A remark on the asymptotic behavior of solutions of the Allen-Cahn equation

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1 Introduction

We consider the reaction-diffusion equation

$$u_t = \Delta u + (u+a)(1-u^2), \quad (x,y) \in \mathbb{R}^2, \tag{1.1}$$

where $u = u(x,y,t)$ is unknown and $a \in (0,1)$ is a constant. Our interest is the asymptotic behavior of solutions of (1.1).

It is well-known that (1.1) has one-dimensional traveling fronts connecting stable equilibria 1 and -1. One of the fronts is given by $u_*(x,y,t) = \Phi(y-kt)$, where $\Phi(\eta) := -\tanh(\eta/\sqrt{2})$ and $k = \sqrt{2}a$. We note that the front propagates in y-direction with speed k and that Φ solves

$$\Phi'' + k\Phi' + (\Phi + a)(1-\Phi^2) = 0. \tag{1.2}$$

It is shown that u_* is stable in appropriate senses (see for instance [3, 4, 8]), and therefore u_* plays an important role to understand front propagation in (1.1). To see this, let us observe the asymptotic behavior of a radially symmetric solution $u(x,y,t) = u(r,t)$, $r = \sqrt{x^2+y^2}$. According to [1, 5, 6, 7, 10, 11, 12], the following hold. For any $\varepsilon > 0$, there exists $r_0 > 0$ such that if

$$\min_{r \le r_0} u(r,0) \ge -a + \varepsilon, \quad \limsup_{r\to\infty} u(r,0) < -a, \quad \liminf_{r\to\infty} u(r,0) > -\frac{1}{\varepsilon},$$

then for some function h satisfying $h(t) \to \infty$ $(t \to \infty)$,

$$u(r,t) - \Phi(r - h(t)) \to 0 \quad \text{uniformly for } r > 0 \text{ as } t \to \infty. \tag{1.3}$$

Furthermore, the asymptotic behavior of h is given by

$$h(t) = kt - \frac{1}{k}\log t + O(1) \quad (t \to \infty). \tag{1.4}$$

From (1.3), we see that the profile of u locally looks like a one-dimensional traveling front. Indeed, one finds that

$$u(x, \eta + kt + \tilde{h}(t), t) \to \Phi(\eta) \quad \text{locally uniformly for } (x,\eta) \in \mathbb{R}^2 \text{ as } t \to \infty, \tag{1.5}$$

where $\tilde{h}(t) = h(t) - kt$. (1.4) and (1.5) mean that the difference of the position of a level set between u and u_* grows logarithmically while u converges locally to u_*. The natural question arises whether it is possible to find a solution which satisfies (1.5) for some $\tilde{h}(t)$ growing polynomially. Our main result is concerned with the existence of such solutions.

Theorem 1. *Let* $0 < \beta < 1/2$ *and* $b > 0$. *Then there exists a solution* u *of* (1.1) *such that for some function* h *satisfying* $h(t) = kt + bt^\beta + o(t^\beta)$ $(t \to \infty)$ *and some time sequence* $\{t_i\}$ *with* $t_1 < t_2 < \cdots \to \infty$,

$$u(x, \eta + h(t_i), t_i) \to \Phi(\eta) \quad \textit{locally uniformly for } (x, \eta) \in \mathbb{R}^2 \textit{ as } i \to \infty. \tag{1.6}$$

Remark 2. The theorem should hold for $\beta < 1$ and without taking a time sequence.

2 Sketch of the proof of Theorem 1

In this section we give a procedure to prove Theorem 1.

2.1 Supersolutions and subsolutions

We find the desired solution by constructing a supersolution and a subsolution. We look for a supersolution in the form

$$u^+(x, y, t) = \Phi\left(\frac{y - \phi(x,t)}{\sqrt{1 + \phi_x(x,t)^2}} - p(t)\right) + q(t),$$

where the functions ϕ, p and q are determined later. A supersolution of this type was first used in [9] to construct conical traveling wave solutions. Put $f(u) = (u + a)(1 - u^2)$ and $\mathcal{F}[u] = u_t - \Delta u - f(u)$. By a direct calculation, we have

$$\begin{aligned}
\mathcal{F}[u^+] = &-\frac{\Phi' \circ \zeta}{\sqrt{1+\phi_x^2}}\left(\phi_t - \phi_{xx} - k\sqrt{1+\phi_x^2}\right) \\
&- \frac{\Phi' \circ \zeta}{1+\phi_x^2}\left[\frac{2\phi_x^2\phi_{xx}}{\sqrt{1+\phi_x^2}} + (\zeta + p)\left\{\phi_x(\phi_t - \phi_{xx})_x - \frac{(1-2\phi_x^2)\phi_{xx}^2}{1+\phi_x^2}\right\}\right] \\
&- \frac{(\zeta+p)(\Phi'' \circ \zeta)\phi_x^2\phi_{xx}}{(1+\phi_x^2)^{3/2}}\left\{2 + \frac{(\zeta+p)\phi_{xx}}{\sqrt{1+\phi_x^2}}\right\} \\
&- (\Phi' \circ \zeta)p_t + f(\Phi \circ \zeta) - f(\Phi \circ \zeta + q) + q_t,
\end{aligned}$$

where we have used (1.2) and put

$$\zeta(x, y, t) = \frac{y - \phi(x,t)}{\sqrt{1 + \phi_x(x,t)^2}} - p(t).$$

Since we are looking for a solution converging locally to a one-dimensional traveling front, the derivatives of ϕ with respect to x should decay as $t \to \infty$. For this reason $\phi_t - \phi_{xx} - k\sqrt{1+\phi_x^2}$ would dominate other terms. Hence it is appropriate to choose ϕ as a solution of the equation

$$\phi_t = \phi_{xx} + k + \frac{k}{2}\phi_x^2, \quad x \in \mathbb{R}. \tag{2.1}$$

As an initial value of ϕ, we take a function with sublinear growth, that is,

$$\phi(x,0) = A|x|^\alpha, \quad x \in \mathbb{R}, \tag{2.2}$$

where $A > 0$ and $\alpha \in (0,1)$. Then the derivatives ϕ_x and ϕ_{xx} indeed decay.

Lemma 3. *Let ϕ be a solution of* (2.1)–(2.2). *Then there is a constant $C > 0$ such that*

$$\|\phi_x(\cdot,t)\|_{L^\infty(\mathbb{R})} \le Ct^{-\frac{1-\alpha}{2-\alpha}}, \qquad \|\phi_{xx}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le Ct^{-\frac{2(1-\alpha)}{2-\alpha}}$$

for all $t \ge 1$.

We omit the proof of this Lemma. Suppose that ϕ satisfies (2.1)–(2.2). Then $\mathcal{F}[u^+]$ is written as

$$\mathcal{F}[u^+] = -(\Phi'\circ\zeta)p_t + f(\Phi\circ\zeta) - f(\Phi\circ\zeta + q) + q_t + R,$$

where

$$\begin{aligned} R = {} & \frac{k(\Phi'\circ\zeta)}{\sqrt{1+\phi_x^2}}\left(\sqrt{1+\phi_x^2} - 1 - \frac{1}{2}\phi_x^2\right) \\ & - \frac{\Phi'\circ\zeta}{1+\phi_x^2}\left[\frac{2\phi_x^2\phi_{xx}}{\sqrt{1+\phi_x^2}} + (\zeta+p)\left\{k\phi_x^2\phi_{xx} - \frac{(1-2\phi_x^2)\phi_{xx}^2}{1+\phi_x^2}\right\}\right] \\ & - \frac{(\zeta+p)(\Phi''\circ\zeta)\phi_x^2\phi_{xx}}{(1+\phi_x^2)^{3/2}}\left\{2 + \frac{(\zeta+p)\phi_{xx}}{\sqrt{1+\phi_x^2}}\right\}. \end{aligned}$$

Note that $\eta\Phi'(\eta)$ and $\eta^2\Phi''(\eta)$ are bounded in $\mathbb{R}$. From this fact and Lemma 3, we deduce that

$$|R| \le K(1+p^2)t^{-\gamma} \tag{2.3}$$

for $t \ge 1$, where $K > 0$ is a constant and $\gamma = 4(1-\alpha)/(2-\alpha)$. For positive constants p_0, q_0 and t_0, we put

$$p(t) = Kp_0(t_0^{-\gamma+1} - t^{-\gamma+1}), \qquad q(t) = Kq_0t^{-\gamma}.$$

We check that under the condition

$$\alpha < \frac{2}{3}, \tag{2.4}$$

u^+ satisfies $\mathcal{F}[u^+] \geq 0$ for $t \geq t_0$ provided that p_0, q_0 and t_0 are chosen appropriately. We first note that (2.4) implies $\gamma > 1$. It is easy to see that

$$\left(\min_{|s| \geq 1-2\delta} (-f'(s)) \right) > 0$$

for some $\delta > 0$. Then we take $q_0 > 0$ such that

$$\left(\min_{|s| \geq 1-2\delta} (-f'(s)) \right) q_0 \geq 3. \tag{2.5}$$

Since $\Phi(\eta) \to \pm 1$ as $\eta \to \mp\infty$, one can pick up $\eta_0 > 0$ such that

$$|\Phi(\eta)| \geq 1 - \delta \quad \text{for all } |\eta| \geq \eta_0. \tag{2.6}$$

The constant $p_0 > 0$ is then chosen so that

$$\left(\min_{|\eta| \leq \eta_0} (-\Phi'(\eta)) \right) (\gamma - 1) p_0 - \left(\max_{s \in \mathbb{R}} f'(s) \right) q_0 \geq 3. \tag{2.7}$$

Finally we choose $t_0 \geq 1$ so large that

$$\max \left\{ K p_0 t_0^{-\gamma+1}, \gamma q_0 t_0^{-1}, \delta^{-1} K q_0 t_0^{-\gamma} \right\} \leq 1. \tag{2.8}$$

This particularly gives $0 \leq p \leq K p_0 t_0^{-\gamma+1} \leq 1$ for $t \geq t_0$. From (2.3), (2.8) and the fact that $0 \leq p \leq 1$, we have

$$\begin{aligned} &\mathcal{F}[u^+] \\ &\geq K t^{-\gamma} \left\{ -(\Phi' \circ \zeta)(\gamma - 1) p_0 + \left(\int_0^1 -f'(\Phi \circ \zeta + \theta q) d\theta \right) q_0 - \gamma q_0 t^{-1} - (1 + p^2) \right\} \\ &= K t^{-\gamma} \left\{ -(\Phi' \circ \zeta)(\gamma - 1) p_0 + \left(\int_0^1 -f'(\Phi \circ \zeta + \theta q) d\theta \right) q_0 - 3 \right\} \end{aligned}$$

for $t \geq t_0$.

Let us verify $\mathcal{F}[u^+] \geq 0$. We consider the case $|\zeta| \geq \eta_0$. We see from (2.6) and (2.8) that

$$|\Phi \circ \zeta + \theta q| \geq |\Phi \circ \zeta| - q \geq 1 - \delta - K q_0 t_0^{-\gamma} \geq 1 - 2\delta$$

for $\theta \in [0, 1]$. Hence, by (2.5) and the fact that $\Phi' < 0$,

$$\mathcal{F}[u^+] \geq K t^{-\gamma} \left\{ \left(\min_{|s| \geq 1-2\delta} (-f'(s)) \right) q_0 - 3 \right\} \geq 0.$$

Moreover if $|\zeta| \le \eta_0$, then (2.7) gives

$$\mathcal{F}[u^+] \ge Kt^{-\gamma} \left\{ \left(\min_{|\eta|\le\eta_0} (-\Phi'(\eta)) \right) (\gamma - 1)p_0 - \left(\max_{s\in\mathbb{R}} f'(s) \right) q_0 - 3 \right\} \ge 0.$$

Thus we conclude that $\mathcal{F}[u^+] \ge 0$.

Let ϕ, p and q are chosen as above. Then from a similar computation we see that

$$u^-(x,y,t) = \Phi \left(\frac{y - \phi(x,t)}{\sqrt{1 + \phi_x(x,t)^2}} + p(t) \right) - q(t)$$

satisfies $\mathcal{F}[u^-] \le 0$ provided $t \ge t_0$. By the monotonicity of Φ, we have $u^- \le u^+$. Consequently the comparison principle shows that there is a solution u of (1.1) satisfying $u^- \le u \le u^+$.

2.2 Asymptotic behavior of ϕ

To determine the asymptotic profile of u^+ and u^-, we need to examine the precise behavior of ϕ. It is well-known that by the putting $v = e^{\phi - kt}$, the problem (2.1)–(2.2) is transformed into the heat equation

$$\begin{cases} v_t = v_{xx}, & x \in \mathbb{R}, t > 0, \\ v(x,0) = \exp(A|x|^\alpha), & x \in \mathbb{R}. \end{cases}$$

Hence ϕ is given by

$$\phi(x,t) = kt + \log v(x,t),$$
$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4t} + A|y|^\alpha \right) dy.$$

Splitting the interval of integration into $[0,\infty)$ and $(-\infty, 0)$ gives

$$\begin{aligned} v(x,t) = &\frac{1}{\sqrt{4\pi t}} \int_0^\infty \exp\left(-\frac{(x-y)^2}{4t} + Ay^\alpha \right) dy \\ &+ \frac{1}{\sqrt{4\pi t}} \int_0^\infty \exp\left(-\frac{(x+y)^2}{4t} + Ay^\alpha \right) dy. \end{aligned}$$

By the change of variables $y = t^{1/(2-\alpha)}Y$, we obtain

$$v(x,t) = I\left(t^{-\frac{1}{2-\alpha}} x, t \right) + I\left(-t^{-\frac{1}{2-\alpha}} x, t \right), \tag{2.9}$$

where

$$I(X,t) := \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \int_0^\infty \exp\left(t^{\frac{\alpha}{2-\alpha}} g(Y)\right) dY,$$

$$g(Y) = g(Y;X) := -\frac{1}{4}(Y-X)^2 + AY^\alpha.$$

It is easily seen that $g(Y)$ has a unique critical point $Y_* = Y_*(X) > 0$. Then we define $M = M(X)$ and $K = K(X)$ by

$$M := g(Y_*) = -\frac{1}{4}(Y_* - X)^2 + AY_*^\alpha,$$

$$K := \frac{1}{\sqrt{-2g''(Y_*)}} = \frac{1}{\sqrt{1 + 2\alpha(1-\alpha)AY_*^{\alpha-2}}}.$$

It is elementary to show that

$$g''(Y) < -\frac{1}{2}, \tag{2.10}$$

$$Y_*(X),\ M(X) \text{ and } K(X) \text{ are increasing}. \tag{2.11}$$

We set

$$U(X,t) := K(X) \exp\left(t^{\frac{\alpha}{2-\alpha}} M(X)\right).$$

The asymptotic behavior of ϕ is described by means of U.

Proposition 4. *The solution ϕ of* (2.1)–(2.2) *satisfies*

$$\left|\phi(x,t) - kt - \log\left(U\left(t^{-\frac{1}{2-\alpha}}x, t\right) + U\left(-t^{-\frac{1}{2-\alpha}}x, t\right)\right)\right| \le Ct^{-\frac{\alpha}{2(2-\alpha)}}$$

for all $x \in \mathbb{R}$ and $t \ge 1$ with some constant $C > 0$.

To prove the proposition we consider the estimate of I.

Lemma 5. *There is a constant $C > 0$ such that*

$$\left|\frac{I(X,t)}{U(X,t)} - 1\right| \le Ct^{-\frac{\alpha}{2(2-\alpha)}}$$

for all $X \ge -1$ and $t \ge 1$.

Proof. Let $X \ge -1$. In the proof, constants in big O notation do not depend on X. Put

$$Y_0 := Y_*(-1), \quad C_1 := \frac{1}{6} g'''\left(\frac{Y_0}{2}\right) = \frac{2^{2-\alpha}}{3}\alpha(1-\alpha)(2-\alpha)AY_0^{\alpha-3},$$

and define $\delta > 0$ by

$$\delta := \min\left\{\frac{Y_0}{2}, \frac{1}{8C_1}\right\}.$$

From (2.11), we see that $0 < \delta < Y_*(X)$. We rewrite I as

$$\begin{aligned} I &= \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) \int_0^\infty \exp\left(t^{\frac{\alpha}{2-\alpha}} (g(Y) - M)\right) dY \\ &= \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) \left(\int_{|Y-Y_*|>\delta, Y\geq 0} + \int_{|Y-Y_*|\leq\delta}\right) \\ &=: \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) (I_1 + I_2). \end{aligned}$$

Let us estimate I_1. From (2.10) and the fact that $g'(Y_*) = 0$,

$$g(Y) - M = \left\{\int_0^1 (1-\theta) g''(Y_* + \theta(Y - Y_*)) d\theta\right\} (Y - Y_*)^2 \leq -\frac{1}{4}(Y - Y_*)^2.$$

For $a, b > 0$,

$$\int_{|x|>a} e^{-b|x|^2} dx = \int_{|x|>a} \frac{1}{|x|} \cdot |x| e^{-b|x|^2} dx \leq \frac{1}{a} \int_{\mathbb{R}} |x| e^{-b|x|^2} dx = \frac{1}{ab}. \tag{2.12}$$

Hence

$$|I_1| \leq \int_{|Y-Y_*|>\delta} \exp\left(-\frac{1}{4} t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY \leq \frac{4}{\delta} t^{-\frac{\alpha}{2-\alpha}}. \tag{2.13}$$

I_2 is handled as follows. We rewrite I_2 as $I_2 = I_{21} + I_{22} + I_{23}$, where

$$\begin{gathered} I_{21} := \int_{\mathbb{R}} \exp\left(\frac{1}{2} g''(Y_*) t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY, \\ I_{22} := \int_{|Y-Y_*|>\delta, Y\geq 0} \exp\left(\frac{1}{2} g''(Y_*) t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY, \\ I_{23} := \int_{|Y-Y_*|\leq\delta} \left(\exp\left(t^{\frac{\alpha}{2-\alpha}} R(Y)\right) - 1\right) \exp\left(\frac{1}{2} g''(Y_*) t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY, \\ R(Y) := g(Y) - M - \frac{1}{2} g''(Y_*)(Y - Y_*)^2. \end{gathered}$$

By a direct computation,

$$I_{21} = \sqrt{4\pi} K t^{-\frac{\alpha}{2(2-\alpha)}}.$$

From (2.10) and (2.12), I_{22} is estimated as

$$|I_{22}| \leq \int_{|Y-Y_*|>\delta} \exp\left(-\frac{1}{4} t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY \leq \frac{4}{\delta} t^{-\frac{\alpha}{2-\alpha}}.$$

We consider I_{23}. Let $|Y - Y_*| \leq \delta$. Then, by (2.11) and the definition of δ, we have $Y_* + \theta(Y - Y_*) \geq Y_0 - \delta \geq Y_0/2$ for all $\theta \in [0, 1]$. This together with the fact that g''' is positive and decreasing yields

$$|g'''(Y_* + \theta(Y - Y_*))| \leq g''' \left(\frac{Y_0}{2} \right) = 6C_1.$$

Therefore

$$|R(Y)| = \frac{1}{2} \left| \int_0^1 (1-\theta)^2 g'''(Y_* + \theta(Y - Y_*)) d\theta \right| |Y - Y_*|^3 \leq C_1 |Y - Y_*|^3. \quad (2.14)$$

Furthermore,

$$C_1 |Y - Y_*| \leq C_1 \delta \leq \frac{1}{8}. \quad (2.15)$$

Combining (2.10), (2.14), (2.15) and the inequality $|e^a - 1| \leq |a| e^{|a|}$ $(a \in \mathbb{R})$, we deduce that

$$\begin{aligned}
|I_{23}| &\leq t^{\frac{\alpha}{2-\alpha}} \int_{|Y-Y_*|\leq\delta} |R(Y)| \exp\left(t^{\frac{\alpha}{2-\alpha}} \left\{ -\frac{1}{4}(Y - Y_*)^2 + |R(Y)| \right\} \right) dY \\
&\leq C_1 t^{\frac{\alpha}{2-\alpha}} \int_{|Y-Y_*|\leq\delta} |Y - Y_*|^3 \exp\left(t^{\frac{\alpha}{2-\alpha}} \left(-\frac{1}{4} + C_1 |Y - Y_*| \right) (Y - Y_*)^2 \right) dY \\
&\leq C_1 t^{\frac{\alpha}{2-\alpha}} \int_{\mathbb{R}} |Y - Y_*|^3 \exp\left(-\frac{1}{8} t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2 \right) dY \\
&= 64 C_1 t^{-\frac{\alpha}{2-\alpha}}.
\end{aligned}$$

From the computations for I_{21}, I_{22} and I_{23}, we obtain

$$I_2 = \sqrt{4\pi} K t^{-\frac{\alpha}{2(2-\alpha)}} \left(1 + O\left(t^{-\frac{\alpha}{2(2-\alpha)}} \right) \right) \quad (t \to \infty), \quad (2.16)$$

where we have used the fact that

$$K(X) \geq K(-1). \quad (2.17)$$

By (2.13), (2.16) and (2.17), we conclude that

$$I(X,t) = \left(1 + O\left(t^{-\frac{\alpha}{2(2-\alpha)}} \right) \right) U(X,t),$$

as claimed. □

Lemma 6. *There is a constant $C > 0$ such that*

$$I(X,t) \leq C t^{-\frac{\alpha}{2(2-\alpha)}} U(-X,t)$$

for all $X < -1$ and $t \geq 1$.

Proof. By (2.10),

$$\begin{aligned}
I &= \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) \int_{\mathbb{R}} \exp\left(t^{\frac{\alpha}{2-\alpha}} (g(Y) - M)\right) dY \\
&\leq \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) \int_{\mathbb{R}} \exp\left(-\frac{1}{4} t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY \\
&= \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right).
\end{aligned}$$

Since $X < -1 < 1 < -X$, we see from (2.11) that

$$\begin{aligned}
\exp\left(t^{\frac{\alpha}{2-\alpha}} M(X)\right) &= \frac{1}{K(-X)} \exp\left(-t^{\frac{\alpha}{2-\alpha}} (M(-X) - M(X))\right) U(-X,t) \\
&\leq \frac{1}{K(1)} \exp\left(-t^{\frac{\alpha}{2-\alpha}} (M(1) - M(-1))\right) U(-X,t).
\end{aligned}$$

It is elementary to show that for any fixed constant $c > 0$,

$$\exp\left(-ct^{\frac{\alpha}{2-\alpha}}\right) = O\left(t^{-\frac{\alpha}{2(2-\alpha)}}\right) \quad (t \to \infty). \tag{2.18}$$

Hence the lemma follows. □

Proof of Proposition 4. In the proof, constants in big O notation do not depend on x. We set $X = t^{-1/(2-\alpha)}x$. Since $\phi(x,t) = kt + \log(I(X,t) + I(-X,t))$, the proof is completed by showing that

$$I(X,t) + I(-X,t) = \left(1 + O\left(t^{-\frac{\alpha}{2(2-\alpha)}}\right)\right)(U(X,t) + U(-X,t)) \quad (t \to \infty). \tag{2.19}$$

In the case $|X| \leq 1$, this is immediately verified by applying Lemma 5 to $I(X,t)$ and $I(-X,t)$.

Assume now that $X < -1$. In this case we use Lemma 5 for $I(-X,t)$ and Lemma 6 for $I(X,t)$ to obtain

$$I(X,t) + I(-X,t) = \left(1 + O\left(t^{-\frac{\alpha}{2(2-\alpha)}}\right)\right) U(-X,t) \quad (t \to \infty).$$

Since (2.11) and (2.18) show that

$$\begin{aligned}
U(X,t) &\leq \frac{K(-1)}{K(1)} \exp\left(-t^{\frac{\alpha}{2-\alpha}} (M(1) - M(-1))\right) U(-X,t) \\
&\leq Ct^{-\frac{\alpha}{2(2-\alpha)}} U(-X,t)
\end{aligned}$$

for some constant $C > 0$, we obtain (2.19). The case $X > 1$ can be handled in the same way as in the case $X < -1$, and therefore the proof is complete. □

2.3 Proof of (1.6)

Finally we show that the solution u constructed above satisfies (1.6). We easily see that as $X \to 0$,

$$M(X) = M_0 + O(X), \qquad K(X) = K_0 + O(X),$$
$$M_0 = (2-\alpha)2^{-\frac{1-\alpha}{2-\alpha}}\alpha^{\frac{\alpha}{2-\alpha}}A^{\frac{2}{2-\alpha}}, \qquad K_0 := (2-\alpha)^{-\frac{1}{2}}.$$

From this we see that for each $x \in \mathbb{R}$,

$$\begin{aligned}
&\left(U\left(t^{-\frac{1}{2-\alpha}}x, t\right) + U\left(-t^{-\frac{1}{2-\alpha}}x, t\right)\right)\exp\left(-t^{\frac{\alpha}{2-\alpha}}M_0\right)\\
&= K\left(t^{-\frac{1}{2-\alpha}}x\right)\exp\left(t^{\frac{\alpha}{2-\alpha}}\left(M\left(t^{-\frac{1}{2-\alpha}}x\right) - M_0\right)\right)\\
&\quad + K\left(-t^{-\frac{1}{2-\alpha}}x\right)\exp\left(t^{\frac{\alpha}{2-\alpha}}\left(M\left(-t^{-\frac{1}{2-\alpha}}x\right) - M_0\right)\right)\\
&\to 2K_0 \quad (t \to \infty).
\end{aligned}$$

Therefore, by Proposition 4,

$$\lim_{t\to\infty}\left(\phi(x,t) - kt - M_0 t^{\frac{\alpha}{2-\alpha}}\right) = \log(2K_0).$$

This together with Lemma 3 implies that for $(x,\eta) \in \mathbb{R}^2$,

$$\begin{aligned}
\limsup_{t\to\infty} u(x,\eta+h_0(t),t) &\le \lim_{t\to\infty} u^+\left(x,\eta+h_0(t),t\right) = \Phi(\eta-\eta_1),\\
\liminf_{t\to\infty} u(x,\eta+h_0(t),t) &\ge \lim_{t\to\infty} u^-\left(x,\eta+h_0(t),t\right) = \Phi(\eta+\eta_1),
\end{aligned} \tag{2.20}$$

where $h_0(t) = kt + M_0 t^{\frac{\alpha}{2-\alpha}} + \log(2K_0)$ and $\eta_1 = Kp_0 t_0^{-\gamma+1}$.

Now we discuss the convergence of the function $w(x,\eta,t) := u(x,\eta+h_0(t),t)$. It is seen that w satisfies

$$w_t = \Delta w + \left(k + \frac{\alpha}{2-\alpha}t^{-\frac{2(1-\alpha)}{2-\alpha}}\right)w_\eta + f(w).$$

The fact that $u_- \le u \le u_+$ shows that $w(\cdot,t)$ is uniformly bounded. By the regularity theory for parabolic partial differential equations and compact embeddings for Sobolev and Hölder spaces, we can take a sequence $\{t_i\}$, $t_1 < t_2 < \cdots \to \infty$ such that $w(\cdot,t_i)$ converges locally uniformly to a solution $W = W(x,\eta)$ of the equation

$$\Delta W + kW_\eta + f(W) = 0 \tag{2.21}$$

as $i \to \infty$. From (2.20), we have

$$\Phi(\eta+\eta_1) \le W(x,\eta) \le \Phi(\eta-\eta_1). \tag{2.22}$$

[2, Theorem 3.1] shows that the function W satisfying (2.21) and (2.22) coincides with $\Phi(\eta - \eta_2)$ for some $\eta_2 \in \mathbb{R}$. Thus we conclude that (1.6) holds with

$$\beta = \frac{\alpha}{2-\alpha}, \quad b = M_0 = (2-\alpha)2^{-\frac{1-\alpha}{2-\alpha}}\alpha^{\frac{\alpha}{2-\alpha}}A^{\frac{2}{2-\alpha}}, \quad h(t) = h_0(t) + \eta_2.$$

The restriction $\beta < 1/2$ comes from the condition (2.4), and $b > 0$ can be chosen arbitrarily since $A > 0$ is arbitrary.

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