On sharpness of the Yang-Yau inequality for the genus two case

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Abstract

This is a survey article for our recent work related to the sharp upper bound of the Yang-Yau inequality for the genus two case.

1 Introduction

Let M be an orientable closed surface. For a Riemannian metric $ds^2 = g_{ij}dx^i dx^j$ on M, the Laplacian is defined as

$$\Delta_{ds^2} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}\, g^{ij}\frac{\partial}{\partial x^j}\right) \qquad ((g^{ij}) = (g_{kl})^{-1},\ \ |g| = \det(g_{ij})).$$

Let

$$\Lambda(ds^2) := \lambda_1(ds^2)\cdot \mathrm{area}(ds^2),$$

where $\lambda_1(ds^2)$ is the first positive eigenvalue of Δ_{ds^2} and area(ds^2) is the area of M with respect to ds^2. The following are well-known results related to the upper bound of $\Lambda(ds^2)$.

Theorem 1.1. (i) (Hersch [Her]) *For any metric ds^2 on the sphere S^2, $\Lambda(ds^2) \leq 8\pi$ holds.*
(ii) (Nadirashvili [N]) *For any metric ds^2 on the torus, $\Lambda(ds^2) \leq 8\pi^2/\sqrt{3}$*

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holds.
(iii) (Yang-Yau [YY]) *If γ is the genus of M, then for any metric ds^2 on M, we have*

$$\Lambda(ds^2) \leq 8\pi \cdot \left\lfloor \frac{\gamma+3}{2} \right\rfloor. \tag{1}$$

The inequality of (i) is sharp since the equality is attained for the standard metric on S^2, and thus the inequality (1) is sharp for $\gamma = 0$. On the other hand, by (ii), the inequality (1) is not sharp when $\gamma = 1$.

For $\gamma = 2$, the inequality (1) implies $\Lambda(ds^2) \leq 16\pi$, and Jakobson-Levitin-Nadirashvili-Nigam-Polterovich [JLNNP] conjectured the following:

Conjecture (Jakobson et al. [JLNNP])**.** Let B be the closed Riemann surface of genus two defined by

$$B = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z(z^4 + 1)\} \cup \{(\infty, \infty)\},$$

called the *Bolza surface.* Then, there exists a singular metric ds_B^2 on B such that $\lambda_1(ds_B^2) = 2$, that is, $\Lambda(ds_B^2) = 16\pi$ holds.

Note that ds_B^2 can be approximated by smooth metrics, which remarked in [JLNNP], and hence the inequality (1) is sharp for $\gamma = 2$.

To see the conjecture, they considered the following deformation of B. For $0 < \theta < \pi/2$, let B_θ be the Riemann surface of genus two defined by

$$B_\theta = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z(z^4 + 2\cos 2\theta \cdot z^2 + 1)\} \cup \{(\infty, \infty)\}.$$

Note that $B_{\pi/4} = B$. Let ds_θ^2 denote the pull-back of the standard metric on $S^2 = \overline{\mathbb{C}}$ by the meromorphic function $g_\theta \colon B_\theta \ni (z, w) \mapsto z \in \overline{\mathbb{C}}$. Then $ds_{\pi/4}^2$ is the desired singular metric ds_B^2 in the conjecture. They used numerical arguments, and thus their result has been treated as conjecture as above. For this open problem, our recent paper [NS] gives an affirmative answer.

Main Theorem (Nayatani and Shoda, 2019)**.** There exists $\theta_1 \approx 0.65$ so that for $\theta_1 \leq \theta \leq \pi/2 - \theta_1$, we have $\lambda_1(ds_{B_\theta}^2) = 2$ and therefore $\Lambda(ds_\theta^2) = 16\pi$.

2 Index and nullity of a holomorphic map

In this section, we first review the arguments in [EK] and [MR], and translate the conjecture into their situations.

Let $g\colon M \to \overline{\mathbb{C}}$ be a non-constant meromorphic function on a compact Riemann surface M, that is, a holomorphic map from M to S^2. Let ds_g^2 be the pull-back metric of the standard metric on $\overline{\mathbb{C}} = S^2$. Note that ds_g^2 is a singular metric, and we can consider the Laplacian Δ_g with respect to ds_g^2 and its eigenvalues in a standard way. The *index* of g (resp. the *nullity* of g) is defined as the number of eigenvalues of Δ_g less than 2 counted with multiplicities (resp. the multiplicity of eigenvalue 2 of Δ_g). Let $\mathrm{ind}(g)$ (resp. $\mathrm{nul}(g)$) be the index of g (resp. the nullity of g).

We now consider the image of g in $S^2 \subset \mathbb{R}^3$. For a constant vector $a \in \mathbb{R}^3$, it is well-known that $\langle a, g\rangle$ gives an eigenfunction of Δ_g of eigenvalue 2, where $\langle *, **\rangle$ is the standard inner product on $\mathbb{R}^3$. Hence we have $\mathrm{nul}(g) \geq 3$. Eigenfunctions of Δ_g of eigenvalue 2 other than $\langle a, g\rangle$-type are called *extra eigenfunctions*. Ejiri-Kotani [EK] and Montiel-Ros [MR] obtained the procedure to determine extra eigenfunctions, which omitted in this report.

Since every constant function is a single eigenfunction of Δ_g of eigenvalue 0, we have $\mathrm{ind}(g) \geq 1$. The conjecture states that, for $g = g_B$, the second least eigenvalue of Δ_{g_B} should equal 2, and so it is equivalent to $\mathrm{ind}(g_B) = 1$.

3 Outline of proof

In this section, we observe behaviors of eigenvalues of Δ_{g_θ} less than 2 as $\theta\colon 0 \to \pi/2$.

By the Montiel-Ros result [MR], we have $\mathrm{Ind}(g_\theta) = 3$ for θ which is sufficiently near to 0. Thus there exists a single eigenvalue 0 and there exist two positive eigenvalues of Δ_{g_θ} less than 2. We next see

Lemma 3.1. *Set*

$$A = \int_0^\infty \frac{dt}{\sqrt{t(t^4 + 2\cos 2\theta \cdot t^2 + 1)}}, \quad B = \int_0^\infty \frac{dt}{\sqrt{t(t^4 - 2\cos 2\theta \cdot t^2 + 1)}},$$
$$C = \int_0^\infty \frac{t^3\, dt}{\sqrt{t(t^4 + 2\cos 2\theta \cdot t^2 + 1)}^3}, \quad D = \int_0^\infty \frac{t^3\, dt}{\sqrt{t(t^4 - 2\cos 2\theta \cdot t^2 + 1)}^3}.$$

Let $\theta_1\,(\approx 0.65)$ be the unique solution of

$$A(B^2 + 16D^2\sin^2 2\theta) + 8(AD + BC)(B\cos 2\theta - 4D\sin^2 2\theta) = 0, \qquad (2)$$

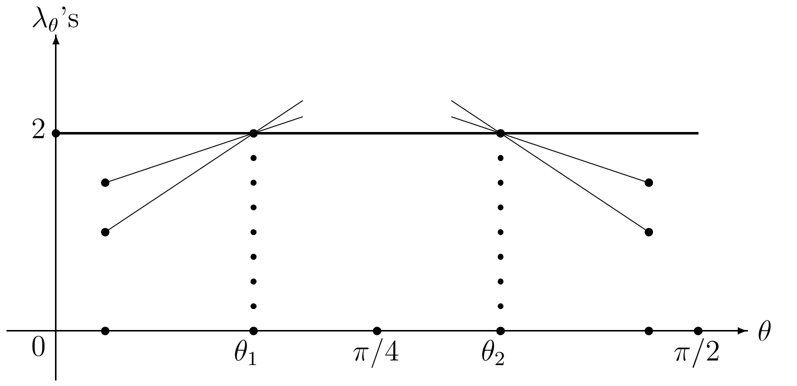

Figure 1: A rough sketch of behaviors of eigenvalues of Δ_{g_θ} less than 2

and set $\theta_2 = \pi/2 - \theta_1$ (≈ 0.91). Then there exist two extra eigenfunctions for $\theta = \theta_1, \theta_2$. Therefore we have

$$\mathrm{Nul}\,(g_\theta) = \begin{cases} 5, & \theta = \theta_1,\, \theta_2, \\ 3, & \theta \neq \theta_1,\, \theta_2. \end{cases}$$

By Lemma 3.1, we can find two eigenvalues 2 of $\Delta_{g_{\theta_i}}$ ($i = 1, 2$) other than three eigenvalues 2 given by $\langle a, g\rangle$-type. By symmetries of the extra eigenfunctions, we can show that their two eigenvalues are monotonically increasing (resp. decreasing) when they pass through $\theta = \theta_1$ (resp. $\theta = \theta_2$). Hence we can see the behaviors of eigenvalues of Δ_{g_θ} less than 2 as $\theta\colon 0 \to \pi/2$ (see Figure 1).

Therefore, we have the following theorem which implies our main result:

Theorem 3.2.

$$\mathrm{ind}(g_\theta) = \begin{cases} 3, & 0 < \theta < \theta_1, \\ 1, & \theta_1 \leq \theta \leq \theta_2, \\ 3, & \theta_2 < \theta < \pi/2. \end{cases}$$

4 Appendix

In this section, we will prove that there exist no solutions of (2) for $\pi/4 \leq \theta < \pi/2$, which omitted in the original paper. In fact, the equation seems to be even more complicated than indicated, since the numbers A, B, C, D

depend in a non-trivial way on θ. So it would be helpful for the readers to give arguments for the claim even if it consists of long calculations.

We first rewrite A, B, C, D by the complete elliptic integrals. Recall that, for $0 < k < 1$, Let

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

$K(k)$ is called the *complete elliptic integral of first kind*, and $E(k)$ the *complete elliptic integral of second kind.*

By **222** in [BF], we have

$$\begin{aligned}
A &= \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{t(t^4 + 2\cos(2\theta)t^2 + 1)}} + \frac{1}{2}\int_0^\infty \frac{t\,dt}{\sqrt{t(t^4 + 2\cos(2\theta)t^2 + 1)}} \\
&\underbrace{=}_{x=\sqrt{t}-1/\sqrt{t}} \int_{-\infty}^\infty \frac{dx}{\sqrt{x^4 + 4x^2 + 2 + 2\cos 2\theta}} = 2\int_0^\infty \frac{dx}{\sqrt{x^4 + 4x^2 + 2 + 2\cos 2\theta}} \\
&= \frac{2}{\sqrt{2(1+\sin\theta)}} K\left(\sqrt{\frac{2\sin\theta}{1+\sin\theta}}\right), \\
B &= 2\int_0^\infty \frac{dx}{\sqrt{x^4 + 4x^2 + 2 - 2\cos 2\theta}} = \frac{2}{\sqrt{2(1+\cos\theta)}} K\left(\sqrt{\frac{2\cos\theta}{1+\cos\theta}}\right), \\
C &= 2\int_0^\infty \frac{dx}{\sqrt{x^4 + 4x^2 + 2 + 2\cos 2\theta}^3} \\
&= \frac{1}{4\sqrt{2(1+\sin\theta)}\sin^2\theta(1-\sin\theta)} \\
&\qquad \times \left(E\left(\sqrt{\frac{2\sin\theta}{1+\sin\theta}}\right) - (1-\sin\theta)K\left(\sqrt{\frac{2\sin\theta}{1+\sin\theta}}\right)\right), \\
D &= 2\int_0^\infty \frac{dx}{\sqrt{x^4 + 4x^2 + 2 - 2\cos 2\theta}^3} \\
&= \frac{1}{4\sqrt{2(1+\cos\theta)}\cos^2\theta(1-\cos\theta)} \\
&\qquad \times \left(E\left(\sqrt{\frac{2\cos\theta}{1+\cos\theta}}\right) - (1-\cos\theta)K\left(\sqrt{\frac{2\cos\theta}{1+\cos\theta}}\right)\right).
\end{aligned}$$

(2) is equivalent to

$$\cos 2\theta(AB^2\cos 2\theta + 8ABD + 8B^2C) + \sin^2 2\theta(AB^2 - 16AD^2 - 32BCD) = 0.$$

Thus, for $\pi/4 \le \theta < \pi/2$, it is sufficient to show

$$AB^2\cos 2\theta + 8ABD + 8B^2C > 0, \quad AB^2 - 16AD^2 - 32BCD < 0,$$

since this claim implies that the left hand side of (2) is negative for $\pi/4 \le \theta < \pi/2$.

For simplicity, we set $k^2 = 2\sin\theta/(1+\sin\theta)$, $l^2 = 2\cos\theta/(1+\cos\theta)$, and we obtain

$$\sin\theta = \frac{k^2}{2-k^2}, \quad 1-\sin\theta = \frac{2(1-k^2)}{2-k^2}, \quad \cos\theta = \frac{l^2}{2-l^2},$$
$$1-\cos\theta = \frac{2(1-l^2)}{2-l^2}, \quad k^4l^4 = 16(1-k^2)(1-l^2).$$

4.1 Proof of $AB^2\cos 2\theta + 8ABD + 8B^2C > 0$

The inequality $AB^2\cos 2\theta + 8ABD + 8B^2C > 0$ follows from the next lemma:

Lemma 4.1. *For $\pi/4 \le \theta < \pi/2$, we have $A\cos 2\theta + 8C > 0$.*

Proof. We first obtain

$$A\cos 2\theta + 8C = \frac{2}{\sqrt{2(1+\sin\theta)}\sin^2\theta(1-\sin\theta)} \times \left(E(k) + (\cos 2\theta\sin^2\theta - 1)(1-\sin\theta)K(k)\right).$$

Next, we find

$$\frac{\pi}{4} \le \theta < \frac{\pi}{2} \iff 2(\sqrt{2}-1) \le k^2 < 1,$$

and the inequality $A\cos 2\theta + 8C > 0$ is equivalent to

$$E(k) + \frac{4(-k^8 + 2k^6 - 10k^4 + 16k^2 - 8)}{(2-k^2)^5} \cdot (1-k^2)K(k) > 0.$$

By **710.02**, **710.04** in [BF], we find

$$\frac{d}{dk}E(k) = \frac{E(k)-K(k)}{k}, \quad \frac{d}{dk}((1-k^2)K(k)) = \frac{dE(k)}{dk} - kK(k).$$

Thus we have

$$\begin{aligned}
&\frac{d}{dk}\left(E(k)+\frac{4(-k^8+2k^6-10k^4+16k^2-8)}{(2-k^2)^5}\cdot(1-k^2)K(k)\right)\\
&=\frac{E(k)-K(k)}{k}-\frac{8k(k^8+4k^6+18k^4-24k^2+8)}{(2-k^2)^6}\cdot(1-k^2)K(k)\\
&\qquad+\frac{4(-k^8+2k^6-10k^4+16k^2-8)}{(2-k^2)^5}\left(\frac{E(k)-K(k)}{k}-kK(k)\right)\\
&=\frac{k(k^8-6k^6+32k^4-40k^2+16)}{(2-k^2)^5}(K(k)-E(k))\\
&\qquad+\frac{4k^3(k^4+4k^2-4)(k^4+6k^2-6)}{(2-k^2)^6}K(k)\\
&\left(=\frac{k(k^8+6k^4(1-k^2)+k^4+(5k^2-4)^2)}{(2-k^2)^5}(K(k)-E(k))\right.\\
&\qquad\left.+\frac{4k^3(k^4+4k^2-4)(k^4+6k^2-6)}{(2-k^2)^6}K(k)\right)\\
&=\frac{k}{(2-k^2)^6}((2-k^2)(k^8-6k^6+32k^4-40k^2+16)(K(k)-E(k))\\
&\qquad+4k^2(k^4+4k^2-4)(k^4+6k^2-6)K(k)).
\end{aligned}$$

Hence, for $\sqrt{15}-3\le k^2<1$, this is automatically positive. For $2(\sqrt{2}-1)\le k^2<\sqrt{15}-3$, we obtain

$$\begin{aligned}
&(2-k^2)(k^8-6k^6+32k^4-40k^2+16)(K(k)-E(k))\\
&\qquad\qquad+4k^2(k^4+4k^2-4)(k^4+6k^2-6)K(k)\\
&\ge\lim_{k^2\to2(\sqrt{2}-1)}(2-k^2)(k^8-6k^6+32k^4-40k^2+16)(K(k)-E(k))\\
&\qquad-0.1\cdot K\left(\sqrt{\sqrt{15}-3}\right)\\
&>(7424-5248\sqrt{2})(K(\sqrt{0.82})-E(\sqrt{0.82}))-0.1\cdot K(\sqrt{0.88})\\
&>2.2(2.305232-1.164798)-0.1\cdot 2.492635=2.25969>0
\end{aligned}$$

(see pp. 324 in [BF]). In fact, by setting $f(x)=(2-x)(x^4-6x^3+32x^2-40x+16)$, $f(x)$ is monotonically increasing for $2(\sqrt{2}-1)\le x<1$. By **710.05** in [BF], we have

$$\frac{d}{dk}(K(k)-E(k))=\frac{kE(k)}{1-k^2}>0,$$

and thus $K(k) - E(k)$ is monotonically increasing.

Hence, for $2(\sqrt{2}-1) \le k^2 < 1$, we find

$$\frac{d}{dk}\left(E(k) + \frac{4(-k^8+2k^6-10k^4+16k^2-8)}{(2-k^2)^5}\cdot(1-k^2)K(k)\right) > 0.$$

It follows that, for $2(\sqrt{2}-1) \le k^2 < 1$,

$$\begin{aligned}
&E(k) + \frac{4(-k^8+2k^6-10k^4+16k^2-8)}{(2-k^2)^5}\cdot(1-k^2)K(k) \\
&\ge E\left(\sqrt{2(\sqrt{2}-1)}\right) - \frac{\sqrt{2}-1}{\sqrt{2}}K\left(\sqrt{2(\sqrt{2}-1)}\right) \\
&> E(\sqrt{0.82}) - 0.3K(\sqrt{0.83}) = 1.164798 - 0.3\cdot 2.331409 = 0.465375 > 0.
\end{aligned}$$

(see again pp.324 in [BF]). □

4.2 Proof of $AB^2 - 16AD^2 - 32BCD < 0$

Since

$$\begin{aligned}
&AB^2 - 16AD^2 - 32BCD \\
&= \begin{cases}
A\left(B - \frac{192}{25}D\right)\left(B + \frac{25}{12}D\right) + BD\left(\frac{1679}{300}A - 32C\right) & (\pi/4 \le \theta \le 5\pi/16) \\
A(B - 10D)\left(B + \frac{8}{5}D\right) + BD\left(\frac{42}{5}A - 32C\right) & (5\pi/16 \le \theta \le 3\pi/8) \\
A(B - 16D)(B + D) + BD(15A - 32C) & (3\pi/8 \le \theta < \pi/2),
\end{cases}
\end{aligned}$$

it is sufficient to prove

$$25B - 192D < 0 \qquad (\pi/4 \le \theta \le 5\pi/16), \tag{3}$$
$$1679A - 9600C < 0 \qquad (\pi/4 \le \theta \le 5\pi/16), \tag{4}$$
$$B - 10D < 0 \qquad (5\pi/16 \le \theta \le 3\pi/8), \tag{5}$$
$$21A - 80C < 0 \qquad (5\pi/16 \le \theta \le 3\pi/8), \tag{6}$$
$$B - 16D < 0 \qquad (3\pi/8 \le \theta < \pi/2), \tag{7}$$
$$15A - 32C < 0 \qquad (3\pi/8 \le \theta < \pi/2) \tag{8}$$

to show $AB^2 - 16AD^2 - 32BCD < 0$.

Proof of (3) Straightforward calculations yield

$$\begin{aligned}
25B - 192D = &\frac{2}{\sqrt{2(1+\cos\theta)}\cos^2\theta(1-\cos\theta)} \\
&\times (25\cos^2\theta(1-\cos\theta)K(l) - 24(E(l) - (1-\cos\theta)K(l))),
\end{aligned}$$

$$\begin{aligned}&25\cos^2\theta(1-\cos\theta)K(l)-24(E(l)-(1-\cos\theta)K(l))\\&\quad=\frac{2}{(2-l^2)^3}((49l^4-96l^2+96)(1-l^2)K(l)-12(2-l^2)^3E(l)).\end{aligned}$$

Thus, we will prove

$$(49l^4-96l^2+96)(1-l^2)K(l)-12(2-l^2)^3E(l)<0$$

for

$$0.714298\approx\frac{2\cos\frac{5}{16}\pi}{1+\cos\frac{5}{16}\pi}\le l^2\le\frac{2\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}\approx 0.828427.$$

We first obtain

$$\begin{aligned}&\frac{d}{dl}((49l^4-96l^2+96)(1-l^2)K(l)-12(2-l^2)^3E(l))\\&\quad=(196l^3-192l)(1-l^2)K(l)+(49l^4-96l^2+96)\left(\frac{E-K}{l}-lK\right)\\&\qquad+72l(2-l^2)^2E(l)-12(2-l^2)^3\cdot\frac{E-K}{l}\\&\quad=l(-(257l^4-507l^2+336)K(l)+(84l^4-311l^2+336)E(l)).\end{aligned}$$

For $0.71\le l^2\le 0.83$, $K(l)$ is positive and monotonically increasing, and $257l^4-507l^2+336$, $84l^4-311l^2+336$, $E(l)$ are positive and monotonically decreasing. We now divide the interval $0.71\le l^2\le 0.83$ into $0.71\le l^2\le 0.81$ and $0.81\le l^2\le 0.83$, and estimate separately. For $0.71\le l^2\le 0.81$, we find

$$\begin{aligned}&-(257l^4-507l^2+336)K(l)+(84l^4-311l^2+336)E(l)\\&\quad\le-(257\cdot 0.81^2-507\cdot 0.81+336)K(\sqrt{0.71})\\&\qquad\quad+(84\cdot 0.71^2-311\cdot 0.71+336)E(\sqrt{0.71})\\&\quad\approx-1.723<0.\end{aligned}$$

For $0.81\le l^2\le 0.83$,

$$\begin{aligned}&-(257l^4-507l^2+336)K(l)+(84l^4-311l^2+336)E(l)\\&\quad\le-(257\cdot 0.83^2-507\cdot 0.83+336)K(\sqrt{0.81})\\&\qquad\quad+(84\cdot 0.81^2-311\cdot 0.81+336)E(\sqrt{0.81})\\&\quad\approx-47.2487<0.\end{aligned}$$

Hence, for $0.71 \le l^2 \le 0.83$, we have

$$\frac{d}{dl}((49l^4 - 96l^2 + 96)(1 - l^2)K(l) - 12(2 - l^2)^3 E(l)) < 0.$$

Thus

$$(49l^4 - 96l^2 + 96)(1 - l^2)K(l) - 12(2 - l^2)^3 E(l)$$

is monotonically decreasing. Since $K(l)$ is positive and monotonically increasing, and $E(l)$ is positive and monotonically decreasing, we have

$$\begin{aligned}
&(49l^4 - 96l^2 + 96)(1 - l^2)K(l) - 12(2 - l^2)^3 E(l) \\
&\quad \le (49 \cdot 0.714^2 - 96 \cdot 0.714 + 96)(1 - 0.714)K(\sqrt{0.714}) \\
&\qquad\quad - 12(2 - 0.714)^3 E(\sqrt{0.714}) \\
&\quad \approx -0.0337104 < 0.
\end{aligned}$$

Therefore the claim follows.

Proof of (4) We first find

$$\begin{aligned}
&1679A - 9600C = \frac{2}{\sqrt{2(1+\sin\theta)}\sin^2\theta(1-\sin\theta)} \\
&\quad \times (1679\sin^2\theta(1-\sin\theta)K(k) - 1200(E(k) - (1-\sin\theta)K(k))), \\
&1679\sin^2\theta(1-\sin\theta)K(k) - 1200(E(k) - (1-\sin\theta)K(k)) \\
&= \frac{2}{(2-k^2)^3}((2879k^4 - 4800k^2 + 4800)(1-k^2)K(k) - 600(2-k^2)^3 E(k)).
\end{aligned}$$

Note that

$$0.828427 \approx \frac{2\sin\frac{\pi}{4}}{1+\sin\frac{\pi}{4}} \le k^2 < 1.$$

Direct calculation yields

$$\begin{aligned}
&\frac{d}{dk}((2879k^4 - 4800k^2 + 4800)(1-k^2)K(k) - 600(2-k^2)^3 E(k)) \\
&\quad = (4 \cdot 2879k^3 - 9600k)(1-k^2)K(k) \\
&\qquad\quad + (2879k^4 - 4800k^2 + 4800)\left(\frac{E-K}{k} - kK\right) \\
&\qquad\qquad + 3600k(2-k^2)^2 E(k) - 600(2-k^2)^3 \cdot \frac{E-K}{k} \\
&\quad = k(-(14995k^4 - 26637k^2 + 16800)K(k) \\
&\qquad\qquad\qquad + (4200k^4 - 15121k^2 + 16800)E(k)).
\end{aligned}$$

We can check that, for $0.82 \le k^2 < 1$, $14995x^2 - 26637x + 16800$ is a positive function which attains the minimum at $x = 26637/(2 \cdot 14995)$ and the maximum at $x = 1$, and $4200x^2 - 15121x + 16800$ is positive and monotonically decreasing. Hence we have

$$\begin{aligned}
&-(14995k^4 - 26637k^2 + 16800)K(k) + (4200k^4 - 15121k^2 + 16800)E(k) \\
&\quad \le -\left(14995 \cdot \left(\frac{26637}{2 \cdot 14995}\right)^2 - 26637 \cdot \frac{26637}{2 \cdot 14995} + 16800\right) K(\sqrt{0.82}) \\
&\qquad\qquad + (4200 \cdot 0.82^2 - 15121 \cdot 0.82 + 16800)E(\sqrt{0.82}) \\
&\quad \approx -3042.79 < 0.
\end{aligned}$$

Thus, $(2879k^4 - 4800k^2 + 4800)(1 - k^2)K(k) - 600(2 - k^2)^3 E(k)$ is monotonically decreasing. We finally obtain

$$\begin{aligned}
&(2879k^4 - 4800k^2 + 4800)(1 - k^2)K(k) - 600(2 - k^2)^3 E(k) \\
&\quad \le (2879 \cdot 0.8284^2 - 4800 \cdot 0.8284 + 4800)(1 - 0.8284)K(\sqrt{0.8284}) \\
&\qquad\qquad - 600(2 - 0.8284)^3 E(\sqrt{0.8284}) \\
&\quad \le (2879 \cdot 0.8284^2 - 4800 \cdot 0.8284 + 4800)(1 - 0.8284)K(\sqrt{0.8284}) \\
&\qquad\qquad - 600(2 - 0.8284)^3 E(\sqrt{0.8284}) \\
&\quad \approx -0.371741 < 0.
\end{aligned}$$

Therefore the claim is proved.

Proof of (5)**.** We first have

$$\begin{aligned}
B - 10D &= \frac{1}{2\sqrt{2(1+\cos\theta)}\cos^2\theta(1-\cos\theta)} \\
&\qquad \times (2\cos^2\theta(1-\cos\theta)K(l) - 5(E(l) - (1-\cos\theta)K(l))),
\end{aligned}$$

$$\begin{aligned}
&2\cos^2\theta(1-\cos\theta)K(l) - 5(E(l) - (1-\cos\theta)K(l)) \\
&\quad = \frac{1}{(2-l^2)^3}(2(7l^4 - 20l^2 + 20)(1 - l^2)K(l) - 5(2 - l^2)^3 E(l)),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dl}(2(7l^4 - 20l^2 + 20)(1 - l^2)K(l) - 5(2 - l^2)^3 E(l)) \\
&\quad = 2(28l^3 - 40l)(1 - l^2)K(l) + 2(7l^4 - 20l^2 + 20)\left(\frac{E - K}{l} - lK\right)
\end{aligned}$$

$$
\begin{aligned}
&+30l(2-l^2)^2E(l)-5(2-l^2)^3\cdot\frac{E-K}{l}\\
&=l(-(75l^4-192l^2+140)K(l)+(35l^4-136l^2+140)E(l)).
\end{aligned}
$$

Note that

$$
0.553537\approx\frac{2\cos\frac{3}{8}\pi}{1+\cos\frac{3}{8}\pi}\le l^2\le\frac{2\cos\frac{5}{16}\pi}{1+\cos\frac{5}{16}\pi}\approx 0.714298,
$$

and $75l^4-192l^2+140$, $35l^4-136l^2+140$ are positive and monotonically decreasing for $0.55\le l^2\le 0.715$.

We now divide the interval $0.55\le l^2\le 0.715$ among $0.55\le l^2\le 0.58$, $0.58\le l^2\le 0.62$, $0.62\le l^2\le 0.66$, $0.66\le l^2\le 0.715$, and estimate separately. For $0.55\le l^2\le 0.58$, we find

$$
\begin{aligned}
&-(75l^4-192l^2+140)K(l)+(35l^4-136l^2+140)E(l)\\
&\le -(75\cdot 0.58^2-192\cdot 0.58+140)K(\sqrt{0.55})\\
&\qquad +(35\cdot 0.55^2-136\cdot 0.55+140)E(\sqrt{0.55})\\
&\approx -1.87479<0.
\end{aligned}
$$

For $0.58\le l^2\le 0.62$, we obtain

$$
\begin{aligned}
&-(75l^4-192l^2+140)K(l)+(35l^4-136l^2+140)E(l)\\
&\le -(75\cdot 0.62^2-192\cdot 0.62+140)K(\sqrt{0.58})\\
&\qquad +(35\cdot 0.58^2-136\cdot 0.58+140)E(\sqrt{0.58})\\
&\approx -0.589051<0.
\end{aligned}
$$

For $0.62\le l^2\le 0.66$,

$$
\begin{aligned}
&-(75l^4-192l^2+140)K(l)+(35l^4-136l^2+140)E(l)\\
&\le -(75\cdot 0.66^2-192\cdot 0.66+140)K(\sqrt{0.62})\\
&\qquad +(35\cdot 0.62^2-136\cdot 0.62+140)E(\sqrt{0.62})\\
&\approx -1.5945<0.
\end{aligned}
$$

For $0.66\le l^2\le 0.715$,

$$
\begin{aligned}
&-(75l^4-192l^2+140)K(l)+(35l^4-136l^2+140)E(l)\\
&\le -(75\cdot 0.715^2-192\cdot 0.715+140)K(\sqrt{0.66})\\
&\qquad +(35\cdot 0.66^2-136\cdot 0.66+140)E(\sqrt{0.66})\\
&\approx -0.115848<0.
\end{aligned}
$$

Hence, $2(7l^4-20l^2+20)(1-l^2)K(l)-5(2-l^2)^3E(l)$ is monotonically decreasing. We finally estimate

$$\begin{aligned}&2(7l^4-20l^2+20)(1-l^2)K(l)-5(2-l^2)^3E(l)\\&\le 2(7\cdot 0.55^2-20\cdot 0.55+20)(1-0.55)K(\sqrt{0.55})-5(2-0.55)^3E(\sqrt{0.55})\\&\approx -1.19735<0,\end{aligned}$$

and the proof is completed.

Proof of (6)

Since $80/21\approx 3.80952>3.8=19/5$, it is sufficient to prove $5A-19C<0$ for $5\pi/16\le\theta<\pi/2$. We first find

$$\begin{aligned}5A-19C&=\frac{1}{4\sqrt{2(1+\sin\theta)}\sin^2\theta(1-\sin\theta)}\\&\quad\times(40\sin^2\theta(1-\sin\theta)K(k)-19(E(k)-(1-\sin\theta)K(k))),\\40\sin^2\theta(1-\sin\theta)&K(k)-19(E(k)-(1-\sin\theta)K(k))\\&=\frac{1}{(2-k^2)^3}(2(59k^4-76k^2+76)(1-k^2)K(k)-19(2-k^2)^3E(k)),\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dk}(2(59k^4-76k^2+76)(1-k^2)K(k)-19(2-k^2)^3E(k))\\&=2(236k^3-152k)(1-k^2)K(k)+2(59k^4-76k^2+76)\left(\frac{E-K}{k}-kK\right)\\&\qquad+114k(2-k^2)^2E(k)-19(2-k^2)^3\cdot\frac{E-K}{k}\\&=k(-7(87k^4-132k^2+76)K(k)+(133k^4-452k^2+532)E(k)).\end{aligned}$$

Note that

$$0.907981\approx\frac{2\sin\frac{5}{16}\pi}{1+\sin\frac{5}{16}\pi}\le k^2<1,$$

and, for $0.9\le x<1$, $87x^2-132x+76$ is positive and monotonically increasing, $133x^2-452x+532$ is positive and monotonically decreasing. Hence, we obtain

$$\begin{aligned}&-7(87k^4-132k^2+76)K(k)+(133k^4-452k^2+532)E(k)\\&\quad\le -7(87\cdot 0.9^2-132\cdot 0.9+76)K(\sqrt{0.9})\end{aligned}$$

$$
\begin{aligned}
&\quad + (133 \cdot 0.9^2 - 452 \cdot 0.9 + 532)E(\sqrt{0.9}) \\
&\approx -242.015 < 0.
\end{aligned}
$$

Thus, $2(59k^4 - 76k^2 + 76)(1-k^2)K(k) - 19(2-k^2)^3E(k)$ is monotonically decreasing. We finally estimate

$$
\begin{aligned}
&2(59k^4 - 76k^2 + 76)(1-k^2)K(k) - 19(2-k^2)^3E(k) \\
&\leq 2(59 \cdot 0.907^2 - 76 \cdot 0.907 + 76)(1 - 0.907)K(\sqrt{0.907}) \\
&\qquad - 19(2 - 0.907)^3E(\sqrt{0.907}) \\
&\approx -0.247829 < 0,
\end{aligned}
$$

and therefore the claim follows.

Proof of (7) Straightforward calculation yields

$$
\begin{aligned}
B - 16D = &\frac{2}{\sqrt{2(1+\cos\theta)}\cos^2\theta(1-\cos\theta)} \\
&\times \left(-2E(l) + (\cos^2\theta + 2)(1-\cos\theta)K(l)\right).
\end{aligned}
$$

Note that

$$
\frac{\pi}{4} \leq \theta < \frac{\pi}{2} \quad \Longleftrightarrow \quad 0 < l^2 \leq 2(\sqrt{2} - 1),
$$

and the inequality $B - 16D < 0$ is equivalent to

$$
-E(l) + \frac{3l^4 - 8l^2 + 8}{(2-l^2)^3} \cdot (1-l^2)K(l) < 0.
$$

By **710.02**, **710.04** in [BF], we have

$$
\begin{aligned}
&\frac{d}{dl}\left(-E(l) + \frac{3l^4 - 8l^2 + 8}{(2-l^2)^3} \cdot (1-l^2)K(l)\right) \\
&= \frac{K(l) - E(l)}{l} + \frac{2l(3l^4 - 4l^2 + 8)}{(2-l^2)^4} \cdot (1-l^2)K(l) \\
&\qquad + \frac{3l^4 - 8l^2 + 8}{(2-l^2)^3}\left(\frac{E(l) - K(l)}{l} - lK(l)\right) \\
&= \frac{l^2(l^4 - 3l^2 + 4)}{(2-l^2)^3} \cdot \frac{E(l) - K(l)}{l} - \frac{3l^7}{(2-l^2)^4}K(l) \\
&= \frac{l^2((l^2 - \frac{3}{2})^2 + \frac{7}{4})}{(2-l^2)^3} \cdot \frac{E(l) - K(l)}{l} - \frac{3l^7}{(2-l^2)^4}K(l) < 0.
\end{aligned}
$$

Hence,

$$-E(l) + \frac{3l^4 - 8l^2 + 8}{(2 - l^2)^3} \cdot (1 - l^2)K(l)$$

is monotonically decreasing. We finally estimate

$$\begin{aligned}
&-E(l) + \frac{3l^4 - 8l^2 + 8}{(2 - l^2)^3} \cdot (1 - l^2)K(l) \\
&\quad < \lim_{l \to 0} \left(-E(l) + \frac{3l^4 - 8l^2 + 8}{(2 - l^2)^3} \cdot (1 - l^2)K(l) \right) = -E(0) + K(0) \\
&\quad = -\frac{\pi}{2} + \frac{\pi}{2} = 0.
\end{aligned}$$

Therefore it completes the proof.

Proof of (8) Note that

$$\frac{3}{8}\pi \le \theta < \frac{\pi}{2} \Longleftrightarrow (2 + \sqrt{2}) \left(2\sqrt{2 + \sqrt{2}} - 2 - \sqrt{2} \right) \le k^2 < 1.$$

We next obtain

$$15A - 32C = \frac{2((15 \sin^2 \theta + 4)(1 - \sin \theta)K(k(\theta)) - 4E(k(\theta)))}{\sqrt{2(1 + \sin \theta)} \sin^2 \theta (1 - \sin \theta)}.$$

Thus it is sufficient to prove

$$\begin{aligned}
&(15 \sin^2 \theta + 4)(1 - \sin \theta)K(k(\theta)) - 4E(k(\theta)) \\
&= \frac{2}{(2 - k^2)^3}((19k^4 - 16k^2 + 16)(1 - k^2)K(k) - 2(2 - k^2)^3 E(k)) < 0.
\end{aligned}$$

To do this, direct calculations yield

$$\begin{aligned}
&\frac{d}{dk}(19k^4 - 16k^2 + 16) = 2k(38k^2 - 16), \\
&\frac{d}{dk}(1 - k^2)K(k) = \frac{E(k)}{k} - \frac{1 + k^2}{k}K(k), \\
&\frac{d}{dk}(2 - k^2)^3 = -6k(2 - k^2)^2, \quad \frac{d}{dk}E(k) = \frac{E(k) - K(k)}{k}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{d}{dk}((19k^4-16k^2+16)(1-k^2)K(k)-2(2-k^2)^3E(k))\\
&=k(-97k^4+117k^2-56)K(k)+k(14k^4-41k^2+56)E(k)\\
&=k((14k^4-41k^2+56)\underbrace{(-K(k)+E(k))}_{<0}+k^2(76-83k^2)K(k)).
\end{aligned}$$

We now observe that

$$(2+\sqrt{2})\left(2\sqrt{2+\sqrt{2}}-2-\sqrt{2}\right)\approx 0.9604\ldots.$$

Set $x=k^2$, and, for $0.96<x<1$, we find

$$14x^2-41x+56>0,\quad 76-83x<0.$$

Thus

$$\frac{d}{dk}((19k^4-16k^2+16)(1-k^2)K(k)-2(2-k^2)^3E(k))<0.$$

It follows that $(19k^4-16k^2+16)(1-k^2)K(k)-2(2-k^2)^3E(k)$ is monotonically decreasing. We finally estimate

$$\begin{aligned}
&(19k^4-16k^2+16)(1-k^2)K(k)-2(2-k^2)^3E(k)\\
&=0.726016\cdot K(\sqrt{0.96})-2.24973\cdot E(\sqrt{0.96})\\
&=0.726016\cdot 3.016112-2.24973\cdot 1.050502=-0.173598<0.
\end{aligned}$$

Therefore, for $3\pi/8\le\theta<\pi/2$, $15A-32C<0$ holds.

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