Positive solutions to semi-linear elliptic problems on metric graphs *

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1 Introduction

We consider positive solutions to semi-linear elliptic problems on a metric graph G:

$$-\epsilon^2 \Delta u + u = f(u) \text{ on } G, \tag{1.1}$$

where $\epsilon > 0$ is a parameter and we assume the Neumann boundary condition or the Dirichlet boundary condition on the ends of G.

Semi-linear elliptic problems on a domain in $\mathbb{R}^n$ has been studied very well. Many authors have obtained various results about the existence and non-existence of solutions, the multiplicity of solutions, the asymptotic behavior of solutions, and so on. Recently, in [1–3, 5, 7], they studied this kind of problems on graphs. Motivated by those results, we study the asymptotic behavior of positive solutions as $\epsilon \to 0$. In this paper, we introduce our recent results and show part of our results. To state our setting and results, we use following notations in graph theory.

- $G = G(V, E)$ is a graph, where V is a set of vertices and E is a set of edges. We always assume that G is connected and the number of edges $\#E$ is finite.
- G is a metric graph if each edge $e \in E$ is isometric to an interval $[0, \ell(e)]$ ($[0, \infty)$ if $\ell(e) = \infty$), where $\ell(e) \in (0, \infty]$ is the length of e. We identify e with $[0, \ell(e)]$.
- A metric graph G is compact if $\ell(e) < \infty$ for each $e \in E$.
- $e \succ v$ means that e is incident to v.
- $\deg v$ is the number of edges that are incident to v. We assume $\deg v \neq 2$ for any $v \in V$.
- V_{int} is the set of all vertices with $\deg v \geq 3$.
- V_{end} is the set of all vertices with $\deg v = 1$, thus $V_{\text{int}} \cup V_{\text{end}} = V$.

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- A loop is an edge that connects a vertex to itself.

Throughout this paper, we always assume the following assumptions:

(f1) $f \in C(\mathbb{R}, \mathbb{R})$ is a locally Lipschitz odd function.

(f2) $\lim_{t\to 0} f(t)/t = 0$.

(f3) $\lim_{t\to\infty} f(t)/t^q = 0$ for some $q > 0$.

(f4) $f(t)/t$ is strictly increasing on $(0, \infty)$.

(f5) $\lim_{t\to\infty} F(t)/t^2 = \infty$, where $F(t) = \int_0^t f(s)\,ds$.

A typical example is $f(u) = |u|^{p-1}u$ $(1 < p < \infty)$. To formulate (1.1), we use a variational structure. Let $H^1(G)$ be the set of every continuous function u on G with $u^{(e)} \in H^1(e)$ for each edge $e \in G$, where $u^{(e)}$ is the restriction of u on e. Then we can check that $H^1(G)$ is a Hilbert space with norm

$$\|u\|_{H^1(G)}^2 := \int_G |\nabla u|^2 + u^2\,dx := \sum_{e\in E} \int_0^{\ell(e)} |\nabla u^{(e)}|^2 + (u^{(e)})^2\,dx,$$

where $\nabla = \frac{d}{dx}$. We define $L^p(G)$-norm similarly. Let J_ϵ be a functional on $H^1(G)$ such that

$$J_\epsilon(u) := \frac{\epsilon}{2}\int_G |\nabla u|^2\,dx + \frac{1}{2\epsilon}\int_G u^2\,dx - \frac{1}{\epsilon}\int_G F(u)\,dx.$$

Then $J_\epsilon \in C^1(H^1(G), \mathbb{R})$. Each critical point u_ϵ of J_ϵ satisfies (1.2) as the Euler-Lagrange equation.

$$\begin{cases} -\epsilon^2 \Delta u_\epsilon^{(e)} + u_\epsilon^{(e)} = f(u_\epsilon^{(e)}) & \text{for each edge } e \in E, \\ \sum_{e\succ v} \partial u_\epsilon^{(e)}(v) = 0 & \text{for each vertex } v \in V_{\mathrm{int}}, \\ \partial u_\epsilon^{(e)}(v) = 0 & \text{for each } e \text{ with } e \succ v \text{ and } v \in V_{\mathrm{end}}, \\ u_\epsilon^{(e)}(v) = u_\epsilon^{(e')}(v) & \text{if } e \succ v \text{ and } e' \succ v, \end{cases} \tag{1.2}$$

where $\Delta = \frac{d^2}{dx^2}$, and $\partial u_\epsilon^{(e)}(v)$ is the outward derivative of $u_\epsilon^{(e)}$ at v. In (1.2), the second line is the Kirchhoff law, the third line is the Neumann boundary condition, and the last line is the continuity condition at v. Put

$$\sigma_\epsilon := \inf_{\substack{u\in H^1(G)\\ u\neq 0}} \sup_{t>0} J_\epsilon(tu).$$

Then, for each $\epsilon > 0$, there exists a positive solution u_ϵ with $J_\epsilon(u_\epsilon) = \sigma_\epsilon$. u_ϵ is called a least energy solution. In [7], they proved that u_ϵ is a constant solution for sufficiently large $\epsilon > 0$. Our results are the asymptotic behavior of least energy solutions as $\epsilon \to 0$.

Theorem 1.1 ([6])**.** *Assume that G is a compact metric graph with $V_{\mathrm{int}} \neq \emptyset$ and $V_{\mathrm{end}} \neq \emptyset$. For each $\epsilon > 0$, let u_ϵ be a least energy positive solution. Then* (i)–(iv) *hold.*

(i) *For sufficiently small $\epsilon > 0$, u_ϵ has exactly one local maximum point x_ϵ. Moreover, $x_\epsilon \in V_{\mathrm{end}}$ holds.*

(ii) *Let* e_ϵ *be the edge with* $e_\epsilon \succ x_\epsilon$. *We use identification* $e_\epsilon = [0, \ell(e_\epsilon)]$ *with* $x_\epsilon = 0$. *Then,*

$$u_\epsilon^{(e_\epsilon)}(\epsilon x) \to \Phi(x) \text{ as } \epsilon \to 0 \text{ in } C^2_{\mathrm{loc}}([0,\infty))$$

holds. On $G \backslash e_\epsilon$, u_ϵ *converges to* 0 *uniformly as* $\epsilon \to 0$, *that is,* $\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^\infty(G_\epsilon \backslash e_\epsilon)} = 0$. *Here,* Φ *is a unique solution to*

$$\begin{cases} -\Delta\Phi + \Phi = f(\Phi), \ \Phi > 0 \text{ on } \mathbb{R}, \\ \nabla\Phi(0) = 0, \ \lim_{|x|\to\infty} \Phi(x) = 0. \end{cases} \tag{1.3}$$

(iii) *For sufficiently small* $\epsilon > 0$, *the edge* e_ϵ *is longest in* $E_{\mathrm{end}} := \{e \in E; e \succ v \in V_{\mathrm{end}}\}$, *that is,*

$$\ell(e_\epsilon) = \ell_{\max} := \max_{e \in E_{\mathrm{end}}} \ell(e).$$

(iv) *It holds that*

$$\sigma_\epsilon = \frac{\sigma}{2} + \exp\left(\frac{-2\ell(e_\epsilon)}{\epsilon}(1 + o(1))\right) \text{ as } \epsilon \to 0,$$

where σ *is the energy of* Φ.

Remark 1.2. For the uniqueness of solutions to (1.3), see [4, Theorem 5].

Moreover, in the typical case with $p > 2$, we can get more precise information about the asymptotic behavior. Let E'_{end} be the set of longest edges in E_{end}, that is,

$$E'_{\mathrm{end}} := \{e \in E_{\mathrm{end}}; \ell(e) = \ell_{\max}\}.$$

Since G is connected, if $V_{\mathrm{int}} \neq \emptyset$, for each $e \in E'_{\mathrm{end}}$, there exists $v \in V_{\mathrm{int}}$ such that $e \succ v$. We denote such vertex by $v(e)$. By Theorem 1.1, $e_\epsilon \in E'_{\mathrm{end}}$ holds for sufficiently small $\epsilon > 0$.

Theorem 1.3 ([9])**.** *Assume that* G *is a compact metric graph with* $V_{\mathrm{int}} \neq \emptyset$. *In addition, we suppose* $f(t) = |t|^{p-1}t$ *for some* $p \in (2, \infty)$. *Then, same conclusions as in Theorem 1.1 hold. Moreover,*

(i) *For sufficiently small* $\epsilon > 0$, $\deg v(e_\epsilon)$ *is a smallest number of* $\deg v(e)$ *among* $e \in E'_{\mathrm{end}}$.

(ii) *It holds that*

$$\sigma_\epsilon = \frac{\sigma}{2} + C_p \frac{\deg v(e_\epsilon) - 2}{\deg v(e_\epsilon)} \exp\left(\frac{-2\ell(e_\epsilon)}{\epsilon}\right)((1 + o(1))) \text{ as } \epsilon \to 0,$$

where C_p *is a positive constant depending only* p.

Remark 1.4. We can show similar results as Theorems 1.1 and 1.3 for (1.2) with the Dirichlet boundary condition. In the Dirichlet case, the maximum point x_ϵ converges the center of a edge which is a longest one in E.

In this paper, we give the outline of the proof for Theorem 1.1. The paper is organized as follows. In Section 2, we give preliminary results. In Section 3, we consider the asymptotic behavior of bounded energy solutions. In Section 4, we get more precise asymptotic behavior of least energy solutions and prove Theorem 1.1.

2 Preliminaries

Hereafter, for simplicity, we assume $f(t) = |t|^{p-1}t$ and $1 < p < \infty$. For a metric graph G, we define a functional I by

$$I(u,G) := \frac{1}{2}\int_G |\nabla u|^2 + u^2\,dx - \frac{1}{p+1}\int_G |u|^{p+1}\,dx = \frac{1}{2}\|u\|_{H^1(G)}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^p$$

for $u \in H^1(G)$. Let $G_\epsilon(V,\hat{E})$ be the dilation image of $G(V,E)$ with a scale factor $1/\epsilon$, that is, each $e \in E$ and $\hat{e} \in \hat{E}$ represent the same edge of the graph, and the length of $\hat{e} \in \hat{E}$ is $\ell(e)/\epsilon$. For $u \in H^1(G)$, we denote $\hat{u} \in H^1(G_\epsilon)$ which satisfies $\hat{u}(\hat{x}) = u(x)$, where $\hat{x}$ is defined by

$$\hat{x} = \begin{cases} v & \text{if } x = v \in V, \\ x/\epsilon \in \hat{e} = [0, \ell(e)/\epsilon] & \text{if } x \in e = [0,\ell(e)]. \end{cases}$$

Then, we have

$$J_\epsilon(u) = I(\hat{u}, G_\epsilon) \text{ for } u \in H^1(G).$$

Moreover,

$$J_\epsilon'(u) = 0 \text{ if and only if } I'(\hat{u}, G_\epsilon) = 0.$$

It is well-known that Φ is uniquely determined and has explicit formula.

$$\Phi(x) = M_0\left(\cosh\frac{p-1}{2}x\right)^{-2/(p-1)},$$

where $M_0 = ((p+1)/2)^{1/(p-1)}$.

Lemma 2.1. *It holds that* $\Phi(x) = \exp(-x(1+o(1)))$ *as* $|x| \to \infty$.

Proof. Using explicit formula, we can check it. □

Next, we recall a characterization of solutions on a interval. For this purpose, we consider an initial value problem

$$-\Delta U + U = |U|^{p-1}U, \quad U(0) = a, \quad \nabla U(0) = b. \tag{2.1}$$

By ODE theory, for any $a, b \in \mathbb{R}$, it has a unique solution on $\mathbb{R}$. Thus, to characterize solutions, it is sufficient to consider

$$-\Delta U + U = |U|^{p-1}U \text{ on } \mathbb{R}. \tag{2.2}$$

Using the phase-plane analysis, we can check the following:

Proposition 2.2. *The initial value problem* (2.1) *has a unique global solution on* $\mathbb{R}$*, hence the solution satisfies* (2.2)*. Let U be a solution of* (2.2)*. Then, one of the following is satisfied.*

(i) *U is a constant solution, that is, $U \equiv 0, \pm 1$.*

(ii) *There exists $d > 0$ such that U is a d-periodic solution.*

(iii) *U is a ground state, that is, there exists $y \in \mathbb{R}$ such that $U \equiv \pm\Phi(\cdot - y)$.*

Remark 2.3. By Proposition 2.2, up to translation, any solution of $-\Delta u + u = |u|^{p-1}u$ on a open interval is the restriction of U which satisfies one of (i)–(iii) in the proposition.

3 Asymptotic behavior of bounded energy solutions.

In this section, we assume that G is a compact metric graph.

3.1 H^1 and L^∞-boundedness

Lemma 3.1. *Let $(\hat{u}_\epsilon)_{\epsilon>0}$ be a family of critical points with bounded energy, that is, for each $\epsilon > 0$, $\hat{u}_\epsilon \in H^1(G_\epsilon)$ is a critical point of $I(\cdot, G_\epsilon)$, and the family satisfies $\limsup_{\epsilon\to 0} I(\hat{u}_\epsilon, G_\epsilon) < \infty$. Then,*

(i) *$(\hat{u}_\epsilon)_{\epsilon>0}$ is $H^1(G_\epsilon)$-bounded, that is, $\limsup_{\epsilon\to 0} \|\hat{u}_\epsilon\|_{H^1(G_\epsilon)} < \infty$.*

(ii) *$(\hat{u}_\epsilon)_{\epsilon>0}$ is $L^\infty(G_\epsilon)$-bounded, that is, $\limsup_{\epsilon\to 0} \|\hat{u}_\epsilon\|_{L^\infty(G_\epsilon)} < \infty$.*

Proof. (i): Since each u_ϵ is a critical point, we have

$$\begin{aligned} I'(u_\epsilon, G_\epsilon)u_\epsilon &= \|u_\epsilon\|^2_{H^1(G_\epsilon)} - \|u_\epsilon\|^{p+1}_{L^{p+1}(G_\epsilon)} = 0, \\ I(u_\epsilon, G_\epsilon) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u_\epsilon\|^2_{H^1(G_\epsilon)} = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u_\epsilon\|^{p+1}_{L^{p+1}(G_\epsilon)}, \end{aligned} \tag{3.1}$$

which mean the claim.

(ii): Let $-\infty \le a < b \le \infty$. If $b - a \ge 1$, by the Sobolev embedding theorem, there exists $C > 0$ which is independent of a and b such that

$$\|u\|_{L^\infty(a,b)} \le C\|u\|_{H^1(a,b)} \text{ for } u \in H^1(a,b). \tag{3.2}$$

We can assume $0 < \epsilon \le \epsilon_0$. For small ϵ_0, the length of each edge $\hat{e} \in G_\epsilon$ is grater than 1. Thus,

$$\|\hat{u}^{(\hat{e})}\|_{L^\infty(\hat{e})} \le C\|\hat{u}^{(\hat{e})}\|_{H^1(\hat{e})} \le C\|\hat{u}\|_{H^1(G_\epsilon)}$$

holds, where C is a constant independent of $\hat{u}$, $\hat{e}$, and ϵ. It implies the conclusion. □

3.2 Asymptotic behavior on edges

Lemma 3.2. *Similarly as in Lemma 3.1, let $(\hat{u}_\epsilon)_{\epsilon>0}$ be a family of solutions with bounded energy. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a subsequence of $\epsilon \to 0$ and fix $\hat{e} = [0, \ell(e)/\epsilon_n] \in G_{\epsilon_n}$. Suppose that there exists a sequence $(\hat{x}_n)_{n\in N}$ such that $\hat{x}_n \in [0, \ell(e)/\epsilon_n]$, $c := \lim_{n\to\infty} \hat{u}^{(\hat{e})}_{\epsilon_n}(\hat{x}_n) > 0$. Then, taking a subsequence if necessary, there exists $y \in \mathbb{R}$ such that*

$$\hat{u}^{(\hat{e})}_{\epsilon_n}(\cdot + \hat{x}_n) \to \Phi(\cdot + y) \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}).$$

Here, if necessary, we extend $\hat{u}^{(\hat{e})}_{\epsilon_n}$ onto $\mathbb{R}$ as a solution of (2.2)*. Moreover, if $\nabla \hat{u}^{(\hat{e})}_{\epsilon_n}(\hat{x}_n) = 0$ $(n \in \mathbb{N})$, $y = 0$ and $c = M_0$ holds.*

Proof. By Lemma 3.1, the assumption $\inf_{n\in\mathbb{N}} \ell_n > 0$, and the regularity of solutions, $\hat{u}^{(\hat{e})}_{\epsilon_n}$ are bounded with respect to $C^2(0, \ell(e)/\epsilon_n)$-norm. If necessary, we extend $\hat{u}^{(\hat{e})}_{\epsilon_n}$ onto $\mathbb{R}$ as a solution of (2.2). Then, $\hat{u}^{(\hat{e})}_{\epsilon_n}$ are bounded with respect to $C^2(\mathbb{R})$-norm. By the Arzelà–Ascoli theorem, there exists $u_\infty \in C^1(\mathbb{R})$ such that

$$\hat{u}^{(\hat{e})}_{\epsilon_n}(\cdot + \hat{x}_n) \to u_\infty \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}) \text{ as } n \to \infty.$$

By the regularity theorem, we have

$$\hat{u}^{(\hat{e})}_{\epsilon_n}(\cdot + \hat{x}_n) \to u_\infty \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}) \text{ as } n \to \infty.$$

In addition, since $\|\hat{u}^{(\hat{e})}_{\epsilon_n}(\cdot + x_n)\|_{H^1(-\hat{x}_n, \ell(e)/\epsilon_n - \hat{x}_n)}$ are bounded, for sufficiently large δ, we get $u_\infty \in H^1(-\infty, -\delta)$ or $u_\infty \in H^1(\delta, \infty)$. The limit u_∞ satisfies (2.2), and it holds that $\lim_{n\to\infty} u_n(x_n) = u_\infty(0) = c$ by the assumption. Thus, Proposition 2.2 implies $u_\infty = \Phi(\cdot + y)$, it means the conclusion. □

3.3 Asymptotic behavior at vertices

Hereafter, for simplicity, we use same notation for subsequences.

Lemma 3.3. *Let $(\hat{u}_\epsilon)_{\epsilon>0}$ be a family of solutions with bounded energy. Assume that, for a subsequence, there exists $v \in V$ such that $\lim_{\epsilon\to 0} \hat{u}_\epsilon(x)(v) = M > 0$. Then*

(i) *For each $\hat{e}_i = [0, \ell(e_i)/\epsilon] \succ v$ $(1 \le i \le \deg v)$, we choose its coordinate satisfying $v = 0$. Then, taking a subsequence if necessary,*

$$\hat{u}^{(\hat{e}_i)}_\epsilon \to \Phi(\cdot \pm y) \text{ in } C^2_{\mathrm{loc}}([0,\infty)) \text{ as } \epsilon \to 0$$

holds. Here $y \ge 0$ is uniquely determined by $\Phi(y) = M$.

(ii) *In (i), the number of edges $\hat{e}_i$ with $\lim_{\epsilon\to 0} \hat{u}^{(\hat{e}_i)}_\epsilon = \Phi(\cdot + y)$ equals the number of edges $\hat{e}_j$ with $\lim_{\epsilon\to 0} \hat{u}^{(\hat{e}_j)}_\epsilon = \Phi(\cdot - y)$. In particular, if $\deg v$ is odd, $y = 0$ and $M = M_0$ hold.*

(iii) $\liminf_{n\to\infty} \sigma_\epsilon \ge (\deg V)\sigma/2$ *holds.*

Proof. For each edge $\hat{e}_i$, we can apply Lemma 3.2 with $\hat{x}_n = 0$ to obtain (i). By the Kirchhoff law at v, we have $\sum_{i=1}^{\deg v} \nabla \hat{u}^{(\hat{e}_i)}_\epsilon(0) = 0$. Let $i_\pm$ be the number of edges such that the limit of $\hat{u}_\epsilon$ on the edge is $\Phi(\cdot \pm y)$, respectively. As $\epsilon \to 0$, we get

$$i_+ \nabla\Phi(y) + i_- \nabla\Phi(-y) = 0,$$

hence (ii) holds. By (3.1), we have

$$\sigma_\epsilon \ge \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u_\epsilon\|^{p+1}_{L^{p+1}(G_\epsilon)} \ge \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i=1}^{\deg v} \int_{\hat{e}_i} |\hat{u}^{(e_i)}_\epsilon|^{p+1}\, dx.$$

Using Fatou's lemma and (i), we obtain.

$$\liminf_{\epsilon\to 0} \sigma_\epsilon \ge \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i=1}^{\deg v} \int_{\hat{e}_i} \Phi^{p+1}(x \pm y)\, dx.$$

By (ii), if $\deg v$ is even, we have

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i=1}^{\deg v} \int_{\hat{e}_i} \Phi^{p+1}(x \pm y)\, dx \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right) \frac{\deg v}{2} \left(\int_0^\infty \Phi^{p+1}(x+y)\, dx + \int_0^\infty \Phi^{p+1}(x-y)\, dx\right) = \frac{(\deg v)\sigma}{2}.
\end{aligned}$$

If $\deg v$ is odd,

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_{i=1}^{\deg v}\int_{\hat{e}_i}\Phi^{p+1}\,dx=\left(\frac{1}{2}-\frac{1}{p+1}\right)(\deg v)\left(\int_0^\infty\Phi^{p+1}\,dx\right)=\frac{(\deg v)\sigma}{2}.$$

□

4 Asymptotic behavior of least energy solutions

Throughout this section, we assume that $V_{\mathrm{int}}\neq\emptyset$, $V_{\mathrm{end}}\neq\emptyset$, and u_ϵ is a least energy solution with its energy σ_ϵ for $\epsilon>0$.

4.1 Upper energy estimate

Lemma 4.1.

$$\sigma_\epsilon\leq\frac{\sigma}{2}+\exp\left(\frac{-2\ell_{\max}}{\epsilon}(1+o(1))\right)\quad as\ \epsilon\to 0$$

holds.

Proof. Choose $e\in E_{\mathrm{end}}$ such that $\ell(e)=\ell_{\max}$. We may assume that $\hat{e}=[0,\ell_{\max}/\epsilon]$ and $0\in V_{\mathrm{end}}$. We define a test function W_ϵ by $W_\epsilon\equiv 0$ on $G_\epsilon\setminus\hat{e}$ and

$$W_\epsilon^{(\hat{e})}(x)=\begin{cases}\Phi(x) & \text{if } 0\leq x\leq \ell_{\max}/\epsilon-1,\\ \Phi(\ell_{\max}/\epsilon-1)(\ell_{\max}/\epsilon-x) & \text{if } \ell_{\max}/\epsilon-1<x\leq\ell_{\max}/\epsilon.\end{cases}$$

Then, we can check that $W_\epsilon\in H^1(G_\epsilon)$ and

$$\sigma_\epsilon\leq\sup_{t>0}J_\epsilon(tW_\epsilon)=\sup_{t>0}I(tW_\epsilon,[0,\ell_{\max}/\epsilon]).$$

Moreover, there exists a global maximum point $t_\epsilon>0$. Since $\frac{d}{dt}I(tW_\epsilon,[0,\ell_{\max}/\epsilon])\big|_{t=t_\epsilon}=0$, $t_\epsilon>0$ is uniquely determined by

$$t_\epsilon^{p-1}=\frac{\|W_\epsilon\|^2_{H^1(0,\ell_{\max}/\epsilon)}}{\|W_\epsilon\|^{p+1}_{L^{p+1}(0,\ell_{\max}/\epsilon)}}.$$

By the definition of W_ϵ, we have

$$t_\epsilon^{p-1}\to\frac{\|\Phi\|^2_{H^1(0,\infty)}}{\|\Phi\|^{p+1}_{L^{p+1}(0,\infty)}}=1\text{ as }\epsilon\to 0,$$

hence $t_\epsilon=1+o(1)$. Using t_ϵ, we get

$$\begin{aligned}\sigma_\epsilon&\leq I(t_\epsilon W_\epsilon,[0,\ell_{\max}/\epsilon])\\&=I(t_\epsilon\Phi,[0,\infty))-I(t_\epsilon\Phi,[\ell_{\max}/\epsilon-1,\infty))+I(t_\epsilon W_\epsilon,[\ell_{\max}/\epsilon-1,\ell_{\max}/\epsilon]).\end{aligned}$$

First, we see

$$I(t_\epsilon\Phi,[0,\infty))\leq\sup_{t>0}I(t\Phi,[0,\infty))=\frac{\sigma}{2}.$$

Next, for sufficiently small ϵ, since Φ is small on $[\ell_{\max}/\epsilon-1,\infty)$, we have $|t_\epsilon\Phi|^{p+1}/(p+1) < t_\epsilon^2\Phi^2/4$ and

$$I(t_\epsilon\Phi,[\ell_{\max}/\epsilon-1,\infty)) \geq \frac{t_\epsilon^2}{4}\int_{\ell_{\max}/\epsilon-1}^{\infty} |\nabla\Phi|^2+\Phi^2\,dx > 0.$$

For sufficiently small $\epsilon>0$, we obtain

$$I(t_\epsilon W_\epsilon,[\ell_{\max}/\epsilon-1,\ell_{\max}/\epsilon]) \leq \frac{t_\epsilon^2}{2}\int_{\ell_{\max}/\epsilon-1}^{\ell_{\max}/\epsilon} |\nabla W_\epsilon|^2+W_\epsilon^2\,dx = \frac{2t_\epsilon^2}{3}\Phi^2(\ell_{\max}/\epsilon-1).$$

Therefore, since $t_\epsilon = 1+o(1)$, we have

$$\sigma_\epsilon \leq \frac{\sigma}{2}+\Phi^2(\ell_{\max}/\epsilon-1).$$

Applying Lemma 2.1, we get the conclusion. □

4.2 Number and position of maximum points

Lemma 4.2. *If $\epsilon>0$ is sufficiently small, u_ϵ has exactly one local maximum point x_ϵ. Moreover, it is an end vertex of G, that is, $x_\epsilon \in V_{\mathrm{end}}$.*

Proof. If $v\in V_{\mathrm{int}}$, then $\lim_{\epsilon\to 0}\hat u_\epsilon(v)=0$ holds. Indeed, if there exists a subsequence with $\hat u_\epsilon(v)\to M>0$, Lemma 3.3 (iii) contradicts Lemma 4.1.

Since the energy estimate and H^1-boundedness, for sufficiently small $\epsilon>0$, $\hat u_\epsilon$ is a non-constant solution. Hence, $\hat u_\epsilon$ has a local maximum point $\hat x_\epsilon$. Extracting a subsequence if necessary, we may assume that there exists $\hat e=[0,\ell(e)/\epsilon]\in G_\epsilon$ such that $\hat x_\epsilon\in\hat e$. Suppose that $\hat x_\epsilon\to\infty$ and $\ell(e)/\epsilon-\hat x_\epsilon\to\infty$. Then by Lemma 3.2 and the argument in the proof of Lemma 3.3, we have

$$\liminf_{\epsilon\to 0}\sigma_\epsilon \geq \left(\frac12-\frac{1}{p+1}\right)\liminf_{\epsilon\to 0}\int_{\hat e}|\hat u_\epsilon^{(e)}|^{p+1}\,dx \geq \left(\frac12-\frac{1}{p+1}\right)\int_{\mathbb{R}}|\Phi|^{p+1}\,dx=\sigma,$$

which contradicts Lemma 4.1. Thus, $(\hat x_\epsilon)_{\epsilon>0}$ or $(\ell(e)/\epsilon-\hat x_\epsilon)_{\epsilon>0}$ is bounded. We may assume $(\hat x_\epsilon)_{\epsilon>0}$ is bounded without loss of generality. Taking a subsequence, we have $\hat x_\epsilon\to\hat x_0$ for some $\hat x_0\geq 0$. By Lemma 3.2, we get

$$\hat u_\epsilon^{(\hat e)} \to \Phi(\cdot-\hat x_0) \text{ in } C^2_{\mathrm{loc}}([0,\infty)).$$

If $0\in V_{\mathrm{int}}$, as mentioned above, $\hat u_\epsilon^{(e)}(0)\to 0$ holds, hence $\Phi(-\hat x_0)=0$, which is a contradiction. Thus, we obtain $0\in V_{\mathrm{end}}$. By the Neumann boundary condition, we have $\Phi'(-\hat x_0)=0$, which implies $\hat x_0=0$. Thus,

$$\hat u_\epsilon^{(\hat e)}\to\Phi \text{ in } C^2_{\mathrm{loc}}([0,\infty))$$

holds. Next, we show $\hat x_\epsilon=0$ for sufficiently small $\epsilon>0$. Contrary, suppose that there exists a subsequence such that $\hat x_\epsilon>0$. By the mean value theorem, there exists $\hat x_\epsilon'\in(0,\hat x_\epsilon)$ such that

$$0=-\frac{\nabla\hat u_\epsilon^{(\hat e)}(\hat x_\epsilon)-\nabla\hat u_\epsilon^{(\hat e)}(0)}{\hat x_\epsilon-0} = -\Delta\hat u_\epsilon^{(\hat e)}(\hat x_\epsilon') = (\hat u_\epsilon^{(\hat e)}(\hat x_\epsilon'))^p-\hat u_\epsilon^{(\hat e)}(\hat x_\epsilon').$$

Taking a limit, since $\hat{x}_\epsilon \to 0$, we get

$$0 = \Phi^p(0) - \Phi(0) = M_0^p - M_0.$$

It contradicts the definition of M_0, hence $\hat{x}_\epsilon = 0 \in V_{\text{end}}$ for small ϵ.

Finally, we show the uniqueness of local maximum points. Suppose that there exist two local maximum point $\hat{x}_\epsilon$ and $\hat{x}'_\epsilon$ for small ϵ. Then, as mentioned above, $\hat{x}_\epsilon, \hat{x}'_\epsilon \in V_{\text{end}}$ holds. It means that there are two different edges $\hat{e}$ and $\hat{e}'$ such that $\hat{x}_\epsilon \in \hat{e}$ and $\hat{x}'_\epsilon \in \hat{e}'$. By similar arguments in Lemma 3.3, we have

$$\begin{aligned}\liminf_{\epsilon\to 0} \sigma_\epsilon &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right) \liminf_{\epsilon \to 0} \int_{\hat{e}} (\hat{u}_\epsilon^{(\hat{e})})^{p+1}\, dx + \left(\frac{1}{2} - \frac{1}{p+1}\right) \liminf_{\epsilon \to 0} \int_{\hat{e}'} (\hat{u}_\epsilon^{(\hat{e}')})^{p+1}\, dx \\ &\geq 2\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_0^\infty \Phi^{p+1}\, dx = \sigma,\end{aligned}$$

which contradicts the energy estimate. □

4.3 Lower energy estimate

To get precise lower energy estimate, we calculate

$$\sigma(0, L) := \inf_{\substack{u \in H^1(0,L) \\ u \neq 0}} \sup_{t>0} I(u, (0, L)).$$

Lemma 4.3. *For $\delta > 0$, there exists a constant $C > 0$ such that*

$$\sigma(0, L) \geq \frac{\sigma}{2} - C \exp\left(-2L(1-\delta)\right) \quad \textit{as } L \to \infty.$$

Proof. We fix $\delta > 0$. Let w_L be a least energy solution for $\sigma(0, L)$. We may assume that w_L is strictly decreasing positive solution on $[0, L]$, and there exists a constant $C > 0$ independent of $L \geq 1$ such that

$$w_L(x) \leq C e^{-(1-\delta)x}. \tag{4.1}$$

In addition, we extend w_L on $[0, \infty)$ by $w_L(x) := w_L(L)e^{L-x}$ for $x \geq L$. Then $w_L \in H^1(0, \infty)$ holds. By the characterization of $\sigma(0, L)$, for any $t > 0$, it holds that

$$\sigma(0, L) \geq I(tw_L, (0, L)) = I(tw_L, (0, \infty)) - I(tw_L, (L, \infty)).$$

Take $t_L > 0$ with $t_L^{p-1} = \|w_L\|^2_{H^1(0,\infty)} / \|w_L\|^{p+1}_{L^{p+1}(0,\infty)}$. Then,

$$I(t_L w_L, (0, \infty)) = \sup_{t>0} I(tw_L, (0, \infty)).$$

By the definition of σ, we have

$$\sup_{t>0} I(tw_L, (0, \infty)) \geq \frac{\sigma}{2}.$$

Thus, we get

$$\sigma(0, L) \geq \frac{\sigma}{2} - I(t_L w_L, (L, \infty)). \tag{4.2}$$

Since $w_L \to \Phi \in H^1(0,\infty)$ because of Lemma 3.2, we have

$$t_L^{p-1} \to \frac{\|\Phi\|^2_{H^1(0,\infty)}}{\|\Phi\|^{p+1}_{L^{p+1}(0,\infty)}} = 1 \text{ as } L \to \infty,$$

hence, $t_L = 1 + o(1)$ as $L \to \infty$. Using $t_L^2 \le 2$ for sufficiently large L, we get

$$I(t_L w_L, (L,\infty)) \le \frac{t_L^2}{2} \int_L^\infty |\nabla w_L|^2 + w_L^2 \, dx \le \int_L^\infty 2w_L^2(L) e^{2L-2x} \, dx = w_L^2(L).$$

By (4.1), we obtain

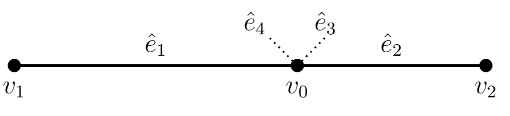
$$I(t_L w_L, (L,\infty)) \le Ce^{-2L(1-\delta)} \text{ as } L \to \infty.$$

Therefore, we get the conclusion. □

Proposition 4.4. *Suppose that, for a subsequence of $\epsilon \to 0$ and $e \in G$, a least energy solution u_ϵ has a local maximal point x_ϵ on e. Then, it holds that*

$$\sigma_\epsilon \ge \frac{\sigma}{2} + \exp\left(\frac{-2\ell(e)}{\epsilon}(1+o(1))\right) \text{ as } \epsilon \to 0.$$

Proof. For simplicity, we assume that G is a star graph. Thus, we may assume that $G = (V, E)$, $E = \{e_1, \ldots, e_k\}$ with $k \ge 3$, $V = \{v_0, \ldots, v_k\}$, $v_0 \in V_{\text{int}}$, $v_i \in V_{\text{end}}$ $(i = 1, \ldots, k)$, and each edge e_i connects v_0 and v_i.



By Lemma 4.2, we may assume that $x_\epsilon = v_1 \in e_1$ without loss of generality. Therefore, we consider a sub-graph $G'_\epsilon = \{\{v_0, v_1, v_2\}, \{\hat{e}_0, \hat{e}_1\}\}$. We estimate similarly as Lemma 4.3. By the characterization of the least energy, for any $t > 0$, it holds that

$$\sigma_\epsilon \ge I(t\hat{u}_\epsilon, G'_\epsilon) + \sum_{i=3}^k I(t\hat{u}_\epsilon, \hat{e}_i).$$

We choose $t_\epsilon > 0$ such that

$$I(t_\epsilon \hat{u}_\epsilon, G'_\epsilon) = \sup_{t>0} I(t\hat{u}_\epsilon, G'_\epsilon).$$

Since G'_ϵ is identified with the interval $[0, (\ell(e_1) + \ell(e_2))/\epsilon]$, by the characterization of $\sigma(0, (\ell(e_1) + \ell(e_2))/\epsilon)$, we have

$$\sup_{t>0} I(t\hat{u}_\epsilon, G'_\epsilon) \ge \sigma(0, (\ell(e_1) + \ell(e_2))/\epsilon).$$

Thus, we get

$$\sigma_\epsilon \ge \sigma(0, (\ell(e_1) + \ell(e_2))/\epsilon) + \sum_{i=3}^k I(t_\epsilon \hat{u}_\epsilon, \hat{e}_i). \tag{4.3}$$

Next, to estimate the second term of the right-hand side, we focus t_ϵ. Since $\hat{u}_\epsilon \to \Phi \in H^1$ on G'_ϵ, we obtain $t_\epsilon = 1 + o(1)$ as $\epsilon \to 0$, similarly as in Lemmas 4.1 and 4.3.

Recall that $\hat{u}_\epsilon$ is a positive solution and $\hat{u}_\epsilon$ has the unique local maximum point v_1. Thus $\hat{u}_\epsilon$ take the global maximum value on $\hat{e}_i$ at v_0 ($i = 3, \ldots, k$). Denote the maximum value by $m_\epsilon := \hat{u}_\epsilon(v_0)$. By the proof of Lemma 4.2, we have $m_\epsilon \to 0$, hence $\|t_\epsilon \hat{u}_\epsilon\|_{L^\infty(\hat{e}_i)} = o(1)$ ($i = 2, \ldots, k$), which implies

$$\frac{1}{p+1} \int_{\hat{e}_i} (t_\epsilon \hat{u}_\epsilon)^{p+1}\, dx \le \frac{1}{4} \int_{\hat{e}_i} t_\epsilon^2 (\hat{u}_\epsilon)^2\, dx \quad (i = 3, \ldots, k)$$

for sufficiently small ϵ. Thus, we get

$$I(t_\epsilon \hat{u}_\epsilon, \hat{e}_i) \ge \frac{t_\epsilon^2}{4} \|\hat{u}_\epsilon\|^2_{H^1(\hat{e}_i)} = \frac{1 + o(1)}{4} \|\hat{u}_\epsilon\|^2_{H^1(\hat{e}_i)} \quad (i = 3, \ldots, k). \tag{4.4}$$

Since $\hat{u}_\epsilon$ is a solution on $\hat{e}_i$ and satisfies the Neumann boundary condition at v_i, using partial integration, we have

$$\begin{aligned}
\|\hat{u}_\epsilon\|^2_{H^1(\hat{e}_i)} &= \int_{\hat{e}_i} \nabla \left(\hat{u}_\epsilon^{(\hat{e}_i)} \nabla \hat{u}_\epsilon^{(\hat{e}_i)} \right) + (-\Delta \hat{u}_\epsilon^{(\hat{e}_i)} + \hat{u}_\epsilon^{(\hat{e}_i)}) \hat{u}_\epsilon^{(\hat{e}_i)}\, dx \\
&= \int_{\hat{e}_i} \nabla \left(\hat{u}_\epsilon^{(\hat{e}_i)} \nabla \hat{u}_\epsilon^{(\hat{e}_i)} \right) + (\hat{u}_\epsilon^{(\hat{e}_i)})^{p+1}\, dx \\
&\ge \int_{\hat{e}_i} \nabla \left(\hat{u}_\epsilon^{(\hat{e}_i)} \nabla \hat{u}_\epsilon^{(\hat{e}_i)} \right)\, dx = \left(\partial \hat{u}_\epsilon^{(\hat{e}_i)}(v_0) \right) \hat{u}_\epsilon^{(\hat{e}_i)}(v_0) \quad (i = 3, \ldots, k).
\end{aligned} \tag{4.5}$$

Next, we estimate the right-hand side. For each $i \in \{2, \ldots, k\}$, we apply Lemma 4.6 below for $\hat{e}_i$. Then, we have

$$\lambda_\epsilon \left(\frac{1}{\tanh \lambda_\epsilon \ell(e_i)/\epsilon} + \frac{-1}{\sinh \lambda_\epsilon \ell(e_i)/\epsilon} \right) \le \frac{\partial \hat{u}_\epsilon^{(\hat{e}_i)}(v_0)}{\hat{u}_\epsilon^{(\hat{e}_i)}(v_0)} \le \frac{1}{\tanh \ell(e_i)/\epsilon} + \frac{1}{\sinh \ell(e_i)/\epsilon},$$

where $\lambda_\epsilon = \sqrt{1 - m_\epsilon^{p-1}}$. Thus, we get

$$\left(\partial \hat{u}_\epsilon^{(\hat{e}_i)}(v_0) \right) \hat{u}_\epsilon^{(\hat{e}_i)}(v_0) = m_\epsilon^2 (1 + o(1)) \text{ as } \epsilon \to 0 \quad (i = 2, \ldots, k). \tag{4.6}$$

Thus, we have

$$\sum_{i=3}^{k} \|\hat{u}_\epsilon\|^2_{H^1(\hat{e}_i)} \ge (k-2) m_\epsilon^2 (1 + o(1)) \ge m_\epsilon^2 (1 + o(1)). \tag{4.7}$$

Finally, we estimate m_ϵ. For the sake of simplicity, we denote $\hat{u}_\epsilon^{(\hat{e}_1)}$ and $\ell(e_1)$ by $\hat{u}_\epsilon$ and ℓ respectively, and assume $\hat{e}_1 = [0, \ell(e_1)/\epsilon]$ with $v_1 = 0$. Put

$$z(x) := \frac{(k+1)e^{-x+\ell/\epsilon} - (k-1)e^{x-\ell/\epsilon}}{(k+1)e^{\ell/\epsilon} - (k-1)e^{-\ell/\epsilon}} \hat{u}_\epsilon(0).$$

Then z is a solution of

$$-\Delta z + z = 0 \text{ on } (0, \ell/\epsilon), \quad z(0) = \hat{u}_\epsilon(0), \quad \nabla z \left(\frac{\ell}{\epsilon} \right) + kz \left(\frac{\ell}{\epsilon} \right) = 0.$$

Putting $w := z - \hat{u}_\epsilon$, we have

$$-\Delta w + w = (-\Delta z + z) - (-\Delta \hat{u}_\epsilon + \hat{u}_\epsilon) = -\hat{u}_\epsilon^p \leq 0 \text{ on } (0, \ell/\epsilon)$$

and

$$w(0) = z(0) - \hat{u}_\epsilon(0) = 0.$$

Using (4.6) and the Kirchhoff law at v_0, we get

$$\frac{\nabla \hat{u}_\epsilon(\ell/\epsilon)}{\hat{u}_\epsilon(\ell/\epsilon)} = \frac{-\sum_{i=2}^k \partial \hat{u}_\epsilon^{(\hat{e}_i)}(v_0)}{m_\epsilon} = -(k-1) + o(1) \geq -k.$$

Thus we get

$$\nabla w\left(\frac{\ell}{\epsilon}\right) + kw\left(\frac{\ell}{\epsilon}\right) = -\nabla \hat{u}_\epsilon\left(\frac{\ell}{\epsilon}\right) - k\hat{u}_\epsilon\left(\frac{\ell}{\epsilon}\right) \leq 0.$$

Applying Lemma 4.5 below to w, we obtain $w \leq 0$ on $(0, \ell/\epsilon)$. Moreover,

$$m_\epsilon = \hat{u}_\epsilon(\ell/\epsilon) \geq z(\ell/\epsilon) = \frac{2\hat{u}_\epsilon(0)}{(k+1)e^{\ell/\epsilon} - (k-1)e^{-\ell/\epsilon}} = \frac{2M_0}{k+1}e^{-\ell/\epsilon}(1 + o(1)) \text{ as } \epsilon \to 0$$

holds. Combining (4.3), (4.4), and (4.7), for $\delta > 0$ with $(\ell(e_1) + \ell(e_2))(1-\delta) > \ell(e_1)$, using Lemma 4.3, we get

$$\begin{aligned}
\sigma_\epsilon &\geq \frac{\sigma}{2} - \exp\left(-\frac{2(\ell(e_1) + \ell(e_2))(1-\delta)}{\epsilon}(1 + o(1))\right) + \frac{M_0^2}{(k+1)^2}\exp\left(\frac{-2\ell(e_1)}{\epsilon}\right)(1 + o(1)) \\
&= \frac{\sigma}{2} + \frac{M_0^2}{(k+1)^2}\exp\left(\frac{-2\ell(e_1)}{\epsilon}(1 + o(1))\right) \text{ as } \epsilon \to 0.
\end{aligned}$$

□

Proof of Theorem 1.1. Lemma 4.2 asserts (i) in Theorem 1.1. Then, we can apply Lemma 3.2 to get (ii). Finally, combining Lemma 4.1 and Proposition 4.4, we obtain (iii) and (iv). □

Lemma 4.5. *Assume that $R > 0$ and $\alpha \geq 0$. Let w be a solution of*

$$-\Delta w + w \leq 0 \text{ on } (0, R), \quad w(0) = 0, \quad \nabla w(R) + \alpha w(R) \leq 0. \tag{4.8}$$

Then $w \leq 0$ holds on $[0, R]$.

Proof. Multiplying (4.8) by $w_+(x) := \max\{w(x), 0\}$ and integrating, since $\nabla w \nabla w_+ = (\nabla w_+)^2$ and $ww_+ = w_+^2$, we have

$$\begin{aligned}
0 &\geq \int_0^R -\Delta w w_+ + ww_+ \, dx = -\nabla w(R)w_+(R) + \nabla w(0)w_+(0) + \int_0^R (\nabla w_+)^2 + w_+^2 \, dx \\
&\geq \alpha w_+^2(R) + \int_0^R (\nabla w_+)^2 + w_+^2 \, dx \geq 0.
\end{aligned}$$

Hence $\int_0^R (\nabla w_+)^2 + w_+^2 \, dx = 0$, which means the conclusion. □

4.4 Estimates for small solutions

Let u be a positive solution of

$$-\Delta u + u = u^p \text{ on } (0, L) \tag{4.9}$$

with $\|u\|_{L^\infty(0,L)} < m$. Putting $\delta := m^{p-1}$, if $0 \le t \le m$, we have $t^p \le \delta t$. Hence,

$$-\Delta u + u \ge 0, \quad -\Delta u + (1-\delta)u \le 0.$$

Thus, we can estimate u by sub-super solutions.

Lemma 4.6. *Let u be a positive solution of* (4.9)*. Suppose that $\|u\|_{L^\infty(0,L)} \le m$, $u(0) = a$, $u(L) = b$, and $\delta = m^{p-1} > 0$. Then*

$$\begin{gathered}
\frac{a\sinh(L-x) + b\sinh x}{\sinh L} \le u(x) \le \frac{a\sinh\lambda(L-x) + b\sinh\lambda x}{\sinh\lambda L} \quad \textit{for } x \in [0, L], \\
\frac{-1}{\tanh L} + \frac{b}{a\sinh L} \le \frac{\nabla u(0)}{u(0)} \le \lambda\left(\frac{-1}{\tanh\lambda L} + \frac{b}{a\sinh\lambda L}\right), \\
\lambda\left(\frac{1}{\tanh\lambda L} - \frac{a}{b\sinh\lambda L}\right) \le \frac{\nabla u(L)}{u(L)} \le \frac{1}{\tanh L} - \frac{a}{b\sinh L},
\end{gathered}$$

where $\lambda := \sqrt{1-\delta}$.

Proof. By the assumption and the maximum principle, we have $\|u\|_{L^\infty(0,L)} = \max\{a, b\}$. Hence,

$$-\Delta u + u \ge 0, \quad -\Delta u + \lambda^2 u \le 0 \text{ on } (0, L).$$

Let $\overline{u}$ and $\underline{u}$ be unique solutions to

$$\begin{cases} -\Delta\overline{u} + \lambda^2\overline{u} = 0 \text{ on } (0, L), \\ -\Delta\underline{u} + \underline{u} = 0 \text{ on } (0, L), \\ \overline{u}(0) = \underline{u}(0) = a, \overline{u}(L) = \underline{u}(L) = b. \end{cases}$$

Then, we have explicit formulas:

$$\underline{u}(x) = \frac{a\sinh(L-x) + b\sinh x}{\sinh L}, \quad \overline{u}(x) = \frac{a\sinh\lambda(L-x) + b\sinh\lambda x}{\sinh\lambda L} \text{ on } [0, L].$$

Moreover, $\underline{u} \le u \le \overline{u}$ holds by the comparison theorem. Since $\underline{u} = \overline{u}$ at $x = 0, L$, we get the estimates for the derivative. □

References

[1] S. Akduman and A. Pankov, *Nonlinear Schrödinger equation with growing potential on infinite metric graphs*, Nonlinear Anal. **184** (2019), 258–272, DOI 10.1016/j.na.2019.02.020.

[2] R. Adami, E. Serra, and P. Tilli, *NLS ground states on graphs*, Calc. Var. Partial Differential Equations **54** (2015), no. 1, 743–761, DOI 10.1007/s00526-014-0804-z.

[3] ———, *Multiple positive bound states for the subcritical NLS equation on metric graphs*, Calc. Var. Partial Differential Equations **58** (2019), no. 1, Art. 5, 16, DOI 10.1007/s00526-018-1461-4.

[4] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345, DOI 10.1007/BF00250555.

[5] C. Cacciapuoti, S. Dovetta, and E. Serra, *Variational and stability properties of constant solutions to the NLS equation on compact metric graphs*, Milan J. Math. **86** (2018), no. 2, 305–327, DOI 10.1007/s00032-018-0288-y.

[6] K. Kurata and M. Shibata, in preparation.

[7] Y. Li, F. Li, and J. Shi, *Ground states of nonlinear Schrödinger equation on star metric graphs*, J. Math. Anal. Appl. **459** (2018), no. 2, 661–685, DOI 10.1016/j.jmaa.2017.10.069.

[8] Y. Li, Z.-Q. Wang, and J. Zeng, *Ground states of nonlinear Schrödinger equations with potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 6, 829–837, DOI 10.1016/j.anihpc.2006.01.003 (English, with English and French summaries).

[9] M. Shibata, in preparation.

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