A GRADIENT-FLOW APPROACH FOR THE CONVERGENCE OF THE ANISOTROPIC ALLEN-CAHN EQUATION

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ABSTRACT. These short notes provide a convergence result for the anisotropic Allen-Cahn equation to anisotropic mean curvature flow which applies to general "energetic" and "kinetic" anisotropies. This extends a previous result by Simon and the author [Comm. Pure Appl. Math. 71.**8** (2018): 1597–1647] to the scalar anisotropic case. For the sake of brevity, the proof is merely sketched, but some additional background on the key tool, the anisotropic tilt excess, is provided. A detailed proof will appear elsewhere.

Keywords: Mean curvature flow, anisotropy, Allen-Cahn equation, gradient flows, tilt excess, phase field.

Mathematical Subject Classification: 53C44, 49Q20, 35A15.

1. INTRODUCTION

These notes explore the connection between the Allen-Cahn equation and mean curvature flow for general anisotropies from a variational viewpoint. The mean curvature flow (MCF) equation

$$V = -\mu H_\sigma \quad \text{on } \Gamma_t \tag{1.1}$$

describes the evolution of hypersurfaces $(\Gamma_t)_{t\in[0,T)}$ in $\mathbb{R}^d$. Here $V = V(x,t)$ denotes the velocity of Γ_t in direction of its normal $\nu = \nu(x,t)$, $\mu = \mu(\nu)$ is called the mobility function, and $H_\sigma = H_\sigma(x,t) = -\nabla_{\Gamma_t}\cdot(D\sigma(\nu))$ denotes the mean curvature with respect to the anisotropic surface tension $\sigma = \sigma(\nu)$. Mean curvature flow models the slow relaxation of grain boundaries in polycrystals, which still remains the major application of and was in fact the first motivation for MCF. The Allen-Cahn equation

$$2g(\nabla u_\varepsilon)\partial_t u_\varepsilon = \nabla\cdot(Df(\nabla u_\varepsilon)) - \frac{1}{\varepsilon^2}W'(u_\varepsilon) \tag{1.2}$$

was introduced by Allen-Cahn [1] in the isotropic case $f(p) = |p|^2$ and $g(p) = 1$ and has received continuous attention ever since. We will use the standard double-well potential $W(u) := \frac{1}{4}(u^2-1)^2$ in these notes, but all statements carry over to double-well potentials with some mild growth and regularity properties. We will restrict our attention to the natural choices

$$f(p) = \sigma^2(p) \quad \text{and} \quad g(p) = \frac{|p|}{\mu(p)}, \tag{1.3}$$

where μ and σ are positively 1-homogeneous functions as in Definition 3.1.

One fundamental structural property of both (1.1) and (1.2), which draws an intimate connection between them—even in the anisotropic case—is their gradient-flow structure. In general, to define a gradient flow one must fix an energy functional and a Riemannian

Date: November 30, 2019.

structure. The solution u_ε of the Allen-Cahn equation (1.2) may be interpreted as the gradient flow of the Cahn-Hilliard (or Ginzburg-Landau) energy functional

$$E_\varepsilon(u) := \frac{1}{2}\int \varepsilon f(\nabla u) + \frac{1}{\varepsilon}W(u)\,dx \tag{1.4}$$

with respect to the inner product $\langle \delta u, \delta u\rangle_u := \int \varepsilon g(\nabla u)(\delta u)^2\,dx$. Formally speaking, also mean curvature flow is a gradient flow, namely of the surface energy functional $E(\Gamma) = \int_\Gamma \sigma(\nu)\,d\mathcal{H}^{d-1}$ with respect to the Riemannian structure $\langle V, V\rangle_\Gamma := \int_\Gamma \frac{1}{\mu(\nu)}V^2\,d\mathcal{H}^{d-1}$; with the caveat that this structure is completely degenerate [22].

In the static case, the variational structure is rigorous: the energy (1.4) approximates the anisotropic area functional

$$E(u) := E(\chi) := \theta\int \sigma(\nu)\,|\nabla\chi| \tag{1.5}$$

defined for functions $u = 2\chi - 1$ taking values only in the wells $-1, 1$ of the double-well potential W for which

$$\int \sigma(\nu)\,|\nabla\chi| := \sup\left\{-\int(\nabla\cdot\xi)\chi\,dx\colon \xi\in C^1([0,1)^d)^d,\ \sigma^\circ(\xi)\le 1\right\} < \infty,$$

see §3 the definition of the polar norm σ°. Indeed, Bouchité [9] showed, for more general integrands, that the functionals E_ε converge (in the sense of Γ-convergence) to E, where the coefficient θ is given by

$$\theta := \int_{-1}^{1}\sqrt{W(s)}ds. \tag{1.6}$$

Regarding the dynamic case under consideration here, it should be pointed out that already in the early 90's, Chen-Giga-Goto [13] proved the existence and uniqueness of viscosity solutions if one allows for the degenerate behaviour called fattening. Bellettini-Paolini [8] set up the problem of anisotropic mean curvature flow very carefully and, using formal asymptotic expansions, show that the (anisotropic) implicit time discretization proposed by Almgren-Taylor-Wang [2] and Luckhaus-Sturzenhecker [21], the level-set formulation [13, 16], and phase-field models produce solutions of MCF.

Using only minimal assumptions on $\sigma(\nu)$, Chambolle-Morini-Ponsiglione [12] proposed a new weak formulation for anisotropic two-phase MCF in the case $\mu(\nu) = \sigma(\nu)$. Their formulation is somewhat reminiscent of the traditional viscosity solution, in the sense that it is based on level-set functions, more precisely, the distance function $d(x,t)$ to $E(t)$. In their setting it is rather easy to prove uniqueness (except for fattening); the more difficult question of existence is solved by proving rigorously that the implicit time discretization [2, 21] converges in the anisotropic case. Using smoothing techniques instead of a time discretization, the same group of authors, together with Novaga, [11] extended this result to arbitrary mobilities $\mu(\nu)$. The interested reader is referred to [19] for the discussion of further related results in the isotropic case.

Throughout the paper, we will work on the d-dimensional cube $[0,1)^d$ with periodic boundary conditions and use the short-hand notation $\int(\,\cdot\,)\,dx := \int_{[0,1)^d}(\,\cdot\,)\,dx$ and $\int(\,\cdot\,)\,|\nabla\chi| := \int_{[0,1)^d}(\,\cdot\,)\,|\nabla\chi|$ for spatial integrals with respect to the d-dimensional Lebesgue measure and the surface measure $|\nabla\chi|$, respectively. The notation $A \lesssim B$ means that there exists a generic constant $C < +\infty$, only depending on the spatial dimension d, the anisotropy σ and mobility μ, such that $A \le CB$. We reserve the notation $D := D_p$ for the derivative with respect to the variable p and write ∇ for the spatial gradient. Throughout,

$a \cdot b := \sum_i a_i b_i$ and $A \colon B := \operatorname{tr}(A^\top B) = \sum_{i,j} a_{ij} b_{ij}$ denote the standard scalar products of vectors and matrices, respectively.

2. Main result

The main result presented in these notes is the following convergence theorem.

Theorem 2.1. *Let σ and μ be an admissible anisotropy and mobility according to Definition 3.1, repectively, and suppose the compatibility conditions* (1.3) *hold. Given a finite time horizon $T < \infty$, and well prepared initial conditions $u_{\varepsilon,0}$ and u_0, i.e.,*

$$u_{\varepsilon,0} \to u_0 \text{ in } L^2 \quad \text{and} \quad E_0 := \lim_{\varepsilon \to 0} E_\varepsilon(u_{\varepsilon,0}) = E(u_0) < \infty,$$

the solutions u_ε of the Allen-Cahn equation (1.2) *with initial data $u_{\varepsilon,0}$ are pre-compact, i.e., there exists a subsequence $\varepsilon \to 0$ and a limit $u = 2\chi - 1$ such that*

$$u_\varepsilon \to u \quad \text{in } L^2.$$

Furthermore, if additionally

$$\lim_{\varepsilon \to 0} \int_0^T E_\varepsilon(u_\varepsilon(t))\, dt = \int_0^T E(u(t))\, dt, \tag{2.1}$$

then u satisfies the MCF equation (1.1) *in the distributional sense: There exists a function V such that $\partial_t \chi = V|\nabla \chi| dt$ and*

$$\int_0^T \int \frac{1}{\mu(\nu)} V X \cdot \nu |\nabla \chi| dt = \int_0^T \int \nabla X \colon \left(\sigma(\nu)\mathrm{Id} - \nu \otimes D\sigma(\nu)\right) |\nabla \chi|\, dt \tag{2.2}$$

for all test vector fields $X \in C^1([0,T] \times [0,1)^d)^d$. Moreover, u satisfies the optimal energy-dissipation relation

$$\int \sigma(\nu) |\nabla \chi(T')| + \int_0^{T'} \int \frac{1}{\mu(\nu)} V^2 |\nabla \chi| dt \le \frac{1}{\theta} E_0 \quad \text{for a.e. } 0 < T' < T. \tag{2.3}$$

Remark 2.2. (i) The integrand on the right-hand side of (2.2) reads

$$\nabla X \colon \left(\sigma(\nu)\mathrm{Id} - \nu \otimes D\sigma(\nu)\right) = \sigma(\nu)(\nabla \cdot X) - (D\sigma(\nu) \cdot \nabla) X \cdot \nu$$

so that for a smooth evolution, an integration by parts reveals

$$\int_0^T \int \nabla X \colon \left(\sigma(\nu)\mathrm{Id} - \nu \otimes D\sigma(\nu)\right) |\nabla \chi|\, dt = - \int_0^T \int H_\sigma \nu \cdot X |\nabla \chi|\, dt.$$

Hence, the theorem indeed provides a distributional solution of the MCF equation (1.1).

(ii) An energy dissipation relation with optimal constant similar to (2.3) is at the heart of weak formulations of general gradient flows [23] and also mean curvature flow in particular [10].

Remark 2.3. (i) In the isotropic case, this convergence result extends naturally to the multiphase case [19]. More precisely, replacing (1.2) by a system of $N \in \mathbb{N}$ reaction-diffusion equation, where the potential W has $P \in \mathbb{N}$ wells, the limit equation is multiphase MCF, i.e., the solution is described by a partition $\chi = (\chi_1, \ldots, \chi_P)$ of the domain $[0,1)^d$ and satisfies a distributional version of multiphase MCF similar to the two-phase version here. This distributional form naturally encodes both (1.1) along each interface and a balance-of-forces condition along triple junctions. In the present more general anisotropic case, this condition reads

$$\sum_{i,j} \big(\sigma_{ij}(\nu_{ij})\, b_{ij} - (b_{ij} \cdot D\sigma_{ij}(\nu_{ij}))\, \nu_{ij}\big) = 0 \quad \text{along triple junctions,}$$

where the sum runs over the three pairs (i,j) representing the three interfaces Γ_{ij} present at the triple junction, and b_{ij} denotes the normal of the triple junction pointing tangential to and away from Γ_{ij}.

(ii) A major difficulty in generalizing the present anisotropic version to the multiphase case lies in the absence of a simple formula to relate the effective anisotropy $\sigma_{ij}(\nu)$ of the interface between Phases i and j to the anisotropy function $f = f(u,p)$ in the vector-valued Allen-Cahn equation. Bellettini-Braides-Riey [7] proved that the anisotropy is given implicitly by solving a cell problem, allowing for small-scale oscillations along the interface. In order to carry out a precise formal asymptotic expansion, Garcke-Nestler-Stoth [17] assumed that the effective energy densities are given by one-dimensional profiles, i.e., their Ansatz reads $\tilde\sigma_{ij}(\nu) := \inf \int_0^1 \sqrt{W(\gamma) f(\gamma, \dot\gamma \times \nu)} ds$. This minimization problem is more restrictive than the minimization for the cell problem in [7], and it is easy to see that $\sigma_{ij}(\nu) \le \tilde\sigma_{ij}(\nu)$. It is, however, possible to construct simple examples of integrands in the case of systems for which this inequality is strict, i.e., $\sigma_{ij}(\nu) < \tilde\sigma_{ij}(\nu)$; see Ambrosio-Fusco-Pallara [3, pp. 312–314; Theorem 5.55]. In other words, the simpler and explicit formula of [17] does not hold in general.

(iii) In the scalar case, Bouchitté [9] proved that the explicit representation formula in [17] does hold indeed. Also in the isotropic vectorial case we have $\sigma_{ij} = \tilde\sigma_{ij}$, as has been shown by Baldo [6]; see also Aviles-Giga [4, 5].

The main difficulty in proving Theorem 2.1 is to derive the weak formulation (2.2) of the MCF equation for the limit. To this end, we fix a vector field $X \in C^1([0,T]\times[0,1)^d)^d$ and test the Allen-Cahn equation (1.2) with $(X\cdot\varepsilon\nabla)u_\varepsilon$ to obtain its appropriate weak form

$$\begin{aligned}\int_0^T \int g(\nabla u_\varepsilon)\partial_t u_\varepsilon (X\cdot\varepsilon\nabla)u_\varepsilon \,dx\,dt \\ = \int_0^T \int \frac{1}{2}\big(\nabla\cdot(Df(\nabla u_\varepsilon)) - \frac{1}{\varepsilon^2}W'(u_\varepsilon)\big)(X\cdot\varepsilon\nabla)u_\varepsilon \,dx\,dt.\end{aligned} \tag{2.4}$$

One can verify the equation for the limit if each side of this equation converges to the corresponding side of the limit equation (2.2). In order to pass to the limit in these nonlinear functions of the weakly converging quantities ∇u_ε and $\partial_t u_\varepsilon$, we will use the energy convergence (2.1) to show that the normals $\nu_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ (or the normalized gradients $\varepsilon\nabla u_\varepsilon$) of the level sets of u_ε in fact converge *strongly.* A caveat is that even in the isotropic multiphase case, the (normalized) gradients $\varepsilon\nabla u_\varepsilon$ do *not* converge strongly. They rather behave like rank-1 matrices $\delta u_\varepsilon \otimes \nu_\varepsilon$, whose second component does not oscillate. However, the first component δu_ε will in general oscillate across the diffuse interface as it follows the tangent field of the geodesic in $\mathbb{R}^N$ between the two corresponding wells of W.

Below, after gathering some basic facts on general anisotropies in §3, we sketch the five steps of the proof of Theorem 2.1 in §4. First, we briefly discuss the compactness. Second, we define an appropriate anisotropic tilt excess for the phase field and show that we can pass to the limit $\varepsilon \to 0$ in this error functional. Third, we use the tilt excess to control the error in the veloctiy-term, the left-hand side of (2.4). Then we apply this method to find an alternative, simple proof of the convergence of the curvature term, a result in the static case due to Cicalese-Nagase-Pisante [14]. Last, in the fifth step, we show that the optimal energy-dissipation relation follows easily from the third step of the proof.

3. Anisotropies

We briefly collect some well-known facts about Finsler metrics. We refer the interested reader to [8, 15, 17, 18] for more details.

Definition 3.1.

(i) An *admissible anisotropy* σ is a function $\sigma\colon \mathbb{R}^d \to [0,\infty)$ with the following properties:
   (a) σ is positively 1-homogeneous, i.e., $\sigma(\lambda p) = \lambda\sigma(p)$ for all $p \in \mathbb{R}^d$ and all $\lambda > 0$.
   (b) σ is smooth, i.e., $\sigma \in C^2(\mathbb{R}^d \setminus \{0\})$.
   (c) σ is positive definite, i.e., $\sigma(p) > 0$ for all $p \in \mathbb{R}^d \setminus \{0\}$.
   (d) σ is (uniformly) conditionally convex, i.e., there exists $\underline{\sigma} > 0$ such that for all $p \in \mathbb{R}^d \setminus \{0\}$, $D^2\sigma(p) > \underline{\sigma}$ on $p^\perp$.

(ii) The dual σ° of an admissible anisotropy function σ is given by the polar norm, i.e.,

$$\sigma^\circ(q) := \sup_{p\neq 0} \frac{p\cdot q}{\sigma(p)} = \sup_{p\colon \sigma(p)\leq 1} p\cdot q.$$

(iii) An *admissible mobility* μ is a function $\mu\colon \mathbb{R}^d \to [0,\infty)$ such that
   (a) μ is positively 1-homogeneous, i.e., $\mu(\lambda p) = \lambda\mu(p)$ for all $p \in \mathbb{R}^d$ and all $\lambda > 0$.
   (b) μ is regular , i.e., $\mu \in C^{0,1}(\mathbb{R}^d)$.
   (c) μ is positive definite, i.e., $\mu > 0$ in $\mathbb{R}^d \setminus \{0\}$.

The following general facts can be found in the references mentioned at the beginning of this section.

Lemma 3.2. *Let σ be an admissible anisotropy. Then the following statements are true.*

(i) The dual anisotropy σ° is admissible.
(ii) σ is a Finsler metric and equivalent to the Euclidean metric, i.e.,

$$|p| \lesssim \sigma(p) \lesssim |p| \quad \text{for all } p \in \mathbb{R}^d.$$

(iii) $p\cdot D\sigma(p) = \sigma(p)$ for all $p \in \mathbb{R}^d$.
(iv) $\sigma^\circ(q)D\sigma(D\sigma^\circ(q)) = q$ for all $q \in \mathbb{R}^d$.

Note that items (iii) & (iv) are trivially true for $p = 0$ and $q = 0$, respectively, even though $D\sigma(0)$ and $D\sigma^\circ(0)$ are not uniquely defined.

The next lemma will be crucial for our proof. It states that the Euclidean distance $|\nu - \tilde\nu|$, restricted to the sphere, is "equivalent" to (the square root of) $\sigma(\nu) - D\sigma(\tilde\nu)\cdot\nu$. In the isotropic case, this simply follows from expanding the square:

$$\frac{1}{2}|\nu - \tilde\nu|^2 = 1 - \tilde\nu\cdot\nu.$$

Lemma 3.3 (Dziuk [15, Proposition 2.2])**.** *Let σ be an admissible anisotropy function and let $\nu, \tilde\nu \in \mathbb{S}^{d-1}$ be two unit vectors. Then*

$$|\nu - \tilde\nu|^2 \lesssim \sigma(\nu) - D\sigma(\tilde\nu)\cdot\nu \lesssim |\nu - \tilde\nu|^2. \tag{3.1}$$

Here, the implicit constant depends on $\sup|D\sigma|$, $\sup|D^2\sigma|$, and $\underline{\sigma}$, where the suprema are taken over the sphere $\mathbb{S}^{d-1}$. We omit the proof here, as the more important lower bound was already shown by Dziuk and the proof of the upper bound is similar.

4. Elements of proof

4.1. **Compactness.** The energy identity

$$E_\varepsilon(u_\varepsilon(T')) + \int_0^{T'} \int \varepsilon g(\nabla u_\varepsilon)(\partial_t u_\varepsilon)^2 \, dx \, dt = E_\varepsilon(u_{\varepsilon,0}) \quad \text{for all } 0 < T' < T, \tag{4.1}$$

which follows from the differential identity $\frac{d}{dt}E_\varepsilon(u_\varepsilon) = -\int \varepsilon g(\nabla u_\varepsilon)(\partial_t u_\varepsilon)^2 \, dx$, will serve as the basic a priori estimate to prove the compactness.

More precisely, one first gains control over the concatenations $\phi(u_\varepsilon)$, where

$$\phi(s) := \int_{-1}^{s} \sqrt{W(s')} ds', \tag{4.2}$$

via Young's inequality for

$$\partial_t(\phi(u_\varepsilon)) = \sqrt{W(u_\varepsilon)}\, \partial_t u_\varepsilon \quad \text{and} \quad \nabla(\phi(u_\varepsilon)) = \sqrt{W(u_\varepsilon)}\, \nabla u_\varepsilon \tag{4.3}$$

and the definiteness of μ and σ, respectively, cf. Definition 3.1. Hence, after passing to a subsequence, we may assume $\phi(u_\varepsilon) \to \phi(u)$ and $u_\varepsilon \to u = 2\chi - 1$ in L^2 and we construct the normal velocity V as the Radon-Nikodym derivative $\frac{\partial_t \chi}{|\nabla \chi|}$.

4.2. **Tilt excess.** To prove the strong convergence of the normals, we define an anisotropic tilt excess for our phase field u_ε. The inspiration comes from the tilt excess in geometric measure theory. In the isotropic case, the tilt excess reads

$$\mathscr{E}_{\text{iso}}(u, \xi) = \frac{1}{2} \int_0^T \int |\nu - \xi|^2 \, |\nabla \chi| \, dt. \tag{4.4}$$

Here ξ is a continuous vector field, which will be chosen to approximate the normal ν. The trivial identity $|\nu - \xi|^2 = 1 + |\xi|^2 - 2\xi \cdot \nu$ implies that, provided $|\xi| = 1$ on $\operatorname{supp} |\nabla \chi|$,

$$\mathscr{E}_{\text{iso}}(u, \xi) = \int_0^T \int (1 - \xi \cdot \nu) \, |\nabla \chi| \, dt. \tag{4.5}$$

Remark 4.1. The crucial properties of the excess are the following three.

(i) The first representation shows that $\mathscr{E}_{\text{iso}}(u, \xi)$ controls oscillations of the normal field ν.

(ii) The second representation is useful to show that $\mathscr{E}_{\text{iso}}(u, \xi)$ is continuous under the convergence $\chi_k \to \chi$ in L^1, $\int |\nabla \chi_k| \to \int |\nabla \chi|$.

(iii) Furthermore, since the normal field ν is measurable, we may approximate it by a continuous (or smooth) vector field ξ so that $\mathscr{E}_{\text{iso}}(u, \xi)$ can be made small, see Lemma 4.3 below for the anisotropic case.

It is worth noting that the two representations (4.4) and (4.5) lead to (at least) two different generalizations of the tilt excess to the anisotropic case. We will generalize the first representation as

$$\mathscr{E}(u, \xi) := \frac{1}{2} \int_0^T \int |\nu - \xi|^2 \sigma(\nu) |\nabla \chi| \, dt \tag{4.6}$$

(which by Lemma 3.2 (ii) is equivalent to $\mathscr{E}_{\text{iso}}$) and, more crucially, the second representation as

$$\tilde{\mathscr{E}}(u, \xi) := \int_0^T \int \big(\sigma(\nu) - D\sigma(\xi) \cdot \nu\big) |\nabla \chi| \, dt. \tag{4.7}$$

Although in the anisotropic case, the two variants do not agree exactly, by the crucial property (3.1) in Lemma 3.3, we know that these two variants are equivalent in the sense that

$$\mathscr{E}(u,\xi) \lesssim \tilde{\mathscr{E}}(u,\xi) \lesssim \mathscr{E}(u,\xi) \quad \text{if } |\xi| = 1 \text{ a.e. on } \operatorname{supp} |\nabla\chi| dt.$$

This can be combined with the trivial inequality

$$|\nu - \xi|^2 \lesssim \left|\nu - \frac{\xi}{|\xi|}\right|^2 + \left|\frac{\xi}{|\xi|} - \xi\right|^2 = \left|\nu - \frac{\xi}{|\xi|}\right|^2 + \left|1 - |\xi|^2\right| \tag{4.8}$$

to relax the constraint $|\xi| = 1$ on the vector field at the expense of an additional lower-order term:

$$\begin{aligned} &\mathscr{E}(u,\xi) - \int_0^T \int (1 - |\xi|^2)\, \sigma(\nu) |\nabla\chi|\, dt \\ &\qquad \lesssim \tilde{\mathscr{E}}(u,\xi) \lesssim \mathscr{E}(u,\xi) + \int_0^T \int (1 - |\xi|^2)\, \sigma(\nu) |\nabla\chi|\, dt \end{aligned} \tag{4.9}$$

for any $\xi \in C([0,1)^d \times [0,T])^d$ with $|\xi| \leq 1$. It is easy to check that the anistropic functionals (partly) inherit the desired properties mentioned above. More precisely, $\mathscr{E}$ satisfies (i) & (iii) in Remark 4.1, and $\tilde{\mathscr{E}}$ satisfies the crucial continuity property (ii).

The phase-field version $\mathscr{E}_\varepsilon(u,\xi)$ of the anisotropic excess will serve as an error functional as it will control oscillations of the normal $\nu_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$. Its necessary key feature is that it needs to be adapted to the particular form of the PDE and its gradient-flow structure so that the basic energy estimate and the natural energy-convergence assumption (2.1) are sufficient to pass to the limit in the excess functional. It turns out that a proper Ansatz in the present anisotropic case is

$$\mathscr{E}_\varepsilon(u_\varepsilon,\xi) = \frac{1}{2} \int_0^T \int |\nu_\varepsilon - \xi|^2 \sqrt{W(u_\varepsilon)}\, |\nabla u_\varepsilon|\, dx\, dt. \tag{4.10}$$

At first sight, this seems somewhat surprising since the anisotropy does not appear explicitly and a priori, the energy convergence (2.1) only gives us the precise limit of the energy density $\varepsilon\sigma^2(\nabla u_\varepsilon) + \frac{1}{\varepsilon}W(u_\varepsilon)$, not its isotropic variant $\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)$. However, by Lemma 3.3, $\mathscr{E}_\varepsilon(u_\varepsilon,\xi)$ is, up to a lower-order term as in (4.9), equivalent to

$$\tilde{\mathscr{E}}_\varepsilon(u_\varepsilon,\xi) = \int_0^T \int \Big(\sigma(\nabla u_\varepsilon)\sqrt{W(u_\varepsilon)} - D\sigma(\xi) \cdot \sqrt{W(u_\varepsilon)}\nabla u_\varepsilon\Big) dx\, dt.$$

The following lemma is based on this observation and states that we can in fact control the excess precisely as $\varepsilon \to 0$.

Lemma 4.2. *Let σ be admissible according to Definition 3.1 and suppose f and is compatible to σ in the sense of* (1.3). *Given a sequence u_ε and u such that*

$$u_\varepsilon \to u \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_0^T E_\varepsilon(u_\varepsilon(t))\, dt = \int_0^T E(u(t))\, dt < \infty,$$

then for any $\xi \in C([0,1)^d \times [0,T])^d$ with $|\xi| \leq 1$

$$\limsup_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u_\varepsilon,\xi) \lesssim \mathscr{E}(u,\xi) + \int_0^T \int (1 - |\xi|^2)\, \sigma(\nu) |\nabla\chi|\, dt.$$

Clearly, the $|\nabla\chi|$-measurability of ν implies that we can approximate it by continuous (or smooth) functions:

Lemma 4.3. *If $\int_0^T E(u)dt < \infty$, then for any $\varepsilon > 0$ there exists $\xi \in C([0,1)^d \times [0,T])^d$ with $|\xi| \leq 1$ such that $\mathscr{E}(u,\xi) < \varepsilon$.*

Of course, by taking u_ε and u independent of t in the previous two lemmas, upon division by $T > 0$, the analogous time-independent results hold as well.

4.3. **Convergence of the velocity term.** We will now employ the tilt excess (4.10) to pass to the limit in the velocity term. More precisely, we will prove the following statement.

Proposition 4.4. *Let μ and σ be admissible according to Definition 3.1 and suppose f and g are compatible to σ and μ in the sense of* (1.3). *Let u_ε and u be such that*

$$u_\varepsilon \to u = 2\chi - 1 \text{ in } L^2 \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_0^T E_\varepsilon(u_\varepsilon(t))\,dt = \int_0^T E(u(t))\,dt < \infty.$$

Then for any test vector field $X \in C([0,1)^d \times [0,T])^d$

$$\lim_{\varepsilon\to 0} \int_0^T \int g(\nabla u_\varepsilon)\partial_t u_\varepsilon (X \cdot \varepsilon\nabla) u_\varepsilon \,dx\,dt = \theta \int_0^T \int \frac{1}{\mu(\nu)} V X \cdot \nu |\nabla\chi| dt.$$

The idea behind the proposition is that, given a continuous vector field ξ with $|\xi| \leq 1$, we write the integrand of the left-hand side as X times

$$g(\nabla u_\varepsilon)\partial_t u_\varepsilon \varepsilon \nabla u_\varepsilon = g(\nu_\varepsilon)\partial_t u_\varepsilon \varepsilon \frac{1}{\sqrt{f(\nu_\varepsilon)}} \sqrt{f(\nabla u_\varepsilon)}\,\nu_\varepsilon.$$

Thinking of $\xi \approx \nu_\varepsilon$, we replace the normal ν_ε on the right-hand side by ξ at the expense of an error which will be controlled by the excess $\mathscr{E}_\varepsilon(u_\varepsilon,\xi)$. Since, by equipartition of energy, the remaining term satisfies

$$\partial_t u_\varepsilon \varepsilon \sqrt{f(\nabla u_\varepsilon)} \approx \partial_t u_\varepsilon \sqrt{W(u_\varepsilon)} = \partial_t(\phi(u_\varepsilon)) \rightharpoonup \partial_t(\phi(u)) = \theta V |\nabla\chi| \quad \text{as } \varepsilon \to 0.$$

Then we undo our previous manipulation by replacing ξ by ν, the (measure theoretic) normal of the limit at the expense of another error controlled by the excess. More precisely, by the above method, we can show

$$\limsup_{\varepsilon\to 0} \left| \int_0^T \int g(\nabla u_\varepsilon)\partial_t u_\varepsilon (X \cdot \varepsilon\nabla) u_\varepsilon \,dx\,dt - \theta \int_0^T \int \frac{1}{\mu(\nu)} V X \cdot \nu |\nabla\chi| dt \right|$$
$$\lesssim (\sup|X|) E_0^{\frac12} (1+T)^{\frac12} \big(\limsup_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u_\varepsilon,\xi) + \mathscr{E}(u,\xi)\big)^{\frac12}.$$

Finally, we may apply Lemmas 4.2 and 4.3 to show that the right-hand side can be made arbitrarily small.

4.4. **Convergence of the curvature term.** The convergence of the right-hand side term in (2.4) has already been proven by Cicalese-Nagase-Pisante [14] who generalized the Reshatnyak-type argument of Luckhaus-Modica [20] to the anisotropic setting. The main result of their paper may be stated as follows.

Proposition 4.5 (Cicalese-Nagasa-Pisante [14, Theorem 3.3])**.** *Suppose that σ is an admissible anisotropy and that the compatibility condition* (1.3) *holds. Given a sequence of functions*

$$u_\varepsilon \to u = 2\chi - 1 \text{ in } L^2 \quad \text{such that} \quad \lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = E(u) < \infty, \tag{4.11}$$

then for all test vector fields $X \in C^1([0,1)^d)^d$

$$\lim_{\varepsilon\to 0}\int \frac{1}{2}\big(\nabla\cdot(Df(\nabla u_\varepsilon)) - \frac{1}{\varepsilon^2}W'(u_\varepsilon)\big)(X\cdot\varepsilon\nabla)u_\varepsilon\,dx \\ = \theta\int \nabla X\colon\, (\sigma(\nu)\mathrm{Id}-\nu\otimes D\sigma(\nu))\,|\nabla\chi|.$$

In other words, this result ensures that the convergence of the energies implies the convergence of their first (inner) variations.

Let us give a brief sketch of a simple alternative proof using the tilt excess. Integration by parts shows that

$$\int \frac{1}{2}\big(\nabla\cdot(Df(\nabla u_\varepsilon)) - \frac{1}{\varepsilon^2}W'(u_\varepsilon)\big)(X\cdot\varepsilon\nabla)u_\varepsilon\,dx = \int \nabla X\colon T_\varepsilon\,dx,$$

where T_ε denotes the energy-stress tensor

$$T_\varepsilon := \frac{1}{2}\big(\varepsilon f(\nabla u_\varepsilon) + \frac{1}{\varepsilon}W(u_\varepsilon)\big)\mathrm{Id} - \varepsilon\nabla u_\varepsilon\otimes\frac{1}{2}Df(\nabla u_\varepsilon).$$

Now we only need to show

$$T_\varepsilon \rightharpoonup \theta\big(\sigma(\nu)\mathrm{Id} - D\sigma(\nu)\otimes\nu\big)|\nabla\chi| \quad \text{as measures on } [0,1)^d.$$

This follows once we have have shown the convergence of the second term

$$\varepsilon\nabla u_\varepsilon\otimes\frac{1}{2}Df(\nabla u_\varepsilon) \rightharpoonup \theta\nu\otimes D\sigma(\nu)|\nabla\chi|, \tag{4.12}$$

since the assumption (4.11) implies the desired convergence of the first term in T_ε.

To convince ourselves of (4.12) we make use of the tilt excess (4.10) once more. Let $\xi\in C([0,1)^d)^d$ with $|\xi|\le 1$ be fixed. By the compatibility (1.3) of f and σ we have $\frac{1}{2}Df = \sigma D\sigma$. Hence, using the positive 1-homogeneity of σ, we may rewrite the left-hand side of (4.12) as

$$\varepsilon\sigma(\nabla u_\varepsilon)\nabla u_\varepsilon\otimes D\sigma(\nu_\varepsilon) \approx \varepsilon\sigma(\nabla u_\varepsilon)\nabla u_\varepsilon\otimes D\sigma(\xi) \approx \sqrt{W(u_\varepsilon)}\nabla u_\varepsilon\otimes D\sigma(\xi),$$

where the first "$\approx$" means up to an error controlled by the excess, and the second one up to an error, which due to equipartition of energy vanishes as $\varepsilon\to 0$. But the resulting nonlinear expression $\sqrt{W(u_\varepsilon)}\nabla u_\varepsilon = \nabla(\phi(u_\varepsilon)$ is compact and, upon replacing ξ by ν after taking the limit $\varepsilon\to 0$ at the expense of another error controlled by the excess, we arrive at

$$\limsup_{\varepsilon\to 0}\Big|\int \frac{1}{2}\big(\nabla\cdot(Df(\nabla u_\varepsilon)) - \frac{1}{\varepsilon^2}W'(u_\varepsilon)\big)(X\cdot\varepsilon\nabla)u_\varepsilon\,dx \\ - \theta\int \nabla X\colon\, (\sigma(\nu)\mathrm{Id}-\nu\otimes D\sigma(\nu))\,|\nabla\chi|\Big| \\ \lesssim (\sup|\nabla X|)E_0^{\frac12}\Big(\limsup_{\varepsilon\to 0}\mathscr{E}_\varepsilon(u_\varepsilon,\xi) + \mathscr{E}(u,\xi)\Big)^{\frac12}.$$

Again, by Lemmas 4.2 and 4.3 the right-hand side can be made arbitrarily small.

4.5. **Optimal energy dissipation.** Let us finally address the optimal energy dissipation relation (2.3).

Lemma 4.6. *Under the same assumptions as in Proposition 4.4, we have*

$$E(u(T')) + \theta\int_0^{T'}\int \frac{1}{\mu(\nu)}V^2|\nabla\chi|dt \le E_0 \quad \textit{for a.e. } 0<T'<T.$$

We give a brief sketch of the proof behind this simple statement. Comparing the claim to the energy-dissipation identity (4.1), we see that the only difficulty is proving

$$\liminf_{\varepsilon\to 0}\frac{1}{2}\int_0^{T'}\int \varepsilon g(\nabla u_\varepsilon)(\partial_t u_\varepsilon)^2\,dx\,dt \geq \frac{\theta}{2}\int_0^{T'}\int\frac{1}{\mu(\nu)}V^2|\nabla\chi|\,dt, \tag{4.13}$$

where we divided through by 2 for convenience in the following argument.

Let $\zeta=\zeta(x,t)$ and $\xi=\xi(x,t)$ be a continuous function and vector field, respectively. Using the trivial inequality $\frac{1}{2}a^2\geq ab-\frac{1}{2}b^2$ with $a=\partial_t u$ and $b=\frac{1}{\varepsilon}\sqrt{W(u_\varepsilon)}\zeta$,

$$\frac{1}{2}\varepsilon g(\nu_\varepsilon)(\partial_t u_\varepsilon)^2\geq g(\nu_\varepsilon)\sqrt{W(u_\varepsilon)}\zeta\partial_t u_\varepsilon-\frac{1}{2}g(\nu_\varepsilon)\frac{1}{\varepsilon}W(u_\varepsilon)\zeta^2.$$

Note that after integration over x and t, the left-hand side is precisely the term on the left-hand side of (4.13) since g is 0-homogeneous, cf. (1.3). Now, as before, we may replace ν_ε on the right-hand side by ξ at the expense of an error controlled by the excess. By equipartition of energy, $\frac{1}{\varepsilon}W(u_\varepsilon)\rightharpoonup\theta|\nabla\chi|$, and as before, $\sqrt{W(u_\varepsilon)}\partial_t u_\varepsilon=\partial_t(\phi(u_\varepsilon))\rightharpoonup\theta V|\nabla\chi|dt$. Hence

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{2}\int_0^{T'}\int\varepsilon g(\nabla u_\varepsilon)(\partial_t u_\varepsilon)^2\,dx\,dt\geq&\,\theta\int_0^{T'}\int g(\xi)\zeta V|\nabla\chi|\,dt-\frac{\theta}{2}\int_0^{T'}\int g(\xi)\zeta^2|\nabla\chi|\,dt\\&-C(1+\sup\zeta^2)(1+T)^{\frac{1}{2}}E_0^{\frac{1}{2}}\big(\limsup_{\varepsilon\to 0}\mathscr{E}(u_\varepsilon,\xi)\big)^{\frac{1}{2}}.\end{aligned}$$

Replacing ξ by the normal in the first two right-hand side terms at the expense of just another error term controlled by the excess, we may then conclude by first appealing to Lemmas 4.2 and 4.3 (i.e., take $\xi\to\nu$) to make the last error term on the right-hand side vanish, and then choose $\zeta\to V$ to maximize the remaining term on right-hand side, which yields precisely (4.13).

Acknowledgement

I would like to thank the organizers, in particular Prof. Yoshihito Kohsaka, for the kind invitation to the RIMS workshop "Geometric Aspects of Solutions to Partial Differential Equations", for fruitful discussions in Kyoto and Kobe, and for generously extending my stay in Japan.

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