Singular limit problem for the Allen-Cahn equation with a zero Neumann boundary condition on non-convex domains

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1 Intrduction

A family of hyper-surfaces is called mean curvature flow if the normal velocity coincides with the mean curvature vector. Since the mean curvature flow is a formal L^2 gradient flow of surface area, the flow basically converges to a point in finite time if the initial surface is closed, compact and embedded. However, singularities can develop in some mean curvature flow before the flow shrinks to a point and various researchers have been interested in the structure of singularities, for example, the blow-up late of the second fundamental form, the behavior of flow around blow-up time and so on. In this paper, our aim is to construct a global-in-time measure-theoretic weak solution to the mean curvature flow with right angle condition taking the analysis on singularities into account. For the details of the connection between our weak solution and the mean curvature flow with right angle condition, see Remark 2.4.

In the following, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. We consider the following Allen-Cahn equation with a zero Neumann boundary condition on the domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \partial_t u_\varepsilon = \Delta u_\varepsilon - \dfrac{W'(u_\varepsilon)}{\varepsilon^2}, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u_\varepsilon(x,0) = u_{\varepsilon,0}(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, $\varepsilon > 0$ is a parameter, ν is the outer unit normal to the boundary $\partial\Omega$ and we assume a double well potential $W \in C^3(\mathbb{R})$ satisfies the following conditions:

(W1) $W(\pm 1) = 0$ and $W(s) > 0$ for $s \neq \pm 1$,
(W2) there exists a constant $-1 < \gamma < 1$ such that $W' < 0$ in $(\gamma, 1)$ and $W' > 0$ in $(-1, \gamma)$,
(W3) there exist constants $0 < \alpha < 1$ and $\beta > 0$ such that $W''(s) \geq \beta$ for $\alpha \leq |s| \leq 1$.

A typical example of such W is $(1-s^2)^2/4$, for which we may set $\alpha = \sqrt{2/3}, \beta = 1$ and $\gamma = 0$.

The Allen-Cahn equation (1.1) is the L^2 gradient flow of

$$E_\varepsilon[u] := \int_\Omega \frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\, dx$$

sped up by the factor $1/\varepsilon$. Heuristically, for a given family of functions $\{u_\varepsilon\}_{0<\varepsilon<1}$ with $\sup_\varepsilon E_\varepsilon[u_\varepsilon] < \infty$, u_ε is close to a characteristic function, with a transition layer of width approximately ε and slope approximately C/ε. Thus Ω is mostly divided into two regions $\{u_\varepsilon \approx 1\}$ and $\{u_\varepsilon \approx -1\}$ for sufficiently small ε. With this heuristic picture, one may expect that the following diffused interface energy

$$\mu_\varepsilon^t := \frac{1}{\sigma}\left(\frac{\varepsilon|\nabla u_\varepsilon(\cdot,t)|^2}{2} + \frac{W(u_\varepsilon(\cdot,t))}{\varepsilon}\right)\mathcal{L}^n\lfloor_\Omega \tag{1.2}$$

behaves more or less like surface measures of moving phase boundaries, where

$$\sigma = \int_{-1}^{1} \sqrt{W(s)}\, ds. \tag{1.3}$$

Furthermore, one may also expect that the motion of the "transition layer" is a mean curvature flow with the right angle condition on $\partial\Omega$ because a formal L^2 gradient flow of the surface area is its mean curvature flow. In order to give a rigorous proof of this kind of singular limit problem for the Allen-Cahn equation (1.1), we have to introduce weak solutions to the mean curvature flow with the right angle condition. For example, Mizuno and Tonegawa [10] constructed Brakke's mean curvature flow with a generalized right angle condition (a measure theoretic weak solution) via the singular limit problem of the Allen-Cahn equation (1.1), and Katsoulakis, Kossioris and Reitich [9] proved a connection of the singular limit problem of (1.1) to the unique viscosity solutions of a level set formulation of the mean curvature flow with the right angle condition. However, they assumed the convexity of the domain in each paper. Accordingly, we prove the convergence of (1.2) to Brakke's mean curvature flow appeared in [10] without the assumption of the convexity of the domain. We note that the connection between (1.1) and the level set formulation of the mean curvature flow with the right angle condition without the assumption of the convexity of the domain was proved by [2, 3]. We also discuss the behavior of the Brakke's mean curvature flow with a generalized right angle condition in Remark 2.4.

2 Notions

We note some notions related geometric measure theory to define Brakke's mean curvature flow with a generalized right angle condition.

2.1 Homogeneous maps and rectifiable measures

Let $\mathbf{G}(n, n-1)$ be the space of $(n-1)$-dimensional subspace of $\mathbb{R}^n$. For $S \in \mathbf{G}(n, n-1)$, we identify S with the corresponding orthogonal projection of $\mathbb{R}^n$ onto S. For two elements A and B of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, we define a scalar product as

$$A \cdot B := \sum_{i,j} A_{ij} B_{ij}.$$

The identity of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is denoted by I.

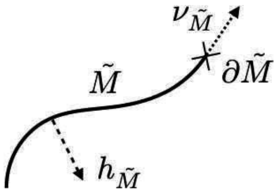

Figure 1: Picture of geometric notions of a smooth manifold.

We recall some notions related to varifold and refer to [1, 13] for more details. We say that a Radon measure μ on $\mathbb{R}^n$ is rectifiable if there exist an $\mathcal{H}^{n-1}$ measurable countably $(n-1)$-rectifiable set $M \subset \mathbb{R}^n$ and a locally $\mathcal{H}^{n-1}$ integrable function θ defined on M such that

$$\mu(\phi) = \theta\mathcal{H}^{n-1}\lfloor_M(\phi) = \int_M \theta(x)\phi(x)\, d\mathcal{H}^{n-1}(x) \quad \text{for} \quad \phi \in C_c(\mathbb{R}^n).$$

Here, we note that the approximate tangent space $\mathrm{Tan}_x M \in \mathbf{G}(n, n-1)$ of M exists $\mathcal{H}^{n-1}$-a.e. on M. Therefore, we can define the first variation

$$\delta\mu(g) := \int_{\mathbb{R}^n} \nabla g(x) \cdot \mathrm{Tan}_x M\, d\mu(x) = \int_M \theta(x)\nabla g(x) \cdot \mathrm{Tan}_x M\, d\mathcal{H}^{n-1}(x) \quad \text{for} \quad g \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$$

if μ is rectifiable. Let $\|\delta\mu\|$ be the total variation when it exists, and if $\|\delta\mu\|$ is locally bounded, we may apply the Riesz representation theorem and the Lebesgue decomposition theorem (see [4, Theorem 1.38, Theorem 1.31]) to $\delta\mu$ with respect to μ. Then, we obtain a μ measurable function $h_\mu : M \to \mathbb{R}^n$, a Borel set $\partial\mu \subset \mathbb{R}^n$ such that $\mu(\partial\mu) = 0$ and a $\|\delta\mu\|\lfloor_{\partial\mu}$ measurable function $\nu_\mu : \partial\mu \to \mathbb{R}^n$ with $|\nu_\mu| = 1$ $\|\delta\mu\|$-a.e. on $\partial\mu$ such that

$$\delta\mu(g) = -\int_{\mathbb{R}^n} \langle h_\mu, g\rangle\, d\mu + \int_{\partial\mu} \langle \nu_\mu, g\rangle\, d\|\delta\mu\| \quad \text{for} \quad g \in C_c^1(\mathbb{R}^n; \mathbb{R}^n). \tag{2.1}$$

The vector field h_μ is called the generalized mean curvature vector of μ, the vector field ν_μ is called the (outer-pointing) generalized co-normal of μ and the Borel set $\partial\mu$ is called the generalized boundary of μ.

Remark 2.1 *For a smooth and oriented hyper-surface $\tilde{M} \subset \mathbb{R}^n$ (with boundary), the divergence theorem*

$$\int_{\tilde{M}} \mathrm{div}_{\tilde{M}} g\, d\mathcal{H}^{n-1} = -\int_{\tilde{M}} \langle h_{\tilde{M}}, g\rangle\, d\mathcal{H}^{n-1} + \int_{\partial\tilde{M}} \langle \nu_{\tilde{M}}, g\rangle\, d\mathcal{H}^{n-2} \quad \textit{for} \quad g \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$$

holds, where $\mathrm{div}_{\tilde{M}}$ *is the divergence on* $\tilde{M}$*,* $h_{\tilde{M}}$ *is the mean curvature vector of* $\tilde{M}$ *and* $\nu_{\tilde{M}}$ *is the co-normal vector of* $\tilde{M}$ *(see Figure 1). Since* $\mathrm{div}_{\tilde{M}} g$ *coincide with* $\nabla g \cdot \mathrm{Tan}_x \tilde{M}$*, we may see that* h_μ, ν_μ *and* $\partial\mu$ *defined by* (2.1) *also coincide with* $h_{\tilde{M}}, \nu_{\tilde{M}}$ *and* $\partial\tilde{M}$*, respectively, if* $\mu = \mathcal{H}^{n-1}\lfloor_{\tilde{M}}$.

We also remark that, for any rectifiable μ *such that* $\|\delta\mu\|$ *is a Radon measure,* h_μ *is perpendicular to* M μ*-a.e. on* M *if the density function* θ *of* μ *is integer* μ*-a.e. on* M *(see [1]).*

In order to discuss a contact angle condition of μ on $\partial\Omega$, we have to introduce a tangential component of $\delta\mu$ on $\partial\Omega$ which is defined by

$$\delta\mu\lfloor_{\partial\Omega}^{\top}(g) := \partial\mu\lfloor_{\partial\Omega}(g - \langle g, \nu\rangle\nu) \quad \text{for} \quad g \in C(\partial\Omega; \mathbb{R}^n)$$

when μ is rectifiable and $\mathrm{spt}\mu \subset \overline{\Omega}$. If the total variation $\|\delta\mu\lfloor_{\partial\Omega}^{\top} + \delta\mu\lfloor_{\Omega}\|$ is absolute continuous with respect to μ, then by the Riesz representation theorem and the Lebesgue decomposition theorem to $\delta\mu\lfloor_{\partial\Omega}^{\top} + \delta\mu\lfloor_{\Omega}$ with respect to μ, we obtain a μ measurable function $h_\mu^b : M \to \mathbb{R}^n$ such that

$$(\delta\mu\lfloor_{\partial\Omega}^{\top} + \delta\mu\lfloor_{\Omega})(g) = -\int_{\mathbb{R}^n} \langle h_\mu^b, g\rangle \, d\mu \quad \text{for} \quad g \in C_c^1(\mathbb{R}^n; \mathbb{R}^n), \tag{2.2}$$

where $M \subset \overline{\Omega}$ is the countably $(n-1)$-rectifiable set associated to μ.

Remark 2.2 *Since $\delta\mu(g)$ coincides with $(\delta\mu\lfloor_{\partial\Omega}^{\top} + \delta\mu\lfloor_{\Omega})(g)$ for any $g \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ with $\langle g, \nu\rangle = 0$ on $\partial\Omega$, we obtain by* (2.1) *and* (2.2)

$$-\int_{\mathbb{R}^n} \langle h_\mu, g\rangle \, d\mu + \int_{\partial\mu} \langle \nu_\mu, g\rangle \, d\|\delta\mu\| = -\int_{\mathbb{R}^n} \langle h_\mu^b, g\rangle \, d\mu$$

for any $g \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ with $\langle g, \nu\rangle = 0$ on $\partial\Omega$ if μ satisfies the following:

(V1) *μ is rectifiable and $\mathrm{spt}\mu \subset \overline{\Omega}$,*
(V2) *$\|\delta\mu\|$ is a Radon measure,*
(V3) *$\|\delta\mu\lfloor_{\partial\Omega}^{\top} + \delta\mu\lfloor_{\Omega}\|$ is absolute continuous with respect to μ.*

By a simple calculation, we may see that

- *the generalized boundary $\partial\mu$ is a subset of $\partial\Omega$,*
- *the generalized co-normal vector field ν_μ is perpendicular to $\partial\Omega$ $\|\delta\mu\|$-a.e. on $\partial\mu$,*
- *the vector field h_μ^b coincides with the generalized mean curvature vector h_μ μ-a.e. in Ω and the projection of h_μ onto the tangent space of $\partial\Omega$ (i.e. $\mathrm{Tan}_x\partial\Omega(h_\mu)$) μ-a.e. on $\partial\Omega$.*

Therefore, we can say μ satisfies a "right angle condition" in the sense of measure if μ fulfills the conditions (V1)–(V3).

2.2 Brakke's mean curvature flow with a generalized right angle condition

We define a measure theoretic weak solution to the mean curvature flow with the right angle condition.

Definition 2.3 *Let $\{\mu^t\}_{t\in[0,\infty)}$ be a family of Radon measures on $\mathbb{R}^n$. We say that $\{\mu^t\}$ is a Brakke's mean curvature flow with a generalized right angle condition if*

(B1) *μ^t satisfies* (V1)–(V3) *and the density function θ^t of μ^t is integer μ^t-a.e. on $\Omega \cap M^t$, where M^t is the countably $(n-1)$-rectifiable set associated to μ^t, for a.e. $t \in [0, \infty)$,*

(B2) *the vector field $h_{\mu^t}^b$ defined by* (2.2) *for μ^t and a.e. $t \in [0, \infty)$ is of the class $L^2_{\mathrm{loc}}(d\mu^t dt)$,*

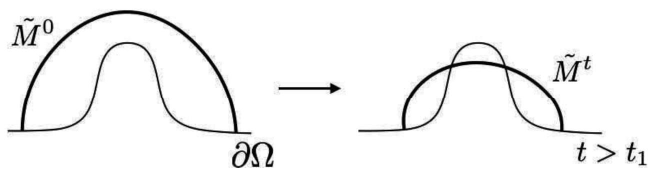

Figure 2: An example of the mean curvature flow with the right angle condition.

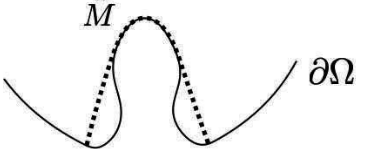

Figure 3: A stationary solution to Brakke's mean curvature flow with a generalized right angle condition such that $\tilde{M} \cap \Omega$ consists of line segments.

(B3) *for any* $\phi \in C_c^1(\mathbb{R}^n \times [0,\infty);\mathbb{R}^+)$ *with* $\langle \nabla\phi, \nu\rangle = 0$ *on* $\partial\Omega \times [0,\infty)$ *and* $0 \le t_1 < t_2 < \infty$,

$$\mu^t(\phi(\cdot,t))\Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2}\int_{\mathbb{R}^n} -\phi|h^b_{\mu^t}|^2 + \langle \nabla\phi, h^b_{\mu^t}\rangle + \partial_t\phi \, d\mu^t dt.$$

Now, we also note the definition of the mean curvature flow with the right angle condition in the classical sense and some relation with the weak solution.

Remark 2.4 *The long time existence of the mean curvature flow with the right angle condition was proved by [14]. Its mean curvature flow is defined as the following: Let $\tilde{M}$ be a compact, smooth and orientable $(n-1)$-dimensional manifold with compact and smooth boundary $\partial\tilde{M}$. If a family of smooth immersions $F : \tilde{M} \times [0,T) \to \mathbb{R}^n$ construct a geometric flow $\{\tilde{M}^t\} = \{F(M,t)\}$ such that*

$$v_{\tilde{M}^t} = h_{\tilde{M}^t} \quad \text{on} \quad \tilde{M}^t, \quad \partial\tilde{M}^t = F(\partial\tilde{M},t) \subset \partial\Omega, \quad \nu_{\tilde{M}^t} \perp \partial\Omega \quad \text{on} \quad \partial\tilde{M}^t,$$

where $v_{\tilde{M}^t}$ is the normal velocity vector of $\tilde{M}^t$, we say that $\tilde{M}^t$ is a mean curvature flow with the right angle condition. Since F is a smooth map, $\tilde{M}^t$ does not change the topology and it is possible that $\tilde{M}^t$ moves to the outside Ω. For example, in Figure 2, the moving hypersurface $\tilde{M}^t$ touch the boundary $\partial\Omega$ at time $t_1 \in (0,T)$ and pass through it. From a physical point of view, we would like to construct a mean curvature flow "only inside Ω" by letting topological changes occur. Since topological changes are ones of the singularities, we study a weak solution to the mean curvature flow in the sense of Brakke.

Here, we discuss the behavior of the Brakke's mean curvature flow defined in Definition 2.3. If we assume that a Brakke's mean curvature flow with a generalized right angle condition μ^t is described as $\mu^t = \mathcal{H}^{n-1}\lfloor_{\tilde{M}^t}$ for some smooth and orientable $(n-1)$-dimensional submanifold $\tilde{M}^t$ in $\mathbb{R}^n$ with compact and smooth boundary $\partial\tilde{M}^t$, we may see that for any $t > 0$

(i) $\tilde{M}^t \subset \overline{\Omega}$,

(ii) $\partial\tilde{M}^t \subset \partial\Omega$ *and* $\nu_{\tilde{M}^t}$ *is perpendicular to* $\partial\Omega$ *on* $\partial\tilde{M}^t$,

(iii) $v_{\tilde{M}^t} = h_{\tilde{M}^t}$ *on* $\tilde{M}^t \cap \Omega$.

The property (i) follows from $\mathrm{spt}\mu^t \subset \overline{\Omega}$ *and we do not know if* $\partial\Omega \cap \tilde{M}^t = \partial\tilde{M}^t$. *We also note that the definition of Brakke's mean curvature flows with a generalized right angle condition do not tell us the behavior of* $\partial\Omega \cap \tilde{M}^t$ *immediately. Indeed,* $\mathcal{H}^{n-1}\lfloor_{\partial\Omega}$ *and* $\mathcal{H}^{n-1}\lfloor_{\tilde{M}}$, *where* $\tilde{M} \subset \overline{\Omega}$ *is a hyper-surface composed of a minimal surface* $\tilde{M} \cap \Omega$ *and the remaining part* $\tilde{M} \cap \partial\Omega$, *are stationary solutions to the Brakke's mean curvature flow with a generalized right angle condition (see Figure 3). The motion of a measure* $\mathcal{H}^{n-1}\lfloor_{\tilde{M}^t}$ *seems possible to converge to the stationary solution* $\mathcal{H}^{n-1}\lfloor_{\tilde{M}}$ *in finite time, and in this case,* $\tilde{M}^t$ *does not change the topology. Therefore, analysis on the behavior of the Brakke's mean curvature flow with a generalized right angle condition, in particular construction of a motion with some topological changes, is a future work.*

We also note that, in broad strokes, the boundary condition of a viscosity solution to a level set formulation of the mean curvature flow with the right angle condition is defined in a "weak sense" and the behavior of the level set flow around boundary is not well known (see [2, 3, 6, 9, 12] for more details). For example, Giga [5] constructed a viscosity solution v *in the case* $n = 2$ *so that the zero level set of* $v(\cdot, t)$ *fattens in finite time* $t_0 > 0$. *By using this solution, we can construct two curvature flows with the right angle condition, which start frow same initial curve, so that one of the flows is separated into two curves for any* $t > t_0$ *and the other does not change the topology.*

3 Assumptions and main result

3.1 Assumptions of initial functions

Hereafter, we assume the following assumptions for the initial function $u_{\varepsilon,0} \in C^1(\overline{\Omega})$ of (1.1):

(A1) $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)} \leq 1$,

(A2) there exists $D_0 > 0$ such that $\sup_{x\in\Omega, r>0} \int_{B_r(x)\cap\Omega} \frac{\varepsilon|\nabla u_{\varepsilon,0}(y)|^2}{2} + \frac{W(u_{\varepsilon,0}(y))}{\varepsilon}\, dy \leq D_0 r^{n-1}$,

(A3) there exists $c_1 > 0$ such that $\sup_{x\in\Omega} \varepsilon|\nabla u_{\varepsilon,0}| \leq c_1$,

(A4) there exist $c_2 > 0$ and $\lambda \in [3/5, 1)$ such that $\sup_{x\in\Omega} \frac{\varepsilon|\nabla u_{\varepsilon,0}(x)|^2}{2} - \frac{W(u_{\varepsilon,0}(x))}{\varepsilon} \leq c_2 \varepsilon^{-\lambda}$,

(A5) $\frac{\partial u_{\varepsilon,0}}{\partial\nu}(x) = 0$ for $x \in \partial\Omega$.

Here, let D_0, c_1, c_2 and $\lambda \in [3/5, 1)$ be some universal constants. By the standard parabolic existence and regularity theory, for each $\varepsilon > 0$, there exists a unique solution u_ε with

$$u_\varepsilon \in C([0,\infty); C^1(\overline{\Omega})) \cap C^\infty(\overline{\Omega} \times (0,\infty)).$$

We also note that the boundedness of the domain Ω and the assumption (A2) imply

$$\sup_i E_{\varepsilon_i}[u_{\varepsilon_i,0}] \leq c_3 \tag{3.1}$$

for some constant c_3 depending only on n, D_0 and the diameter of Ω. Only the conditions (A1), (3.1) and the regularity $u_0 \in H^1(\Omega)$ are assumed in [10]. Therefore, we note a choice of initial functions satisfying the assumptions (A1)–(A5) in the following remark.

Remark 3.1 *We note that for a surface* Γ *with* 90 *degree contact angles on* $\partial\Omega$ *it is possible to construct diffuse approximations that satisfy the assumptions* (A1)–(A5) *as the following. Our construction is standard as in [7, 11]. Let* Ω_d *be*

$$\Omega_d := \{(y_1, y') \in \mathbb{R}^n : y_1 \in \mathbb{R}, |y'| < d\}$$

for $d > 0$ *and define* $\tilde{\Gamma} := \overline{\Omega}_d \cap \{y_1 = 0\}$. *By the standard existence theory for ordinary differential equations, we may choose the unique function* $q \in C^4(\mathbb{R})$ *such that*

$$q(0) = 0, \quad \lim_{s\to\pm\infty} q(s) = \pm 1, \quad q'(s) = \sqrt{2W(q(s))} \quad \text{in } \mathbb{R}.$$

Then it is easy to see that the C^4 *function* $v_{\varepsilon_i}(y) := q(y_1/\varepsilon_i)$ *defined on* $\overline{\Omega}_d$ *satisfies*

$$\begin{aligned} &\int_{B_r(y_0)\cap\Omega_d} \frac{\varepsilon_i |\nabla v_{\varepsilon_i}|^2}{2} + \frac{W(v_{\varepsilon_i})}{\varepsilon_i}\, dy \le \sigma\omega_{n-1} r^{n-1} \quad \text{for } r > 0,\ y_0 \in \mathbb{R}^n, \\ &\varepsilon_i |\nabla v_{\varepsilon_i}(y)| \le \max_{|s|\le 1} \sqrt{2W(s)}, \ \frac{\varepsilon_i |\nabla v_{\varepsilon_i}(y)|^2}{2} = \frac{W(v_{\varepsilon_i}(y))}{\varepsilon_i} \quad \text{for } y \in \overline{\Omega}_d, \\ &\langle \nabla v_{\varepsilon_i}, \nu_d \rangle = 0 \quad \text{on } \partial\Omega_d, \end{aligned} \tag{3.2}$$

where $\sigma := \int_{-1}^1 \sqrt{2W(s)}\, dx$ *and* ν_d *is the out ward unit normal to* $\partial\Omega_d$. *Now we assume that* $\tilde{U}$ *is a neighborhood of* $\tilde{\Gamma}$ *and that* ϕ *is a bijective* C^1 *map from* $\tilde{U}$ *onto* $U := \phi(\tilde{U})$ *such that*

$$\phi(\Omega_d \cap \tilde{U}) = \Omega \cap U, \quad \phi(\partial\Omega_d \cap \tilde{U}) = \partial\Omega \cap U, \quad \sup_{x\in U} \|\nabla\phi^{-1}(x)\| \le 1, \quad \sup_{y\in\tilde{U}} \|\nabla\phi(y)\| \le C$$

for a suitable $d > 0$ *and a constant* $C > 0$, *where* $\|\cdot\|$ *is the operator norm. By using this mapping,* (3.2) *implies that* $u_{\varepsilon_i,0}(x) := v_{\varepsilon_i} \circ \phi^{-1}(x)$ *satisfies the assumptions (A1)–(A5) with a positive constant* D_0 *depending only on* σ, n *and* C, $c_1 = 1$ *and* $c_2 = 0$ *on the set* $\overline{\Omega} \cap U$. *By expanding* $u_{\varepsilon_i,0}$ *as a mostly constant function to satisfy the assumptions outside of* U, *we may see the possibility of the initial assumptions in the present paper. In this construction, the diffused interface energy for* $u_{\varepsilon_i,0}$ *should behave like the surface measure of the surface* $\Gamma := \phi(\tilde{\Gamma})$ *and* Γ *intersects* $\partial\Omega$ *with 90 degrees.*

3.2 Main result

Our goal is to extend the convergence theory in [10] to remove the assumption of the convexity of the domain as the following.

Theorem 3.2 ([8]) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary. Assume (A1)–(A5) and let* u_ε *be the unique solution of* (1.1) *for* $\varepsilon > 0$. *Define a Radon measure* μ_ε^t *by* (1.2). *Then, there exist a sub-sequence* $\{\varepsilon_i\}_{i\in\mathbb{N}}$ *converging to* 0 *as* $i \to \infty$ *and a set of Radon measures* μ^t *on* $\mathbb{R}^n$ *such that* $\mu_{\varepsilon_i}^t \rightharpoonup \mu^t$ $(i \to \infty)$ *in the sense of measure for all* $t \ge 0$. *Furthermore,* μ^t *is a Brakke's mean curvature flow with a generalized right angle condition defined by Definition 2.3.*

Remark 3.3 *The integrality of the limit Radon measures* μ^t *in the interior of* Ω *follows from [16].*

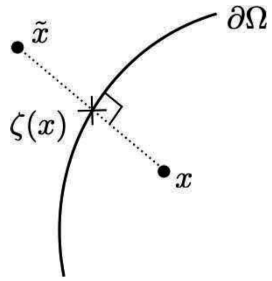

Figure 4: Picture of $\zeta(x)$ and $\tilde{x}$.

4 Outline of proof

As we mentioned in Section 1, the equation (1.1) is a L^2-gradient flow of E_ε, therefore we obtain the uniformly boundedness of $E_\varepsilon[u_\varepsilon(\cdot,t)]$ with respect to $t>0$ and $\varepsilon>0$ by applying (3.1). Roughly speaking, this fact and the compactness of Radon measure imply the convergence $\mu^t_{\varepsilon_i} \rightharpoonup \mu^t$ $(i\to\infty)$. Here, we discuss the rectifiability of μ^t (i.e. the condition (V1)). We note that the condition $\mathrm{spt}\mu \subset \overline{\Omega}$ obviously follows from the convergence $\mu^t_{\varepsilon_i} \rightharpoonup \mu^t$ and the inclusion $\mathrm{spt}\mu^t_\varepsilon \subset \overline{\Omega}$ for any $\varepsilon>0$.

One of the key arguments to prove the rectifiability of μ^t is a characterization by the $(n-1)$-dimensional backward heart kernel. For $y\in\mathbb{R}^n$ and $s>0$, let $\rho_{(y,s)}$ be the $(n-1)$-dimensional backward hear kernel, namely,

$$\rho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}} \quad \text{for} \quad x\in\mathbb{R}^n,\ t<s. \tag{4.1}$$

Roughly speaking, the heart kernel $\rho_{(y,s)}(\cdot,t)$ converges to $(n-1)$-dimensional delta function on $(n-1)$-dimensional hyper-surface as $t\to s$ in the sense of distribution. For example, if M is a smooth k-dimensional sub-manifold in $\mathbb{R}^n$ such that y is a interior point of M, then

$$\lim_{t\uparrow s}\int_M \rho_{(y,s)}(x,t)\, d\mathcal{H}^k(x) = \begin{cases} 0 & \text{if} \quad k=n, \\ 1 & \text{if} \quad k=n-1, \\ \infty & \text{if} \quad k\le n-2. \end{cases}$$

Therefore, the "dimension" of μ^t can be analyzed by $\mu^t(\rho_{(y,s)}(\cdot,t))$ and this analysis is a first step to prove the rectifiablity of μ^t. The Huisken or Ilmanen type monotonicity formula is an inequality to control the time development of $\mu^t(\rho_{(y,s)}(\cdot,t))$, thus we define some notions to present the statement of the monotonicity formula.

The following notions are related to the reflection argument. Define κ as

$$\kappa := \|\text{principal curvature of } \partial\Omega\|_{L^\infty(\partial\Omega)}.$$

For $s>0$, define a subset N_s of $\mathbb{R}^n$ by

$$N_s := \{x\in\mathbb{R}^n : \mathrm{dist}(x,\partial\Omega) < s\}.$$

There exists a sufficiently small

$$c_4 \in (0, (6\kappa)^{-1}]$$

depending only on $\partial\Omega$ such that all points $x \in N_{6c_4}$ have a unique point $\zeta(x) \in \partial\Omega$ such that $\mathrm{dist}(x, \partial\Omega) = |x - \zeta(x)|$ (see also Figure 4). By using this $\zeta(x)$, we define the reflection point $\tilde{x}$ of x with respect to $\partial\Omega$ as

$$\tilde{x} := 2\zeta(x) - x.$$

We also fix a function $\eta \in C^\infty(\mathbb{R})$ such that

$$0 \leq \eta \leq 1, \quad \frac{d\eta}{dr} \leq 0, \quad \mathrm{spt}\eta \subset [0, c_4/2), \quad \eta = 1 \text{ on } [0, c_4/4].$$

For $s > t > 0$ and $x, y \in N_{c_4}$, we define the truncated version of the $(n-1)$-dimensional backward heat kernel and the reflected backward heat kernel as

$$\rho_{1,(y,s)}(x,t) := \eta(|x-y|)\rho_{(y,s)}(x,t), \quad \rho_{2,(y,s)}(x,t) := \eta(|\tilde{x}-y|)\rho_{(y,s)}(\tilde{x},t),$$

where $\rho_{(y,s)}$ is defined as in (4.1). For $x \in N_{2c_4} \setminus N_{c_4}$ and $y \in N_{c_4/2}$, we have

$$|\tilde{x} - y| \geq |\tilde{x} - \zeta(y)| - |\zeta(y) - y| > c_4 - \frac{c_4}{2} = \frac{c_4}{2}.$$

Thus we may smoothly define $\rho_{2,(y,s)} = 0$ for $x \in \mathbb{R}^n \setminus N_{c_4}$ and $y \in N_{c_4/2}$. We also define the discrepancy function ξ_{ε_i} as

$$\xi_{\varepsilon_i}(x,t) := \frac{\varepsilon_i |\nabla u_{\varepsilon_i}(x,t)|^2}{2} - \frac{W(u_{\varepsilon_i}(x,t))}{\varepsilon_i} \quad \text{for} \quad (x,t) \in \overline{\Omega} \times [0,\infty).$$

Proposition 4.1 (Boundary monotonicity formula [10]) *There exist constants $0 < c_5, c_6 < \infty$ depending only on n, c_3 and $\partial\Omega$ such that*

$$\begin{aligned} &\frac{d}{dt}\left(\sigma e^{c_5(s-t)^{\frac{1}{4}}} \int_\Omega \rho_{1,(y,s)}(x,t) + \rho_{2,(y,s)}(x,t)\, d\mu_{\varepsilon^i}^t(x)\right) \\ &\leq e^{c_5(s-t)^{\frac{1}{4}}} \left(c_6 + \int_\Omega \frac{\rho_{1,(y,s)}(x,t) + \rho_{2,(y,s)}(x,t)}{2(s-t)} \xi_{\varepsilon_i}(x,t)\, dx\right) \end{aligned} \tag{4.2}$$

for all $s > t > 0, y \in N_{c_4/2}$ and $i \in \mathbb{N}$,

$$\frac{d}{dt}\left(\sigma e^{c_5(s-t)^{\frac{1}{4}}} \int_\Omega \rho_{1,(y,s)}(x,t)\, d\mu_{\varepsilon^i}^t(x)\right) \leq e^{c_5(s-t)^{\frac{1}{4}}} \left(c_6 + \int_\Omega \frac{\rho_{1,(y,s)}(x,t)}{2(s-t)} \xi_{\varepsilon_i}(x,t)\, dx\right) \tag{4.3}$$

for all $s > t > 0, y \in \mathbb{R}^n \setminus N_{c_4/2}$ and $i \in \mathbb{N}$, where σ is the constant defined by (1.3).

The proof of Proposition 4.1 in [10] does not require the convexity of Ω, thus we can apply this monotonicity formula to our problem. In order to control the time evolution of $\mu^t(\rho_{(y,s)}(\cdot,t)) (\approx \mu^t(\rho_{1,(y,s)}(\cdot,t) + \rho_{2,(y,s)}(\cdot,t)))$, we have to take the limit $i \to \infty$ for both inequalities (4.2) and (4.3). Therefore, analysis on the behavior of the discrepancy function ξ_{ε_i} with respect to i is one of the key arguments. In the following, we study the upper bound of the discrepancy function.

4.1 Preparation

In this section, we note some lemmas to discuss estimates on the upper bound of the discrepancy function. A key lemma is the following equality to control the normal derivative of the discrepancy function.

Lemma 4.2 *Let A_x be the second fundamental form of $\partial\Omega$ at $x \in \partial\Omega$. Then*

$$\frac{\partial}{\partial\nu}\frac{|\nabla u_{\varepsilon_i}|^2}{2} = A_x(\nabla u_{\varepsilon_i}, \nabla u_{\varepsilon_i}) \quad \text{for} \quad (x,t) \in \partial\Omega \times (0,\infty).$$

This equality can be proved by using only the Neumann boundary condition of (1.1). We also note that Lemma 4.2 and the Neumann boundary condition of (1.1) imply that for any $(x,t) \in \partial\Omega \times (0,\infty)$

$$\frac{\partial}{\partial\nu}\xi_{\varepsilon_i} \le \begin{cases} 0 & \text{if } \Omega \text{ is convex,} \\ \kappa\varepsilon_i|\nabla u_{\varepsilon_i}|^2 & \text{even if } \Omega \text{ is not convex.} \end{cases} \tag{4.4}$$

Another key lemma is an estimate which follows from the scaling argument. Let

$$\Omega_{\varepsilon_i} = \{y \in \mathbb{R}^n : \varepsilon_i y \in \Omega\}$$

and define the function

$$v_{\varepsilon_i}(y,\tau) := u_{\varepsilon_i}(\varepsilon_i y, \varepsilon_i^2\tau) \quad \text{for} \quad y \in \overline{\Omega}_{\varepsilon_i}, \quad \tau \in [0,\infty).$$

We note that

$$\kappa_{\varepsilon_i} := \|\text{principal curvature of } \partial\Omega_{\varepsilon_i}\|_{L^\infty(\partial\Omega_{\varepsilon_i})} = \varepsilon_i\kappa \tag{4.5}$$

holds and v_{ε_i} satisfies

$$\begin{cases} \partial_\tau v_{\varepsilon_i} = \Delta v_{\varepsilon_i} - W'(v_{\varepsilon_i}) \quad \text{in} \quad \Omega_{\varepsilon_i} \times (0,\infty), \\ \langle \nabla v_{\varepsilon_i}, \nu_{\varepsilon_i}\rangle = 0 \quad \text{on} \quad \partial\Omega_{\varepsilon_i} \times (0,\infty), \end{cases}$$

where ν_{ε_i} is the outward unit normal to $\partial\Omega_{\varepsilon_i}$. The standard gradient estimate depends on the second fundamental form of the boundary of the domain. Therefore, "uniformly gradient estimate" of v_{ε_i} holds by (4.5), namely, $|\nabla v_{\varepsilon_i}|$ is uniformly bounded with respect to x, t and ε_i if $\sup_{x\in\overline{\Omega}_{\varepsilon_i}, i\in\mathbb{N}} |\nabla v_{\varepsilon_i}(x,0)|$ is finite. Since the boundedness of ∇v_{ε_i} at initial time is equivalent to the assumption (A3), we obtain the following estimate.

Lemma 4.3 *There exists a constant c_7 depending only on c_1, c_4 and W such that*

$$\sup_{\Omega\times[0,\infty)} \varepsilon_i|\nabla u_{\varepsilon_i}| \le c_7$$

for all $0 < \varepsilon_i < 1$.

Remark 4.4 *By the scaling argument, we can obtain the uniformly boundedness of the second derivatives of v_{ε_i} if we assume the uniformly boundedness of its derivatives at initial time. Therefore, roughly speaking, the estimate $|\nabla^2 u_{\varepsilon_i}| \lesssim \varepsilon_i^{-2}$ follows from the scaling argument under suitable assumptions, which gives the estimate $|\langle \nabla \xi_{\varepsilon_i}, \nu \rangle| \lesssim \varepsilon_i^{-2}$. On the other hand, by combining* (4.4) *and Lemma 4.3, we obtain*

$$\frac{\partial}{\partial \nu} \xi_{\varepsilon_i} \leq \kappa c_7^2 \varepsilon_i^{-1} \quad \text{for} \quad (x,t) \in \partial\Omega \times [0,\infty) \tag{4.6}$$

which is better than the estimate following from the scaling argument in the viewpoint of the oder of ε_i.

We also note that the estimate $\xi_{\varepsilon_i} \lesssim \varepsilon_i^{-1}$ can be obtained by Lemma 4.3 since $\sup_{x,t} |u_{\varepsilon_i}| \leq 1$ *follows from the maximum principle and the assumption (A1). Our aim is to obtain a better estimate of the upper bound of the discrepancy function in the viewpoint of the oder of ε_i.*

4.2 Upper bound of discrepancy function on CONVEX domains

First, we discuss the upper bound of discrepancy in the case that Ω is convex. By the Allen-Cahn equation (1.1) and a simple calculation, we obtain

$$\partial_t \xi_{\varepsilon_i} - \Delta \xi_{\varepsilon_i} \leq -\frac{2\langle W'(u_{\varepsilon_i}) \nabla u_{\varepsilon_i}, \nabla \xi_{\varepsilon_i} \rangle}{\varepsilon_i^2 |\nabla u_{\varepsilon_i}|^2} \quad \text{on} \quad \{(x,t) \in \Omega \times (0,\infty) : |\nabla u_{\varepsilon_i}| \neq 0\}. \tag{4.7}$$

Here, we have used the Cauchy-Schwarz inequality

$$|\nabla^2 u_{\varepsilon_i}|^2 |\nabla u_{\varepsilon_i}|^2 \geq |\nabla^2 u_{\varepsilon_i} \nabla u_{\varepsilon_i}|^2.$$

We note that ξ_{ε_i} is obviously non-positive if $|\nabla u_{\varepsilon_i}| = 0$. Therefore, if Ω is convex, the maximum principle for the discrepancy function works well by virtue of (4.4) and (4.7), and Mizuno and Tonegawa [10] proved the uniformly boundedness $\xi_{\varepsilon_i} \leq C$ for some $C > 0$ being independent of x, t and ε_i via this argument.

4.3 Upper bound of discrepancy function on NON-CONVEX domains

Our aim is to extend the convergence theory in [10] to remove the assumption of the convexity of the domain. Therefore, we estimate the upper bound of the discrepancy function without the assumption of the convexity of the domain as the following.

Proposition 4.5 *There exists a constant c_8 depending only on $n, \kappa, c_1, c_2, c_4, W$ and Ω such that*

$$\sup_{\Omega \times [0,\infty)} \frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2} - \frac{W(u_{\varepsilon_i})}{\varepsilon_i} \leq c_8 \varepsilon_i^{-\lambda} \tag{4.8}$$

for any $0 < \varepsilon_i < 1$, where λ is the constant in the assumption (A4).

In the following, we assume $3/5 < \lambda < 1$ for simplicity. We define a function $\phi_{\varepsilon_i} \in C^\infty(\overline{\Omega})$ based on the distance function $\mathrm{dist}(\partial\Omega, \cdot)$ from $\partial\Omega$ by

$$\phi_{\varepsilon_i}(x) := \kappa(c_7^2 + 1)\psi(\mathrm{dist}(\partial\Omega, x)/\varepsilon_i),$$

where $\psi \in C^\infty([0,\infty);\mathbb{R}^+)$ satisfies

$$\psi(s) = s \quad \text{for } s \in [0, c_4/2], \quad \psi'(s) = 0 \quad \text{for } s \in [c_4, \infty), \quad |\psi'| \leq 1, \quad |\psi''| \leq 4/c_4.$$

By applying the standard estimates of the derivatives of the distance function $\mathrm{dist}(\partial\Omega, \cdot)$, we obtain

$$\frac{\partial}{\partial \nu}\phi_{\varepsilon_i} = -\frac{\kappa(c_7^2+1)}{\varepsilon_i} \quad \text{on} \quad \partial\Omega \tag{4.9}$$

and

$$0 < \phi_{\varepsilon_i} \leq M_1, \quad |\nabla \phi_{\varepsilon_i}| \leq M_1/\varepsilon_i, \quad \Delta\phi_{\varepsilon_i} \leq M_1/\varepsilon_i^2 \quad \text{in} \quad \Omega \tag{4.10}$$

for some positive constant M_1 depending only on n, κ, c_4 and c_7. Define $G \in C^\infty(\mathbb{R})$ by $G(s) := 1 - (s-\gamma)^2/8$, where γ is the constant in the assumption (W2). We note that $G(u_{\varepsilon_i})$ satisfies

$$0 < G(u_{\varepsilon_i}) < 1, \quad G'(u_{\varepsilon_i})W'(u_{\varepsilon_i}) \geq 0, \quad G''(u_{\varepsilon_i}) = -\frac{1}{4}. \tag{4.11}$$

Let $\tilde{\xi}_{\varepsilon_i}$ is a modified discrepancy function defined by

$$\tilde{\xi}_{\varepsilon_i}(x,t) := \xi_{\varepsilon_i}(x,t) - \varepsilon_i^{-\lambda} G(u_{\varepsilon_i}) + \phi_{\varepsilon_i}(x).$$

Then, by virtue of (4.10) and (4.11), we may see the equivalence

$$\xi_{\varepsilon_i} \lesssim \varepsilon_i^{-\lambda} \quad \text{in} \quad \Omega \times [0,\infty) \iff \tilde{\xi}_{\varepsilon_i} \lesssim \varepsilon_i^{-\lambda} \quad \text{in} \quad \Omega \times [0,\infty).$$

Therefore, it is sufficient to prove the estimate on the left hand side of the equivalence. By a similar argument for (4.7), we obtain

$$\begin{aligned}
\partial_t \tilde{\xi}_{\varepsilon_i} - \Delta \tilde{\xi}_{\varepsilon_i} \leq & -\frac{2\langle \nabla \tilde{\xi}_{\varepsilon_i}, (W'(u_{\varepsilon_i}) + \varepsilon_i^{1-\lambda} G'(u_{\varepsilon_i}))\nabla u_{\varepsilon_i} - \nabla \phi\rangle}{\varepsilon_i^2 |\nabla u_{\varepsilon_i}|^2} - \varepsilon_i^{-\lambda-2} G'(u_{\varepsilon_i}) W'(u_{\varepsilon_i}) \\
& + \varepsilon_i^{-\lambda} G''(u_{\varepsilon_i}) |\nabla u_{\varepsilon_i}|^2 + \frac{2(W'(u_{\varepsilon_i}) + \varepsilon_i^{1-\lambda} G'(u_{\varepsilon_i}))}{\varepsilon_i^2 |\nabla u_{\varepsilon_i}|^2} \langle \nabla u_{\varepsilon_i}, \nabla\phi\rangle + \Delta\phi \\
& \text{on} \quad \{(x,t) \in \Omega \times (0,\infty) : |\nabla u_{\varepsilon_i}| \neq 0\}.
\end{aligned}$$

By applying the inequalities (4.10) and (4.11), it implies

$$\begin{aligned}
\partial_t \tilde{\xi}_{\varepsilon_i} - \Delta \tilde{\xi}_{\varepsilon_i} \leq & -\frac{2\langle \nabla \tilde{\xi}_{\varepsilon_i}, (W'(u_{\varepsilon_i}) + \varepsilon_i^{1-\lambda} G'(u_{\varepsilon_i}))\nabla u_{\varepsilon_i} - \nabla \phi\rangle}{\varepsilon_i^2 |\nabla u_{\varepsilon_i}|^2} \\
& + \frac{M_2}{\varepsilon_i^3 |\nabla u_{\varepsilon_i}|} - \frac{\varepsilon_i^{-\lambda}}{4} |\nabla u_{\varepsilon_i}|^2 + \frac{M_1}{\varepsilon_i^2} \quad \text{on} \quad \{(x,t) \in \Omega \times (0,\infty) : |\nabla u_{\varepsilon_i}| \neq 0\},
\end{aligned} \tag{4.12}$$

where M_2 is a positive constant depending only on M_1 and $\sup_{|s|\leq 1} |W'(s)|$. On the other hand, (4.6) and (4.9) imply

$$\frac{\partial}{\partial \nu} \tilde{\xi}_{\varepsilon_i} < 0 \quad \text{on} \quad \partial\Omega \times (0,\infty).$$

Therefore, we can apply a modified maximum principle for the modified discrepancy function. Indeed, if we assume

$$\tilde{\xi}_{\varepsilon_i}(y,\tau) = \sup_{(x,t)\in\Omega\times(0,T)} \tilde{\xi}_{\varepsilon_i}(x,t) = C\varepsilon_i^{-\lambda}$$

for sufficiently large $C > 0$ and a fixed time $T > 0$, then y is a interior point of Ω and (4.12) at the point (y,τ) shows

$$0 \leq \tilde{C}_1(\varepsilon_i^{\frac{\lambda-5}{2}} + \varepsilon_i^{-2}) - \tilde{C}_2\varepsilon_i^{-1-2\lambda},$$

where the constants $\tilde{C}_1$ and $\tilde{C}_2$ are positive and independent of ε_i and $T > 0$. However this is a contradiction for sufficiently large $i \in \mathbb{N}$ because the right hand side diverges to $-\infty$ as $i \to \infty$. Therefore, we have the conclusion.

Remark 4.6 *Roughly speaking, Lemma 4.2 and* (4.7) *give improved estimates for the order of ε_i, which are better than the estimates following from the scaling argument (see Remark 4.4). The inequality* (4.8) *corresponds to one kind of "interpolation inequality" between the inequalities* (4.6) *and* (4.7)*, thus the fractional exponent λ appears in* (4.8)*.*

4.4 Vanishing of the discrepancy

By applying the inequality (4.8), we can prove

$$\int_{t_1}^{t_2} e^{c_5(s-t)^{\frac{1}{4}}} \int_{\Omega} \frac{\rho_{1,(y,s)}(x,t) + \rho_{2,(y,s)}(x,t)}{2(s-t)} \xi_{\varepsilon_i}(x,t)\, dxdt \to 0 \quad \text{as} \quad i \to \infty$$

for $0 \leq t_1 < t_2 \leq s$. Therefore, we can take the limit $i \to \infty$ for (4.2). The proof is based on the argument by [15]. Here, we note that we have to modify the argument to include the reflection argument.

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