.9}$$

$$4\partial_+\partial_-\omega + \mathrm{e}^{2\omega}\cosh\left(2\eta(\phi-1)\right) = 0\,, \tag{3.4.10}$$

$$-\mathrm{e}^{2\omega}\partial_+(\mathrm{e}^{-2\omega}\partial_+\phi) = 0\,, \tag{3.4.11}$$

$$-\mathrm{e}^{2\omega}\partial_-(\mathrm{e}^{-2\omega}\partial_-\phi) = 0\,. \tag{3.4.12}$$

By solving the above equations, the general vacuum solution has been discussed in [20]. However, as we will show below, the deformed model (3.4.2) has a nice property, with which we can discuss classical solutions in a more systematic way.

In conformal gauge, the classical action for ω_1 and ω_2 is further simplified as

$$S_\phi = \frac{1}{8\pi G\eta}\int \mathrm{d}^2x\left[-\left(\partial_+\omega_1\,\partial_-\omega_1 - \frac{1}{4}\mathrm{e}^{2\omega_1}\right) + \left(\partial_+\omega_2\,\partial_-\omega_2 - \frac{1}{4}\mathrm{e}^{2\omega_2}\right)\right]\,. \tag{3.4.13}$$

The classical equations of motion take the standard forms of the Liouville equation

$$4\partial_+\partial_-\omega_1 + \mathrm{e}^{2\omega_1} = 0\,, \qquad 4\partial_+\partial_-\omega_2 + \mathrm{e}^{2\omega_2} = 0\,. \tag{3.4.14}$$

The general solutions of Liouville equation are given by

$$\mathrm{e}^{2\omega_1} = \frac{4\partial_+ X_1^+ \partial_- X_1^-}{\left(X_1^+ - X_1^-\right)^2}\,, \qquad \mathrm{e}^{2\omega_2} = \frac{4\partial_+ X_2^+ \partial_- X_2^-}{\left(X_2^+ - X_2^-\right)^2}\,, \tag{3.4.15}$$

where $X_i^+ = X_i^+(x^+)$ and $X_i^- = X_i^-(x^-)$ are arbitrary holomorphic and anti-holomorphic functions, respectively.

Note that the equations (3.4.14) can be expressed by using the metric and dilaton.

$$\begin{aligned} &4\partial_+\partial_-(\omega + \eta(\phi - 1)) + \mathrm{e}^{2\omega + 2\eta(\phi-1)} = 0\,, \\ &4\partial_+\partial_-(\omega - \eta(\phi - 1)) + \mathrm{e}^{2\omega - 2\eta(\phi-1)} = 0\,. \end{aligned} \tag{3.4.16}$$

By summing and subtracting them each other, the equations of motion (3.4.9) and (3.4.10) can be reproduced.

By taking $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ in (3.4.8), the energy-momentum tensors are also rewritten as

$$\tilde{T}^{(1)}_{+-} + \tilde{T}^{(2)}_{+-} = \frac{-1}{16\pi G\eta}(\partial_+\partial_-\omega_1 + \frac{1}{4}\mathrm{e}^{2\omega_1} - \partial_+\partial_-\omega_2 - \frac{1}{4}\mathrm{e}^{2\omega_2})\,, \tag{3.4.17}$$

$$\tilde{T}^{(1)}_{\pm\pm} + \tilde{T}^{(2)}_{\pm\pm} = \frac{1}{16\pi G\eta}(\partial_\pm\partial_\pm\omega_1 - \partial_\pm\omega_1\partial_\pm\omega_1 - \partial_\pm\partial_\pm\omega_2 + \partial_\pm\omega_1\partial_\pm\omega_1)\,. \tag{3.4.18}$$

The first condition (3.4.17) vanishes automatically due to the equations of motion. Two conditions in (3.4.18) give rise to nontrivial constraints for the solutions of the equations of motion (3.4.14). By using the definitions of ω_1 and ω_2 in (3.4.3), it is easy to directly see that the constraints in (3.4.17) and (3.4.18) are equivalent to the ones in (3.4.11) and (3.4.12).

By using the general solutions (3.4.15), the constraint conditions for the holomorphic (antiholomorphic) functions X_i^+ (X_i^-) can be rewritten as

$$\begin{aligned} &\mathrm{Sch}\{X_1^+, x^+\} - \mathrm{Sch}\{X_2^+, x^+\} = 0\,, \\ &\mathrm{Sch}\{X_1^-, x^-\} - \mathrm{Sch}\{X_2^-, x^-\} = 0\,. \end{aligned} \tag{3.4.19}$$

These constraints mean that the holomorphic (antiholomorphic) functions should be the same functions, up to linear fractional transformations

$$
\begin{aligned}
X_2^+(x^+) &= \frac{aX_1^+ + b}{cX_1^+ + d} \qquad (a,b,c,d \in \mathbb{R})\,, \\
X_2^-(x^-) &= \frac{a'X_1^- + b'}{c'X_1^- + d'} \qquad (a',b',c',d' \in \mathbb{R})\,.
\end{aligned} \tag{3.4.20}
$$

Because $\mathrm{e}^{2\omega_1} > 0$ and $\mathrm{e}^{2\omega_2} > 0$, determinants of the transformations must be positive:

$$
ad - bc > 0\,, \qquad a'd' - b'c' > 0\,. \tag{3.4.21}
$$

This ambiguity comes from the appearance of Schwarzian derivatives.

3.4.2 General vacuum solutions

In this subsection, let us revisit the vacuum solutions by employing a couple of the new variables (3.4.3) . Before going to the detail, it is helpful to recall that the original metric and dilaton can be reconstructed from ω_1 and ω_2 through the following relations:

$$
\mathrm{e}^{2\omega} = \sqrt{\mathrm{e}^{2\omega_1}\mathrm{e}^{2\omega_2}}\,, \qquad \phi = 1 + \frac{1}{2\eta}(\omega_1 - \omega_2) = 1 + \frac{1}{4\eta}\log\left(\frac{\mathrm{e}^{2\omega_1}}{\mathrm{e}^{2\omega_2}}\right)\,. \tag{3.4.22}
$$

Here let us take a parametrization for the linear fractional transformations, which come from (3.4.19) as follows:[3]

$$
\begin{aligned}
X_1^+(x^+) &= \frac{(1-\eta\beta)X^+(x^+) - 2\eta\alpha}{-2\eta\gamma X^+(x^+) + (1+\eta\beta)}\,, \qquad X_1^-(x^-) = X^-(x^-)\,, \\
X_2^+(x^+) &= \frac{(1+\eta\beta)X^+(x^+) + 2\eta\alpha}{2\eta\gamma X^+(x^+) + (1-\eta\beta)}\,, \qquad X_2^-(x^-) = X^-(x^-)\,.
\end{aligned} \tag{3.4.23}
$$

Because of the constraint (3.4.21), we have to work in a restricted parameter region with

$$
1 - \eta^2(\beta^2 + 4\alpha\gamma) > 0\,. \tag{3.4.24}
$$

Then the solutions in (3.4.15) are expressed as

$$
\begin{aligned}
\mathrm{e}^{2\omega_1} &= \frac{4\,(1-\eta^2(\beta^2+4\alpha\gamma))\,\partial_+X^+\partial_-X^-}{(X^+ - X^- - \eta(2\alpha + \beta(X^+ + X^-) - 2\gamma X^+X^-))^2}\,, \\
\mathrm{e}^{2\omega_2} &= \frac{4\,(1-\eta^2(\beta^2+4\alpha\gamma))\,\partial_+X^+\partial_-X^-}{(X^+ - X^- + \eta(2\alpha + \beta(X^+ + X^-) - 2\gamma X^+X^-))^2}\,.
\end{aligned} \tag{3.4.25}
$$

[3]Note that we can take this parametrization without loss of generality.

Thus the general solution of ω and ϕ are also determined through the relation (3.4.22). Given that $X^{\pm}(x^{\pm}) = x^{\pm}$, the deformed metric and dilaton become

$$
\begin{aligned}
\mathrm{e}^{2\omega} &= \frac{1-\eta^2(\beta^2+4\alpha\gamma)}{z^2-\eta^2\left(\alpha+\beta t+\gamma(-t^2+z^2)\right)^2}\,, \\
\phi &= 1+\frac{1}{2\eta}\log\left|\frac{z+\eta(\alpha+\beta t+\gamma(-t^2+z^2))}{z-\eta(\alpha+\beta t+\gamma(-t^2+z^2))}\right|\,.
\end{aligned} \tag{3.4.26}
$$

Here the condition (3.4.24) is consistent with the positivity of $\mathrm{e}^{2\omega_1}$ and $\mathrm{e}^{2\omega_2}$. This metric is the same as the result obtained in [20] as a Yang-Baxter deformation of AdS_2, up to a scaling factor.

For concreteness, let consider a simple case of (3.4.23) with $\alpha = 1,\ \beta = \gamma = 0$. Then conformal factors of the metrics for X_1 and X_2 are given by, respectively,

$$
\mathrm{e}^{2\omega_1} = \frac{1}{(z-\eta)^2}\,, \qquad \mathrm{e}^{2\omega_2} = \frac{1}{(z+\eta)^2}\,. \tag{3.4.27}
$$

For each of the AdS_2 factors, the origin of the z-direction is shifted by $\pm\eta$. Another example is the case with $\alpha = 1/2,\ \beta = 0,\ \gamma = \mu/2$ (where μ is a positive), in which we have considered a deformed black hole solution [20]

3.5 Solutions with matter fields

In this section, we shall include additional matter fields. Then the action is given by the sum of the dilaton part S_ϕ and the matter part S_{matter} like

$$
S = S_\phi + S_{\mathrm{matter}}\,. \tag{3.5.1}
$$

Note here that we have not specified the concrete expression of the matter action S_{matter} yet. In general, S_{matter} may depend on the metric, dilaton as well as additional matter fields. Hence the inclusion of matter fields leads to the modified equations:

$$
\begin{aligned}
4\partial_+\partial_-\phi + \frac{1}{\eta}\mathrm{e}^{2\omega}\sinh\left(2\eta(\phi-1)\right) &= 32\pi G\, T_{+-}\,, \\
4\partial_+\partial_-\omega + \mathrm{e}^{2\omega}\cosh\left(2\eta(\phi-1)\right) &= -16\pi G\,\frac{\delta S_{\mathrm{matter}}}{\delta\phi}\,, \\
-\mathrm{e}^{2\omega}\partial_+(\mathrm{e}^{-2\omega}\partial_+\phi) &= 8\pi G\, T_{++}\,, \\
-\mathrm{e}^{2\omega}\partial_-(\mathrm{e}^{-2\omega}\partial_-\phi) &= 8\pi G\, T_{--}\,.
\end{aligned} \tag{3.5.2}
$$

Here the energy-momentum tensor $T_{\mu\nu}$ defined as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}\,. \tag{3.5.3}$$

Furthermore, one needs to take account of the equation of motion for the matter fields, which is provided as the conservation law of the energy-momentum tensor $T_{\mu\nu}$. So far, it seems difficult to treat the general expression of $T_{\mu\nu}$. Hence we will impose some conditions for $T_{\mu\nu}$ hereafter.

3.5.1 A certain class of matter fields

For simplicity, let us consider a certain class of matter fields by supposing the following properties:

$$T_{+-} = 0\,, \qquad \frac{\delta S_{\rm matter}}{\delta \phi} = 0\,. \tag{3.5.4}$$

This case is very special because the equations of motion for ω_1 and ω_2 remain to be a pair of Liouville equations because the right-hand sides of the first and second equations in (3.5.2) vanish. Hence one can still use the general solutions (3.4.15). The constraints are also still written in terms of Schwarzian derivatives, but slightly modified like

$$\mathrm{Sch}\{X_1^{\pm}, x^{\pm}\} - \mathrm{Sch}\{X_2^{\pm}, x^{\pm}\} = -32\pi G\,\eta\,T_{\pm\pm}\,. \tag{3.5.5}$$

That is, the right-hand side does not vanish.

To solve the set of equations, it is helpful to introduce new functions $\varphi_{\pm} = \varphi_{\pm}(x^{\pm})$ defined as

$$\varphi_{\pm} \equiv \frac{1}{\sqrt{|\partial_{\pm}X_2^{\pm}|}}\,. \tag{3.5.6}$$

Note here that $X_2^{\pm}$ only have been utilized. Then by using $\varphi_{\pm}$, the Schwarzian derivatives can be rewritten as

$$\mathrm{Sch}\{X_2^{\pm}, x^{\pm}\} = -2\frac{\partial_{\pm}^2\varphi_{\pm}}{\varphi_{\pm}}\,. \tag{3.5.7}$$

When the coordinates are taken as

$$X_1^{\pm}(x^{\pm}) = x^{\pm}\,, \tag{3.5.8}$$

the constraints become Schrödinger equations as follows:

$$\left(-\partial_\pm^2 - 16\pi G\eta T_{\pm\pm}\right)\varphi_\pm(x^\pm) = 0\,. \tag{3.5.9}$$

Thus, for the simple class of matter fields, the constraints have been drastically simplified.

3.5.2 A solution describing formation of a black hole

As an example in the simple class, let us consider an ingoing matter pulse of energy $E/(8\pi G)$:

$$T_{--} = \frac{E}{8\pi G}\,\delta(x^-)\,, \qquad T_{++} = T_{+-} = 0\,. \tag{3.5.10}$$

Note here that $T_{\mu\nu}$ does not depend on the dilaton ϕ and hence this case belongs to the simple class (3.5.4). This pulse causes a shock-wave traveling on the null curve $x^- = 0$.

Then the constraint for the anti-holomorphic part is written as

$$\left(-\partial_-^2 - 2E\eta\delta(x^-)\right)\varphi_-(x^-) = 0\,. \tag{3.5.11}$$

By solving this equation, we obtain the following solution:

$$\varphi_-(x^-) = \varphi_-(0)\left(1 - 2E\eta x^-\theta(x^-)\right)\,. \tag{3.5.12}$$

Assuming the continuity, X_2^- is given by

$$X_2^-(x^-) = \begin{cases} (1-4\eta^2 Ea)x^- - 2\eta a & (\text{for } \ x^- < 0) \\ \frac{x^- - 2\eta a}{1-2E\eta x^-} & (\text{for } \ x^- > 0) \end{cases}\,. \tag{3.5.13}$$

Here a is an arbitrary integral constant and the scaling factor $\varphi_-(0)$ is fixed as

$$\varphi_-(0)^2 = \frac{1}{1-4\eta^2 Ea}\,. \tag{3.5.14}$$

Note here that for simplicity, we dropped and tuned some integration constants in (3.5.12) and (3.5.13).

The remaining task is to determine $X_2^+(x^+)$. The constraint for $\varphi_+(x^+)$ is given by

$$-\partial_+^2\varphi_+(x^+) = 0\,. \tag{3.5.15}$$

Thus one can determine $\varphi_+(x^+)$ and $\partial_+ X_2^+(x^+)$ as

$$\varphi_+(x^+) = \gamma x^+ + \delta\,, \qquad \partial_+ X_2^+(x^+) = \frac{1}{(\gamma x^+ + \delta)^2}\,,$$

where γ and δ are constants. Hence X_2^+ is obtained as

$$X_2^+(x^+) = \frac{\alpha x^+ + \beta}{\gamma x^+ + \delta} \qquad (\alpha\delta - \beta\gamma = 1) \tag{3.5.16}$$

with new constants α and β. For simplicity, we will set $\alpha = \delta = 1,\ \beta = \gamma = 0$. That is, $X_2^+ = x^+$.

Thus one can obtain a solution of the two Liouville equations as follows:

$$\begin{aligned} \mathrm{e}^{2\omega_1} &= \frac{4}{(x^+ - x^-)^2}\,, \\ \mathrm{e}^{2\omega_2} &= \begin{cases} \frac{4(1-4\eta^2 Ea)}{(x^+ - x^- + 2\eta a(1+2\eta E x^-))^2} & (\text{for} \quad x^- < 0) \\ \frac{4(1-4\eta^2 Ea)}{(x^+ - x^- + 2\eta(a - E x^+ x^-))^2} & (\text{for} \quad x^- > 0) \end{cases}\,. \end{aligned} \tag{3.5.17}$$

As a result, the original metric and dilaton are given by

$$\begin{aligned} \mathrm{e}^{2\omega} &= \begin{cases} \frac{4\sqrt{1-4\eta^2 Ea}}{(x^+-x^-)(x^+ - x^- + 2\eta a(1+2\eta E x^-)} & (\text{for} \quad x^- < 0) \\ \frac{4\sqrt{1-4\eta^2 Ea}}{(x^+-x^-)(x^+ - x^- + 2\eta(a - E x^+ x^-))} & (\text{for} \quad x^- > 0) \end{cases}\,, \\ \phi &= \begin{cases} 1 + \frac{1}{2\eta}\log\left|1 + \frac{2\eta a(1+2E\eta x^-)}{x^+-x^-}\right| - \frac{1}{2\eta}\log(1-4\eta^2 Ea) & (\text{for} \quad x^- < 0) \\ 1 + \frac{1}{2\eta}\log\left|1 + \frac{2\eta(a - E x^+ x^-)}{x^+-x^-}\right| - \frac{1}{2\eta}\log(1-4\eta^2 Ea) & (\text{for} \quad x^- > 0) \end{cases}\,. \end{aligned} \tag{3.5.18}$$

The undeformed limit $\eta \to 0$ leads to a solution describing formation of a black hole in the undeformed model [24]. Note here that the energy-dependent constant in ϕ vanishes in the undeformed limit. At least so far, we have no idea for the physical interpretation of this constant.

3.5.3 The deformed system with a conformal matter

In this section, we will consider conformal matters, which do not belong to the previous class (3.5.4), and discuss the effect of them to thermodynamic quantities associated with a black hole solution.

Let us study a conformal matter whose dynamics is governed by the classical action:

$$S_{\text{matter}} = -\frac{N}{24\pi}\int \mathrm{d}^2x\,\sqrt{-g}\left[\chi(R - 2\eta\nabla^2\phi) + (\nabla\chi)^2\right] - \frac{N}{12\pi}\int \mathrm{d}t\,\sqrt{-\gamma_{tt}}\,\chi K \tag{3.5.19}$$

Here N denotes the central charge of χ. It is worth noting that the conformal matter couples to dilaton as well as the Ricci scalar, in comparison to the undeformed case [24]. Then the energy-momentum tensor and a variation of S_{matter} with respect to the dilaton are given by

$$\begin{aligned} T_{+-} &= \frac{N}{12\pi}\partial_+\partial_-\chi\,, \\ T_{\pm\pm} &= \frac{N}{12\pi}(-\partial_\pm\partial_\pm\chi + \partial_\pm\chi\partial_\pm\chi + 2\partial_\pm\chi\partial_\pm\omega_1)\,, \\ \frac{\delta S_{\text{matter}}}{\delta\phi} &= -\frac{N}{6\pi}\eta\,\partial_+\partial_-\chi\,. \end{aligned} \tag{3.5.20}$$

Hence the equations of motion are given by

$$\begin{aligned} \partial_+\partial_-(\omega_1+\chi) &= 0\,, \\ 4\partial_+\partial_-\omega_1 + \mathrm{e}^{2\omega_1} &= \frac{16}{3}GN\eta\partial_+\partial_-\chi\,, \\ 4\partial_+\partial_-\omega_2 + \mathrm{e}^{2\omega_2} &= 0\,, \\ \mathrm{e}^{\omega_1}\partial_\pm\partial_\pm\mathrm{e}^{-\omega_1} - \mathrm{e}^{\omega_2}\partial_\pm\partial_\pm\mathrm{e}^{-\omega_2} &= \frac{2}{3}GN(-\partial_\pm\partial_\pm\chi + \partial_\pm\chi\partial_\pm\chi + 2\partial_\pm\chi\partial_\pm\omega_1)\,. \end{aligned} \tag{3.5.21}$$

Note here that the third equation is still the Liouville equation, while the second equation acquired the source term due to the matter contribution.

As we will see below, the system of equations (3.5.21) is still tractable and one can readily find out a black hole solution including the back-reaction from the conformal matter χ.

A black hole solution with a conformal matter

Let us derive a black hole solution.

Given that the solution is static, χ can be expressed as

$$\chi = -\omega_1 - \sqrt{\mu}\,(x^+ - x^-)\,. \tag{3.5.22}$$

By eliminating χ from the other equations, one can derive a couple of Liouville equations and the constraint conditions:

$$
\begin{aligned}
4\left(1+\frac{4}{3}GN\eta\right)\partial_+\partial_-\omega_1+\mathrm{e}^{2\omega_1} &= 0\,,\\
4\partial_+\partial_-\omega_2+\mathrm{e}^{2\omega_2} &= 0\,,\\
\left(1+\frac{2}{3}GN\right)\mathrm{e}^{\omega_1}\partial_\pm\partial_\pm\mathrm{e}^{-\omega_1}-\mathrm{e}^{\omega_2}\partial_\pm\partial_\pm\mathrm{e}^{-\omega_2} &= \frac{2}{3}GN\mu\,.
\end{aligned}
\tag{3.5.23}
$$

Note that a numerical coefficient in the first equation is shifted by a certain constant as a non-trivial contribution of the conformal matter.

Still, we can use the general solutions of Liouville equations given by

$$
\begin{aligned}
\mathrm{e}^{2\omega_1} &= \frac{4(1+\frac{4}{3}GN\eta)}{\left(X_1^+-X_1^-\right)^2}\partial_+X_1^+\partial_-X_1^-\,,\\
\mathrm{e}^{2\omega_2} &= \frac{4}{\left(X_2^+-X_2^-\right)^2}\partial_+X_2^+\partial_-X_2^-\,.
\end{aligned}
\tag{3.5.24}
$$

By using $X_i^\pm$ $(i=1,2)$ and the Schwarzian derivative, the constraints can be rewritten as

$$
\begin{aligned}
\left(1+\frac{2}{3}GN\right)\mathrm{Sch}\{X_1^+,x^+\}-\mathrm{Sch}\{X_2^+,x^+\} &= -\frac{4}{3}GN\mu\,,\\
\left(1+\frac{2}{3}GN\right)\mathrm{Sch}\{X_1^-,x^-\}-\mathrm{Sch}\{X_2^-,x^-\} &= -\frac{4}{3}GN\mu\,.
\end{aligned}
\tag{3.5.25}
$$

It is an easy task to see that the hyperbolic-type coordinates

$$
X_{1,2}^\pm = L_{1,2}^\pm\left(\tanh(\sqrt{\mu}x^\pm)\right)\,,
\tag{3.5.26}
$$

satisfy the constraints (3.5.25), where $L_{1,2}^\pm$ denote linear fractional transformations as in (3.4.20). Note that each of $X_{1,2}^\pm$ covers a partial region of the original spacetime. Hence the coordinate transformations (3.5.26) may lead to a black hole solution [20, 24]. In fact, the Schwarzian derivatives have particular values like

$$
\mathrm{Sch}\{L_{1,2}^\pm\left(\tanh(\sqrt{\mu}\,x^\pm)\right),\,x^\pm\} = -2\mu\,,
\tag{3.5.27}
$$

and hence these coordinates satisfy the constraints.

Here we choose the following linear transformations $L^{\pm}_{1,2}$:

$$X^+_1(x^+) = \frac{\tanh(\sqrt{\mu}\,x^+) - \eta\sqrt{\mu}}{-\eta\mu\tanh(\sqrt{\mu}\,x^+) + \sqrt{\mu}}\,,$$

$$X^+_2(x^+) = \frac{\tanh(\sqrt{\mu}\,x^+) + \eta\sqrt{\mu}}{\eta\mu\tanh(\sqrt{\mu}\,x^+) + \sqrt{\mu}}\,,$$

$$X^-_1(x^-) = X^-_2(x^-) = \frac{1}{\sqrt{\mu}}\tanh(\sqrt{\mu}x^-)\,, \tag{3.5.28}$$

one can derive a deformed black hole solution with conformal matters:

$$\mathrm{e}^{2\omega} = \frac{4\mu(1-\eta^2\mu)\sqrt{1+\frac{4}{3}GN\eta}}{\sinh^2(2\sqrt{\mu}\,Z) - \eta^2\mu\cosh^2(2\sqrt{\mu}\,Z)}\,, \tag{3.5.29}$$

$$\phi = 1 + \frac{1}{2\eta}\log\left|\frac{1+\eta\sqrt{\mu}\,\coth(2\sqrt{\mu}\,Z)}{1-\eta\sqrt{\mu}\,\coth(2\sqrt{\mu}\,Z)}\right| + \frac{1}{4\eta}\log\left(1+\frac{4}{3}GN\eta\right)\,. \tag{3.5.30}$$

The matter effect just changes the overall factor of the metric and shifts the dilaton by a constant. In the undeformed limit $\eta \to 0$, this solution reduces to a black hole solution with conformal matters presented in [24]:

$$\mathrm{e}^{2\omega} = \frac{4\mu}{\sinh^2(2\sqrt{\mu}\,Z)}\,, \qquad \phi = 1 + \sqrt{\mu}\,\coth(2\sqrt{\mu}\,Z) + \frac{1}{3}GN\,. \tag{3.5.31}$$

Black hole entropy

In this subsection, we shall compute the entropy of the black hole solution with a conformal matter given in (3.5.29) and (3.5.30) from two points of view: 1) the Bekenstein-Hawking entropy and 2) the boundary stress tensor with a certain counter-term.

1) the Bekenstein-Hawking entropy

Let us first compute the Bekenstein-Hawking entropy. From the metric (3.5.29), one can compute the Hawking temperature as

$$T_{\mathrm{H}} = \frac{\sqrt{\mu}}{\pi}\,. \tag{3.5.32}$$

From the classical action, the effective Newton constant G_{eff} is determined as

$$\frac{1}{G_{\mathrm{eff}}} = \frac{\phi}{G} - \frac{2}{3}N\chi\,. \tag{3.5.33}$$

Note that the presence of the conformal matter fields is reflected as a shift of G_{eff}. Given that the horizon area A is 1, the Bekenstein-Hawking entropy S_{BH} is computed as

$$
\begin{aligned}
S_{\text{BH}} &= \left.\frac{A}{4G_{\text{eff}}}\right|_{Z\to\infty} \\
&= \frac{1+\frac{2}{3}GN\eta}{4G\eta}\operatorname{arctanh}(\pi\, T_{\text{H}}\,\eta) + \frac{N}{6}\log(T_{\text{H}}) \\
&\quad + \frac{1}{4G} + \frac{N}{6}\log(4\pi) + \left(\frac{N}{12}+\frac{1}{16G\eta}\right)\log\left(1+\frac{4}{3}GN\eta\right). \qquad (3.5.34)
\end{aligned}
$$

The terms in the last line are constants independent of the Hawking temperature. It is worth noting that the first two constants are exactly the same as the ones in the undeformed case [24], while the last constant is modified due to the deformation. At least so far, we have no idea for the physical interpretation of this modification. It would be important to seek for it in the future study.

2) the boundary stress tensor

The next is to evaluate the entropy by computing the boundary stress tensor with a certain counter-term.

In conformal gauge, the total action including the Gibbons-Hawking term can be rewritten as

$$
\begin{aligned}
S_\phi &= \frac{1}{8\pi G}\int d^2x\left[-4\partial_{(+}\phi\partial_{-)}\omega + \frac{1}{2\eta}\sinh(2\eta(\phi-1))\,e^{2\omega}\right], \\
S_{\text{matter}} &= \frac{N}{6\pi}\int d^2x\left[\partial_+\chi\partial_-\chi + 2\partial_{(+}\chi\partial_{-)}\omega + 2\eta\partial_{(+}\chi\partial_{-)}\phi\right]. \qquad (3.5.35)
\end{aligned}
$$

By using the explicit expression of the black hole solution in (3.5.29) and (3.5.30), the on-shell bulk action can be evaluated on the boundary,

$$
S_\phi + S_{\text{matter}} = \int dt\,\left.\frac{-(1+\frac{2}{3}GN\eta)\mu - \frac{\sqrt{\mu}GN}{3}(1-\eta^2\mu)\sinh(4\sqrt{\mu}Z)}{2\pi G(1+\eta^2\mu+(-1+\eta^2\mu)\cosh(4\sqrt{\mu}Z))}\right|_{Z\to Z_0}. \qquad (3.5.36)
$$

As argued in [20] the singularity of (3.5.29) is identified as the boundary Z_0:

$$
Z_0 \equiv \frac{1}{2\sqrt{\mu}}\operatorname{arctanh}(\eta\sqrt{\mu}). \qquad (3.5.37)
$$

As the bulk action approaches the boundary $(Z \to Z_0)$, the bulk action (3.5.36) diverges and hence one needs to introduce a cut-off. When the regulator ϵ is introduced such that $Z - Z_0 = \epsilon$, the on-shell action is expanded as

$$S_\phi + S_{\text{matter}} = \int \mathrm{d}t \left[\frac{1 + \frac{4}{3} G N \eta}{16\pi G \, \eta \, \epsilon} - \frac{1 + \eta^2 \mu}{16\pi G \, \eta^2} + O(\epsilon^1) \right] . \tag{3.5.38}$$

To cancel the divergence, it is appropriate to add the following counter-term:

$$\begin{aligned} S_{\text{ct}} = \frac{-1}{8\pi G} \int \mathrm{d}t \, \frac{\sqrt{-\gamma_{tt}}}{L} \, & \left[\sqrt{F(\phi) - \frac{1}{\eta^2} \log(1 - \eta^2 \mu)} \right. \\ & \left. + \frac{4}{3} G N \sqrt{G(\phi) - \frac{1}{2\eta^2} \log(1 - \eta^2 \mu)} \right] . \end{aligned} \tag{3.5.39}$$

Here L is the overall factor of the metric defined as

$$L^2 \equiv (1 - \eta^2 \mu) \sqrt{1 + \frac{4}{3} G N \eta} \, , \tag{3.5.40}$$

and scalar functions F and G are defined as

$$\begin{aligned} F(\phi) &\equiv \frac{-1 - \eta^2 \mu + (1 - \eta^2 \mu) \cosh \left[2\eta \left(\phi - 1 - \frac{\log\left(1 + \frac{4}{3} G N \eta\right)}{4\eta} \right) \right]}{2\eta^2} \, , \\ G(\phi) &\equiv \frac{8\eta\sqrt{\mu} + (1 - \eta^2 \mu) \exp \left[2\eta \left(\phi - 1 - \frac{\log\left(1 + \frac{4}{3} G N \eta\right)}{4\eta} \right) \right]}{4\eta^2} \, . \end{aligned} \tag{3.5.41}$$

The extrinsic metric γ_{tt} on the boundary is evaluated as

$$\gamma_{tt} = -\mathrm{e}^{2\omega} \big|_{Z \to Z_0} \, .$$

In the undeformed limit $\eta \to 0$, this counter-term reduces to

$$S_{\text{ct}}^{(\eta=0)} = \int \mathrm{d}t \, \sqrt{-\gamma_{tt}} \left(-\frac{\phi - 1}{8\pi G} - \frac{N}{24\pi} \right) . \tag{3.5.42}$$

This is nothing but the counter-term utilized in the undeformed model [24].

It is straightforward to check that the sum $S = S_\phi + S_{\text{matter}} + S_{\text{ct}}$ becomes finite on the boundary by using the expanded form of the counter-term (3.5.39):

$$S_{\text{ct}} = \int \mathrm{d}t \left[-\frac{1 + \frac{4}{3} G N \eta}{16\pi G \eta \, \epsilon} + \frac{1 + \eta^2 \mu - \frac{16}{3} G N \eta^2 + 2(1 + \frac{2}{3} G N \eta) \log(1 - \eta^2 \mu)}{16\pi G \, \eta^2} + O(\epsilon) \right] .$$

In a region near the boundary, the warped factor of the metric (3.5.29) is expanded as

$$e^{2\omega} = \frac{L}{\eta\,\epsilon} + O(\epsilon^0)\,. \tag{3.5.43}$$

Hence, by normalizing the boundary metric as

$$\hat{\gamma}_{tt} = \frac{\eta\,\epsilon}{L}\,\gamma_{tt}\,,$$

the boundary stress tensor is defined as

$$\langle \hat{T}_{tt} \rangle \equiv \frac{-2}{\sqrt{-\hat{\gamma}_{tt}}}\,\frac{\delta S}{\delta \hat{\gamma}^{tt}} = \lim_{\epsilon\to 0}\sqrt{\frac{\eta\,\epsilon}{L}}\,\frac{-2}{\sqrt{-\gamma_{tt}}}\,\frac{\delta S}{\delta \gamma^{tt}}\,. \tag{3.5.44}$$

After all, $\langle \hat{T}_{tt} \rangle$ is evaluated as

$$\langle \hat{T}_{tt} \rangle = \frac{-(1+\frac{2}{3}GN\eta)\log(1-\eta^2\mu)}{8\pi G\,\eta^2} + \frac{N\sqrt{\mu}}{6\pi}\,. \tag{3.5.45}$$

To compute the associated entropy, $\langle \hat{T}_{tt} \rangle$ should be identified with energy E like

$$E = \frac{-(1+\frac{2}{3}GN\eta)\log(1-\pi^2 T_{\rm H}^2\eta^2)}{8\pi G\,\eta^2} + \frac{N}{6}T_{\rm H}\,, \tag{3.5.46}$$

where we have used the expression of the Hawking temperature (3.6.16) . Then by solving the thermodynamic relation,

$$\mathrm{d}E = \frac{\mathrm{d}S}{T_{\rm H}}\,, \tag{3.5.47}$$

the associated entropy is obtained as

$$S = \frac{(1+\frac{2}{3}GN\eta)}{4G\eta}\mathrm{arctanh}(\pi T_{\rm H}\eta) + \frac{N}{6}\log(T_{\rm H}) + S_{T_{\rm H}=0}\,. \tag{3.5.48}$$

Here $S_{T_{\rm H}=0}$ has appeared as an integration constant that measures the entropy at zero temperature. Thus the resulting entropy precisely agrees with the Bekenstein-Hawking entropy (3.6.18) , up to the temperature-independent constant.

3.6 The Liouville dilaton gravity

In this section, we shall introduce a new proper frame, which was originally utilized by Frolov and Zelnikov [74]. One can see that in this proper frame, the deformed JT model can be recaptured as a Liouville dilaton gravity model with a cosmological constant term, while solutions are still given by $\tilde{\omega}_1$ and $\tilde{\omega}_2$.

3.6.1 Weyl transformation depending on the dilaton

The proper frame can be introduced through the following Weyl transformation:

$$g_{\mu\nu} = \mathrm{e}^{-2\eta\,\phi}\tilde{g}_{\mu\nu}\,. \tag{3.6.1}$$

In the conformal gauge, $\tilde{\omega}_1$ plays the role of the conformal factor in front of the metric:

$$\mathrm{d}\tilde{s}^2 = \tilde{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{e}^{2\tilde{\omega}_1}\mathrm{d}x^+\mathrm{d}x^-\,. \tag{3.6.2}$$

In terms of the new metric $\tilde{g}_{\mu\nu}$, the classical action of the deformed JT model (3.5.19) can be rewritten as

$$\tilde{S}_\phi = \frac{1}{16\pi G}\int \mathrm{d}^2x\sqrt{-\tilde{g}}\left[\phi\tilde{R} - 2\eta(\tilde{\nabla}\phi)^2 - \frac{1}{2\eta L^2}\left(\mathrm{e}^{-4\eta\phi}-1\right)\right]\,. \tag{3.6.3}$$

Note here that the kinematic term of ϕ is well-defined because we have assumed that η is a positive real constant. The potential is now bounded from below, but it is the run-away type potential. Note here that ϕ (instead of ϕ) appears in the classical action (4.4.1) and ϕ should be definitely positive. Hence this is not the usual Liouville gravity but rather a *constrained* Liouville gravity. Interestingly, this constrained system can also be derived from Einstein-Hilbert action with a cosmological constant [75].

It is remarkable that the equations of motion for ω and ϕ are equivalent to the equations for $\tilde{\omega}_1$ and $\tilde{\omega}_2$. Thus, the solutions of the eom are obtained by $\tilde{\omega}_1$ and $\tilde{\omega}_2$ as (3.4.15) . From $\tilde{\omega}_1$ and $\tilde{\omega}_2$, the dilation is determined by (3.4.22) again. However, in the proper frame, the metric is given by $\tilde{\omega}_1$ directly.

In summary, the general vacuum solutions of (4.4.1) are given by

$$\begin{aligned}
\mathrm{e}^{2\tilde{\omega}_1} &= \frac{4L^2\partial_+X_1^+\partial_-X_1^-}{\left(X_1^+ - X_1^-\right)^2}\,,\\
\phi &= \frac{1}{4\eta}\log\left|\frac{\partial_+X_1^+\partial_-X_1^-}{\left(X_1^+ - X_1^-\right)^2}\,\frac{\left(X_2^+ - X_2^-\right)^2}{\partial_+X_2^+\partial_-X_2^-}\right|\,.
\end{aligned} \tag{3.6.4}$$

Note here that the metric is given by the solution of Liouville equation and the rigid AdS_2 geometry is preserved in the new frame (i.e., proper frame). This result indicates that the Weyl transformation carried out here has undone the Yang-Baxter deformation from the metric, while the deformations effect has been encoded into only the dilaton part. It should be remarked that this is a rather natural result, noticing that the Yang-Baxter deformation effect has been factored out as shown in [20].

3.6.2 A black hole solution and its thermodynamics

In this section, we present a new black hole solution with a conformal matter. The proper frame (3.6.1) enables us to construct an AdS_2 black hole solution (i.e., the metric is the same as the undeformed case [24]) After that, we compute the entropy of the black hole solution in two manners: 1) the Bekenstein-Hawking entropy and 2) the boundary stress tensor with a certain counter-term.

A new black hole solution

In the proper frame, the classical action of the matter (3.5.19) is given by

$$\tilde{S}_\chi = -\frac{N}{24\pi}\int \mathrm{d}^2x\,\sqrt{-\tilde{g}}\,\left[\chi\tilde{R} + (\tilde{\nabla}\chi)^2\right] - \frac{N}{12\pi}\int \mathrm{d}t\,\sqrt{-\tilde{\gamma}_{tt}}\,\chi\,K\,. \tag{3.6.5}$$

Note here that in the proper frame, χ couples to only the Ricci scalar, while in the old frame $g_{\mu\nu}$, χ coupled to both Ricci scalar and dilaton. This point is the same as in the undeformed case [24].

We have already known that solutions are given by (3.4.15). Hence, the remaining task is to determine how to choose parameters of linear fractional transformations (3.5.26) so as to realize a black hole solution.

In order to find out a black hole solution of the system (3.6.5), we first employ the black hole coordinates for $X_1^\pm$ like

$$X_1^\pm(x^\pm) = \frac{1}{\sqrt{\mu}}\tanh(\sqrt{\mu}x^\pm)\,, \tag{3.6.6}$$

by following the argument in [24]. Then, for $X_2^\pm$, let us take the following linear fractional transformation.

$$\begin{aligned} X_2^+(x^+) &= \frac{\tanh(\sqrt{\mu}x^+) + 2\eta\sqrt{\mu}}{2\eta\mu\tanh(\sqrt{\mu}x^+) + \sqrt{\mu}}\,, \\ X_2^-(x^-) &= \frac{1}{\sqrt{\mu}}\tanh(\sqrt{\mu}x^-)\,. \end{aligned} \tag{3.6.7}$$

Thus, we have obtained the static solutions for $\tilde{\omega}_1$, $\tilde{\omega}_2$:

$$\mathrm{e}^{2\tilde{\omega}_1} = \frac{4\mu(1+\frac{4}{3}GN\eta)L^2}{\sinh^2(2\sqrt{\mu}Z)}\,, \tag{3.6.8}$$

$$\mathrm{e}^{2\tilde{\omega}_2} = \frac{4\mu(1-4\eta^2\mu)L^2}{\left(\sinh(2\sqrt{\mu}Z) + 2\eta\sqrt{\mu}\cosh(2\sqrt{\mu}Z)\right)^2}\,, \tag{3.6.9}$$

and hence a black hole solution with conformal matters has been derived as follows:

$$
\begin{aligned}
\mathrm{e}^{2\tilde{\omega}_1} &= \frac{4\mu(1+\frac{4}{3}GN\eta)L^2}{\sinh^2(2\sqrt{\mu}Z)}\,, & (3.6.10)\\
\phi &= \frac{1}{2\eta}\log\left|1+2\eta\sqrt{\mu}\,\coth(\sqrt{\mu}\,Z)\right| + \phi_0\,. & (3.6.11)
\end{aligned}
$$

Here ϕ_0 is the constant part of the dilaton:

$$
\phi_0 \equiv \frac{1}{2\eta}\log\left(\frac{1+\frac{4}{3}GN\eta}{1-4\eta^2\mu}\right)\,. \tag{3.6.12}
$$

The matter contribution just rescales the metric and shifts the dilaton by a constant. Note here that the allowed region of μ is restrcted like

$$
0 \leq \sqrt{\mu} \leq \frac{1}{2\eta} \tag{3.6.13}
$$

so as to make the value of ϕ_0 well-defined and prserve the positivity of (3.6.9) .

Note here that this solution is different from the previous black hole solution with (3.5.29) and (3.5.30), though the two solutions are quite similar but the μ-dependence of the metric and the range of $\sqrt{\mu}$ are different.

By taking the undeformed limit $\eta \to 0$, this solution goes to the black hole solution with conformal matters in the undeformed case [24]:

$$
\begin{aligned}
\mathrm{e}^{2\omega} &= \frac{4\mu L^2}{\sinh^2(2\sqrt{\mu}Z)}\,, & (3.6.14)\\
\phi &= \sqrt{\mu}\,\coth(\sqrt{\mu}\,Z) + \frac{1}{3}GN\,. & (3.6.15)
\end{aligned}
$$

In the following, let us evaluate the black hole entropy associated with (3.6.10) and (3.6.11) in two manners.

1) Bekenstein-Hawking entropy

Let us first compute the Bekenstein-Hawking entropy. From the black hole metric (3.6.10), one can compute the Hawking temperature as

$$
T_{\mathrm{H}} = \frac{\sqrt{\mu}}{\pi}\,. \tag{3.6.16}
$$

This is the same as in the undeformed case [24]. From the classical action, one can read off the effective Newton constant G_{eff} as

$$\frac{1}{G_{\text{eff}}} = \frac{\phi}{G} - \frac{2N\chi}{3}\,. \tag{3.6.17}$$

Given that the horizon area A is 1, the Bekenstein-Hawking entropy S_{BH} is computed as

$$\begin{aligned} S_{\text{BH}} &= \left.\frac{A}{4G_{\text{eff}}}\right|_{Z\to\infty} \\ &= \frac{\operatorname{arctanh}(2\pi\, T_{\text{H}}\,\eta)}{8G\eta} + \frac{N}{6}\log(T_{\text{H}}) + \text{constant}\,. \end{aligned} \tag{3.6.18}$$

The last term is a constant term independent of the Hawking temperature. Note here that the argument of arctanh should be less than 1. This means that

$$0 \leq T_{\text{H}} \leq \frac{1}{2\pi\eta}\,.$$

This range agrees with the possible values of $\sqrt{\mu}$ given in (3.6.13) .

2) Boundary stress tensor

In the conformal gauge, the total action, including the Gibbons-Hawking term, can be rewritten as

$$\begin{aligned} \tilde{S}_\phi &= \frac{1}{8\pi G}\int \mathrm{d}^2x \left[-4\partial_{(+}\phi\partial_{-)}\tilde{\omega}_1 + 4\eta\partial_+\phi\partial_-\phi - \frac{1}{4\eta L^2}\mathrm{e}^{2\tilde{\omega}_1}(\mathrm{e}^{-4\eta\phi}-1)\right]\,, \\ \tilde{S}_\chi &= \frac{N}{6\pi}\int \mathrm{d}^2x \left[\partial_+\chi\partial_-\chi + 2\partial_{(+}\chi\partial_{-)}\tilde{\omega}_1\right]. \end{aligned} \tag{3.6.19}$$

By using the explicit expression of the black hole solution, the on-shell bulk action can be evaluated on the boundary.

The on-shell action diverges at the boundary $Z = 0$, hence one needs to introduce a cut-off as $Z = \epsilon\ (> 0)$, where ϵ is an infinitesimal quantity. Then the on-shell action can be expanded with respect to ϵ like

$$\tilde{S}_\phi + \tilde{S}_\chi = \int \mathrm{d}t \left[\frac{1+\frac{4}{3}GN\eta}{16\pi G\,\eta\,\epsilon} + O(\epsilon^1)\right]. \tag{3.6.20}$$

We have igonored the terms which vanish in the $\epsilon \to 0$ limit, and only the divergent term has explicitly been written down. To cancel out the divergence, it is necessary to add an appropriate counter-term.

Our proposal for the counter-term is the following:

$$\tilde{S}_{\rm ct} = \int {\rm d}t\, \frac{\sqrt{-\tilde{\gamma}_{tt}}}{L'} \left[\frac{-1}{16\pi G\,\eta}\Big(1+(2\eta\,\phi_0^2-1){\rm e}^{-2\eta(\phi-\phi_0^2)}\Big) - \frac{N}{24\pi}\right]\,. \tag{3.6.21}$$

Here ϕ_0 is the constant defined in (3.6.12) and L' is the rescaled AdS radius defined as

$$L'^2 \equiv L^2\left(1+\frac{4}{3}GN\eta\right)\,. \tag{3.6.22}$$

Then the extrinsic metric $\tilde{\gamma}_{tt}$ on the boundary is defined as

$$\tilde{\gamma}_{tt} \equiv -{\rm e}^{2\tilde{\omega}_1}\big|_{Z=\epsilon}\,.$$

In the undeformed limit $\eta \to 0$, the counter-term (3.6.21) reduces to

$$\tilde{S}_{\rm ct}^{(\eta=0)} = \int {\rm d}t\, \frac{\sqrt{-\gamma_{tt}}}{L}\left(-\frac{\phi}{8\pi G} - \frac{N}{24\pi}\right)\,. \tag{3.6.23}$$

This is nothing but the counter-term utilized in the undeformed case [24].

It is straightforward to check that the sum $\tilde{S} = \tilde{S}_\phi + \tilde{S}_\chi + \tilde{S}_{\rm ct}$ becomes finite on the boundary by using the expanded form of the counter-term (3.6.21):

$$\tilde{S}_{\rm ct} = \int {\rm d}t\left[-\frac{1+\frac{4}{3}GN\eta}{16\pi G\eta\,\epsilon} + \frac{1-2\eta\phi_0^2}{16\pi G\,\eta^2} + O(\epsilon)\right]\,.$$

In a region near the boundary, the warped factor of the metric can be expanded as

$$e^{2\tilde{\omega}_1} = \frac{L'^2}{\epsilon^2} + O(\epsilon^0)\,. \tag{3.6.24}$$

Hence, by normalizing the boundary metric as

$$\hat{\gamma}_{tt} = \frac{\epsilon^2}{L'^2}\,\tilde{\gamma}_{tt}\,,$$

the boundary stress tensor is defined as

$$\langle \hat{T}_{tt}\rangle \equiv \frac{-2}{\sqrt{-\hat{\gamma}_{tt}}}\frac{\delta S}{\delta\hat{\gamma}^{tt}} = \lim_{\epsilon\to 0}\frac{\epsilon}{L'}\frac{-2}{\sqrt{-\tilde{\gamma}_{tt}}}\frac{\delta S}{\delta\tilde{\gamma}^{tt}}\,. \tag{3.6.25}$$

After all, $\langle \hat{T}_{tt}\rangle$ has been evaluated as

$$\langle \hat{T}_{tt}\rangle = -\frac{\log(1-4\eta^2\mu)}{32\pi G\,\eta^2} + \frac{\log\left(1+\frac{4}{3}GN\eta\right)}{32\pi G\,\eta^2} + \frac{N\sqrt{\mu}}{6\pi}\,. \tag{3.6.26}$$

To compute the associated entropy, $\langle \hat{T}_{tt} \rangle$ is identified with energy E like

$$E = -\frac{\log(1-4\pi^2 T_{\mathrm{H}}^2 \eta^2)}{32\pi G\,\eta^2} + \frac{\log\left(1+\frac{4}{3}GN\eta\right)}{32\pi G\,\eta^2} + \frac{N}{6}T_{\mathrm{H}}\,, \tag{3.6.27}$$

where we have used the expression of the Hawking temperature (3.6.16) .

Then, by solving the thermodynamic relation,

$$\mathrm{d}E = \frac{\mathrm{d}S}{T_{\mathrm{H}}}\,, \tag{3.6.28}$$

the entropy is obtained as

$$S = \frac{\operatorname{arctanh}(2\pi T_{\mathrm{H}}\eta)}{8G\eta} + \frac{N}{6}\log(T_{\mathrm{H}}) + S_{T_{\mathrm{H}}=0}\,. \tag{3.6.29}$$

Here $S_{T_{\mathrm{H}}=0}$ has appeared as an integration constant that measures the entropy at zero temperature. Thus the resulting entropy precisely agrees with the Bekenstein-Hawking entropy (3.6.18) , up to the temperature-independent constant.

Chapter 4

Gravitational perturbation as $T\bar{T}$-deformation

In this chapter, we will discuss the relationship between a gravitational perturbation and a $T\bar{T}$-deformation in 2D dilaton gravity models. We will consider dilaton gravity models with general potential, and evaluate a quadratic action by solving the equations of motion. We will provide certain conditions under which a gravitational perturbation can be recast as a $T\bar{T}$-deformation of original matter action. In particular, in the case of the Liouville dilaton gravity derived in chapter 3, a finite $T\bar{T}$-deformation is realized as a gravitational perturbation on AdS_2

4.1 Perturbing 2D dilaton gravity systems

4.1.1 Our set-up and notation

In the following, we will consider a 2D dilaton gravity system coupled with an arbitrary matter field ψ . We will work with the Lorentzian signature, and the coordinates are described as $x^\mu = (x^0, x^1) = (t, x)$. The metric field and dilaton are given by $g_{\mu\nu}(x^\mu)$ and $\phi(x^\mu)$, respectively.

The classical action is given by

$$S[g_{\mu\nu},\phi,\psi] = S_{\rm dg}[g_{\mu\nu},\phi] + S_m[\psi,g_{\mu\nu},\phi]\,, \tag{4.1.1}$$

$$\begin{aligned} S_{\rm dg}[g_{\mu\nu},\phi] &= \frac{1}{16\pi G_N}\int {\rm d}^2x\,\sqrt{-g}\;\left[\,\phi\,R - U(\phi)\,\right]\,, \\ S_m[\psi,g_{\mu\nu},\phi] &= \int {\rm d}^2x\,\sqrt{-g}\,F(\phi)\,\mathcal{L}_m[g_{\mu\nu},\psi]\,, \end{aligned} \tag{4.1.2}$$

where G_N is the Newton constant in two dimensions and $U(\phi)$ is a dilaton potential. The matter action S_m may include a non-trivial dilaton coupling $F(\phi)$ in general in front of the matter Lagrangian $\mathcal{L}_m$. In the following, we assume that $F(\phi)$ is constant and normalized as $F(\phi) = 1$, for simplicity.

The equations of motion of this system are given by

$$R - U'(\phi) = 0\,, \tag{4.1.3}$$

$$\frac{1}{2}g_{\mu\nu}U(\phi) - \left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi\right) = 8\pi G_N\,T_{\mu\nu}\,, \tag{4.1.4}$$

where we have defined the energy-momentum tensor $T_{\mu\nu}$ for the matter field ψ as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\,. \tag{4.1.5}$$

We do not discuss the dynamics of the matter field itself (nor the backreaction of the metric and dilaton to the matter field). Therefore the equation of motion for ψ is not included. In deriving the equations of motion (4.1.4) for the metric, we have used the fact that the Einstein tensor $G_{\mu\nu}$ in two dimensions vanishes:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\,. \tag{4.1.6}$$

In our later discussion, we are interested in studying gravitational perturbations around a vacuum solution (i.e., a solution obtained when $T_{\mu\nu} = 0$)[1]. Hence, it would be useful to write down some relations for the dilaton ϕ in an arbitrary vacuum solution. When $T_{\mu\nu} = 0$, the equation of motion (4.1.4) for the metric takes a simple form

$$\nabla_\mu\nabla_\nu\phi = g_{\mu\nu}\left(\nabla^2\phi + \frac{1}{2}U(\phi)\right)\,. \tag{4.1.7}$$

[1]Note here that the dilaton is not regarded as a matter field but a part of the metric. This viewpoint would be rather natural as some dilaton gravities are obtained by dimensional reduction of higher-dimensional theories.

The trace of (4.1.7) is given by

$$\nabla^2\phi + U(\phi) = 0\,. \tag{4.1.8}$$

By using (4.1.8) and (4.1.7), the dilaton potential $U(\phi)$ can be deleted. The resulting expression is

$$\nabla_\mu \nabla_\nu \phi = \frac{1}{2} g_{\mu\nu} \nabla^2 \phi\,. \tag{4.1.9}$$

Comment on the flat-space JT gravity

In the work [17], S. Dubovsky et al considered a special case called the flat-space JT gravity. This case corresponds to the following dilaton potential

$$U(\phi) = \Lambda\,, \tag{4.1.10}$$

where Λ is a constant. The vacuum solution is uniquely determined (up to trivial ambiguities) as

$$\mathrm{d}^2 s = -2\mathrm{d}x^+ \mathrm{d}x^-\,, \qquad \phi = \frac{\Lambda}{2} x^+ x^-\,, \tag{4.1.11}$$

where the light-cone coordinates are defined as

$$x^\pm \equiv \frac{1}{\sqrt{2}}(t \pm x)\,.$$

In [17], the dilaton gravity system coupled with an arbitrary matter field has been expanded around this vacuum solution and the quadratic fluctuations have been recast into a form of $T\bar{T}$-deformation. We will return to this point as a special example later after we carry out general computation.

4.1.2 Evaluation of the quadratic action

By starting from the classical action (4.1.1), let us consider a gravitational perturbation around a vacuum solution. In the following, we will slightly change our notation. The original metric, dilaton and matter field are described as $g_{\mu\nu}$, ϕ and ψ, respectively. The vacuum solution, which is taken as an expansion point, is specified by $\bar{g}_{\mu\nu}$ and $\bar{\phi}$. Since we assume that the expansion point is a vacuum solution with $T_{\mu\nu} = 0$, the matter field ψ should be regarded as a fluctuation (i.e., ψ should be expanded around zero). In summary, a gravitational perturbation around a vacuum solution is described as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} , \qquad \phi = \bar{\phi} + \sigma , \qquad \psi = 0 + \psi , \tag{4.1.12}$$

where $h_{\mu\nu}$ and σ are fluctuations of metric and dilaton, respectively, and ψ in the right hand side is treated as a fluctuation with a slight abuse of notations. Note here that since $\bar{g}_{\mu\nu}$ and $\bar{\phi}$ describe a vacuum solution, the equations of motion (4.1.7) should be satisfied.

It is an easy practice to derive a vacuum solution explicitly by specifying a dilaton potential at the beginning. However, we will not do that here and keep an abstract form of the vacuum solution so as to argue in a *covariant* way. If we use a concrete expression of the vacuum solution, covariance of the expression is not manifest like in [17].

The quadratic action

Let us expand the classical action $S[g_{\mu\nu}, \phi, \psi]$ in (4.1.1) by the fluctuations (4.1.12). The classical action can be expanded as

$$\begin{aligned} S[g_{\mu\nu}, \phi, \psi] &= S^{(0)} + S^{(1)} + S^{(2)} + \cdots \\ &= S^{(0)}_{\rm dg}[\bar{g}_{\mu\nu}, \bar{\phi}] + S^{(1)}_{\rm dg}[\bar{g}_{\mu\nu}, \bar{\phi}; h_{\mu\nu}, \sigma] + S^{(2)}_{\rm dg}[\bar{g}_{\mu\nu}, \bar{\phi}; h_{\mu\nu}, \sigma] \\ &\quad + S^{(1)}_{\rm m}[\bar{g}_{\mu\nu}; \psi] + S^{(2)}_{\rm m}[\bar{g}_{\mu\nu}; \psi, h_{\mu\nu}] + \cdots , \end{aligned} \tag{4.1.13}$$

where the superscript of $S^{(n)}$ denotes the order of fluctuations. The zeroth order part $S^{(0)}_{\rm dg}$ is the classical value of $S_{\rm dg}$ with the vacuum configuration. It is just a constant in the case of [17] but in general depends on the coordinates as we will see later. Then the first order action $S^{(1)}_{\rm dg}$ should vanish since the vacuum solution satisfies the equations of motion with $\bar{\psi} = 0$. For the matter sector, $S^{(1)}_{\rm m}$ describes the matter field action on the classical background specified by the metric of the vacuum solution. The second-order contribution $S^{(2)}_{\rm m}$ is evaluated as

$$S^{(2)}_{\rm m} = \delta g^{\mu\nu} \left. \frac{\delta S_{\rm m}}{\delta g^{\mu\nu}} \right|_{g_{\mu\nu} = \bar{g}_{\mu\nu}} = \frac{1}{2} \int {\rm d}^2 x \, \sqrt{-\bar{g}} \, h^{\mu\nu} t_{\mu\nu} , \tag{4.1.14}$$

where $t_{\mu\nu}$ is the energy-momentum tensor for the matter theory described by $S_{\rm m}^{(1)}$. Note here that

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)\,.$$

where the indices in the perturbations are raised, lowered, and contracted with the background metric $\bar{g}_{\mu\nu}$: $h^{\mu\nu} \equiv \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}$.

After carrying out a lengthy calculation, we obtain the explicit expression of the quadratic action $S^{(2)} \equiv S_{\rm dg}^{(2)} + S_{\rm m}^{(2)}$. By ignoring total derivative terms, this is given by

$$\begin{aligned}
S^{(2)} = \frac{1}{16\pi G_N}\int \mathrm{d}^2x\,\sqrt{-\bar{g}}\bigg(&\bigg[\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2}h\,U'(\bar{\phi}) - \frac{1}{2}U''(\bar{\phi})\sigma\bigg]\,\sigma \\
&- \frac{1}{8}\bar{\nabla}^2\bar{\phi}\,h_{\mu\nu}h^{\mu\nu} - \bar{\nabla}^\rho\bar{\phi}\left[-\frac{1}{2}h_{\rho\sigma}\bar{\nabla}_\mu h^{\mu\sigma} + \frac{1}{4}h\bar{\nabla}^\mu h_{\mu\rho} + \frac{3}{4}h_{\rho\mu}\bar{\nabla}^\mu h\right]\bigg) \\
&+ \frac{1}{2}\int \mathrm{d}^2x\,\sqrt{-\bar{g}}\,h^{\mu\nu}t_{\mu\nu}\,. \qquad (4.1.15)
\end{aligned}$$

where $h \equiv \bar{g}^{\mu\nu}h_{\mu\nu}$. The derivation of the above expression is given in Appendix B.

Equations of motion for the fluctuations

Taking the variation of the quadratic action (4.1.15), or expanding the equations of motion (4.1.3) and (4.1.4), we obtain the equations of motion for the fluctuations as

$$\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2}U'(\bar{\phi})\,h - U''(\bar{\phi})\sigma = 0\,, \qquad (4.1.16)$$

$$\begin{aligned}
&\left(-\bar{\nabla}_\mu\bar{\nabla}_\nu\sigma + \bar{g}_{\mu\nu}\bar{\nabla}^2\sigma + \frac{1}{2}\bar{g}_{\mu\nu}\,U'(\bar{\phi})\,\sigma\right) + \frac{1}{2}\bar{\nabla}^2\phi\,(h_{\mu\nu} - \bar{g}_{\mu\nu}\,h) \\
&+ \frac{1}{2}\bar{\nabla}^\rho\bar{\phi}\left[\left(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}\right) - 2\bar{g}_{\mu\nu}\left(\bar{\nabla}^\sigma h_{\rho\sigma} - \frac{1}{2}\bar{\nabla}_\rho h\right)\right] = 8\pi G_N\,t_{\mu\nu}\,. \qquad (4.1.17)
\end{aligned}$$

Taking the trace of (4.1.17) leads to

$$\left(\bar{\nabla}^2 + U'(\bar{\phi})\right)\sigma + \frac{1}{2}U(\bar{\phi})\,h - \bar{\nabla}^\rho\bar{\phi}\left(\bar{\nabla}^\sigma h_{\rho\sigma} - \frac{1}{2}\bar{\nabla}_\rho h\right) = 8\pi G_N\,t^\mu_\mu\,. \qquad (4.1.18)$$

Subtracting the trace part (4.1.18) from (4.1.17), we obtain

$$\begin{aligned}
&\bar{\nabla}_\mu\bar{\nabla}_\nu\sigma + \frac{1}{2}\bar{g}_{\mu\nu}\,U'(\bar{\phi})\,\sigma \\
&= -8\pi G_N\,(t_{\mu\nu} - \bar{g}_{\mu\nu}t^\rho_\rho) - \frac{1}{2}U(\bar{\phi})\,h_{\mu\nu} + \frac{1}{2}\bar{\nabla}^\rho\bar{\phi}\left(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}\right)\,, \qquad (4.1.19)
\end{aligned}$$

which may also be useful in our discussion later.

Simplification of the quadratic action

It is worth noting that the quadratic action (4.1.15) can be further simplified by using the equations of motion obtained in Sec. 4.1.2.

Let us assume that the metric fluctuation takes a covariant expression of the one employed in the flat-space JT case [17] as follows:

$$h_{\mu\nu} = -16\pi G_N (t_{\mu\nu} - \bar{g}_{\mu\nu}\, t^{\rho}_{\rho})\, k\,. \tag{4.1.20}$$

Here k is an arbitrary constant. Then this ansatz leads to the relation $\bar{\nabla}^{\mu} h_{\mu\nu} = \bar{\nabla}_{\nu} h$ and

$$\begin{aligned} -\frac{1}{2} h_{\rho\sigma} \bar{\nabla}_{\mu} h^{\mu\sigma} + \frac{1}{4} h \bar{\nabla}^{\sigma} h_{\sigma\rho} + \frac{3}{4} h_{\rho\mu} \bar{\nabla}^{\mu} h &= \frac{1}{4} h_{\rho\mu} \bar{\nabla}^{\mu} h + \frac{1}{4} h \bar{\nabla}_{\rho} h \\ &= \frac{1}{4} \bar{\nabla}_{\mu} \left(h^{\mu}{}_{\rho}\, h \right)\,. \end{aligned} \tag{4.1.21}$$

Thus, by using this relation and (4.1.16), the quadratic action (4.1.15) can be simplified as

$$\begin{aligned} S^{(2)} &= \frac{1}{2\kappa} \int \mathrm{d}^2 x\, \sqrt{-\bar{g}} \left[\frac{1}{2} U''\left(\bar{\phi}\right) \sigma^2 - \frac{1}{8} \bar{\nabla}^2 \bar{\phi}\, h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} \left(\bar{\nabla}^{\rho} \bar{\nabla}^{\sigma} \bar{\phi} \right) h_{\rho\sigma} h + \kappa\, h^{\mu\nu} t_{\mu\nu} \right] \\ &= \frac{1}{2\kappa} \int \mathrm{d}^2 x\, \sqrt{-\bar{g}} \left[\frac{1}{2} U''\left(\bar{\phi}\right) \sigma^2 + \frac{1}{8} U\left(\bar{\phi}\right) \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) + \kappa\, h^{\mu\nu} t_{\mu\nu} \right] \\ &= \int \mathrm{d}^2 x\, \sqrt{-\bar{g}} \left[\frac{1}{4\kappa} U''\left(\bar{\phi}\right) \sigma^2 - \kappa \left(k - \frac{k^2}{4} U\left(\bar{\phi}\right) \right) \left(t_{\mu\nu} t^{\mu\nu} - t^2 \right) \right]. \end{aligned} \tag{4.1.22}$$

The second term is proportional to the $T\bar{T}$ operator, though the coefficient depends on the background dilaton $\bar{\phi}$ and in general has space-time coordinate dependence.

In conclusion, if $U''(\bar{\phi}) = 0$ and the metric fluctuation $h_{\mu\nu}$ satisfies the ansatz (4.1.20), the quadratic action can be regarded as a $T\bar{T}$ deformation of the original matter action, up to the background dilaton dependence.

However, we still need to check the existence of a solution to the equations of motion. This remaining task will be discussed in the next section.

4.2 Gravitational perturbations as $T\bar{T}$-deformations

In the previous section, we have shown the gravitational perturbations can be seen as $T\bar{T}$-deformations under some conditions. Here, let us check the consistency of these conditions with the equations of motion. A general treatment seems difficult, so we will consider some simple cases of the dilation potential like flat space, AdS, dS, and then construct the explicit solutions of the metric and dilaton fluctuations.

4.2.1 The case of the flat-space JT gravity

As the first example, let us revisit the case of the flat-space JT gravity considered in [17]. This case is realized by taking a constant dilaton potential

$$U'(\phi) = 0\,, \qquad U(\phi) = \Lambda\,, \tag{4.2.1}$$

where Λ is a constant. The background dilaton $\bar{\phi}$ should satisfy the following conditions

$$\bar{R} = 0\,, \qquad \bar{\nabla}^2\bar{\phi} = -U(\bar{\phi}) = -\Lambda\,, \tag{4.2.2}$$

which follow from (4.1.3) and (4.1.7).

The first is to solve the equation of motion (4.1.16) , which in the present case is simplified to

$$\partial^\nu(\partial^\mu h_{\mu\nu} - \partial_\nu h) = 0\,. \tag{4.2.3}$$

A possible solution to the equation (4.2.3) is given by

$$h_{\mu\nu} = -16\pi G_N(t_{\mu\nu} - \bar{g}_{\mu\nu}t^\rho_\rho)\,k\,, \tag{4.2.4}$$

where k is an overall constant. It is easy to see that the above $h_{\mu\nu}$ indeed solves the equation (4.2.3) by noting that the conservation law of the energy-momentum tensor $t_{\mu\nu}$, $\bar{\nabla}_\mu t^{\mu\nu} = 0$, leads to the relation

$$\partial^\mu h_{\mu\nu} = \partial_\nu h\,. \tag{4.2.5}$$

The next is to construct an explicit solution of the dilaton fluctuation σ under the metric solution (4.2.4). By using the conservation law of the energy-momentum tensor, the equations of motion (4.1.19) can be rewritten as

$$\begin{aligned}\partial_\mu\partial_\nu\sigma &= -8\pi G_N\,(t_{\mu\nu} - \bar{g}_{\mu\nu}t^\rho_\rho) - \frac{1}{2}U(\bar{\phi})\,h_{\mu\nu} + \frac{1}{2}\partial^\rho\bar{\phi}\,(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}) \\ &= -8\pi G_N\left[(1-k\Lambda) - \frac{k\Lambda}{2}x^\rho\partial_\rho\right](t_{\mu\nu} - \bar{g}_{\mu\nu}\,t^\sigma_\sigma)\,.\end{aligned} \tag{4.2.6}$$

It is possible to construct explicitly a non-local solution to the equations (4.2.6). To see this, let us first decompose the dilaton into two parts as

$$\sigma(x^+, x^-) = \sigma_0(x^+, x^-) + \sigma_{\text{non-local}}(x^+, x^-)\,. \tag{4.2.7}$$

Here the first term $\sigma_0(x^+, x^-)$ corresponds to the sourceless part,

$$\sigma_0(x^+, x^-) = a_1 + a_2\, x^+ + a_3\, x^-\,, \qquad a_i\ (i = 1, 2, 3)\text{: arbitrary real consts.}, \tag{4.2.8}$$

and obviously satisfies $\partial_\mu\partial_\nu\sigma = 0$. The second term $\sigma_{\text{non-local}}(x^+, x^-)$ describes the non-local part,

$$\begin{aligned}\sigma_{\text{non-local}} = 4\pi G_N \Bigg[& k\Lambda \int_0^{x^+} \mathrm{d}s\, s\, t_{++}(s, x^-) + k\Lambda \int_0^{x^-} \mathrm{d}s\, s\, t_{--}(x^+, s) \\ & - 2\,(k\Lambda - 1) \int_0^{x^+} \mathrm{d}s \int_0^{x^-} \mathrm{d}s'\, t_{+-}(s, s') \\ & + (k\Lambda - 2) \left(\int_{u_1^+}^{x^+} \mathrm{d}s \int_{u_2^+}^{s} \mathrm{d}s'\, t_{++}(s', 0) + \int_{u_1^-}^{x^-} \mathrm{d}s \int_{u_2^-}^{s} \mathrm{d}s'\, t_{--}(0, s') \right) \Bigg]\,,\end{aligned} \tag{4.2.9}$$

where $u^{\pm}_{1,2}$ are arbitrary constants.[2] It is easy to check that the non-local solution (4.2.7) satisfies the equations of motion (4.2.6) by using the conservation law of the energy-momentum tensor. Note here that the sign of the deformation depends on the values of Λ and k.

After substituting the solutions (4.2.4) and (4.2.7) into (4.1.15), the resulting quadratic

[2]The domain of integration may change due to the shift symmetry of the background, $x^+ \to x^+ - a^+$ and $x^- \to x^- - a^-$. After making this shift, the background dilaton is transformed like

$$\bar{\phi} = \frac{\Lambda}{2}(x^+ - a^+)(x^- - a^-)\,. \tag{4.2.10}$$

Then the non-local part (4.2.9) is modified as

$$\begin{aligned}\sigma_{\text{non-local}} = 4\pi G_N \Bigg[& k\Lambda \int_{a^+}^{x^+} \mathrm{d}s\, (s - a^+)\, t_{++}(s, x^-) + k\Lambda \int_{a^-}^{x^-} \mathrm{d}s\, (s - a^-)\, t_{--}(x^+, s) \\ & - 2\,(k\Lambda - 1) \int_{a^+}^{x^+} \mathrm{d}s \int_{a^-}^{x^-} \mathrm{d}s'\, t_{+-}(s, s') \\ & + (k\Lambda - 2) \left(\int_{u_1^+}^{x^+} \mathrm{d}s \int_{u_2^+}^{s} \mathrm{d}s'\, t_{++}(s', a^-) + \int_{u_1^-}^{x^-} \mathrm{d}s \int_{u_2^-}^{s} \mathrm{d}s'\, t_{--}(a^+, s') \right) \Bigg]\,.\end{aligned} \tag{4.2.11}$$

action is given by (up to the total derivative terms)

$$\begin{aligned} S^{(2)} &= \frac{1}{16\pi G_N}\int \mathrm{d}^2x\left(-\frac{1}{8}\partial^2\bar{\phi}\,h_{\mu\nu}h^{\mu\nu}-\frac{1}{4}\partial^\rho\bar{\phi}\,(h\partial_\rho h+h_{\rho\mu}\partial^\mu h)+8\pi G_N\,h^{\mu\nu}t_{\mu\nu}\right) \\ &= \frac{1}{16\pi G_N}\int \mathrm{d}^2x\left(-\frac{1}{8}\partial^2\bar{\phi}\,h_{\mu\nu}h^{\mu\nu}+\frac{1}{4}\partial^\mu\partial^\rho\bar{\phi}\,h_{\rho\mu}h+8\pi G_N\,h^{\mu\nu}t_{\mu\nu}\right) \\ &= -16\pi G_N\left(\frac{1}{2k}-\frac{\Lambda}{8}\right)k^2\int \mathrm{d}^2x\left[\,t_{\mu\nu}t^{\mu\nu}-(t^\mu_\mu)^2\,\right]\,. \end{aligned} \tag{4.2.12}$$

Thus the quadratic action can be regarded as a $T\bar{T}$ deformation of $S^{(1)}_{\mathrm{m}}$. It should be remarked here that the quadratic action (4.2.12) can be derived by using only the expression of $h_{\mu\nu}$, without using the explicit expression of σ. It is because the first line proportional to σ in the action (4.1.15) vanishes identically under the condition (4.2.5) and then the action is independent of σ, though the existence of σ as a consistent solution to (4.2.6) is crucial as carefully discussed in [17].

If we set $k=\frac{2}{\Lambda}$ as in [17], then the resulting action is simplified to

$$S^{(2)} = -\frac{8\pi G_N}{\Lambda}\int \mathrm{d}^2x\left[\,t_{\mu\nu}t^{\mu\nu}-(t^\mu_\mu)^2\,\right]\,. \tag{4.2.13}$$

This is nothing but the result obtained in [17].

Finally, we should argue the signature of (4.2.12). It depends on the value of k. If this signature is negative, it is well known that some pathologies like complex energies appear or closed time-like curve may appear as the onset of the breakdown of gravitational physics as discussed in [45]. In our current analysis, there is no argument to determine the signature of (4.2.12). It would be possible to fix it or obtain a bound for k by considering another property like causality or S-matrix. We will leave it as a future problem.

4.2.2 The $U'(\phi)\neq 0$ and $U''(\phi)=0$ case

The next case is a more general class of 2D dilaton gravity systems with dilaton potentials satisfying the following conditions:

$$U'(\phi)\neq 0\,,\qquad U''(\phi)=0\,. \tag{4.2.14}$$

Under these conditions, the equations of motion for the fluctuations are simplified to

$$\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2} U'(\bar{\phi})\, h = 0\,, \tag{4.2.15}$$

$$\begin{aligned}
&\left(-\bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \bar{g}_{\mu\nu} \bar{\nabla}^2 \sigma + \frac{1}{2} \bar{g}_{\mu\nu}\, U'(\bar{\phi})\, \sigma \right) + \frac{1}{2} \bar{\nabla}^2 \bar{\phi}\, (h_{\mu\nu} - \bar{g}_{\mu\nu}\, h) \\
&+ \frac{1}{2} \bar{\nabla}^\rho \bar{\phi} \left[(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}) - 2\bar{g}_{\mu\nu} \left(\bar{\nabla}^\sigma h_{\rho\sigma} - \frac{1}{2} \bar{\nabla}_\rho h \right) \right] = 8\pi G_N\, t_{\mu\nu}\,.
\end{aligned} \tag{4.2.16}$$

As in the previous case, let us solve the first equation (4.2.15). Suppose that the matter ψ is taken to be a conformal matter i.e. $t^\mu_\mu = 0$. Then it is easy to find out a solution to the equation (4.2.15),

$$h_{\mu\nu} = -k \cdot 16\pi G_N\, t_{\mu\nu}\,, \qquad h = -k \cdot 16\pi G_N\, t^\mu_\mu = 0\,, \tag{4.2.17}$$

with the help of the conservation law of the energy-momentum tensor. Finally, the quadratic action (4.1.15) can be rewritten as

$$S^{(2)} = (16\pi G_N)\, \frac{k^2}{8} \int \mathrm{d}^2 x\, \sqrt{-\bar{g}}\, \left(U(\bar{\phi}) - \frac{4}{k} \right) t_{\mu\nu} t^{\mu\nu}\,. \tag{4.2.18}$$

Thus this may also be regarded as a $T\bar{T}$ deformation.

However, we should make some comments here for the interpretation of (4.2.18). First of all, the coefficient of $t_{\mu\nu}t^{\mu\nu}$ depends on the background dilaton $\bar{\phi}$ and especially on the space-time coordinates in general. Hence we need to consider the physical interpretation of this coefficient. Then we have imposed the conformal matter condition, so we cannot interpret our result as a $T\bar{T}$-flow.We need to devise to remove the conformal matter condition somehow in order to apply the $T\bar{T}$-flow interpretation as in the flat-space JT gravity. However, our result can be understood as the exact solution to the linear-order pertubation of the AP model, and there should be some potential application in the context of $\mathrm{AdS}_2/\mathrm{CFT}_1$.

Concrete examples

In the following, we show two examples for the present case.

(i) the Almheiri-Polchinski model

An interesting example of 2D dilaton gravity systems satisfying the condition (4.2.14) is the Almheiri-Polchinski (AP) model [24][3]. This model has the dilaton potential[4]

$$U(\phi) = \Lambda - \frac{2}{L^2}\phi\,. \tag{4.2.19}$$

Let us construct the explicit form of σ. In the following, we will employ conformal gauge and the metric is given by

$$\mathrm{d}^2 s = \bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -2\mathrm{e}^{2\bar{\omega}}\mathrm{d}x^+\mathrm{d}x^-\,. \tag{4.2.20}$$

The general vacuum solution incorporates the AdS_2 metric and a non-constant dilaton

$$\mathrm{e}^{2\bar{\omega}} = \frac{2\,L^2}{(x^+ - x^-)^2}\,, \qquad \bar{\phi} = \frac{L^2}{2}\left(\Lambda + \frac{a + b\,(x^+ + x^-) + c\,x^+x^-}{x^+ - x^-}\right)\,, \tag{4.2.21}$$

where L is the AdS radius and a, b and c is an arbitrary constant.

Let us solve the equations of motion (4.2.16). The $(++)$ and $(--)$ components of (4.2.16) are evaluated as

$$\mathrm{e}^{2\bar{\omega}}\partial_+\left(\mathrm{e}^{-2\bar{\omega}}\partial_+\sigma\right) = -\,8\pi G_N \mathcal{T}_+(x^+)\,, \tag{4.2.22}$$

$$\mathrm{e}^{2\bar{\omega}}\partial_-\left(\mathrm{e}^{-2\bar{\omega}}\partial_-\sigma\right) = -\,8\pi G_N \mathcal{T}_-(x^-)\,, \tag{4.2.23}$$

where $\mathcal{T}_\pm(x^\pm)$ are defined as

$$\mathcal{T}_\pm(x^\pm) \equiv (1 \mp b\,k \mp c\,k\,x^\pm)\,t_{\pm\pm} \mp \frac{k}{4}\left(a + 2b\,x^\pm + c\,(x^\pm)^2\right)\partial_\pm t_{\pm\pm}\,. \tag{4.2.24}$$

By following [24], it is useful to express σ with a scalar function $M(x^+\,, x^-)$ as

$$\sigma(x^+, x^-) \equiv \frac{M(x^+\,, x^-)}{x^+ - x^-}\,. \tag{4.2.25}$$

Then the left-hand sides of (4.2.22) and (4.2.23) can be written as

$$\mathrm{e}^{2\bar{\omega}}\partial_\pm\left(\mathrm{e}^{-2\bar{\omega}}\partial_\pm\sigma\right) = \frac{\partial_\pm\partial_\pm M(x^+\,, x^-)}{x^+ - x^-}\,. \tag{4.2.26}$$

[3]Here, we dare to call the JT gravity with a conformal matter field as the AP model so as to respect the analysis on the conformal matter in [24] which plays a crucial role in our analysis here.

[4]As for the notation, note that our $\bar{\phi}$ corresponds to Φ^2 in [24].

By integrating (4.2.22) and (4.2.23), the general solution can be derived as

$$M(x^+\,,x^-) = I_0(x^+,x^-) + I^+(x^+,x^-) - I^-(x^+,x^-)\,. \tag{4.2.27}$$

Here I_0 is the sourceless solution,

$$I_0(x^+,x^-) \equiv A + B\,(x^+ + x^-) + C\,x^+x^-\,, \qquad A,\ B,\ C\text{: arbitrary real consts.,}$$

and $I^\pm(x^+,x^-)$ correspond to the non-local parts of dilaton and are given by

$$I^\pm(x^+,x^-) \equiv 8\pi G_N \int_{u^\pm}^{x^\pm} \mathrm{d}s\,(s-x^+)(s-x^-)\,\mathcal{T}_\pm(s)\,. \tag{4.2.28}$$

The $(+-)$ component of (4.1.19) is drastically simplified due to the traceless condition $t_{+-} = 0$ and is given by

$$\partial_+\partial_-\sigma + \frac{2\,\sigma}{(x^+ - x^-)^2} = 0\,. \tag{4.2.29}$$

It is an easy task to see that σ with (4.2.27) satisfies the above condition (4.2.29).

It should be remarked that this *non-local* solution to (4.2.29) might be epochal. One would usually try to employ hypergeometric functions or Gegenbauer polynomials to solve it by assuming that the solution is local. But this solution is non-local and it has not been presented at least as far as we know. This non-local solution may play an important role in resolving the long-standing issue of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence.

A flat-space limit

It is intriguing to consider a flat-space limit of the vacuum solution in the AP model (See also Appendix B of [17] for the limit with the embedding coordinates).

Let us first take a constant shift of $x^\pm$ and introduce new coordinates $X^\pm$ defined as

$$X^\pm \equiv x^\pm \mp \frac{L}{\sqrt{2}}\,. \tag{4.2.30}$$

By taking the large radius limit $L \to \infty$, the AdS_2 metric goes to the Minkowski metric,

$$\mathrm{d}s^2 = \frac{-4\,L^2\mathrm{d}x^+\mathrm{d}x^-}{(x^+ - x^-)^2} \quad \longrightarrow \quad -2\,\mathrm{d}X^+\mathrm{d}X^-\,. \tag{4.2.31}$$

For the background dilaton $\bar{\phi}$, it is helpful to take a particular choice of a, b and c as

$$a = -\frac{\Lambda L}{\sqrt{2}}\,, \qquad b = 0\,, \qquad c = \frac{\sqrt{2}\Lambda}{L}\,. \tag{4.2.32}$$

Then the dilaton is rewritten as

$$\bar{\phi} = \frac{\Lambda L^2}{2}\left(1 - \frac{1}{\sqrt{2}\,L}\frac{L^2 - 2\,x^+x^-}{x^+ - x^-}\right)\,. \tag{4.2.33}$$

After taking the limit $L \to \infty$, the dilaton reduces to the one (4.1.11) in the flat-space JT gravity:

$$\bar{\phi} \quad \longrightarrow \quad \frac{\Lambda}{2}X^+X^-\,. \tag{4.2.34}$$

It may be worth noting that the choice (4.2.32) corresponds to a black hole solution discussed in [24]. In particular, the parameter c is basically associated with the Hawking temperature and eventually this part gives rise to the dilaton in the flat-space JT gravity. It would be intriguing to try to get much deeper understanding for this connection.

(ii) 2D de Sitter space

Another interesting example is a de Sitter version of the AP model. This case is realized by taking the following dilaton potential,

$$U(\phi) = \Lambda + \frac{2}{L^2}\phi\,. \tag{4.2.35}$$

For the recent progress on the dS_2 in the JT gravity, see [76,77]. There might be a potential application in the context of dS/dS correspondence [78].

The general vacuum solution incorporates the dS_2 metric and a non-constant dilaton

$$\mathrm{e}^{2\bar{\omega}} = \frac{2\,L^2}{(x^+ + x^-)^2}\,, \qquad \bar{\phi} = \frac{L^2}{2}\left(-\Lambda + \frac{a + b\,(x^+ - x^-) + c\,x^+x^-}{x^+ + x^-}\right)\,, \tag{4.2.36}$$

where L is the curvature radius and a, b and c are arbitrary constants.

Again, let us examine the equations of motion (4.1.19). The $(++)$ and $(--)$ components of (4.1.19) are evaluated as

$$\mathrm{e}^{2\bar{\omega}}\partial_+\left(\mathrm{e}^{-2\bar{\omega}}\partial_+\sigma\right) = -\,8\pi G_N \mathcal{T}_{\mathrm{dS}+}(x^+)\,, \tag{4.2.37}$$

$$\mathrm{e}^{2\bar{\omega}}\partial_-\left(\mathrm{e}^{-2\bar{\omega}}\partial_-\sigma\right) = -\,8\pi G_N \mathcal{T}_{\mathrm{dS}-}(x^-)\,, \tag{4.2.38}$$

where $\mathcal{T}_{\mathrm{dS}\pm}(x^{\pm})$ are defined as

$$\mathcal{T}_{\mathrm{dS}\pm}(x^{\pm}) \equiv (1 \pm b\,k - c\,k\,x^{\pm})\,t_{\pm\pm} + \frac{k}{4}\left(a \pm 2b\,x^{\pm} - c\,(x^{\pm})^2\right)\partial_{\pm}t_{\pm\pm}\,. \tag{4.2.39}$$

Similarly to the AdS case, it is useful to represent σ by using a scalar function $M_{\mathrm{dS}}(x^+\,,x^-)$:

$$\sigma(x^+,x^-) \equiv \frac{M_{\mathrm{dS}}(x^+\,,x^-)}{x^+ + x^-}\,. \tag{4.2.40}$$

By integrating (4.2.37) and (4.2.38), the general solution can be derived as

$$M_{\mathrm{dS}}(x^+\,,x^-) = J_0(x^+,x^-) + J^+(x^+,x^-) + J^-(x^+,x^-)\,. \tag{4.2.41}$$

Here J_0 is the sourceless solution

$$J_0(x^+,x^-) \equiv A + B\,(x^+ - x^-) + C\,x^+x^-\,, \qquad A,\ B,\ C\text{: arbitrary real consts.},$$

and $J^{\pm}(x^+,x^-)$ correspond to the non-local part of dilaton and are given by

$$J^{\pm}(x^+,x^-) \equiv 8\pi G_N \int_{u^{\pm}}^{x^{\pm}} \mathrm{d}s\,(s \mp x^+)(s \pm x^-)\,\mathcal{T}_{\mathrm{dS}\pm}(s)\,. \tag{4.2.42}$$

The $(+-)$ component of (4.1.19) is drastically simplified due to the traceless condition $t_{+-} = 0$ and is given by

$$\partial_+\partial_-\sigma - \frac{2\,\sigma}{(x^+ + x^-)^2} = 0\,. \tag{4.2.43}$$

σ with (4.2.41) also satisfies the above condition (4.2.43).

A flat-space limit

In a similar way to the AdS_2 case, it is easy to consider a flat-space limit for the vacuum solution in the dS model.

Let us take a constant shift of $x^{\pm}$ and introduce new coordinates $X^{\pm}$ defined as

$$X^{\pm} \equiv x^{\pm} \pm \frac{L}{\sqrt{2}}\,. \tag{4.2.44}$$

By taking the large radius limit $L \to \infty$, the dS_2 metric goes to the Minkowski metric,

$$\mathrm{d}s^2 = \frac{-4\,L^2\mathrm{d}x^+\mathrm{d}x^-}{(x^+ + x^-)^2} \quad \longrightarrow \quad -2\,\mathrm{d}X^+\mathrm{d}X^-\,. \tag{4.2.45}$$

For the background dilaton $\bar{\phi}$, it is helpful to take a particular choice of a, b and c as

$$a = \frac{\Lambda\, L}{\sqrt{2}}\,, \qquad b = 0\,, \qquad c = \frac{\sqrt{2}\Lambda}{L}\,. \tag{4.2.46}$$

Then the dilaton is rewritten as

$$\bar{\phi} = \frac{\Lambda\, L^2}{2}\left(-1 + \frac{1}{\sqrt{2}\,L}\frac{L^2 + 2\,x^+ x^-}{x^+ + x^-}\right)\,. \tag{4.2.47}$$

After taking the limit $L \to \infty$, the dilaton reduces to the one (4.1.11) in the flat-space JT gravity:

$$\bar{\phi} \quad \longrightarrow \quad \frac{\Lambda}{2} X^+ X^-\,. \tag{4.2.48}$$

4.3 Derivation of the gravitationally dressed S-matrix

We shall derive here a gravitational dressing factor of the S-matrix in the case of the flat-space JT gravity (4.2.1). This was originally derived in [17] in the light-cone coordinates without the explicit solution of σ. It is instructive to reproduce the factor by using our exact solution of σ with the Cartesian coordinates.

Introducing the dynamical coordinates

Let us first introduce the dynamical coordinates X^μ defined as

$$X^\mu \equiv -\frac{2}{\Lambda}\partial^\mu \phi = x^\mu + Y^\mu\,, \qquad Y^\mu \equiv -\frac{2}{\Lambda}\partial^\mu \sigma\,. \tag{4.3.1}$$

The components of Y^μ are explicitly given by

$$\begin{aligned} Y^t(t,x) =& \frac{2}{\Lambda}a_2 + \kappa\, k\left[x\, T^{(0)}_{tx}(t,x) + t\, T^{(0)}_{xx}(t,x)\right] \\ &+ \frac{\kappa}{\Lambda}(k\Lambda - 2)\left(\int_0^x \mathrm{d}x'\, T^{(0)}_{tx}(t,x') + \int_{t_2}^t \mathrm{d}t'\, T^{(0)}_{xx}(t',0)\right)\,, \end{aligned} \tag{4.3.2}$$

$$\begin{aligned} Y^x(t,x) =& -\frac{2}{\Lambda}a_3 - \kappa\, k\left[x\, T^{(0)}_{tt}(t,x) + t\, T^{(0)}_{tx}(t,x)\right] \\ &- \frac{\kappa}{\Lambda}(k\Lambda - 2)\left(\int_0^t \mathrm{d}t'\, T^{(0)}_{tx}(t',x) + \int_{x_2}^x \mathrm{d}x'\, T^{(0)}_{tt}(0,x')\right)\,, \end{aligned} \tag{4.3.3}$$

where the indices have been lowered in the right-hand side. Then Y^μ satisfies

$$\partial_\mu Y^\nu = -\frac{2}{\Lambda}\partial_\mu\partial^\nu\sigma = \frac{2\kappa}{\Lambda}\left((1 - k\Lambda) - \frac{k\Lambda}{2}\, x^\rho\partial_\rho\right)({T^{(0)}}^\nu{}_\mu - \delta^\nu_\mu T^{(0)})\,, \tag{4.3.4}$$

where we have used (4.1.4).

In the standard manner, the conserved charge is given by

$$P_\mu \equiv \int_{-\infty}^{\infty} \mathrm{d}x\, T^{(0)}_{t\mu}(t,x)\,. \tag{4.3.5}$$

The total energy P_t and momentum P_x are given by, respectively,

$$P_t = \int_{-\infty}^{\infty} \mathrm{d}x\, T^{(0)}_{tt}(t,x)\,, \qquad P_x = \int_{-\infty}^{\infty} \mathrm{d}x\, T^{(0)}_{tx}(t,x)\,. \tag{4.3.6}$$

The conservation law of the energy momentum tensor is given as

$$\partial^\mu T^{(0)}_{\mu\nu} = 0\,. \tag{4.3.7}$$

In the Cartesian coordinates, it is expressed as

$$\partial_t T^{(0)}_{tt} = \partial_x T^{(0)}_{tx}\,, \qquad \partial_t T^{(0)}_{tx} = \partial_x T^{(0)}_{xx}\,. \tag{4.3.8}$$

Using the relations in (4.3.8) and the invariance under the parity-transformation,

$$T^{(0)}_{tx}(t,\infty) = T^{(0)}_{tx}(t,-\infty)\,, \qquad T^{(0)}_{xx}(t,\infty) = T^{(0)}_{xx}(t,-\infty)\,, \tag{4.3.9}$$

the conservation of the charges P_μ is shown as follows;

$$\partial_t P_t = \int_{-\infty}^{\infty} \mathrm{d}x\, \partial_x T^{(0)}_{tx}(t,x) = 0\,, \qquad \partial_t P_x = \int_{-\infty}^{\infty} \mathrm{d}x\, \partial_x T^{(0)}_{xx}(t,x) = 0\,. \tag{4.3.10}$$

Note here that Y^μ still contains four arbitrary parameters a_2, a_3, t_2 and x_2. In order to fix the expression of Y^μ definitely, we need to impose some boundary conditions for Y^μ. Then, as a result, (4.3.2) and (4.3.3) can be expressed in terms of the conserved charges P_μ.

Let us first impose a boundary condition for the energy momentum tensor as follows:

$$x\, T^{(0)}_{\mu\nu}(t,x) \to 0 \qquad (x \to \pm\infty)\,. \tag{4.3.11}$$

By using the conservation of $T^{(0)}_{\mu\nu}$ in (4.3.8), one can obtain the following relations:

$$\int_{t_2}^{t} \mathrm{d}t'\, T^{(0)}_{xx}(t',0) = \int_{-\infty}^{0} \mathrm{d}x'\, T^{(0)}_{tx}(t,x') - \int_{-\infty}^{0} \mathrm{d}x'\, T^{(0)}_{tx}(t_2,x') + \int_{t_2}^{t} \mathrm{d}t'\, T^{(0)}_{xx}(t',-\infty)\,, \tag{4.3.12}$$

$$\int_{0}^{t} \mathrm{d}t'\, T^{(0)}_{tx}(t',x) = \int_{-\infty}^{x} \mathrm{d}x'\, T^{(0)}_{tt}(t,x') - \int_{-\infty}^{x} \mathrm{d}x'\, T^{(0)}_{tt}(0,x') + \int_{0}^{t} \mathrm{d}t'\, T^{(0)}_{tx}(t',-\infty)\,. \tag{4.3.13}$$

Then Y^μ can be rewritten as

$$
\begin{aligned}
Y^t(t,x) =& \frac{2}{\Lambda} a_2 + \kappa\, k \left[x\, T^{(0)}_{tx}(t,x) + t\, T^{(0)}_{xx}(t,x) \right] \\
&+ \frac{\kappa}{\Lambda}(k\Lambda - 2) \left(\int_{-\infty}^{x} \mathrm{d}x'\, T^{(0)}_{tx}(t,x') - \int_{-\infty}^{0} \mathrm{d}x'\, T^{(0)}_{tx}(t_2,x') \right), \qquad (4.3.14) \\
Y^x(t,x) =& -\frac{2}{\Lambda} a_3 - \kappa\, k \left[x\, T^{(0)}_{tt}(t,x) + t\, T^{(0)}_{tx}(t,x) \right] \\
&- \frac{\kappa}{\Lambda}(k\lambda - 2) \left(\int_{-\infty}^{x} \mathrm{d}x'\, T^{(0)}_{tt}(t,x') - \int_{-\infty}^{x_2} \mathrm{d}x'\, T^{(0)}_{tt}(0,x') \right). \qquad (4.3.15)
\end{aligned}
$$

Now the unknown constants a_2 and a_3 are determined by using the boundary condition (4.3.11) as follows:

$$
\begin{aligned}
a_2 &= \frac{\kappa}{2}(k\Lambda - 2) \int_{-\infty}^{0} dx'\, T^{(0)}_{tx}(t_2,x') + \frac{\Lambda}{2} Y^t_{(-)}\,, \qquad (4.3.16) \\
a_3 &= -\frac{\kappa}{2}(k\Lambda - 2) \int_{-\infty}^{x_2} dx'\, T^{(0)}_{tt}(0,x') - \frac{\Lambda}{2} Y^x_{(-)}\,, \qquad (4.3.17)
\end{aligned}
$$

where we have defined $Y^\mu_{(\pm)} \equiv Y^\mu|_{x\to\pm\infty}$. Using these expression of $a_{2,3}$, we find that

$$
\begin{aligned}
Y^t(t,x) &= Y^t_{(-)} + \kappa\, k \left[x\, T^{(0)}_{tx}(t,x) + t\, T^{(0)}_{xx}(t,x) \right] + \frac{\kappa}{\Lambda}(k\Lambda - 2) \int_{-\infty}^{x} \mathrm{d}x'\, T^{(0)}_{tx}(t,x')\,, \qquad (4.3.18) \\
Y^x(t,x) &= Y^x_{(-)} - \kappa\, k \left[x\, T^{(0)}_{tt}(t,x) + t\, T^{(0)}_{tx}(t,x) \right] - \frac{\kappa}{\Lambda}(k\Lambda - 2) \int_{-\infty}^{x} \mathrm{d}x'\, T^{(0)}_{tt}(t,x')\,. \qquad (4.3.19)
\end{aligned}
$$

Taking $x \to \infty$ and using (4.3.6) leads to the following relations:

$$
Y^t_{(+)} - Y^t_{(-)} = \frac{\kappa}{\Lambda}(k\Lambda - 2)\, P_x\,, \qquad Y^x_{(+)} - Y^x_{(-)} = -\frac{\kappa}{\Lambda}(k\Lambda - 2)\, P_t\,. \qquad (4.3.20)
$$

By employing a parity symmetric prescription, we obtain that

$$
Y^t_{(\pm)} = \mp \frac{\kappa}{2\Lambda}(k\Lambda - 2)\, P_x\,, \qquad Y^x_{(\pm)} = \pm \frac{\kappa}{2\Lambda}(k\Lambda - 2)\, P_t\,. \qquad (4.3.21)
$$

It is useful to introduce a new quantity $\widetilde{P}_\mu$ defined as

$$
\widetilde{P}_\mu \equiv 2 \int_{-\infty}^{x} \mathrm{d}x\, T^{(0)}_{t\mu}(t,x) - P_\mu\,. \qquad (4.3.22)
$$

In the spacial infinity region $x \to \pm\infty$, $\widetilde{P}_\mu$ becomes the conserved charge $\widetilde{P}_\mu \to \pm P_\mu$.

Finally, the dynamical coordinates in (4.3.1) are expressed in terms of $T^{(0)}_{\mu\nu}$ as follows:

$$
X^\mu = x^\mu - \kappa\, k\, ({T^{(0)}}^\mu{}_\nu - \delta^\mu_\nu T^{(0)})\, x^\nu - \frac{\kappa}{2\Lambda}\, (k\,\Lambda - 2)\, \epsilon^{\mu\nu} \widetilde{P}_\nu\,. \qquad (4.3.23)
$$

Here $\epsilon^{\mu\nu}$ is an antisymmetric tensor normalized as $\epsilon^{tx} = -1$. For simplicity, we will set $k = 0$ in the following discussion. Then, the metric fluctuation $h_{\mu\nu}$ and the quadratic action vanish while σ does not. The dynamical coordinates in (4.3.23) are simplified as

$$X^\mu = x^\mu + \frac{\kappa}{\Lambda}\epsilon^{\mu\nu}\widetilde{P}_\nu \,. \tag{4.3.24}$$

This corresponds to the one obtained in [17].

The gravitationally dressed S-matrix

A significant implication of the dynamical coordinates in (4.3.23) is the gravitationally dressed S-matrix [17].

Let us consider a scattering process in a scalar field theory. Here the detail of the interaction potential is not necessary. In the infinite past $t \to -\infty$, N_{in} particles are prepared and each of them has a momentum $p_{(i)}$. Then the asymptotic field (in-field) is given by

$$\psi = \int \frac{\mathrm{d}p}{\sqrt{2E}}\frac{1}{2\pi}\left[a^\dagger_{\text{in}}(p)\,\mathrm{e}^{-ip_\mu x^\mu} + \text{h.c.}\right] . \tag{4.3.25}$$

It is known that a $T\bar{T}$-deformed QFT on the undeformed background is equivalent to the undeformed QFT with the dynamical coordinates [4, 14, 15, 17]. This statement means that the deformation effect for the asymptotic state can be evaluated by replacing the original coordinates x^μ with the dynamical ones X^μ.

As a result, a creation operator $a^\dagger_{\text{in}}$ gets a extra-phase factor $\mathrm{e}^{ip_\mu Y^\mu}$ and a dressed creation operator can be defined as

$$A^\dagger_{\text{in}}(p) \equiv a^\dagger_{\text{in}}(p)\,\mathrm{e}^{ip_\mu Y^\mu} . \tag{4.3.26}$$

By employing this dressed operator $A^\dagger_{\text{in}}(p)$ (instead of $a^\dagger_{\text{in}}(p)$), the associated dressed in-state can be defined as

$$\begin{aligned} \big|\{p_{(i)}\},\,\text{in}\big\rangle_{\text{dressed}} &\equiv \prod_{i=1}^{N_{\text{in}}} A^\dagger_{\text{in}}(p_{(i)})\,|0\rangle \\ &= \exp\left(i\sum_{i=1}^{N_{\text{in}}} {p_{(i)}}_\mu Y^\mu(x_{(i)})\right)\big|\{p_{(i)}\},\,\text{in}\big\rangle \,. \end{aligned} \tag{4.3.27}$$

In the infinite past, $Y^\mu(x_{(i)})$ can be evaluated as follows:

$$\begin{aligned}
Y^\mu(x_{(i)}) &= \frac{\kappa}{\Lambda}\epsilon^{\mu\nu}\left[2\int_{-\infty}^{x_{(i)}} \mathrm{d}x'\, T^{(0)}_{t\nu}(t,x') - P_\nu\right] \\
&= \frac{\kappa}{\Lambda}\epsilon^{\mu\nu}\left[2\left(\frac{1}{2}p_{(i)_\nu} + \sum_{j<i} p_{(j)_\nu}\right) - \sum_{i=1}^{N_{in}} p_{(i)_\nu}\right] \\
&= \frac{\kappa}{\Lambda}\epsilon^{\mu\nu}\left(p_{(i)_\nu} + \sum_{j<i} p_{(j)_\nu} - \sum_{j>i} p_{(j)_\nu}\right) . \qquad (4.3.28)
\end{aligned}$$

From the first line to the second line, we have assumed the mid-point prescription. Finally, one can write the dressed state in terms of $p_{(i)}$.

$$\big|\{p_{(i)}\},\,\mathrm{in}\big\rangle_{\mathrm{dressed}} = \exp\left(2i\,\frac{\kappa}{\Lambda}\sum_{i=1}^{N_{\mathrm{in}}}\sum_{i<j}\epsilon^{\mu\nu}p_{(i)_\mu}p_{(j)_\nu}\right)\big|\{p_{(i)}\},\,\mathrm{in}\big\rangle\ . \qquad (4.3.29)$$

The phase factor in front of the original in-state is nothing but the gravitational dressing factor. Similarly, the phase factor for the out-state can be evaluated and then the dressed S-matrix can be derived as shown in [17].

4.4 Liouville gravity and $T\bar{T}$-deformation

So far, we have considered at most a linear potential like (3.1.6) in order to solve the condition $U''(\phi) = 0$. Note however that the condition we have to solve is that $U''(\bar{\phi}) = 0$ and the argument is the background dilaton $\bar{\phi}$ rather than ϕ. Hence it is enough to consider the behavior of the dilaton potential around the vacuum solution and it is possible to take account of more complicated dilaton potentials.

As such an example, we will consider a classical Liouville gravity with a negative cosmological constant.[5] Remarkably, the quadratic action is recast into a $T\bar{T}$-deformation of the original matter action with a finite coupling (i.e., the conformal matter condition is not necessary).

[5]The classical Liouville gravity can be derived from pure Einstein gravity in $2+\epsilon$ dimensions [75].

4.4.1 Classical Liouville gravity

The classical action of the Liouville gravity with a negative cosmological constant is

$$S=\frac{1}{2\kappa}\int \mathrm{d}^2x\,\sqrt{-G}\left(\phi R_{(\mathrm{G})}-\frac{2\eta}{L^2}(\nabla_{(\mathrm{G})}\phi)^2-\frac{1}{2\eta}\mathrm{e}^{2\eta\left(\Lambda-\frac{2}{L^2}\phi\right)}+\frac{1}{2\eta}\right)+S_m\,[\psi,\,G_{\mu\nu}]\,, \quad (4.4.1)$$

where $R_{(G)}$ and $\nabla_{(G)}$ are the Ricci scalar and covariant derivative, respectively, defined with the metric $G_{\mu\nu}$. Then η is a new constant parameter with dimension (length)^2. When η is negative ($\eta<0$), the kinetic term of the dilaton has the wrong sign and the potential of (4.4.1) is not bounded from below. Hence, we take η to be positive as a natural choice,

$$\eta>0\,. \quad (4.4.2)$$

The Liouville gravity action (4.4.1) has the dilaton kinetic term. Hence, we have to remove the dilaton kinetic term by performing an appropriate Weyl transformation.

Let us consider the following Weyl transformation depending on the dilaton [21],

$$G_{\mu\nu}=\mathrm{e}^{-\eta\left(\Lambda-\frac{2}{L^2}\phi\right)}g_{\mu\nu}\,. \quad (4.4.3)$$

In the frame with $g_{\mu\nu}$, the kinetic term has been removed as follows:

$$S=\frac{1}{2\kappa}\int \mathrm{d}^2x\sqrt{-g}\left(\phi R-\frac{1}{\eta}\sinh\left[\eta\left(\Lambda-\frac{2\phi}{L^2}\right)\right]\right)+S_m\,\left[\psi,\,\mathrm{e}^{-\eta\left(\Lambda-\frac{2}{L^2}\phi\right)}g_{\mu\nu}\right]\,. \quad (4.4.4)$$

Thus the dilaton potential $U(\phi)$ is identified with the following hyperbolic potential:

$$U(\phi)=\frac{1}{\eta}\sinh\left[\eta\left(\Lambda-\frac{2\phi}{L^2}\right)\right]\,. \quad (4.4.5)$$

Note here that the matter action S_m now depends on the dilaton explicitly through the Weyl factor of the metric $g_{\mu\nu}$.

Originally, this hyperbolic-type potential was introduced in [20] so as to support Yang-Baxter deformations [25–27] of AdS_2, where η corresponds to the deformation parameter. In the $\eta\to 0$ limit, the JT model (3.1.6) is reproduced.

It is known that the AdS_2 metric with a constant dilaton is one of the vacuum solutions (For the general solution, see Appendix B). In the conformal gauge , this solution is given by

$$\mathrm{e}^{2\bar{\omega}}=\frac{2\,L^2}{(x^+-x^-)^2}\,,\qquad \bar{\phi}=\frac{\Lambda L^2}{2}\,. \quad (4.4.6)$$

In the following, we will consider fluctuations around this vacuum solution. For this constant dilaton background, one can show that

$$U(\bar{\phi}) = 0\,, \qquad U'(\bar{\phi}) = -\frac{2}{L^2}\,, \qquad U''(\bar{\phi}) = 0\,. \tag{4.4.7}$$

Thus this hyperbolic dilaton potential (4.4.5) indeed satisfies the condition $U''(\bar{\phi}) = 0\,.$

4.4.2 The quadratic action

Let us then consider the quadratic action for the hyperbolic potential (4.4.5) . By supposing the ansatz (4.1.20) , the equation in (4.1.16) is simplified as

$$\frac{2\kappa}{L^2}\,T^{(0)}\,k + \frac{2\kappa}{\sqrt{-\bar{g}}}\frac{\delta S_m^{(1)}}{\delta\phi}\,(\bar{\phi}) = 0\,. \tag{4.4.8}$$

The dilaton dependence in the matter action has been determined in (4.4.4), and the second term in (4.4.8) is replaced by the trace of the energy-momentum tensor as follows:

$$\frac{\delta S_m^{(1)}}{\delta\phi}(\bar{\phi}) = -\frac{2\eta}{L^2}\,g^{\mu\nu}\frac{\delta S_m^{(1)}}{\delta g^{\mu\nu}} = \frac{\eta}{L^2}\sqrt{-\bar{g}}\,T^{(0)}\,. \tag{4.4.9}$$

As a result, (4.4.8) reduces to a simple equation,

$$\frac{2\kappa}{L^2}\,(k+\eta)\;T^{(0)} = 0\,. \tag{4.4.10}$$

A possible solution is the conformal matter case $T^{(0)} = 0$. Then the matter action S_m is invariant under the Weyl transformation and its dilaton dependence disappears. Hence, it is the same as the AP model case discussed in Section 2.

Unless the matter action is conformal, k is directly connected to η like

$$k = -\eta\,. \tag{4.4.11}$$

Originally, k has been introduced as an arbitrary constant of the metric ansatz (4.1.20). In comparison to the flat-space JT case where k is completely free, in the present case k is determined completely by the initial set-up of the Liouville action.

With the condition (4.4.11), the quadratic action (4.1.22) leads to the form of $T\bar{T}$-deformation on the AdS_2 background,

$$S^{(2)} = \kappa\,\eta\int \mathrm{d}^2x\,\sqrt{-\bar{g}}\,\left(T^{(0)}_{\mu\nu}T^{(0)\mu\nu} - T^{(0)^2}\right)\,. \tag{4.4.12}$$

Note here that the deformation is measured by $\kappa\,\eta$. It is significant to see the signature of the deformation because it is sensitive to the physics of the deformed theory. Recall that both κ and η are positive. Hence the deformation (4.4.12) corresponds to the negative sign in the convention of [42]. Then the deformed theory should have a UV cut-off (at least) in the flat-space limit, because the energy becomes complex in the UV region. Hence the above result would have an intimate connection with the cut-off AdS geometry [30, 31] or the random boundary geometry [40].

On the other hand, a negative η corresponds to a positive-sign $T\bar{T}$-deformation. Then the deformed theory does not have the UV cut-off. However, if η is negative, then the potential of the dilaton is not bounded from below and the dilaton becomes unstable. This case may be interpreted as a quantum Liouville theory and then be related to the Little String Theory scenario proposed in [42].

4.4.3 The explicit solution of σ

The remaining task is to derive a non-trivial solution to the equations of motion (4.1.16) and (4.2.16). For this purpose, let us start from considering some properties of the energy-momentum tensor.

The energy momentum tensor $T^{(0)}_{\mu\nu}$ should satisfy the conservation law.

$$\bar{\nabla}^{\mu}T^{(0)}_{\mu\nu} = 0\,. \tag{4.4.13}$$

In the conformal gauge , the components of the conservation law are given by

$$\partial_{-}T^{(0)}_{++} = -\partial_{+}T^{(0)}_{+-} - \frac{2}{x^{+}-x^{-}}T^{(0)}_{+-}\,, \qquad \partial_{+}T^{(0)}_{--} = -\partial_{-}T^{(0)}_{+-} + \frac{2}{x^{+}-x^{-}}T^{(0)}_{+-}\,. \tag{4.4.14}$$

The trace of the energy-momentum tensor ${T^{(0)}}_{+-}$ is not zero and gives rise to a no-trivial contribution. Moreover, in comparison to the conformal matter case, the $(++)$ component of the energy-momentum tensor ${T^{(0)}}_{++}$ is no longer a holomorphic function and it depends on x^{-} as well. This is also the same for $T^{(0)}_{--}$.

Thus, the equations of motion are rewritten as

$$\begin{aligned} e^{2\bar{\omega}}\partial_+\left(e^{-2\bar{\omega}}\partial_+\sigma\right) =& -\kappa\, T^{(0)}_{++}(x^+,\, x^-)\,, \\ e^{2\bar{\omega}}\partial_-\left(e^{-2\bar{\omega}}\partial_-\sigma\right) =& -\kappa\, T^{(0)}_{--}(x^+,\, x^-)\,, \\ \partial_+\partial_-\sigma + \frac{2}{(x^+-x^-)^2}\sigma =& \kappa\, T^{(0)}_{+-}(x^+,\, x^-)\,. \end{aligned} \tag{4.4.15}$$

It is useful to introduce a scalar function $M(x^+\,, x^-)$ as

$$\sigma = \frac{M(x^+\,, x^-)}{x^+-x^-}\,. \tag{4.4.16}$$

The equations in (4.4.15) are further rewritten as

$$\begin{aligned} \partial_+^2 M =& -\kappa\,(x^+-x^-)\,T^{(0)}_{++}(x^+,\, x^-)\,, \\ \partial_-^2 M =& -\kappa\,(x^+-x^-)\,T^{(0)}_{--}(x^+,\, x^-)\,, \\ (x^+-x^-)\,\partial_+\partial_- M + \partial_+ M - \partial_- M =& 2\kappa\,(x^+-x^-)\,T^{(0)}_{+-}(x^+,\, x^-)\,. \end{aligned} \tag{4.4.17}$$

The solution is given by

$$M(x^+\,, x^-) = I_0(x^+\,, x^-) + \mathcal{I}^+(x^+,\, x^-) - \mathcal{I}^-(x^+,\, x^-)\,, \tag{4.4.18}$$

where $I_0(x^+\,, x^-)$ is the sourceless solution given in in section 2. $\mathcal{I}^+(x^+,\, x^-)$ and $\mathcal{I}^-(x^+,\, x^-)$ are defined as

$$\mathcal{I}^+(x^+,\, x^-) \equiv \frac{\kappa}{2}\int_{u^+}^{x^+} ds\,(s-x^+)(s-x^-)\,T^{(0)}_{++}(s,\, x^-)\,, \tag{4.4.19}$$

$$\mathcal{I}^-(x^+,\, x^-) \equiv \frac{\kappa}{2}\int_{u^-}^{x^-} ds\,(s-x^+)(s-x^-)\,T^{(0)}_{--}(x^+,\, s)\,. \tag{4.4.20}$$

This solution resembles the one in the AP case (4.2.27). However, the energy-momentum tensor is not (anti-)holomorphic, hence be careful for calculating partial derivatives of $\mathcal{I}^\pm$.

It would be instructive to demonstrate, for example, the calculation of the partial derivative of $\mathcal{I}^-$:

$$\begin{aligned} &\partial_+\mathcal{I}^-(x^+,\, x^-) \\ &= \frac{\kappa}{2}\int_{u^-}^{x^-} ds\left[-(s-x^-)T^{(0)}_{--}(x^+,\, s) + (s-x^+)(s-x^-)\partial_+ T^{(0)}_{--}(x^+,\, s)\right] \\ &= \frac{\kappa}{2}\int_{u^-}^{x^-} ds\left[-(s-x^-)T^{(0)}_{--}(x^+,\, s) - (s-x^+)(s-x^-)\partial_s T^{(0)}_{+-}(x^+,\, s) - 2(s-x^-)T^{(0)}_{+-}(x^+,\, s)\right] \\ &= \frac{\kappa}{2}\int_{u^-}^{x^-} ds\left[-(s-x^-)T^{(0)}_{--}(x^+,\, s) - (x^+-x^-)T^{(0)}_{+-}(x^+,\, s)\right]. \end{aligned} \tag{4.4.21}$$

From the second line to the third line, we have used the conservation law (4.4.14) and also assumed that the boundary terms vanish. Similarly, one can also evaluate the second-order derivative as follows:

$$
\begin{aligned}
&\partial_+^2 \mathcal{I}^-(x^+,\, x^-) \\
&= \frac{\kappa}{2}\int_{u^-}^{x^-} \mathrm{d}s\, \Big[-(s-x^-)\partial_+ T^{(0)}_{--}(x^+,\, s) - T^{(0)}_{+-}(x^+,\, s) - (x^+-x^-)\partial_+ T^{(0)}_{+-}(x^+,\, s)\Big] \\
&= \frac{\kappa}{2}\int_{u^-}^{x^-} \mathrm{d}s\, \Big[-(s-x^-)\partial_s T^{(0)}_{+-}(x^+,\, s) - 2\frac{(s-x^-)}{x^+-s}T^{(0)}_{+-}(s,\, x^-) \\
&\qquad\qquad - T^{(0)}_{+-}(x^+,\, s) - (x^+-x^-)\partial_+ T^{(0)}_{+-}(x^+,\, s)\Big] \\
&= \frac{\kappa}{2}\int_{u^-}^{x^-} \mathrm{d}s\, (x^+-x^-)\Big[-\frac{2}{x^+-s}T^{(0)}_{+-}(x^+,\, s) - \partial_+ T^{(0)}_{+-}(x^+,\, s)\Big] \\
&= \frac{\kappa}{2}(x^+-x^-)T^{(0)}_{++}(x^+,\, x^-)\,, \\
&\partial_+\partial_- \mathcal{I}^-(x^+,\, x^-) = \frac{\kappa}{2}\int_{u^-}^{x^-} \mathrm{d}s\, \Big[T^{(0)}_{--}(x^+,\, s) - T^{(0)}_{+-}(x^+,\, s)\Big]\,.
\end{aligned}
\tag{4.4.22}
$$

Thus, one can directly confirm the solution (4.4.18) satisfies the equations in (4.4.17).

Chapter 5

Conclusion

5.1 Summary of the thesis

In this thesis, we have discussed the generalization of the relationship between a $T\bar{T}$-deformation and a gravitational perturbation for curved space-times.

Using a technique of an integrable deformation, called the Yang-Baxter deformation, we discussed the deformation of AdS_2 space-time and obtained the deformed JT model. This deformed model has a hyperbolic-type dilaton potential. The space-time near the boundary is dramatically changed and the singularity surface is generated by the deformation. Such a singularity surface is one of characteristic feature of the deformed backgrounds derived by using a technique of the YB-deformations. (For example, see [72] in the cases of the deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring.) We constructed general solutions of the deformed model and showed that a deformed black hole solution is contained as a solution. We calculated the energy and the entropy of the deformed black hole and showed that the thermodynamic quantities are reproduced from the physical quantities on the singularity surface. That means that the holographic principle is realized even in the deformed space-time, and that the singularity surface functions as a holographic screen originally proposed in [72]. We also discussed the deformed system coupling to a conformal matter and similarly constructed a deformed black hole solution. Again, we can calculate the thermodynamic quantities and reproduce from ones on the singularity surface. As an interesting property, the deformed model is classically equivalent to the Liouville dilaton gravity with a negative cosmological constant moving into a proper frame by a certain Weyl transformation depending on the dilaton. This Liouville dilaton gravity supports AdS_2 space-time as a vacuum solution. The Liouville gravity can be embedded in higher-dimensional Einstein gravity, we expect that above results are also applicable to higher-dimensional theory.

W also discussed a gravitational perturbation in two-dimensional dilaton gravity models. We first reviewed Dubovsky et al. discussion of the flat-space JT model and demonstrated that a gravitational perturbation can be interpreted as a $T\bar{T}$-deformation of the original matter action. In order to generalize this result for curved space-times, we considered general 2D dilaton gravity models coupled to matters and derived a quadratic action. We found the conditions under which a gravitational perturbation can be interpreted as a $T\bar{T}$-deformation. The JT gravity satisfy the conditions and we solved the equations of motion perturbatively. However, in this cases, one can consider only infinitesimally deformations from the JT gravity coupling to conformal matters. However, the $T\bar{T}$-deformation is irrelevant and breaks the conformal symmetry. Thus, one cannot obtain the finite $T\bar{T}$-flow in the theory space. We resolved this problem by considering the Liouville gravity model on AdS space-time discussed in Chapter 3, and we showed the correspondence between a $T\bar{T}$-deformation and a gravitational perturbation for a finite deformation parameter. We also discussed the relationship between the parameter of the $T\bar{T}$-deformation and one of the Yang-Baxter deformation. This indicates that the sign of the deformation parameter is directly related to the spectrum of the deformed theory. We also provide a exact gravitational solution.

5.2 Open problems

At the end of this thesis, we would like to mention some open problems.

Recently, the relationship between the matrix model and two-dimensional gravity has been refocused to understand gauge/gravity correspondence at the quantum level. This is an attempt to apply the theory of the matrix integral discussed in the '90s to the cases in space-times with boundaries. In recent works [92, 93], it has been discussed that the partition function of the Jackiw-Teitelboim gravity can be reproduced from the large N limit of the matrix model. Quantum aspects of various 2D gravity models are investigated, for example, JT gravity at a finite cut-off [94], deformed JT gravity modified by a certain vertex operator [95, 96]. It is expected that these quantum aspects play an important role to prove gauge/gravity correspondence.

Analyzing the integrable deformation of the Jackiw-Teitelboim gravity from the viewpoint of matrix integrals, it would be possible to understand the correspondence between 2D gravity and 1D quantum system, and to construct a new exact model for gauge/gravity correspondence at the quantum level. In particular, we derived the deformed JT model by using a technique of the integrable deformation, thus we expect that some physical variables such as the partition function may be calculated exactly. Generalization of the $T\bar{T}$-deformation in curved space-time has also been discussed and evaluated a partition function [6, 7]. It is interesting to consider the relation between our work and these results.

Our research also reveals that the Liouville gravity supporting AdS space-time is closely related to $T\bar{T}$-deformation. Our deformed JT model have been discussed in the context of non-critical string theories and matrix models [97, 98]. We would like to comment the significance of the non-local solution (4.4.18). It is interesting to understand the origin of such a non-locality in the view point of matrix models and 1D quantum mechanics. One of the most significant models is Sachdev-Ye-Kitaev (SYK) model [94]. SYK model is a fermionic many body system with quenched interaction. In large N limit, this model is solvable and the spectrum and correlation functions have been evaluated [82–84]. (In detail, see review [85].) There are many generalizations of SYK model like q-points interaction [86], supersymmetric model [87], and a model without disorder [88–90]. In particular, a conformal SYK model [91], which has a non-local kinetic term to preserve conformal invariance, may relate to our model rather than an original SYK model.

On the other hand, the $T\bar{T}$-deformation is expected to be closely related to the gravity. As we explained above, the $T\bar{T}$-deformation of original matters action is interpreted as a gravitational perturbation under certain conditions. In this thesis, we have focused only on the classical $T\bar{T}$-deformation, which means we considered on-shell variables. It is interesting to investigate $T\bar{T}$-deformation at the quantum level. In [99, 100], it was conjectured that the $T\bar{T}$-deformation is equivalent to the non-critical string theory. However, dealing with renormalization of the deformation parameter, negative norm states appear in general [101]. It has been pointed out that 2D diffeomorphism invariance is a key to avoid such a negative norm states. The quantum aspect of the $T\bar{T}$-deformation seems an important clue to understand quantum gravity.

We hope that our deformed model would provide a new arena to study the correspondence between gravity theories and integrable systems.

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Appendix A

Useful formulae

It would be helpful for readers to summarize formulae useful in computing some quantities in this paper.

We consider a small perturbation around a given metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{A.0.1}$$

and expand some geometric quantities in terms of the metric fluctuation up to and including the second-order in $h_{\mu\nu}$. In the following we will drop the higher order terms. The explicit expressions of the perturbed quantities are useful in deriving the quadratic action (4.1.15).

We start by expanding the inverse and the determinant of the perturbed metric $g_{\mu\nu}$. These are given by

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\rho} h^{\rho\nu} \,, \tag{A.0.2}$$

$$\sqrt{-g} = \sqrt{-\bar{g}} \left(1 + \frac{1}{2} h - \frac{1}{4} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) \right) \,, \tag{A.0.3}$$

where $h^{\mu\nu} = \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} h_{\rho\sigma}$, $h = \bar{g}^{\mu\nu} h_{\mu\nu} = h^{\mu}_{\mu}$.

The Christoffel symbol is defined as

$$\Gamma^{\rho}_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} (\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}) \,, \tag{A.0.4}$$

and can be expanded as

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \Gamma^{(1)\rho}_{\mu\nu} + \Gamma^{(2)\rho}_{\mu\nu} \,, \tag{A.0.5}$$

where the first and second order terms in the fluctuation are

$$\Gamma^{(1)\rho}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}(\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu})\,, \tag{A.0.6}$$

$$\Gamma^{(2)\rho}_{\mu\nu} = -\frac{1}{2}h^{\rho\sigma}(\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu})\,. \tag{A.0.7}$$

The Riemann tensor and the Ricci tensor are defined as

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\nu\alpha}\,, \tag{A.0.8}$$

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu}\,. \tag{A.0.9}$$

The Ricci tensor can be expanded as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}\,, \tag{A.0.10}$$

where $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\bar{\nabla}^\rho\left(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}\right) - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h\,, \tag{A.0.11}$$

$$\begin{aligned} R^{(2)}_{\mu\nu} = {} & \frac{1}{2}\bar{\bar{\nabla}}_\nu(h^{\rho\sigma}\bar{\bar{\nabla}}_\mu h_{\rho\sigma}) - \frac{1}{2}\bar{\bar{\nabla}}_\rho\left[h^{\rho\sigma}\left(\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}\right)\right] \\ & + \frac{1}{4}\bar{\nabla}^\rho h\left(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}\right) \\ & - \frac{1}{4}\bar{g}^{\alpha\beta}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_\mu h_{\alpha\rho} + \bar{\nabla}_\alpha h_{\rho\mu} - \bar{\nabla}_\rho h_{\alpha\mu}\right)\left(\bar{\nabla}_\sigma h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\sigma} - \bar{\nabla}_\beta h_{\sigma\nu}\right)\,. \end{aligned} \tag{A.0.12}$$

The Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ can be expanded as

$$R = \bar{R} + R^{(1)} + R^{(2)}\,, \tag{A.0.13}$$

where $R^{(1)}$ and $R^{(2)}$ are given by

$$\begin{aligned} R^{(1)} &= \bar{g}^{\mu\nu}R^{(1)}_{\mu\nu} - h^{\mu\nu}\bar{R}_{\mu\nu} \\ &= \bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - h^{\mu\nu}\bar{R}_{\mu\nu}\,, \end{aligned} \tag{A.0.14}$$

$$R^{(2)} = \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} - h^{\mu\nu}R^{(1)}_{\mu\nu} + h^{\mu}_{\rho}h^{\rho\nu}\bar{R}_{\mu\nu}\,, \tag{A.0.15}$$

with

$$\begin{aligned}
\bar{g}^{\mu\nu} R^{(2)}_{\mu\nu} = & \frac{1}{2}\bar{\nabla}^{\mu}(h^{\rho\sigma}\bar{\nabla}_{\mu}h_{\rho\sigma}) - \bar{\nabla}_{\rho}\left[h^{\rho\sigma}\left(\bar{\nabla}^{\mu}h_{\sigma\mu} - \frac{1}{2}\bar{\nabla}_{\sigma}h\right)\right] \\
& + \frac{1}{2}\bar{\nabla}^{\rho}h\left(\bar{\nabla}^{\mu}h_{\rho\mu} - \frac{1}{2}\bar{\nabla}_{\rho}h\right) \\
& - \frac{1}{4}\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_{\mu}h_{\alpha\rho} + \bar{\nabla}_{\alpha}h_{\rho\mu} - \bar{\nabla}_{\rho}h_{\alpha\mu}\right)\left(\bar{\nabla}_{\sigma}h_{\beta\nu} + \bar{\nabla}_{\nu}h_{\beta\sigma} - \bar{\nabla}_{\beta}h_{\sigma\nu}\right) \\
= & \bar{\nabla}^{\rho}\left(\frac{3}{4}h^{\mu\nu}\bar{\nabla}_{\rho}h_{\mu\nu} - h^{\sigma}_{\rho}\left(\bar{\nabla}^{\mu}h_{\sigma\mu} - \frac{1}{2}\bar{\nabla}_{\sigma}h\right)\right) + \frac{1}{2}\bar{\nabla}^{\rho}h\left(\bar{\nabla}^{\mu}h_{\rho\mu} - \frac{1}{2}\bar{\nabla}_{\rho}h\right) \\
& - \frac{1}{2}\bar{\nabla}_{\mu}h_{\alpha\beta}\bar{\nabla}^{\alpha}h^{\beta\mu} - \frac{1}{4}h^{\mu\nu}\bar{\nabla}^{2}h_{\mu\nu}\,.
\end{aligned} \tag{A.0.16}$$

Appendix B

A derivation of the quadratic action

We explain how to derive the quadratic action (4.1.15) in detail.

By using (A.0.2) and (A.0.3), it is straightforward to derive the following quadratic action:

$$\begin{aligned} S^{(2)} = \frac{1}{16\pi G_N} \int \mathrm{d}^2 x \sqrt{-\bar{g}} \Bigg(& \left[\frac{1}{2} h \, \left(\bar{R} - U'(\bar{\phi}) \right) + R^{(1)} - \frac{1}{2} U''(\bar{\phi}) \sigma \right] \sigma \\ & + \frac{1}{4} U(\bar{\phi}) \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) + \bar{\phi} \left[R^{(2)} + \frac{1}{2} h R^{(1)} - \frac{1}{4} \bar{R} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) \right] \Bigg) \\ & + \frac{1}{2} \int \mathrm{d}^2 x \sqrt{-\bar{g}} \, h^{\mu\nu} t_{\mu\nu} \, . \end{aligned} \tag{B.0.1}$$

By using the explicit expression (A.0.14) of $R^{(1)}$ and the vanishing of the two dimensional Einstein tensor (4.1.6), this can be rewritten as

$$\begin{aligned} S^{(2)} = \frac{1}{16\pi G_N} \int \mathrm{d}^2 x \sqrt{-\bar{g}} \Bigg(& \left[\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2} h \, U'(\bar{\phi}) - \frac{1}{2} U''(\bar{\phi}) \sigma \right] \sigma \\ & + \frac{1}{4} U(\bar{\phi}) \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) + \bar{\phi} \left[R^{(2)} + \frac{1}{2} h R^{(1)} - \frac{1}{4} \bar{R} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) \right] \Bigg) \\ & + \frac{1}{2} \int \mathrm{d}^2 x \sqrt{-\bar{g}} \, h^{\mu\nu} t_{\mu\nu} \, . \end{aligned} \tag{B.0.2}$$

We then rewrite the terms proportional to the background dilaton $\bar{\phi}$ in (B.0.2). For this purpose, it is helpful to employ the following identity:

$$0 = h^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \, . \tag{B.0.3}$$

The identity (B.0.3) gives at $\mathcal{O}(h_{\mu\nu}^2)$

$$-\frac{1}{4}\bar{R}\left(h_{\mu\nu}h^{\mu\nu}-\frac{1}{2}h^2\right)=-\frac{1}{2}h^{\mu\nu}R^{(1)}_{\mu\nu}+\frac{1}{4}h\left(\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}-\bar{\nabla}^2 h\right)\,. \tag{B.0.4}$$

By using the identity (B.0.4) and the formula (A.0.15) together with the fact that the Einstein tensor vanishes in two dimensions (4.1.6), the part proportional to the background dilaton $\bar{\phi}$ in (B.0.2) can be rewritten as

$$\begin{aligned}&\bar{\phi}\left[R^{(2)}+\frac{1}{2}hR^{(1)}-\frac{1}{4}\bar{R}\left(h_{\mu\nu}h^{\mu\nu}-\frac{1}{2}h^2\right)\right]\\&=\bar{\phi}\left[\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}-\frac{1}{2}h^{\mu\nu}R^{(1)}_{\mu\nu}+\frac{1}{4}h\left(\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}-\bar{\nabla}^2 h\right)\right]\,.\end{aligned} \tag{B.0.5}$$

Substituting (A.0.11) and (A.0.16) into (B.0.5), the above expression becomes

$$\begin{aligned}\text{(B.0.5)}=\;&\bar{\phi}\bar{\nabla}^\rho\bigg(\frac{3}{4}h^{\mu\nu}\bar{\nabla}_\rho h_{\mu\nu}-\frac{1}{4}h\bar{\nabla}_\rho h-h_{\rho\sigma}\bar{\nabla}_\mu h^{\mu\sigma}\\&\qquad-\frac{1}{2}h^{\mu\nu}\bar{\nabla}_\mu h_{\rho\nu}+\frac{1}{4}h\bar{\nabla}^\sigma h_{\sigma\rho}+\frac{3}{4}h_{\rho\mu}\bar{\nabla}^\mu h\bigg)\\=\;&\frac{1}{4}\bar{\nabla}^2\bar{\phi}\left(h_{\mu\nu}h^{\mu\nu}-\frac{1}{2}h^2\right)+\frac{1}{8}\bar{\nabla}^2\bar{\phi}\,h_{\mu\nu}h^{\mu\nu}\\&+\bar{\phi}\bar{\nabla}^\rho\bigg(-h_{\rho\sigma}\bar{\nabla}_\mu h^{\mu\sigma}-\frac{1}{2}h^{\mu\nu}\bar{\nabla}_\mu h_{\rho\nu}+\frac{1}{4}h\bar{\nabla}^\sigma h_{\sigma\rho}+\frac{3}{4}h_{\rho\mu}\bar{\nabla}^\mu h\bigg)\\&+\bar{\nabla}^\rho\left(\bar{\phi}\left[\frac{3}{4}h^{\mu\nu}\bar{\nabla}_\rho h_{\mu\nu}-\frac{1}{4}h\bar{\nabla}_\rho h\right]-\bar{\nabla}_\rho\bar{\phi}\left[\frac{3}{8}h^{\mu\nu}-\frac{1}{8}h^2\right]\right)\,.\end{aligned} \tag{B.0.6}$$

Furthermore, it should be useful to rewrite the second line from the end in (B.0.6) so that it contains only the terms containing $\bar{\nabla}_\mu h$ and $\bar{\nabla}^\mu h_{\mu\nu}$. By using the fact that the background dilaton $\bar{\phi}$ satisfies (4.1.9), the second term in that line can be rewritten as

$$\begin{aligned}-\frac{1}{2}\bar{\phi}\bar{\nabla}^\rho(h^{\mu\nu}\bar{\nabla}_\mu h_{\rho\nu})&=-\frac{1}{2}(\bar{\nabla}_\mu\bar{\nabla}^\rho\bar{\phi})h^{\mu\nu}h_{\rho\nu}-\frac{1}{2}\bar{\nabla}^\rho\bar{\phi}\,h_{\rho\nu}\bar{\nabla}_\mu h^{\mu\nu}\\&\quad-\frac{1}{2}\bar{\nabla}^\rho(\bar{\phi}\,h^{\mu\nu}\bar{\nabla}_\mu h_{\rho\nu})+\frac{1}{2}\bar{\nabla}_\mu(\bar{\nabla}^\rho\bar{\phi}\,h^{\mu\nu}h_{\rho\nu})\\&=-\frac{1}{4}\bar{\nabla}^2\bar{\phi}\,h_{\mu\nu}h^{\mu\nu}-\frac{1}{2}\bar{\nabla}^\rho\bar{\phi}\,h_{\rho\nu}\bar{\nabla}_\mu h^{\mu\nu}\\&\quad-\frac{1}{2}\bar{\nabla}^\rho(\bar{\phi}\,h^{\mu\nu}\bar{\nabla}_\mu h_{\rho\nu})+\frac{1}{2}\bar{\nabla}_\mu(\bar{\nabla}^\rho\bar{\phi}\,h^{\mu\nu}h_{\rho\nu})\,.\end{aligned} \tag{B.0.7}$$

As a result, we obtain

$$
\begin{aligned}
&\bar{\phi}\left[R^{(2)}+\frac{1}{2}hR^{(1)}-\frac{1}{4}\bar{R}\left(h_{\mu\nu}h^{\mu\nu}-\frac{1}{2}h^2\right)\right]\\
&=\frac{1}{4}\bar{\nabla}^2\bar{\phi}\left(h_{\mu\nu}h^{\mu\nu}-\frac{1}{2}h^2\right)-\frac{1}{8}\bar{\nabla}^2\bar{\phi}\,h_{\mu\nu}h^{\mu\nu}-\frac{1}{2}\bar{\nabla}^\rho\bar{\phi}\,h_{\nu\rho}\bar{\nabla}_\mu h^{\mu\nu}\\
&\quad+\bar{\phi}\bar{\nabla}^\rho\left(-h_{\rho\sigma}\bar{\nabla}_\mu h^{\mu\sigma}+\frac{1}{4}h\bar{\nabla}^\sigma h_{\sigma\rho}+\frac{3}{4}h_{\rho\mu}\bar{\nabla}^\mu h\right)-\frac{1}{2}\bar{\nabla}^\rho\bar{\phi}\,h_{\rho\nu}\bar{\nabla}_\mu h^{\mu\nu}\\
&\quad+\bar{\nabla}^\rho\left(\bar{\phi}\left[\frac{3}{4}h^{\mu\nu}\bar{\nabla}_\rho h_{\mu\nu}-\frac{1}{4}h\bar{\nabla}_\rho h-\frac{1}{2}h^{\mu\nu}\bar{\nabla}_\mu h_{\nu\rho}\right]-\bar{\nabla}_\rho\bar{\phi}\left[\frac{3}{8}h^{\mu\nu}-\frac{1}{8}h^2\right]\right)\\
&\quad+\frac{1}{2}\bar{\nabla}_\mu(\bar{\nabla}^\rho\bar{\phi}\,h^{\mu\nu}h_{\nu\rho})\,. \qquad \text{(B.0.8)}
\end{aligned}
$$

Finally, by using (B.0.8) and the on-shell condition of the background dilaton (4.1.9) and doing partial integration, the quadratic action $S^{(2)}$ becomes

$$
\begin{aligned}
S^{(2)}=\frac{1}{16\pi G_N}\int\mathrm{d}^2x\,\sqrt{-\bar{g}}\,\Bigg(&\left[\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}-\bar{\nabla}^2h-\frac{1}{2}h\,U'(\bar{\phi})-\frac{1}{2}U''(\bar{\phi})\sigma\right]\sigma\\
&-\frac{1}{8}\bar{\nabla}^2\bar{\phi}\,h_{\mu\nu}h^{\mu\nu}-\bar{\nabla}^\rho\bar{\phi}\left[-\frac{1}{2}h_{\rho\sigma}\bar{\nabla}_\mu h^{\mu\sigma}+\frac{1}{4}h\bar{\nabla}^\sigma h_{\sigma\rho}+\frac{3}{4}h_{\rho\mu}\bar{\nabla}^\mu h\right]\Bigg)\\
&+\frac{1}{2}\int\mathrm{d}^2x\,\sqrt{-\bar{g}}\,h^{\mu\nu}t_{\mu\nu}\,, \qquad \text{(B.0.9)}
\end{aligned}
$$

where we have ignored the total derivative terms. This is the quadratic action in (4.1.15).

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