Upper bounds for higher-order Poincaré constants

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1 Introduction

Estimating eigenvalues of the Laplacian in terms of geometric quantities such as diameter, isoperimetric constants, etc were studied various mathematicians ([2]). In this survey we explain several results obtained in [9] and their background. In [9] the author together with Yohei Sakurai gave upper bounds of eigenvalues of the (p-)Laplacian in terms of subsets of compact Riemannian manifolds. Although these estimates also hold in the setting of weighted Riemannian manifolds we treat only unweighted Riemannian manifolds as usual for simplicity.

1.1 The case of closed Riemannian manifolds

Let M be a closed Riemannian manifold and μ be the Riemannian volume measure of M normalized as $\mu(M) = 1$. In order to estimate eigenvalues of the p-Laplacian, we introduce the *k-th order p-Poincaré constant* as follows:

$$\nu_{k,p}(M) := \inf_{L_k} \sup_{\phi \in L_k \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, d\mu}{\int_M |\phi - \int_M \phi \, d\mu|^p \, d\mu}. \tag{1}$$

Here L_k runs over k-dimensional subspaces in the $(1,p)$ Sobolev space $W^{1,p}(M)$ which do not contain nonzero constant functions. In the case of $p = 2$ $\nu_{k,2}(M)$ coincides with the k-th nontrivial eigenvalue of the Laplacian $\lambda_k(M)$. Also $\nu_{1,1}(M)$ is equivalent to the *Cheeger isoperimetric constant* (see Section 4). In the case of $p > 1$ the above p-Pincaré constants are closely related with eigenvalues of the p-Laplacian (see Section 4).

For a family $\{A_i\}_{i=0}^k$ of subsets of M we set

$$\mathcal{D}(\{A_i\}) := \min_{i \neq j} d(A_i, A_j),$$

where $d(A_i, A_j) := \inf\{d(x,y) \mid x \in A_i, y \in A_j\}$.

Theorem 1.1. *Let M be a closed Riemannian manifold. For a family $\{A_i\}_{i=0}^k$ of Borel subsets of M we have*

$$\nu_{k,p}(M)^{\frac{1}{p}} \leq \frac{2}{\mathcal{D}(\{A_i\})} \max_{i=0,\dots,k} \log \frac{e(1 - \sum_{j \neq i} \mu(A_j))}{\mu(A_i)}. \tag{2}$$

1.2 The case of compact Riemannian manifolds with boundary

Let M be a compact Riemannian manifold with boundary.

We define the *p-th Dirichlet p-Poincaré constant* as

$$\nu_{k,p}^D(M) := \inf_{L_k} \sup_{\phi \in L_k \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, d\mu}{\int_M |\phi|^p \, d\mu}, \tag{3}$$

where L_k runs over k-dimensional subspaces of the $(1,p)$ Sobolev space $W_0^{1,p}(M)$. When p = 2, this constant is equal to the k-th Dirichlet eigenvalue of the Laplacian. When $p = 2$, this constant is equal to the k-th Dirichlet eigenvalue of the Laplacian. For $p \in (1, \infty)$ the value $\nu_{1,p}^D(M)$ coincides with the first Dirichlet eigenvalue of the p-Laplacian and $\nu_{1,1}^D(M)$ the Dirichlet isoperimetric constant (see Section 4).

For a family $\{A_i\}_{i=1}^k$ of subsets of M we define

$$\mathcal{D}^\partial\big(\{A_i\}\big) := \min\{ \min_{i \neq j} d(A_i, A_j), \ \min_i d(A_i, \partial M) \}. \tag{4}$$

Theorem 1.2. *Let M be a compact Riemannian manifold with boundary. For a family $\{A_i\}_{i=1}^k$ of Borel subsets of M we have*

$$\nu_{k,p}^D(M)^{\frac{1}{p}} \leq \frac{2}{\mathcal{D}^\partial(\{A_i\})} \max_{i=1,\ldots,k} \log \frac{e(1 - \sum_{j \neq i} \mu(A_j))}{\mu(A_i)}. \tag{5}$$

Before we explain the proof of Theorems 1.1 and 1.2 we explain the former results.

2 Results by Gromov-Milman and Chung-Grigory'an-Yau

2.1 The result by Gromov-Milman

Gromov-Milman obtained a similar result of Theorem 1.1 in the case of $k = 1$ and $p = 2$. The result asserts in terms of the concentration inequality and we will see that it is essentially equivalent to Theorem 1.1 in the case where $k = 1$ and $p = 2$. In the proof of Theorems 1.1 and 1.2 we use their argument. For a subset A of a Riemmanian manifold M and a positive number $r > 0$ $U_r(A)$ denotes the r-neighborhood of A.

Theorem 2.1 ([13, Theorem 4.1], [16, Theorem 3.1])**.** *Let M be a closed Riemannian manifold and A be a subset of M such that $\mu(A) \geq 1/2$. For $r > 0$ we have*

$$\mu(M \setminus U_r(A)) \leq e^{-\sqrt{\lambda_1(M)}r/3}. \tag{6}$$

Sketch. For a subset B of M we can show the following inequality by using the min-max principle:

$$\mu(M \setminus U_\varepsilon(B)) \leq \Big(1 + \frac{\lambda_1(M)\varepsilon^2}{2}\Big)^{-1} \mu(M \setminus B) \tag{7}$$

([16, Theorem 3.1]). Set $\varepsilon := \sqrt{2/\lambda_1(M)}$. If $\varepsilon < r$ we take a natural number k such that $k\varepsilon \le r < (k+1)\varepsilon$. Using (7) repeatedly we get

$$\begin{aligned}\mu(M \setminus U_r(A)) &\le \mu(M \setminus U_{k\varepsilon}(A)) \\ &\le 2^{-1}\mu(M \setminus U_{(k-1)\varepsilon}(A)) \\ &\quad \cdots \\ &\le 2^{-k}\mu(M \setminus A) \\ &\le 2^{-k-1} \\ &\le 2^{-\varepsilon^{-1}r} \\ &\le e^{-\sqrt{\lambda_1(M)}r/3}.\end{aligned}$$

If $r \le \varepsilon$ we have

$$\mu(M \setminus U_r(A)) \le 2^{-1} \le 2^{-\varepsilon^{-1}r} \le e^{-\sqrt{\lambda_1(M)}r/3}.$$

This completes the proof. □

For A in the claim of the above theorem we set $B := M \setminus U_r(A)$. Then (6) can be written as

$$\lambda_1(M) \le \frac{3}{d(A,B)} \log \frac{1}{\mu(B)} = \frac{3}{d(A,B)} \max\Big\{ \log \frac{1}{\mu(A)}, \log \frac{1}{\mu(B)} \Big\}. \tag{8}$$

This is similar to Theorem 1.1 in the case of $k = 1$. One might think that A is constrained as $\mu(A) \ge 1/2$ but if (6) holds then (8) holds for any A, B up to universal constants. Conversely (8) implies the exponential concentration inequality (6). Refer to [7, Lemma 4.3].

In the case where $p \ge 1$ an exponential concentration inequality in terms of $\nu_{1,p}(M)^{1/p}$ is known, for instance, see [17, Remark2.8] (One can fix the above proof to get the inequality).

2.2 The result by Chung-Grigoriy'an-Yau

In this subsection we explain the proof of the following result by Chung-Grigoriy'an-Yau. The proof is based on some heat kernel estimate.

Theorem 2.2 ([3, 4], [10, Theorem 12.5])**.** Let M be a closed Riemannian manifold and $A_0, A_1, \cdots, A_k$ be subsets of M. Then we have

$$\lambda_k(M)^{\frac{1}{2}} \le \frac{1}{\mathcal{D}(\{A_i\})} \max_{i \ne j} \log \frac{e}{\mu(A_i)\mu(A_j)}. \tag{9}$$

Sketch. We use the Davis-Gaffney type heat kernel estimate and the heat kernel expansion Laplacian. Here, the Davis-Gaffney (integral) heat kernel is

$$\int_{A_i}\int_{A_j} p_t(x,y)d\mu(x)d\mu(y) \le \sqrt{\mu(A_i)\mu(A_j)}e^{-\frac{d(A_i,A_j)^2}{4t}} \tag{10}$$

([10, Corollary 12.4]). Also the heat kernel expansion means

$$p_t(x,y) = \sum_{l=0}^{\infty} e^{-\lambda_l t}\varphi_l(x)\varphi_l(y) \tag{11}$$

([10, Theorem 10.13]), where φ_l is the eigenfunction which corresponds to λ_l and normalized as $\int_M \varphi_l^2 d\mu = 1$. Subsetituting (11) to the left-hand side of (10) we obtain

$$\begin{aligned} \int_{A_i}\int_{A_j} p_t(x,y)d\mu(x)d\mu(y) &= \sum_{l=0}^{\infty} e^{-\lambda_l t}\int_{A_i}\varphi_l(x)d\mu(x)\int_{A_j}\varphi_l(y)d\mu(y) \\ &\geq \mu(A_i)\mu(A_j) + \sum_{l=1}^{k-1} e^{-\lambda_l t}\int_{A_i}\varphi_l(x)d\mu(x)\int_{A_j}\varphi_l(y)d\mu(y) \\ &\quad - \|1_{A_i}\|_{L^2(\mu)}\|1_{A_j}\|_{L^2(\mu)}e^{-\lambda_k t}. \end{aligned} \tag{12}$$

Here we used that $\varphi_0 \equiv 1$. Chung-Grigory'an-Yau proved that the second term of the right-hanf side of the above inequality is nonnegative for some pair i, j. In fact, define an inner product of $\mathbb{R}^{k-1}$ as

$$(u,v) := \sum_{l=1}^{k-1} e^{-\lambda_l t}u_i v_i.$$

Then one can find a pair among $k+1$ vectors in $\mathbb{R}^{k-1}$ whose inner product is nonnegative. For such a pair i, j such that the second term of the right-hand side of (12) is nonnegative the heat kernel estimate (10) implies

$$\sqrt{\mu(A_i)\mu(A_j)}e^{-\frac{d(A_i,A_j)^2}{4t}} \geq \mu(A_i)\mu(A_j) - \sqrt{\mu(A_i)\mu(A_j)}e^{-\lambda_k t}.$$

Choosing an appropriate t we obtain (9). □

In Proposition 2.2 of [11] Gozlan-Herry imposed the following assumption for k Borel subsets $A_1, A_2, \dots, A_k$ of M and obtained a similar statement as Theorems 1.1 and 2.2;

$$\mu(A_i) + \sum_{j=1}^{k}\mu(A_j) \geq 1 \quad (i = 1, 2, \cdots, k).$$

Under this assumption putting $A_0 := M \setminus U_r(\bigcup_{i=1}^k A_i)$ they showed the following estimate of the k-th eigenvalue of the Laplacian:

$$\lambda_k(M)^{\frac{1}{2}} \leq \frac{2}{\mathcal{D}(\{A_i\})}\phi\Big(\frac{1}{c}\log\frac{1-\sum_{i=1}^k \mu(A_i)}{\mu(A_0)}\Big), \tag{13}$$

where $\phi(x) := \max\{\sqrt{x}, x\}$ and $c > 0$ is some constant. In [9] we compare our estimate (2) with the estimates (9) and (13). Our inequality is better than their inequalities upon the choice of $A_0, A_1, \cdots, A_k$. . Comparing (9) the numerator in logarithm of our inequality (2) can be small for some families $\{A_i\}$. From this one can show the sharpness of our inequality (2) with respect to the order of k ([9, Subsection 6.1]). However, we do not know whether the following inequality holds or not:

$$\nu_{k,p}(M)^{1/p} \lesssim \frac{1}{\mathcal{D}(\{A_i\}) \log k} \max_i \log \frac{1}{\mu(A_i)}.$$

The above estimate was conjectured in [6]. If the conjecture is affirmative we can prove some universal inequality for succesive eigenvalues of the Neumann Laplacian of bounded convex domains in Euclidean spaces, see [6, 7].

In the proof of 2.2 Chung-Grigor'yan-Yau used the fact that eigenfunctions correponding to the eigenvalue 0 are constant functions. Thus one cannot apply the method for Dirichlet eigenvalues of the Laplacian. Also it seems the heat kernel method cannot apply for the p-Laplacian in the case of $p \neq 2$. In [4] Chung-Grigor'yan-Yau generalize the method of the proof of Theorem 2.2.

3 Outline of the proof of Theorem 1.1

Let M be a closed Riemannian manifold. In stead of working with $\nu_{k,p}(M)$ we consider the following quantity:

$$\widehat{\nu}_{k,p}(M) := \inf_{L_{k+1}} \sup_{\phi \in L_{k+1} \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, d\mu}{\int_M |\phi|^p \, d\mu}. \tag{14}$$

Here the infimum is taken all over $k+1$-dimensional subspace of $W^{1,p}(M)$. We find the follwing:

Proposition 3.1 ([9, Proposition 2.2])**.**

$$\nu_{k,p}(M) \leq \widehat{\nu}_{k,p}(M) \leq 2^p \, \nu_{k,p}(M).$$

Theorem 1.1 follows from the above proposition and the following theorem:

Theorem 3.2. *Let M be a closed Riemannian manifold. For a family $\{A_i\}_{i=0}^k$ of Borel subsets of M we have*

$$\widehat{\nu}_{k,p}(M)^{\frac{1}{p}} \leq \frac{2}{\mathcal{D}(\{A_i\})} \max_{\alpha=0,\dots,k} \log \frac{e(1 - \sum_{j \neq i} \mu(A_j))}{\mu(A_i)}.$$

In the proof of Theorem 3.2 we use the following two theorems and proposition:

Lemma 3.3 (Domain monotonicity)**.** *For a subdomain Ω of a closed Riemannian manifold we have*

$$\widehat{\nu}_{k,p}(M) \leq \widehat{\nu}^D_{k+1,p}(\Omega).$$

Lemma 3.4 (Domain decomposition principle)**.** *Let $\{\Omega_i\}_{i=0}^k$ be a family of pairwise disjoint subdomains of a closed Riemannian manifold M. Then we have*

$$\widehat{\nu}^D_{k+1,p}\left(\bigsqcup_{i=0}^k \Omega_i\right) \leq \max_{i=0,\dots,k} \widehat{\nu}^D_{1,p}(\Omega_i).$$

Proposition 3.5 (Boundary concentration inequality)**.** *Let M be a compact Riemannian manifold with boundary. Then for any $r > 0$ we have*

$$\mu(M \setminus U_r(\partial M)) \leq \exp(1 - \widehat{\nu}^D_{1,p}(M)^{\frac{1}{p}}\, r). \tag{15}$$

The idea of the above boundary concentration is Theorem 2.1. In the proof in stead of (7) we prove

$$(1 + \varepsilon^p\, \widehat{\nu}^D_{1,p}(M))\mu(M \setminus U_{r+\varepsilon}(\partial M)) \leq \mu(M \setminus U_r(\partial M)).$$

See [8], [9, Proposition 2.3].

Proof of Theorem 3.2. Let $A_0, A_1, \cdots, A_k$ be subsets of M and put $D := D(\{A_i\})$, $\Omega_i := U_{\frac{D}{2}}(A_i)$. Note that Ω_i are pairwise disjoint. Using Lemma 3.3, 3.4 we get

$$\widehat{\nu}_{k,p}(M) \leq \widehat{\nu}^D_{k,p}\Big(\coprod_{i=1}^{k} \Omega_i\Big) \leq \max_i \widehat{\nu}^D_{k,p}(\Omega_i). \tag{16}$$

Let $\widehat{\nu}^D_{k,p}(\Omega_{i_0}) = \max_i \widehat{\nu}^D_{k,p}(\Omega_i)$. Setting $\mu|_{\Omega_{i_0}} := (1/\mu(\Omega_{i_0}))\mu|_{\Omega_{i_0}}$ we have

$$\mu_{\Omega_{i_0}}(A_{i_0}) = \frac{\mu(A_{i_0})}{\mu(\Omega_{i_0})} \geq \frac{\mu(A_{i_0})}{1 - \sum_{i \neq i_0} \mu(A_i)} =: \alpha_{i_0}$$

If we assume $\mu_{\Omega_{i_0}}(\Omega_{i_0} \setminus U_r(\partial\Omega_{i_0})) < \alpha_{i_0}$ then we have $A_{i_0} \cap U_r(\partial\Omega_{i_0}) \neq \emptyset$ and $\frac{D}{2} \leq r$. From the boundary concentration inequality (15)

$$\mu_{\Omega_{i_0}}(\Omega_{i_0} \setminus U_r(\partial\Omega_{i_0})) \leq \exp(1 - \widehat{\nu}^D_{1,p}(\Omega_{i_0})^{\frac{1}{p}} r) < \alpha_{i_0}$$

holds as long as r satisfies

$$r > \frac{1}{\widehat{\nu}^D_{1,p}(\Omega_{i_0})^{\frac{1}{p}}} \log \frac{e}{\alpha_{i_0}}.$$

Thus (16) implies

$$\frac{D}{2} \leq \frac{1}{\widehat{\nu}^D_{1,p}(\Omega_{i_0})^{\frac{1}{p}}} \log \frac{e}{\alpha_{i_0}} \leq \frac{1}{\widehat{\nu}^D_{k,p}(M)^{\frac{1}{p}}} \log \frac{e}{\alpha_{i_0}}.$$

This completes the proof. □

The proof of Theorem 1.2 follows from the same argument and we omit it.

4 Relation to the isoperimetric constant and eigenvalues of the p-Laplacian

Let M be a closed Riemannian manifold and $p \in (1, \infty)$. We define the p-Laplacian Δ_p as

$$\Delta_p := -\operatorname{div}\left(\|\nabla \cdot\|^{p-2} \nabla \cdot\right).$$

In the case of $p = 2$ this coincises with the usual Laplacian. If $p \neq 2$ then Δ_p becomes a nonlinear operator (refer to [1] for the spectral theory of the p-Laplacian). If $\lambda_{1,p}(M)$ denotes the least positive eigenvalue of Δ_p then we have the following variational formula ([12, Corollary 2.1]):

$$\lambda_{1,p}(M) = \inf_{\phi \in W^{1,p}(M) \setminus \mathcal{C}} \frac{\int_M \|\nabla \phi\|^p \, d\mu}{\inf_{c \in \mathbb{R}} \int_M \|\phi - c\|^p \, d\mu},$$

where $\mathcal{C}$ is the set of constant functions on M.

It turns out that

$$\lambda_{1,p}(M)^{\frac{1}{p}} \simeq \nu_{1,p}(M)^{\frac{1}{p}} \simeq \widehat{\nu}_{1,p}(M)^{\frac{1}{p}},$$

where $C_1 \simeq C_2$ means that C_1 and C_2 are equivalent up to universal constants.

Let us compare $\widehat{\nu}_{k,p}(M)$ with eigenvalues of the p-Laplacian in the case where $p \neq 2$. Refer to [1] for eigenvalues of the p-Laplacian. We explain some construction of eigenvalues of the p-Laplacian. Let us $\mathcal{B}_p$ denote the set of symmetric closed subsets of $\{\phi \in W^{1,p}(M) \mid \|\phi\|_{L^p} = 1\}$. For $B \in \mathcal{B}_p$ its *g*enus $\gamma^+(B)$ is defined as the supremum of $l \geq 1$ such that there exists an odd continuous map from l dimensional Euclidean sphere $\mathbb{S}^l$ to B.

The *c*ogenus (*K*rasnosel'skii genus) $\gamma^-(B)$ is defined as the least $l \geq 1$ such that there exists an odd continuous map from B to $\mathbb{S}^l$ ([15]). Set

$$\lambda^{\pm}_{k,p}(M) := \inf_{B \in \mathcal{B}^{\pm}_{k+1,p}} \sup_{\phi \in B} \int_M \|\nabla \phi\|^p d\mu,$$

where $\mathcal{B}^{\pm}_{k+1,p} := \{B \in \mathcal{B}_p \mid \gamma^{\pm}(B) \geq k+1\}$. These two sequences $\{\lambda^{\pm}_{k,p}(M)\}_k$ are known to be increasing and unbounded sequences of eigenvalues of the p-Laplacian and it is known that $\lambda_{1,p}(M)$ coincides with $\lambda^{+}_{1,p}(M)$ ([5]). $\lambda^{+}_{k,p}(M)$ is introduced by Drábek-Robinson ([5]). From the Borsuk-Ulam theorem one can find the following relation between these quantities and $\widehat{\nu}_{k,p}(M)$:

$$\lambda^{\pm}_{k,p}(M) \leq \widehat{\nu}_{k,p}(M)$$

(see [9, Remark 2.1]). Theorem 3.2 implies the following.

Theorem 4.1. *Let M be a closed Riemannian manifold. For a family $\{A_i\}_{i=0}^k$ of Borel subsets of M we have*

$$\nu^{\pm}_{k,p}(M)^{\frac{1}{p}} \leq \frac{2}{\mathcal{D}(\{A_i\})} \max_{\alpha=0,\ldots,k} \log \frac{e(1 - \sum_{j \neq i} \mu(A_j))}{\mu(A_i)}.$$

For a subset A of a closed Riemannian manifold M we define its (Minkowski) boundary measure as

$$\mu^+(A) := \liminf_{r \to 0} \frac{\mu(U_r(A)) - \mu(A)}{r}.$$

The Cheeger's isoperimetric constant of M is defined as

$$h(M) := \inf_{A,B} \max \Big\{ \frac{\mu^+(A)}{\mu(A)}, \frac{\mu^+(B)}{\mu(B)} \Big\},$$

where the infimum is taken over pairwise disjoint Borel subsets A, B of M. Due to Feferer-Fleming $h(M)$ coincides with the best constant of the (1,1)-Poincaré inequality ([17, Lemma 2.2]), and hence $\nu_{1,1}(M) \simeq h(M)$.

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